Stable Conjugacy: Definitions and Lemmas*

The purpose of the present note is to introduce some notions useful for applications of the trace formula to the study of the principle of functoriality, including base change, and to the study of zeta-functions of Shimura varieties. In order to avoid disconcerting technical digressions I shall work with reductive groups over fields of characteristic zero, but the second assumption is only a matter of convenience, for the problems caused by inseparability are not serious.

The difficulties with which trace formula confronts us are manifold. Most of them arise from the non-compactness of the quotient and will not concern us here. Others are primarily arithmetic and occur even when the quotient is compact. To see how they arise, we consider a typical problem.

Suppose G is a quasi-split group over a global field F and G' is a group obtained from G by an inner twisting. Thus there is an isomorphism φ: $G' \to G$ defined over a finite Galois extension K of F which is such that $\sigma(\psi)\psi^{-1}$ is inner for all $\sigma \in \mathrm{Gal}(K/F)$. Apart from the contributions from the cusps, the trace formula for G, insofar as it is available, expressed the trace as a sum over semi-simple elliptic conjugacy classes, and the trace formula for G' expresses the trace as a sum over the semi-simple elliptic conjugacy classes of $G'(F)$. The traces of which we speak are those of $r(f)$ or of $r'(f')$ where f and f' are suitable functions on $G(\mathbf{A}(F))$ or $G'(\mathbf{A}(F))$ and r and r' are representations of $G(\mathbf{A}(F))$ and $G'(\mathbf{A}(F))$ on suitable spaces of automorphic forms, which it will be safer not to define precisely.

In its naive form the principle of functoriality suggests, and even affirms, that there is an injection of the set of automorphic representations of $G'(\mathbf{A}(F))$ into the set of automorphic representations of $G(\mathbf{A}(F))$. This is not so, and if we attempt to prove it by following the standard paradigm ([6], §16), we will discover why. We must show that the traces of $r(f)$ and $r(f')$ are equal for suitable test functions. The best procedure is to consider first the contributions from the elliptic conjugacy classes, and then, confidence gained, to pass to the cuspidal terms or, our misapprehensions revealed, to modify our expectations.

To compare the two traces one considers the two trace formulae and compares them term-by-term. If γ' is a semi-simple element in $G'(F)$ then the conjugacy class of $\psi(\gamma')$ is defined over F because $\sigma(\psi(\gamma')) = \sigma(\psi)\psi^{-1}(\psi(\gamma'))$ and $\sigma(\psi)\psi^{-1}$ is inner. A theorem of Steinberg [11] then assures us that the conjugacy class of $\psi(\gamma')$ in $G(\bar{F})$ contains an element γ in $G(F)$. If conjugacy within $G(\bar{F})$, called *stable* conjugacy, is the same as conjugacy within $G(F)$, and if conjugacy within $G'(\bar{F})$ is the same as conjugacy within $G'(F)$ then we obtain an injection of the elliptic conjugacy classes of $G'(F)$ into the elliptic conjugacy classes of $G(F)$ and can hope, by means of a supplementary study of the local harmonic analysis, to show that the terms corresponding to associated classes $\{\gamma\}$ and $\{\gamma'\}$ are equal, and that, for the functions being tested, the term of the trace formula

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corresponding to a class $\{\gamma\}$ not associated to a conjugacy class in $G'(F)$ is zero. This is the method used for $\mathrm{GL}(2)$ [6] and $\mathrm{GL}(3)$ [4], and one expects that it will eventually deal with $\mathrm{GL}(n)$.

For other groups stable conjugacy will be different from conjugacy, but at first glance this appears to be no serious obstacle. One should simply group together those terms corresponding to the conjugacy classes lying within a stable conjugacy class, obtaining thereby sums over stable conjugacy classes which can then be compared term-by-term. But the comparison of terms has to be carried out by an analysis of local orbital integrals to which the sum over a global stable conjugacy class is not directly amenable. Indeed the two terms to be compared are unlikely to be equal. A further adelic stabilization is necessary, but this can only be done by adding terms not present in the trace formula, and so they must be again subtracted, as an error term. The fully stabilized trace formulae will probably be amenable to comparison by a local study of orbital integrals, but a supplementary analysis of the error term is now necessary.

It may be possible to effect this by a procedure which may strike the more prosaic of our readers as extravagant. Regarding the stabilized trace formula as basic, we try to express the error term as a sum of stabilized trace formulae for lower-dimensional groups H, whose representation theory is related to that of G by the principle of functoriality.

All this will take time, and the efforts of more than one. My purpose here is simply to give the definitions of the groups H which intervene in the error term, together with their elementary properties. The definitions emerged from a close examination of a special case, $\mathrm{SL}(2)$, for which the procedure outlined has been carried out in detail ([7], [14]).

The groups H can also be introduced locally, where their purpose is to reduce the harmonic analysis of invariant distributions to the analysis of stably invariant distributions, and the local problems must be solved previously to, or simultaneously with, the global problems. For $\mathrm{SL}(2)$ they are either easy or had already been treated. For other groups this is not so, and even over the field of real numbers they are novel and difficult, but are yielding to the efforts of Shelstad ([12], [13]), whose work does much to dispel our doubts about the value of the definitions below.

At first F can be any field of characteristic 0 and G a reductive group over it. Let $T = T^G$ be a Cartan subgroup of G. Let $\mathfrak{A}(T)$ or $\mathfrak{A}(T, F)$ be the set of all g in $G(\bar{F})$ for which $T' = g^{-1}Tg$ and the morphism $t \to t' = g^{-1}tg$ are both defined over F and let

$$\mathfrak{D}(T, F) = \mathfrak{D}(T) = T(\bar{F})\backslash\mathfrak{A}(T)/G(F)\ .$$

An element g in $G(\bar{F})$ lies in $\mathfrak{A}(T)$ if and only if $a_\sigma = \sigma(g)g^{-1}$ lies in $T(\bar{F})$ for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$. The collection $\{a_\sigma \mid \sigma \in \mathrm{Gal}(\bar{F}/F)\}$ defines a cohomology class in $H^1(F, T)$ and the map $g \to \{a_\sigma\}$ yields an injection

$$\mathfrak{D}(T) \hookrightarrow H^1(F, T)\ .$$

The image is the kernel of

$$H^1(F,T) \to H^1(F,G)$$

and is not always a group. If $T' = g^{-1}Tg$ with $g \in \mathfrak{A}(T)$ then T and T' are said to be *stably conjugate*. The set $\mathfrak{D}(T)$ parametrizes the conjugacy classes within the stable conjugacy class of T.

If G_{sc} is the simply-connected covering group of the derived group of G and T_{sc} the inverse image of T in G_{sc} then

$$\mathfrak{D}(T_{\mathrm{sc}}) \to \mathfrak{D}(T)$$

is surjective. We define $\mathfrak{E}(T)$ or $\mathfrak{E}(T,F)$ to be the image of $H^1(F,T_{\mathrm{sc}})$ in $H^1(F,T)$. It is a group and $\mathfrak{D}(T)$ is a subset of it. If F is local and non-archimedean then $H^1(F,G_{\mathrm{sc}}) = \{1\}$ and $\mathfrak{D}(T) = \mathfrak{E}(T)$.

Let $X_*(T)$ and $X_*(T_{\mathrm{sc}})$ be the lattices of coweights of T and T_{sc}. $X_*(T_{\mathrm{sc}})$ may be identified with the sublattice of $X_*(T)$ generated by the coroots. If E is a local field and K a large but finite Galois extension then, by the Tate-Nakayama theory, $\mathfrak{E}(T)$ is canonically isomorphic to the quotient of

$$\left\{\lambda \in X_*(T_{\mathrm{sc}}) \middle| \sum_{\mathrm{Gal}(K/F)} \omega_{T/G}(\sigma)\lambda = 0\right\}$$

by

$$\left\{\lambda \in X_*(T_{\mathrm{sc}}) \middle| \lambda = \sum_{\mathrm{Gal}(K/F)} \omega_{T/G}(\sigma)\mu(\sigma) - \mu(\sigma),\ \mu(\sigma) \in X_*(T)\right\}\ .$$

Here $\omega_{T/G}(\sigma)$ is the natural action of σ on $X_*(T)$. If F is any field we let κ be a homomorphism of $X_*(T_{\mathrm{sc}})$ into $\mathbf{C}^\times$ which is 1 on the second of these modules.

I am now going to associate to the pair $(T,\kappa) = (T^G,\kappa)$ a quasi-split group H over F, and a family $\{T^H,\varphi)\}$ where T^H is a Cartan subgroup of H, and $\varphi : T^G \to T^H$ is an isomorphism over F. If (T_1^H,φ_1) and (T_2^H,φ_2) are two pairs in the family then there is an h in $\mathfrak{A}(T_1^H)$ for which

$$T_2^H = h^{-1}T_1^H h$$

and

$$\varphi_2(t) = h^{-1}\varphi_1(t)h\ .$$

To define H one needs the associate group of [10], which following Borel [1] I denote ${}^LG$. Its connected component ${}^LG^0$ is furnished with a distinguished Borel subgroup ${}^LB^0$ and a distinguished Cartan subgroup ${}^LT^0$. The group ${}^LT^0$ is contained in ${}^LB^0$. To define ${}^LG^0$ concretely we need to choose an isomorphism ψ of G with a quasi-split G_1. G_1 is defined over F, but ψ is only defined over $\bar{F}$. We also need to choose a Borel subgroup B^{G_1} and a Cartan subgroup T^{G_1} lying in B^{G_1}, both groups being defined over F. Choose g_1 in $G_1(\bar{F})$ so that

$$\psi' = \mathrm{ad}\, g_1 \circ \psi$$

takes T^G to T^{G_1}. Then g_1 is determined up to left multiplication with an element of Norm (T^{G_1}).

ψ' also yields an isomorphism

$$\psi' : X_*(T^G) \to X_*(T^{G_1})$$

and by the construction of ${}^LG$

$$X_*(T^{G_1}) = X^*({}^LT^0)$$

if $X^*({}^LT^0)$ is the lattice of weights of ${}^LT^0$. Define κ' by

$$\kappa'(\psi'(\lambda)) = \kappa(\lambda)$$

and let ${}^LH^0$ be the connected subgroup of ${}^LG^0$ generated by ${}^LT^0$ and the one-parameter root groups $U_{\alpha^\vee}$ associated to $\alpha^\vee$ with

$$\kappa'(\alpha^\vee) = 0 \ .$$

${}^LH^0$ is furnished with a distinguished Cartan subgroup, namely ${}^LT^0$, and a distinguished Borel subgroup

$${}^LH^0 \cap {}^LB^0 \ .$$

We transfer the operators $\omega_{T/G}(\sigma)$ to $X_*({}^LT^0)$ and write

$$\omega_{T/G}(\sigma) = \omega^1(\sigma)\omega^2(\sigma)$$

where $\omega^2(\sigma)$ is given by an element of the Weyl group of ${}^LT^0$ in ${}^LH^0$ and $\omega^1(\sigma)$ leaves the set of positive roots of ${}^LT^0$ in ${}^LH^0$ invariant. If for each simple root $X_{\alpha^\vee}$ we choose the $X_{\alpha^\vee} \neq 0$ in the Lie algebra of $U_{\alpha^\vee}$ used in the definition of ${}^LG$ we may associate to $\omega^1(\sigma)$ a unique automorphism of ${}^LH^0$, again denoted by $\omega^1(\sigma)$, with the following properties:

$$\omega^1(\sigma)\lambda(\omega^1(\sigma)t) = \lambda(t),\ \ \lambda \in X^*({}^LT^0),\ \ t \in {}^L(T^0);\ \omega^1(\sigma)X_{\alpha^\vee} = X_{\omega^1(\sigma)\alpha^\vee} \ .$$

If we choose the Galois extension K of F sufficiently large, then $\sigma \to \omega^1(\sigma)$ is a homomorphism of $\mathrm{Gal}(K/F)$ into the group of automorphisms of ${}^LH^0$. By means of $W_{K/F} \to \mathrm{Gal}(K/F)$ we let the Weil group act and form the semi-direct product

$${}^LH = {}^LH^0 \times W_{K/F} \ .$$

As in [10] we associate to ${}^LH^0$, ${}^LH^0 \cap {}^LB^0$, ${}^LT^0$, $\{X_{\alpha^\vee}\}$, and $\{\omega^1(\sigma)\}$ a quasi-split group H, furnished with a Cartan subgroup T_0^H, and a Borel subgroup B^H, all defined over F, so that

$$X_*(T_0^H) = X^*({}^LT^0)$$

and

$$\omega_{T_0^H/H}(\sigma) = \omega^1(\sigma) \ .$$

The concatenation of

$$X_*(T^G) \to X_*(T^{G_1}) = X^*({}^LT^0) = X_*(T_0^H)$$

yields an isomorphism

$$\varphi' : X_*(T^G) \to X_*(T_0^H)$$

and hence an isomorphism

$$\varphi' : T^G \to T_0^H \ .$$

However φ' is not defined over F. By a theorem of Steinberg [11] there is at least one Cartan subgroup T^H of H over F given by

$$T^H = h^{-1}T_0^H h, \qquad h \in H(\bar{F})$$

such that the composition

$$\varphi : T^G \xrightarrow{\varphi'} T_0^H \xrightarrow{\mathrm{ad}\, h^{-1}} T^H$$

is defined over F. The pairs (T^H, φ) obtained in this way form the family I mentioned. I forego for now a close examination of the manner in which H and the family $\{(T^H, \varphi)\}$ depend on the choices required for their construction, merely stressing that once ψ is fixed one must still choose g_1; hence $H = H(T, \kappa, g_1)$. The triple (T_2, κ_2, g_2) will be called a *companion* to (T_1, κ_1, g_1) if the associated homomorphisms κ_1', κ_2' of $X_*(T_{\mathrm{sc}}^{G_1})$ to $\mathbf{C}^\times$ and the associated operators $\omega_1^1(\sigma)$, $\omega_2^1(\sigma)$ on $X^*({}^LT^0)$ are equal. Then $H(T_1, \kappa_1, g_1)$ and $H(T_2, \kappa_2, g_2)$ are canonically isomorphic.

By its construction we have an imbedding $\xi : {}^LH^0 \hookrightarrow {}^LG^0$. In order to bring the principle of functoriality in the dual group into play we need to extend it to an imbedding $\xi : {}^LH \hookrightarrow {}^LG$ which commutes with the projections on $W_{K/F}$. This is not always possible, but it is possible in sufficiently many cases that the groups H can be used for the purpose for which they were intended, the study of L-indistinguishability.

Proposition 1. *Suppose F is a global or a local field and the center of ${}^LG^0$ is connected. Then the imbedding $\xi : {}^LH^0 \hookrightarrow {}^LG^0$ extends to an imbedding $\xi : {}^LH \hookrightarrow {}^LG$ which commutes with the projections on $W_{K/F}$.*

I shall argue by induction on the dimension of ${}^LG^0$. The statement is certainly clear if the dimension is zero. The center of ${}^LG^0$ is connected if and only if the lattice $X_*(T_{\mathrm{sc}}^{G_1})$ is primitive in $X_*(T^{G_1})$, that is, if any rational linear combination of the coroots which is a coweight is in fact an integral linear combination.

There is an integer m such that the image of $X_*(T_{\mathrm{sc}}^{G_1})$ under $\lambda \to \kappa'(\lambda)^m$ is torsion free. Choose independent homomorphisms $\eta_1, \dots, \eta_s$ of $X_*(T_{\mathrm{sc}}^{G_1})$ into $\mathbf{Z}$ and complex numbers $\zeta_1, \dots, \zeta_s$ so that

$$\kappa'(\lambda)^m = \prod_{i=1}^{s} \zeta_i^{\eta_i(\lambda)} \ .$$

Let $\mathfrak{X}$ be the set of roots $\alpha^\vee$ for which the first non-zero $\eta_i(\alpha^\vee)$ is positive and $\mathfrak{X}_0 \subseteq \mathfrak{X}$ the set of roots $\alpha^\vee$ for which all $\eta_i(\alpha^\vee)$ are zero. The group ${}^LP^0$ generated by ${}^LT^0$ and the one-parameter root groups $U_{\alpha^\vee}$ for which $\alpha^\vee \in \mathfrak{X}$

is a parabolic subgroup of ${}^LG^0$ and a Levi factor ${}^LM^0$ is generated by ${}^LT^0$ and the $U_{\alpha^\vee}$ with $\alpha^\vee \in \mathfrak{X}_0$. Since $\mathfrak{X}$ is invariant under $\omega_{T/G}(\sigma)$, $\sigma \in \mathfrak{G}(\bar{F}/F)$, the normalizer ${}^LP$ of ${}^LP^0$ in ${}^LG$ is a parabolic subgroup of ${}^LG$ in the sense of [10]. The normalizer ${}^LM$ of ${}^LM^0$ in ${}^LG$ is itself an L-group. Certainly ${}^LH^0 \subseteq {}^LM^0$. Hence, if

$$dim({}^LM^0) < dim({}^LG^0)$$

and the center of ${}^LM^0$ is connected, we may apply induction. However, the center is connected because it is defined by

$$\alpha^\vee(t) = 0$$

for each simple root $\alpha^\vee$ of ${}^LT^0$ in ${}^LM^0$. These simple roots generate a primitive lattice in $X^*({}^LT^0) = X_*(T^{G_1})$.

The upshot of the preceding analysis is that we need only consider κ' that are of finite order. Some preparation is necessary.

Lemma 2. *Suppose R is an indecomposable, reduced root system and D a subset of R with the following two properties.*

(i) *If $\alpha^\vee$, $\beta^\vee$ lie in D then $\alpha^\vee - \beta^\vee$ is not a root.*

(ii) *Every root of R is an integral linear combination of the elements of D.*

Then D is either a base of R or a base together with the negative of the corresponding highest root.

This lemma is implicit in [3]. As one expects, a proof can also be extracted from the thesaurus of Bourbaki [2]. If $\alpha^\vee \neq \beta^\vee$ both lie in D then certainly $(\alpha^\vee, \beta^\vee) \leqq 0$. One defines a Coxeter matrix $(m_{\alpha^\vee,\beta^\vee})$, $\alpha^\vee, \beta^\vee$ in D, by taking

$$\pi - \frac{\pi}{m_{\alpha^\vee,\beta^\vee}}$$

to be the angle between $\alpha^\vee$, $\beta^\vee$. Let $\{e_{\alpha^\vee}\}$ be the standard basis of

$$E = \bigoplus_{\alpha^\vee \in D} \mathbf{R}$$

and define an inner product $B(\,\cdot\,,\,\cdot\,)$ on E by

$$B(e_{\alpha^\vee}, e_{\beta^\vee}) = \cos\left(\pi - \frac{\pi}{m_{\alpha^\vee,\beta^\vee}}\right) = \frac{(\alpha^\vee, \beta^\vee)}{(\alpha^\vee, \alpha^\vee)^{1/2}(\beta^\vee, \beta^\vee)^{1/2}}$$

Since

$$B\Big(\sum a(\alpha^\vee)e_{\alpha^\vee}, \sum a(\beta^\vee)e_{\beta^\vee}\Big) = \Big(\sum \frac{a(\alpha^\vee)}{(\alpha^\vee,\alpha^\vee)^{1/2}}\,\alpha^\vee,\ \sum \frac{b(\beta^\vee)}{(\beta^\vee,\beta^\vee)^{1/2}}\,\beta^\vee\Big)$$

the inner product is positive semi-definite.

Suppose the form is definite or, what amounts to the same thing, that the roots $\alpha_1^\vee, \ldots, \alpha_l^\vee$ of D are linearly independent. If $1 \leq k \leq l$ let R_k be the set of the roots in the real linear span V_k of $\{\alpha_1^\vee, \ldots, \alpha_k^\vee\}$. R_k is a root system and every element of R_k is an integral linear combination of $\alpha_1^\vee, \dots, \alpha_k^\vee$. We show by induction that $\alpha_1^\vee, \dots, \alpha_k^\vee$ is a base of R_k or, more precisely, that if we define an order on V_k by

$$\sum_{i=1}^{k} a_i \alpha_i^\vee > 0$$

if and only if the last non-zero a_i is positive, then $\alpha_1^\vee, \dots, \alpha_k^\vee$ are the minimal elements of R_k with respect to this order. This is clear for $k = 1$.

The induction assumption will be that the assertion is true for a given k regardless of the initial numeration of the roots in D. If $k < l$ it is clear that the minimal elements in R_{k+1} are then $\alpha_1^\vee, \dots, \alpha_k^\vee$ together with some $\beta^\vee$. Moreover

$$\alpha_{k+1}^\vee = a_1 \alpha_1^\vee + \cdots + a_k \alpha_k^\vee + a\beta^\vee$$

with $a_1, \dots, a_k$ and a integral and non-negative. Solving for $\beta^\vee$ we see that $a = 1$; so we write

$$a_{k+1}^\vee = \gamma^\vee + \beta^\vee$$

with

$$\gamma^\vee = a_1 \alpha_1^\vee + \cdots + a_k \alpha_k^\vee \ .$$

We suppose that $\gamma^\vee \neq 0$ and derive a contradiction. Since $\alpha_{k+1}^\vee - \alpha_j^\vee$, $1 \leqq j \leqq k$, is not a root, $\gamma^\vee = \alpha_{k+1}^\vee$ is. This implies that $k > 1$, for if k were 1 then a_1 would be 1 and $\alpha_2^\vee - a_1^\vee = \beta^\vee$ would be a root. We observe that

$$(\alpha_{k+1}^\vee, \alpha_{k+1}^\vee) = \sum_{i=1}^{k} a_i (\alpha_{k+1}^\vee, \alpha_i^\vee) + (\alpha_{k+1}^\vee, \beta^\vee) \leqq (\alpha_{k+1}^\vee, \beta^\vee) \ .$$

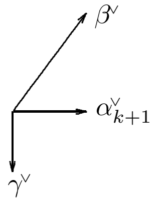
If $(\alpha_{k+1}^\vee, \alpha_{k+1}^\vee)$ were equal to $(\beta^\vee, \beta^\vee)$ we would conclude from this inequality that $\alpha_{k+1}^\vee = \beta^\vee$. However, this cannot be so, for $\gamma^\vee$ is supposed not to be zero. We infer, therefore, from the above inequality combined with the Schwarz inequality that

$$(\alpha_{k+1}^\vee, \alpha_{k+1}^\vee) < (\beta^\vee, \beta^\vee) \ .$$

Since the rank of R is greater than 2 and R is irreducible, we must have

$$2(\alpha_{k+1}^\vee, \alpha_{k+1}^\vee) = (\beta^\vee, \beta^\vee) \ .$$

The geometrical situation is:



Hence

$$(\alpha_{k+1}^\vee, \alpha_{k+1}^\vee) = (\alpha_{k+1}^\vee, \beta^\vee) \ .$$

Consequently

$$a_i(\alpha_{k+1}^\vee, \alpha_i^\vee) = 0$$

for $1 \leqq i \leqq k$ and

$$a_i(\gamma^\vee, \alpha_i^\vee) = -a_i(\beta^\vee, \alpha_i^\vee) \geqq 0 \ .$$

However, our initial assumption that our assertion is valid at the kth stage regardless of the initial numeration implies that each $a_i > 0$. Moreover, since $\gamma^\vee \neq 0$ there is one root $\alpha_i^\vee$ with $(\beta^\vee, \alpha_i^\vee) < 0$. Then

$$(\beta^\vee, \alpha_i^\vee) = -(\alpha_i^\vee, \alpha_i^\vee)$$

if $\alpha_i^\vee$ is short, and

$$(\beta^\vee, \alpha_i^\vee) = -\frac{1}{2}(\alpha_i^\vee, \alpha_i^\vee)$$

if $\alpha_i^\vee$ is long. However, $\gamma^\vee$ is short. Therefore,

$$(\gamma^\vee, \alpha_i^\vee) = \frac{1}{2}(\alpha_i^\vee, \alpha_i^\vee)$$

regardless of the length of $\alpha_i^\vee$. We conclude that $\alpha_i^\vee$ is long and that

$$0 < (\alpha_{k+1}^\vee, \beta^\vee) \leqq a_i(\alpha_i^\vee, \beta^\vee) + (\beta^\vee, \beta^\vee) = (1 - a_i/2)(\beta^\vee, \beta^\vee) \ .$$

It follows that $a_i = 1$.

There is another root $\alpha_j^\vee$, $1 \leqq j \leqq k$, with $(\alpha_j^\vee, \alpha_i^\vee) < 0$. Since the Dynkin diagram of R_{k+1} contains no cycles, $(\alpha_j^\vee, \beta^\vee) = 0$ and

$$0 = (\alpha_{k+1}^\vee, \alpha_i^\vee) \leqq a_j(\alpha_j^\vee, \alpha_i^\vee) + a_i(\alpha_i^\vee, \alpha_i^\vee) + (\alpha_i^\vee, \beta^\vee) = a_j(\alpha_j^\vee, \alpha_i^\vee) + \frac{1}{2}(\alpha_i^\vee, \alpha_i^\vee) \ .$$

Since

$$(\alpha_j^\vee, \alpha_i^\vee) = -\frac{1}{2}(\alpha_i^\vee, \alpha_i^\vee)$$

no matter whether $\alpha_j^\vee$ is long or short, we conclude that $a_j = 1$.

Suppose we have a path leading out from $\beta^\vee$ in the Dynkin diagram with at least three vertices besides $\beta^\vee$ in it. Suppose moreover that we have shown that the coefficient a_m of $\alpha_m^\vee$ is 1 for all vertices of the path except perhaps the last and that all vertices except perhaps for the last two are long. Let the last vertices be $\alpha_u^\vee, \alpha_v^\vee, \alpha_w^\vee$. Then

$$0 = (\alpha_{k+1}^\vee, \alpha_v^\vee) \leqq a_w(\alpha_w^\vee, \alpha_v^\vee) + (\alpha_v^\vee, \alpha_v^\vee) + (\alpha_u^\vee, \alpha_v^\vee) \ .$$

If $\alpha_v^\vee$ were short, then

$$(\alpha_v^\vee, \alpha_v^\vee) + (\alpha_u^\vee, \alpha_v^\vee) = 0$$

and the contradiction

$$0 \leqq (\alpha_w^\vee, \alpha_v^\vee) < 0$$

results. Thus $\alpha_v^\vee$ is long and

$$0 \leqq a_w(\alpha_w^\vee, \alpha_v^\vee) + \frac{1}{2}(\alpha_v^\vee, \alpha_v^\vee) \ .$$

We conclude that $a_w = 1$. By induction all the coefficients are 1. Then if $\alpha_m^\vee$ is an extreme point of the Dynkin diagram we infer from Cor. 3 to Prop. 19 of Chap. VI of [2] that $\alpha_{k+1}^\vee = \alpha_m^\vee$ is a root. This is a contradiction.

We must still prove the lemma when the form $B(\ \cdot\ ,\ \cdot\ )$ is degenerate. In general the collection D may be partitioned into subsets $D_1, \dots, D_r$ corresponding to the connected components of the graph of the Coxeter matrix we have introduced. If V_i is the space over $\mathbf{R}$ spanned by D_i then $V_1, \dots, V_r$ are mutually orthogonal and any root in V_i is a linear combination of elements of D and hence of D_i. Let $R_i = R \cap V_i$.

Suppose next that the graph of our Coxeter matrix is connected but that the form $B(\ \cdot\ ,\ \cdot\ )$ is degenerate. It is a consequence of Th. 4 of Chap. VI of [2] that the Coxeter matrix $(m_{\alpha^\vee,\beta^\vee})$ is defined by the Weyl group of a completed Dynkin diagram. Hence there is only one relation between the roots $\alpha_1^\vee, \dots, \alpha_l^\vee$ of D and with a suitable numeration it is

$$\left(\sum_1^{l-1} a_i \alpha_i^\vee\right) + \alpha_l^\vee = 0 \ .$$

We may remove $\alpha_l^\vee$ from D without destroying either of the two properties demanded of it. Since $\alpha_1^\vee, \dots, \alpha_{l-1}^\vee$ are linearly independent, the previous discussion implies that they form a base of R.

Returning to the general case, we see that we can select from any of the D_i a base for the corresponding R_i. Putting these bases together, we find a collection that satisfies (i) and (ii) of the lemma and is in addition linearly independent. We conclude from the first part of the proof that it is a base. Since R is indecomposable, we infer that $r = 1$. The lemma is now proved.

We return to the proof of Proposition 1, supposing now that κ' is of finite order m. Let

$$\zeta = e^{2\pi i/m} \ .$$

Define $\mathfrak{Y}_k$ for $0 \leqq k < m$ by

$$\mathfrak{Y}_k = \{\alpha^\vee | \kappa'(\alpha^\vee) = \zeta^k\} \ .$$

Let $\mathfrak{X}_0$ be the set of simple roots of ${}^LT^0$ in ${}^LH^0$ with respect to ${}^LH^0 \cap^L B^0$. If $\alpha^\vee, \beta^\vee$ lie in $\mathfrak{Y}_k$ write $\alpha^\vee < \beta^n$ if $\beta^\vee - \alpha^\vee$ is an integral linear combination of roots of $\mathfrak{X}_0$ with non-negative coefficients. Let $\mathfrak{Z}_k$ consist of those elements of $\mathfrak{Y}_k$ which are not integral linear combinations of elements in

$$\bigcup_0^{k-1} \mathfrak{Y}_j$$

and let $\mathfrak{X}_k$, $k \geqq 1$ consist of the minimal elements in $\mathfrak{Z}_k$. Clearly $\mathfrak{X}_0, \mathfrak{X}_1, \ldots, \mathfrak{X}_{m-1}$ are disjoint and span $X_*(T_{\mathrm{sc}}^{G_1})$ over **Z**. Moreover, each of these sets is invariant under $\omega^1(\sigma)$, $\sigma \in \mathfrak{G}(\bar{F}/F)$. If $\alpha^\vee \in \mathfrak{X}_k$, $\beta^\vee \in \mathfrak{X}_j$ and $j \leqq k$ then $\alpha^\vee - \beta^\vee$ is not a root. Suppose otherwise. If $0 < j < k$ then $\alpha^\vee - \beta^\vee \in \mathfrak{Y}_{k-j}$, and so $\alpha^\vee$ is not in $\mathfrak{Z}_k$. If $j = 0$ then $\alpha^\vee - \beta^\vee \in \mathfrak{Z}_k$ and $\alpha^\vee$ is not minimal. Lemma 2 may be applied to

$$D = \bigcup_0^{m-1} \mathfrak{X}_j$$

when ${}^L G^0$ is simple, and yields an important tool for the study of ${}^L H^0$. The Dynkin diagram of D together with the action $\omega^1(\sigma)$ of $\mathfrak{G}(\bar{F}/F)$ on it will be called the $diagram$ of (T, κ) or of (T, κ, g_1). Each vertex is labelled with an integer k, $0 \leqq k < m$.

Choose an $X_{\alpha^\vee}$ for each $\alpha^\vee$ in $\mathfrak{X}_0$. We denote a typical element of $W_{K/F}$ by w and its image in $\mathrm{Gal}(K/F)$ by σ. For each w we may choose $\xi'(w)$ in ${}^L G$ so that $\xi'(w)$ project to w in $W_{K/F}$ and so that

$$\begin{aligned} \xi'(w) t \xi'(w)^{-1} &= \omega^1(\sigma)(t), \qquad\qquad t \in {}^L T^0, \\ \xi'(w) X_{\alpha^\vee} \xi'(w)^{-1} &= X_{\omega^1(\sigma)\alpha^\vee} \ . \end{aligned}$$

$\xi'(w)$ is not uniquely determined, but we may modify it only by left multiplication with elements from Z, the center of ${}^L H^0$. Thus

$$\xi'(w_1)\xi'(w_2) = a_{w_1,w_2} \xi'(w_1 w_2)$$

with $a_{1,w_2} \in Z$. Clearly $\{a_{w_1 w_2}\}$ defines a 2-cocycle of $W_{K/F}$ with values in Z. Since, as a first try, we can even take $\xi'(w)$ to depend only on σ, it is continuous. Our problem is to show that it is trivial, that

$$a_{w_1,w_2} = b(w_1)\omega^{-1}(\sigma_1)(b(w_2))b^{-1}(w_1 w_2)$$

with $w \to b(w) \in Z(K)$ continuous.

The first step is to show that if we take $\xi'(w)$ to depend only on σ so that $\{a_{w_1,w_2}\} = \{a_{\sigma_1,\sigma_2}\}$ is a cocycle of $\mathrm{Gal}(K/F)$ with values in $Z(K)$ then it is trivial modulo $Z^0(K)$, Z^0 being the connected component of the identity in Z. Since the center of ${}^L G^0$ is connected, we may divide by it and assume that ${}^L G^0$ is adjoint. Then an application of Shapiro's lemma allows us to assume that ${}^L G^0$ is simple.

If the diagram of (T, κ) is ordinary and not extended then the roots in $\mathfrak{X}_0$ generate a primitive lattice in

$$X_*(T_{\mathrm{sc}}^{G_1})$$

and Z is connected. For now we take the diagram to be extended. We write the vertices of it as $\alpha^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee$ and the one relation as

$$\alpha^\vee + \sum_{i=1}^{l} a_i \alpha_i^\vee = 0 \ .$$

Here the a_i are positive integers. It is clear that Z will be connected unless $\alpha^\vee \in \mathfrak{X}_0$ and

$$\text{g.c.d.}\{a_i | \alpha_i^\vee \notin \mathfrak{X}_0\} > 1 \ .$$

We shall examine the possible diagrams individually. Given ${}^LG^0$, ${}^LB^0$, ${}^LT^0$ and $\{X_{\alpha^\vee} | \alpha^\vee \text{ simple}\}$, we forget their origins in G but take the group $A({}^LG^0, {}^LB^0, {}^LT^0, \{X_{\alpha^\vee}\})$ as on p. 4 of [10] and build the semi-direct product

$${}^LG' = {}^LG^0 \times A({}^LG^0, {}^LB^0, {}^LT^0, \{X_{\alpha^\vee}\}) \ .$$

Recall that $A({}^LG^0, {}^LB^0, {}^LT^0, \{X_{\alpha^\vee}\})$ is the group of automorphisms of ${}^LG^0$ which leave ${}^LB^0$ and ${}^LT^0$ invariant and permute the $X_{\alpha^\vee}$ amongst themselves. In addition we consider an extended Dynkin diagram of ${}^LT^0$ in ${}^LG^0$ with vertices $D = \{\alpha^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee\}$ where $\alpha_1^\vee, \ldots, \alpha_l^\vee$ are the simple positive roots with respect to another ordering of the roots than that defining ${}^LB^0$ and a subset $\mathfrak{X}_0$ of D whose elements are positive with respect to the original order. We suppose that $\alpha^\vee \in \mathfrak{X}_0$ and that

$$\text{g.c.d.}\{a_i | \alpha_i^\vee \notin \mathfrak{X}_0\} > 1 \ .$$

We let A be the subgroup of $\mathrm{Norm}_{{}^LG'}({}^LT^0)/{}^LT^0$ formed by those ω that leave D and $\mathfrak{X}_0$ invariant. We assign to each ω a representative $\epsilon(\omega)$ in $\mathrm{Norm}_{{}^LG'}({}^LT^0)$ so that

$$\begin{aligned} \epsilon(\omega) t \epsilon(\omega)^{-1} &= \omega(t), && t \in {}^LT^0, \\ \epsilon(\omega) X_{\alpha^\vee} \epsilon(\omega)^{-1} &= X_{\omega\alpha^\vee}, && \alpha^\vee \in \mathfrak{X}_0 \ . \end{aligned}$$

Then

$$\epsilon(\omega_1)\epsilon(\omega_2) = a_{\omega_1,\omega_2}\epsilon(\omega_1\omega_2) \ .$$

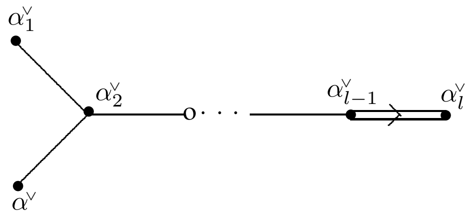
$\{a_{\omega_1,\omega_2}\}$ is a cocycle of A with coefficients in

$$Z = \{t \in {}^LT^0 | \alpha^\vee(t) = 1 \text{ for } \alpha^\vee \in \mathfrak{X}_0\} \ .$$

We shall show that this cocycle is trivial modulo Z^0. It is enough to show that its restriction to a Sylow subgroup A_p of A is trivial for each p.

We now check this by examining the possible diagrams one-by-one, excluding those for which $A = \{1\}$ or Z is connected because the assertion is then trivial. Diagrams of type A_l do not appear because all the a_i are then 1, and for diagrams of type E_8, F_4, G_2 the group A is $\{1\}$.

1) B_l, $l \geqq 3$.



$$\alpha^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_l^\vee = 0$$

Here $\alpha^\vee$ and $\alpha_1^\vee$ must belong to $\mathfrak{X}_0$ and $A = \{1, \omega\}$ where ω interchanges $\alpha^\vee$ and $\alpha_1^\vee$ and fixes $\alpha_2^\vee, \dots, \alpha_l^\vee$. With the standard representation of the root system of type B_l

$$\alpha_1^\vee = x_1 - x_2, \quad \alpha_2^\vee = x_2 - x_3, \dots, \alpha_{l-1}^\vee = x_{l-1} - x_l, \quad \alpha_l^\vee - x_l, \quad \alpha^\vee = -x_1 - x_2.$$

Thus ω is the reflection with respect to $\beta^\vee = x_1$. If β is the corresponding root then $\beta = 2x_1$ and $\langle \beta, \lambda^\vee \rangle$ is even for all $\lambda^\vee \in X^*({}^LT^0)$. If $\varphi_{\beta^\vee}$ is the map of $\mathrm{SL}(2)$ into ${}^LG^0$ given in the usual way once $X_{\beta^\vee}$ is chosen, we may take

$$\epsilon(\omega) = \varphi_{\beta^\vee}\left(\begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix}\right)$$

then

$$\epsilon(\omega) X_{\gamma^\vee} \epsilon(\omega)^{-1} = X_{\gamma^\vee} \quad \gamma^\vee \in \mathfrak{X}_0 \quad \gamma^\vee \neq \alpha^\vee, \alpha_1^\vee$$

and for a suitable choice of x

$$\epsilon(\omega) X_{\alpha_1^\vee} \epsilon(\omega)^{-1} = X_{\alpha^\vee}.$$

Since

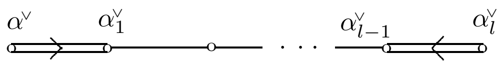
$$\lambda^\vee(\epsilon(\omega)^2) = (-1)^{\langle \beta, \lambda^\vee \rangle} = 1$$

we conclude that $\epsilon(\omega)^2 = 1$ and that

$$\epsilon(\omega) X_{\alpha^\vee} \epsilon(\omega)^{-1} = X_{\alpha_1^\vee}.$$

The cocycle therefore splits.

2) C_l, $1 \geqq 2$.



$$\alpha^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_l^\vee = 0$$

Here $\alpha^\vee$ and $\alpha_1^\vee$ must belong to $\mathfrak{X}_0$ and $A = \{1, \omega\}$ where ω reflects the diagram in its center. We realize ${}^LG^0$ as usual as the symplectic group in $2l$ variables modulo its center. With the usual representation of the roots $\alpha^\vee = -2x_1$, $\alpha_1^\vee = x_1 - x_2, \dots, \alpha_{l-1}^\vee = x_{l-1} - x_l, \alpha_l^\vee = 2x_l$.

Suppose first that $l = 2k$ is even. Then ω fixes the roots

$$x_1 - x_{2k}, \qquad x_2 - x_{2k-1}, \quad \dots, \qquad x_k - x_{k+1}.$$

The only roots orthogonal to all of these are

$$\beta_1' = x_1 + x_{2k}, \qquad \beta_2' = x_2 + x_{2k-1}, \quad \dots, \qquad x_k + x_{k+1}$$

and if

$$\delta(\omega) = \prod_{i=1}^{k} \varphi_{\beta_j^\vee}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

then $\delta(\omega)$ is a representtive of ω. The only root of D fixed by ω is $\alpha_{k^\vee}$ and

$$\begin{aligned} \operatorname{Ad} \varphi_{\beta_j^\vee}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)&: X_{\alpha_k^\vee} \to X_{\alpha_k^\vee}, \qquad\qquad j \neq k, \\ \operatorname{Ad} \varphi_{\beta_k^\vee}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)&: X_{\alpha_k^\vee} \to -X_{\alpha_k^\vee}. \end{aligned}$$

Thus

$$\operatorname{Ad}\delta(\omega)\colon X_{\alpha_k^\vee} \to -X_{\alpha_k^\vee}.$$

Moreover

$$\lambda^\vee(\delta(\omega)^2) = (-1)^{\left\langle \sum \beta_i, \lambda^\vee \right\rangle}.$$

Since

$$\left\langle \sum \beta_i, \lambda^\vee \right\rangle = \lambda_1^\vee + \cdots + \lambda_l^\vee,$$

if $\lambda_1^\vee, \ldots, \lambda_l^\vee$ are the coordinates of $\lambda^\vee$, it is always even and $\delta(\omega)^2 = 1$. Thus if $\beta^\vee \neq \omega\beta^\vee$ lies in D and

$$\operatorname{Ad}\delta(\omega)\colon X_{\beta^\vee} \to c(\beta^\vee) X_{\omega\beta^\vee}$$

then

$$\operatorname{Ad}\delta(\omega)\colon X_{\omega\beta^\vee} \to c(\beta^\vee)^{-1} X_{\beta^\vee}.$$

Define t in ${}^L T^0$ by

$$\beta^\vee(t) = \begin{cases} 1 & \beta^\vee \in D, \quad \beta^\vee \notin \mathfrak{X}_0 \\ c(\beta^\vee)^{-1} & \beta^\vee \in \mathfrak{X}_0. \end{cases}$$

These demands are consistent and we may take $\epsilon(\omega) = \delta(\omega)t$. Then

$$\epsilon(\omega)^2 = 1.$$

If $l = 2k+1$ is odd we take

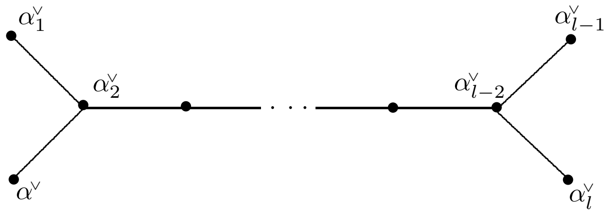
$$\beta_1^\vee = x_1 + x_{2l+1}, \quad \beta_2^\vee = x_2 + x_{2l}, \quad \ldots, \beta_k^\vee = x_k + x_{k+2}$$

and argue as before with

$$\delta(\omega) = \prod_{i-1}^{k} \varphi_{\beta_i^\vee}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

Since ω fixes no root of $\mathfrak{X}_0$ or even of D the argument is easier.

3) D_l, $l \geqq 4$.



$$\alpha^{\vee} + \alpha_1^{\vee} + 2\alpha_2^{\vee} \cdots + 2\alpha_{l-2}^{\vee} + \alpha_{l-1}^{\vee} + \alpha_l^{\vee} = 0$$

$\mathfrak{X}_0$ must contain $\alpha^{\vee}$, $\alpha_1^{\vee}$, $\alpha_{l-1}^{\vee}$, and $\alpha_l^{\vee}$. If $l > 4$ then A is at most of order 8; if $l = 4$ then A_2 is again at most of order 8 while A_3 is $\{1\}$ or $\mathbf{Z}_3$. Since Z/Z^0 is of order two we need in any case only consider the cocycle on A_2. With the usual representation of the roots

$$\alpha^{\vee} = -x_1 - x_2, \quad \alpha_1^{\vee} = x_1 - x_2, \quad \alpha_2^{\vee} = x_2 - x_3, \quad \dots, \qquad \alpha_{l-1}^{\vee} = x_{l-1} - x_1, \quad \alpha_l^{\vee} = x_{l-1} + x_l.$$

If $l = 2k$ then the roots

$$x_2 - x_{l-1}, \quad x_3 - x_{l-2}, \quad \dots, \qquad x_k - x_{k+1}$$

are fixed by all the elements of A_2. The roots orthogonal to this set are

$$\beta^{\vee} = x_1 - x_l, \quad \beta_1^{\vee} = x_1 + x_l, \quad \beta_2^{\vee} = x_2 + x_{l-1}, \quad \dots, \qquad \beta_k^{\vee} = x_k + x_{k+1}.$$

The group A_2 is generated by $\omega_1, \omega_2, \omega_3$ where ω_1 interchanges $\alpha_1^{\vee}$ and $\alpha^{\vee}$ but fixes the other roots of D, while ω_2 interchanges $\alpha_{l-1}^{\vee}$ and $\alpha_l^{\vee}$ and fixes the other roots. Finally ω_3 interchanges $\alpha^{\vee}$ with $\alpha_l^{\vee}$, $\alpha_1^{\vee}$ with $\alpha_{l-1}^{\vee}$, $\alpha_2^{\vee}$ with $\alpha_{l-2}^{\vee}$, and so on, and fixes $\alpha_k^{\vee}$. The defining relations are

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = 1$$
$$\omega_1\omega_2 = \omega_2\omega_1$$
$$\omega_3\omega_1 = \omega_2\omega_3.$$

By its construction there is in ${}^LG'$ an element δ_2 normalizing ${}^LT^0$, representing ω_2, and satisfying

$$\delta_2 X_{\alpha_i^{\vee}} \delta_2^{-1} = X_{\alpha_i^{\vee}} \qquad 1 \leqq i \leqq l-2$$
$$\delta_2 X_{\alpha_{l-1}^{\vee}} \delta_2^{\vee} = X_{\alpha_l^{\vee}}$$
$$\delta_2 X_{\alpha_l^{\vee}} \delta_2^{-1} = X_{\alpha^{\vee}_{i-1}}.$$

In addition

$$\delta_2^2 = 1\ .$$

We recall a fact which is verified in [10].

Lemma 3. *Suppose $\epsilon \in A({}^LG^0, {}^LB^0, {}^LT^0, \{X_{\alpha^\vee}\})$ and $\beta^\vee$ is a root fixed by ϵ. Let $\beta^\vee = \sum a(\alpha)\alpha^\vee$ be its expression as a sum of simple roots and let a be the sum over the pairs $\{\alpha^\vee, \epsilon\alpha^\vee\}$ with $\alpha^\vee \neq \epsilon\alpha^\vee$ and $(\alpha^\vee, \epsilon\alpha^\vee) \neq 0$ of $a(\alpha) = a(\epsilon\alpha)$. Then*

$$\epsilon(X_{\beta^\vee}) = (-1)^a X_{\epsilon\beta^\vee} \ .$$

It follows that

$$\delta_2(X_{\alpha^\vee}) = X_{\alpha^\vee} \ .$$

We take $\epsilon_2 = \delta_2$. The lemma also implies that

$$\epsilon_2 X_{\beta_1^\vee} \epsilon_2^{-1} = X_{\beta_i^\vee}, \qquad 2 \leqq i \leqq k \ .$$

Set

$$\delta_3 = \prod_{i=1}^{k} \varphi_{\beta_i^\vee} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \ .$$

Because ${}^LG^0$ is an adjoint group

$$\lambda^\vee(\delta_3^2) = (-1)^{\langle \sum \beta_i, \lambda^\vee \rangle} = 1, \qquad \lambda^\vee \in X^*({}^LT^0) \ ,$$

and $\delta_3^2 = 1$. Let

$$\delta_3 X_{\gamma^\vee} \delta_3^{-1} = c(\gamma^\vee) X_{\omega_3\gamma^\vee}, \qquad \gamma^\vee \in \mathfrak{X}_0 \ .$$

Certainly

$$c(\omega_3\gamma^\vee) = c(\gamma^\vee)^{-1}$$

and, because $a_k = 2$, we may define t_3 by

$$\gamma^\vee(t_3) = \begin{cases} 1, & \gamma^\vee \in D, \quad \gamma^\vee \notin \mathfrak{X}_0, \\ c(\gamma^\vee)^{-1}, & \gamma^\vee \in \mathfrak{X}_0 \ . \end{cases}$$

Then we take $\epsilon_3 = \delta_3 t_3$. Finally we take $\epsilon_1 = \epsilon_3\epsilon_2\epsilon_3^{-1} = \epsilon_3\epsilon_2\epsilon_3$. To show that the cocycle splits it is enough to show that

$$\epsilon_1\epsilon_2 = \epsilon_2\epsilon_1 \ .$$

The left side is

$$\epsilon_3\epsilon_2\epsilon_3\epsilon_2$$

and the right

$$\epsilon_2\epsilon_3\epsilon_2\epsilon_3 \ .$$

Now

$$\epsilon_2\epsilon_3\epsilon_2 = \varphi_{\beta^\vee} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \prod_{i=2}^{k} \varphi_{\beta_i^\vee} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \omega_2(t_3) \ .$$

Since any two of the roots $\beta^\vee, \beta_1^\vee, \dots, \beta_l^\vee$ are strongly orthogonal we are reduced to verifying the equality

$$t_3\omega_3\omega_2(t_3) = \omega_2(t_3)\omega_2\omega_3\omega_2(t_3)$$

or

$$c(\gamma^\vee)c(\omega_2\omega_3\gamma^\vee) = c(\omega_2\gamma^\vee)c(\omega_2\omega_3\omega_2\gamma^\vee)$$

for all $\gamma^\vee \in \mathfrak{X}_0$. This is clear if $\gamma^\vee \in \{\alpha_2^\vee, \dots, \alpha_{l-2}^\vee\}$ for then ω_2 may be removed without affecting either side. If $\gamma^\vee \in \{\alpha^\vee, \alpha_1^\vee\}$ then

$$\gamma^\vee = \omega_2\gamma^\vee$$

and the equation is trivially valid. If $\gamma^\vee \in \{\alpha_{l-1}^\vee, \alpha_l^\vee\}$ then

$$\omega_2\omega_3\gamma^\vee = \omega_3\gamma^\vee, \quad \omega_2\omega_3\omega_3\gamma^\vee = \omega_3\omega_2\gamma^\vee$$

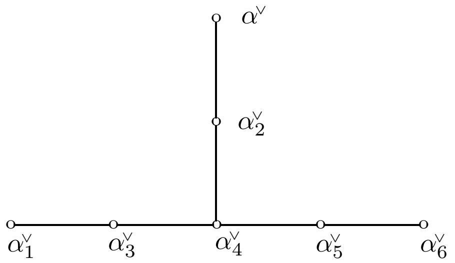
and both sides are equal to 1.

If $l = 2k+1$ we start with

$$\beta^\vee = x_1 - x_l, \quad \beta_1^\vee = x_1 + x_l, \quad \dots, \qquad \beta_k^\vee = x_k + x_{k+2}\ ,$$

but the argument is otherwise the same.

4) E_6.



$$\alpha^\vee + \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 3\alpha_4^\vee + 2\alpha_5^\vee + \alpha_6^\vee = 0$$

$\mathfrak{X}_0$ must contain $\alpha^\vee$, $\alpha_1^\vee$ and $\alpha_6^\vee$. The group A_2 is $\{1\}$ or $\mathbf{Z}_2$ and the group A_3 is $\{1\}$ or $\mathbf{Z}_3$. If the group A_2 is $\{1, \omega\}$ then we may with no loss of generality assume that ω fixes $\alpha^\vee$ and $\alpha_2^\vee$. By construction there is an ϵ in ${}^LG'$ of order two such that ϵ acts on ${}^LT^0$ as ω and such that

$$\epsilon X_{\alpha_1^\vee}\epsilon^{-1} = X_{\omega\alpha_1^\vee} \qquad 1 \leqq i \leqq 6\ .$$

Lemma 3 again implies that

$$\epsilon X_{\alpha^\vee}\epsilon^{-1} = X_{\alpha^\vee}\ .$$

The cocycle is therefore trivial on A_2.

We now consider A_3, which we suppose is $\mathbf{Z}_3$. If Z/Z^0 has order prime to 3 the cocycle is certainly trivial. Thus we may assume that $\alpha_2^\vee$, $\alpha_3^\vee$, $\alpha_5^\vee$ belong to $\mathfrak{X}_0$ but that $\alpha_4^\vee$ does not. We are going to realize A_3 in the centralizer of α_4, regarded as an element of the Lie algebra of ${}^LT^0$.

Running through the table of positive roots of E_6 given in [5], we find that the following are orthogonal to α_4:

$$\begin{matrix} & & \begin{matrix}0\\10000\end{matrix} & & \begin{matrix}0\\00001\end{matrix} & & \\ & \begin{matrix}1\\01100\end{matrix} & & \begin{matrix}1\\01110\end{matrix} & & \begin{matrix}1\\00110\end{matrix} & \\ \begin{matrix}1\\11100\end{matrix} & & \begin{matrix}0\\11110\end{matrix} & & \begin{matrix}0\\01111\end{matrix} & & \begin{matrix}1\\00111\end{matrix} \end{matrix}$$

$$\begin{matrix} & \begin{matrix}0\\11111\end{matrix} & \\ \begin{matrix}1\\12210\end{matrix} & & \begin{matrix}1\\01221\end{matrix} \\ \begin{matrix}1\\12211\end{matrix} & & \begin{matrix}1\\11221\end{matrix} \\ & \begin{matrix}2\\12321\end{matrix} & \end{matrix}$$

Thus the centralizer is of type A_5. With the standard representation of the root system of type A_5

$$\begin{array}{rcl} x_1 - x_2 & \leftrightarrow & \begin{matrix}1\\01100\end{matrix} \\[1em] x_2 - x_3 & \leftrightarrow & \begin{matrix}0\\10000\end{matrix} \\[1em] x_3 - x_4 & \leftrightarrow & \begin{matrix}0\\01110\end{matrix} \\[1em] x_4 - x_5 & \leftrightarrow & \begin{matrix}0\\00001\end{matrix} \\[1em] x_5 - x_6 & \leftrightarrow & \begin{matrix}1\\00110\end{matrix} \end{array}$$

Thus a generating element of A_3 corresponds to the permutation

$$1 \to 2 \to 4 \to 1, \qquad 3 \to 5 \to 6 \to 3\ .$$

This we can realize in $\mathrm{SL}(6)$ and therefore certainly in ${}^LG^0$ by an element δ of order 3. As usual we let

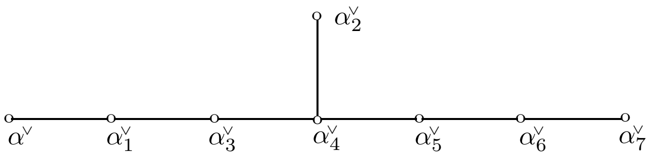
$$\delta X_{\gamma^\vee} = c(\gamma^\vee) X_{\omega\gamma^\vee}, \qquad \gamma^\vee \in \mathfrak{X}_0\ ,$$

and define t by

$$\gamma^\vee(t) = \begin{cases} 1, & \gamma^\vee \in D, \quad \gamma^\vee \notin \mathfrak{X}_0, \\ c(\gamma^\vee)^{-1}, & \gamma^\vee \in \mathfrak{X}_0. \end{cases}$$

Then $\epsilon = \delta t$ has order 3 and can serve as a representative of ω.

5) E_7.



$$\alpha^\vee + 2\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_5^\vee + 2\alpha_6^\vee + \alpha_7^\vee = 0$$

$\mathfrak{X}_0$ must contain $\alpha^\vee$ and $\alpha_7^\vee$. Moreover if A is not trivial then it is $\mathbf{Z}_2$ and $[Z : Z^0]$ is prime to 2 unless $\alpha_3^\vee$ and $\alpha_5^\vee$ also belong to $\mathfrak{X}_0$. In order to construct a representative of the generator ω of A we work within the centralizer of α_2 and α_4.

We work through the list of positive roots given in [5] and find that those orthogonal to α_2 and α_4 are the following:

$$\begin{array}{ccccc}
\begin{matrix}0\\000001\end{matrix} & & \begin{matrix}0\\000010\end{matrix} & & \begin{matrix}0\\100000\end{matrix} \\[1ex]
 & & \begin{matrix}0\\000011\end{matrix} & & \\[1ex]
 & \begin{matrix}1\\012210\end{matrix} & & \begin{matrix}1\\122100\end{matrix} & \\[1ex]
\begin{matrix}1\\012211\end{matrix} & & \begin{matrix}1\\112210\end{matrix} & & \begin{matrix}1\\122110\end{matrix} \\[1ex]
\begin{matrix}1\\012221\end{matrix} & & \begin{matrix}1\\112211\end{matrix} & & \begin{matrix}1\\122111\end{matrix} \\[1ex]
 & & \begin{matrix}1\\112221\end{matrix} & & \\[1ex]
 & & \begin{matrix}2\\134321\end{matrix} & & \\[1ex]
 & & \begin{matrix}2\\234321\end{matrix} & &
\end{array}$$

We obtain a system of type A_5 and

$$\begin{aligned}
x_1 - x_2 &\leftrightarrow \begin{matrix} & & 1 & & & \\ 1&2&2&1&0&0 \end{matrix} \\
x_2 - x_3 &\leftrightarrow \begin{matrix} & & 0 & & & \\ 0&0&0&0&1&0 \end{matrix} \\
x_3 - x_4 &\leftrightarrow \begin{matrix} & & 0 & & & \\ 0&0&0&0&0&1 \end{matrix} \\
x_4 - x_5 &\leftrightarrow \begin{matrix} & & 1 & & & \\ 1&1&2&2&1&0 \end{matrix} \\
x_5 - x_6 &\leftrightarrow \begin{matrix} & & 0 & & & \\ 1&0&0&0&0&0 \end{matrix}
\end{aligned}$$

The image δ of

$$\begin{pmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \end{pmatrix}$$

in $\mathrm{SL}(6)$ is a representative of ω. δ^2 is the image of

$$\begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$$

and it is easily checked that this is 1. Since the coefficients a_2 and a_4 are even we can define t and ϵ as usual.

The proof of Proposition 1 will be completed by arguments which have nothing to do with semi-simple groups but rely rather on our knowledge of Galois cohomology. We now have a cocycle $\{a_{\sigma_1,\sigma_2}\}$ with values in Z^0 and we want to show that the inflated cocycle $\{a_{w_1,w_2}\}$ is trivial.

Lemma 4 *Suppose F is a local or a global field and K a finite Galois extension. Let $S = \mathrm{Hom}(X, \mathbf{C}^\times)$ be a torus over $\mathbf{C}$. Let $\mathrm{Gal}(K/F)$ act on X and hence on S. Let the Weil group $W_{K/F}$ act on S through its projection on $\mathrm{Gal}(K/F)$. If $\{a_{\sigma_1,\sigma_2}\}$ is a 2-cocycle of $\mathrm{Gal}(K/F)$ with values in S then there is a continuous function $b(w)$ on $W_{K/F}$ with values in S such that*

$$a_{\sigma_1,\sigma_2} = b(w_1)w_1(b(w_2))b(w_1w_2)^{-1}$$

for all $w_1 w_2$.

Variants of this lemma had been drawn to my attention by both Deligne and Hoechsmann, who proved them by means of the dualities of Poitou and Tate. Such methods may well work in general, but it is easier for me to draw on a theorem from [8].

If K is local let C_K be the multiplicative group of K and if K is global let it be the group of idèle classes. If $H^1_c(W_{K/F}, S)$ denotes the group of continuous 1-cocycles of $W_{K/F}$ in S modulo coboundaries then, according to Theorem 1 of [8], there is a canonical isomorphism of $H^1_c(W_{K/F}, S)$ with the group of characters of the topological group

$$\mathrm{Hom}_{\mathrm{Gal}(K/F)}(X^\vee, C_K) \ .$$

Here

$$X^\vee = \mathrm{Hom}(X, \mathbf{Z}) \ .$$

The characters are not necessarily of absolute value 1. The isomorphism is functorial in S.

If we have an exact sequence

$$0 \to X_2 \to X_1 \to X \to 0$$

in which X_1, X_2 are also $\mathrm{Gal}(K/F)$-modules free over $\mathbf{Z}$, then

$$1 \to \mathrm{Hom}_{\mathrm{Gal}(K/F)}(X_2^\vee, C_K) \to \mathrm{Hom}_{\mathrm{Gal}(K/F)}(X_1^\vee, C_K) \to \mathrm{Hom}_{\mathrm{Gal}(K/F)}(X^\vee, C_K)$$

is also exact. Passing to the group of characters we infer from standard facts about extensions of characters that

$$H^1_c(W_{K/F}, S_1) \to H^1_c(W_{K/F}, S_2) \tag{1}$$

is surjective.

To deduce the lemma from this we need only choose X_1 correctly. We can clearly choose it to be free over the group ring $\mathbf{Z}(\mathrm{Gal}(K/F))$. Then S_1 is also induced and hence homologically trivial. Consequently in S_1

$$a_{\sigma_1,\sigma_2} = c(\sigma_1)\sigma_1(c(\sigma_2))c(\sigma_1\sigma_2)^{-1} \ .$$

If $\bar{c}(\sigma)$ denotes the image of $c(\sigma)$ in S_2 then $\{\bar{c}(\sigma)\}$ is a 1-cocycle of $\mathrm{Gal}(K/F)$ and thus of $W_{K/F}$. By the surjectivity of (1) there is a continuous 1-cocycle $d(w)$ of $W_{K/F}$ with values in S_1 such that

$$\bar{d}(w) = \bar{c}(\sigma)\bar{a}^{-1}\sigma(\bar{a})$$

if $w \to \sigma$. Here $\bar{a}$ is a fixed element of S_2. Since $S_1 \to S_2$ is surjective, $\bar{a}$ is the image of some $a \in S_1$. We may replace $c(\sigma)$ by $c(\sigma)a^{-1}\sigma(a)$ and suppose that

$$\bar{d}(w) = \bar{c}(\sigma) \ .$$

Then $w \to b(w) = d^{-1}(w)c(\sigma)$ is a continuous 1-cochain with values in S whose boundary is the inflation of $\{a_{\sigma_1,\sigma_2}\}$.

With the proof of Proposition 1 the principal purpose of this note is achieved, but there are some supplementary remarks to be made. First a comment on the role of g_1. The influence of g_1 on groups ${}^LH^0$ and ${}^LH$ is through ψ'. Suppose we replace g_1 by $\bar{g}_1 = wg_1$ and ψ' by $\bar{\psi}' = \operatorname{ad} w \circ \psi'$, with w in the normalizer of T^{G_1} in G_1. If ω is the element of $\Omega(T_1, G_1) \simeq \Omega({}^LT^0, {}^LT^0,)$ represented by w then $\omega_{T/G}(\sigma)$ is replaced by

$$\bar{\omega}_{T/G}(\sigma) = \omega\omega_{T/G}(\sigma)\omega^{-1}$$

and κ' by $\bar{\kappa}' = \omega(\bar{\kappa}')$. Let ${}^LH^0$ be replaced by ${}^L\bar{H}^0$. We write $\omega = \omega_1\omega_2$ where ω_2 lies in the Weyl group of ${}^LH^0$ and ω_1 takes positive roots of ${}^LT^0$ in ${}^LH^0$ to positive roots of ${}^LT^0$ in ${}^L\bar{H}^0$. Then

$$\bar{\omega}_{T/G}(\sigma) = (\omega_1\omega^1(\sigma)\omega_1^{-1})(\omega_1\omega^1(\sigma)^{-1}\omega_2\omega^1(\sigma)\omega^2(\sigma)\omega_2^{-1}\omega_1^{-1}) \ .$$

The expression within the second parentheses lies in the Weyl group of ${}^LT^0$ in ${}^L\bar{H}^0$, and the expression within the first parentheses takes positive roots of ${}^LT^0$ in ${}^LH^0$ to positive roots. Thus

$$\bar{\omega}^1(\sigma) = \omega_1\omega^1(\sigma)\omega_1^{-1} \ .$$

In order to interpret ${}^LH^0$ or ${}^L\bar{H}^0$ as the connected components of associate groups we need to choose in addition root vectors $X_{\alpha^\vee}$ and $X_{\bar{\alpha}^\vee}$ corresponding to the simple roots. Let u be a representative of ω_1 in ${}^LG^0$ such that

$$\operatorname{Ad} u(X_{\alpha^\vee}) = X_{\bar{\alpha}^\vee}$$

if $\bar{\alpha}^\vee = \omega_1\alpha^\vee$. The isomorphism $\zeta\colon\, h \to uhu^{-1}$ of ${}^LH^0$ with ${}^L\bar{H}^0$ may then be extended in a natural fashion to

$$\zeta\colon\, {}^LH \to {}^L\bar{H} \ .$$

We extend the imbedding $\bar{\xi}\colon\, {}^L\bar{H}^0 \to {}^LG^0$ to $\bar{\xi}\colon\, {}^L\bar{H} \to \bar{G}^L$ by setting

$$\bar{\xi}(h) = u(\xi\zeta^{-1}(h))u^{-1} \ .$$

The conclusion is that g_1 has no real influence. To each choice Proposition 1 assigns a set of ξ. It is not the individual ξ which matter but only the orbits under conjugation by elements of ${}^LG^0$, and the preceding discussion yields a canonical bijection between the collections of orbits arising from two different choices.

In order to apply Proposition 1 and the hypothetical principle of functoriality in the associate group effectively, we shall need a way of reducing the study of irreducible representations or of automorphic forms to groups G for which ${}^LG^0$ has a connected center.

We start from a given G and set

$$P = X_*(T^{G_1})/X_*(T^{G_1}_{\mathrm{sc}}) \ .$$

The difficulty arises when P is not torsion-free. We represent the $\operatorname{Gal}(K/F)$-module P as a quotient

$$0 \to M' \to Q' \to P \to 0$$

with Q' torsion-free. We then introduce an imbedding

$$0 \to M' \xrightarrow{\eta} M_*$$

with M_* induced and M_*/M' torsion free. Set

$$Q = M_* \oplus Q'/\{(\eta(m), m)\} \ .$$

Q is again torsion free and we clearly have an exact sequence

$$0 \to M_* \to Q \xrightarrow{\epsilon_2} P \to 0 \ .$$

If ϵ_1 is the homomorphism $X_*(T^{G_1}) \to P$ set

$$\widetilde{X}_* = \{(\lambda, \mu) \in X_*(T^{G_1}) \oplus Q | \epsilon_1(\lambda) = \epsilon_2(\mu)\}$$

and set

$$\widetilde{X}_* = \mathrm{Hom}(\widetilde{X}_*, \mathbf{Z}) \ .$$

We certainly have

$$X_*(T^G_{\mathrm{sc}}) \hookrightarrow \widetilde{X}_*$$

by means of the map to the first factor as well as a surjection

$$\widetilde{X}_* \to X_* \to 0$$

with kernel M_*. Dual to this we have

$$0 \to X^* \to \widetilde{X}^* \to M^* \to 0 \ .$$

Here

$$M^* = \mathrm{Hom}(M_*, \mathbf{Z}) \ .$$

There is clearly a central extension $\widetilde{G}_1$ of G_1 over F such that if $\widetilde{T}^{G_1}$ is the inverse image of T^{G_1} then

$$X^*(\widetilde{T}^{G_1}) = \widetilde{X}^*$$

and the contragredient to $\widetilde{T}^{G_1} \to G^{G_1}$ is

$$X^*(T^{G_1}) = X^* \to \widetilde{X}^* = X^*(\widetilde{T}^{G_1}) \ .$$

We define $\widetilde{G}$ over F by twisting $\widetilde{G}_1$ by the cocycle $\{\psi\circ(\psi^{-1})\}$ in the adjoint group of G_1 and obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\tilde{\psi}} & \widetilde{G}_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ G & \xrightarrow[\psi]{} & G_1 \end{array}$$

with vertical arrows defined over F. If Z is the kernel of φ or of φ_1, for they are isomorphic, then

$$M^* = X^*(Z)\ .$$

Since M^* is induced we infer from the Tate-Nakayama theory and Hilbert's Theorem 90 that if F is a local or a global field then

$$\widetilde{G}(F) \to G(F)$$

is surjective and that if it is global then

$$\widetilde{G}(\mathbf{A}_F) \to G(\mathbf{A}_F)$$

is surjective as well. This allows us when F is local to identify representations of $G(F)$ as representations of $\widetilde{G}(F)$ which are trivial on $Z(F)$, and when F is global to identify automorphic representations of $G(\mathbf{A}_F)$ with automorphic representations of $\widetilde{G}(\mathbf{A}_F)$ trivial on $Z(\mathbf{A}_F)$.

If T is any Cartan subgroup of G with inverse image $\widetilde{T}$ then

$$1 \to Z \to \widetilde{T} \to T \to 1$$

is exact and

$$H^1(\mathrm{Gal}(\bar{F}/F), \widetilde{T}(\bar{F})) \to H^1(\mathrm{Gal}(\bar{F}/F), T(\bar{F}))$$

is injective. Consequently

$$\mathfrak{D}(\widetilde{T}) = \mathfrak{D}(T)$$

and

$$\mathfrak{E}(\widetilde{T}) = \mathfrak{E}(T)\ .$$

There is another way of expressing the last relation.

Certainly $X_*(\widetilde{T}_{\mathrm{sc}}) = X_*(T_{\mathrm{sc}})$. I claim that the groups

$$\Big\{\lambda \in X_*(\widetilde{T}_{\mathrm{sc}}) | \lambda = \sum_{\mathrm{Gal}(K/F)} \omega_{\widetilde{T}/G}(\sigma)\tilde{\mu}(\sigma) - \tilde{\mu}(\sigma), \tilde{\mu}(\sigma) \in X_*(\widetilde{T})\Big\}$$

and

$$\Big\{\lambda \in X_*(T_{\mathrm{sc}}) | \lambda = \sum_{\mathrm{Gal}(K/G)} \omega_{T/G}(\sigma)\mu(\sigma) - \mu(\sigma), \mu(\sigma) \in X_*(T)\Big\}$$

are also equal. Since

$$X_*(\widetilde{T}) = \{(\lambda, \mu) \in X_*(T) \oplus Q | \epsilon_1(\lambda) = \epsilon_2(\mu)\} \ ,$$

the first group is certainly contained in the second. On the other hand, if $\mu(\sigma) \in X_*(T)$ and

$$\lambda = \sum \omega_{T/G}(\sigma)\mu(\sigma) - \mu(\sigma) \in X_*(T_{\mathrm{sc}}) \ ,$$

then $\epsilon_1\mu$: $\sigma \to \epsilon_1(\mu(\sigma))$ defines an element of $H^{-2}(\mathrm{Gal}(K/F), P)$. Since

$$H^{-2}(\mathrm{Gal}(K/F), Q) \to H^{-2}(\mathrm{Gal}(K/F), P)$$

is surjective, there exists $\nu(\sigma)$ in Q such that

$$\sum \sigma\nu(\sigma) - \nu(\sigma) = 0$$

and

$$\epsilon_2(\mu) - \epsilon_1(\nu) = \delta\eta$$

is a boundary. Here η: $(\sigma_1, \sigma_2) \to \eta(\sigma_1, \sigma_2)$ is a 2-chain. If η' is a 2-chain with values in $X_*(T)$ with $\epsilon_1(\eta)' = \eta$, we may replace μ by $\mu - \delta\eta'$ without affecting λ and hence assume that

$$\epsilon_2(\mu) = \epsilon_1(\nu) \ .$$

Then

$$\tilde{\mu} = \mu \oplus \nu$$

defines a chain of

$$X_*(\widetilde{T}) \subseteq X_*(T) + Q$$

and

$$\sum \omega_{\widetilde{T}/G}(\sigma)\tilde{\mu}(\sigma) - \tilde{\mu}(\sigma) = \lambda \ .$$

Finally, I add a few remarks that it is useful to bear in mind when applying the constructions of this paper to groups in whose definition a restriction of scalars intervenes. Suppose F is a finite extension of E and $\bar{G}$ is the group over E obtained from G by restriction of scalars. Then

$$T^{\bar{G}} = \mathrm{Res}_{F/E} T^G$$

is a Cartan subgroup of $\bar{G}$ over E. Moreover, $\bar{G}_1 = \mathrm{Res}_{F/E} G_1$ is quasi-split, and $T^{\bar{G}_1} = \mathrm{Res}_{F/E} T^{G_1}$ is a Cartan subgroup of it.

Once we have fixed an imbedding of F in $\bar{E}$ we may identify $\bar{G}(\bar{E})$ with the set of functions φ from $\mathrm{Gal}(\bar{E}/E)$ to $G(\bar{E}) = G(\bar{F})$ satisfying

$$\varphi(\sigma\tau) = \sigma(\varphi(\tau)) \qquad \sigma \in \mathrm{Gal}(\bar{E}/F)\ .$$

$\bar{G}_1(\bar{E})$ is obtained in a similar fashion. Recall that in Lemma 2.3 of [10] we have associated to ψ: $G \to G_1$ an isomorphism $\bar{\psi}$: $\bar{G}_1$ over $\bar{E}$. If I is a set of representatives for the cosets of $\mathrm{Gal}(\bar{E}/F)$ in $\mathrm{Gal}(\bar{E}/E)$, then $\bar{\psi}$ takes φ to φ_1 with

$$\varphi_1(\tau) = \psi(\varphi(\tau)), \qquad \tau \in I\ .$$

If $\bar{g}_1$ is the function in $\bar{G}_1(\bar{E})$ which takes $\tau \in I$ to g_1, then

$$\bar{\psi}' = \mathrm{ad}\,\bar{g}_1 \circ \bar{\psi}$$

is obtained from ψ' in just the same way that $\bar{\psi}$ is obtained from ψ. It does depend on the choice of coset representatives, but that is not important. Let us fix I for now.

It was observed in [10] that, as $\mathrm{Gal}(\bar{E}/E)$-modules,

$$X_*(T^{\bar{G}}) = \mathrm{Ind}(\mathrm{Gal}(\bar{E}/E),\ \mathrm{Gal}(\bar{E}/F),\ X_*(T^G))$$
$$X_*(T^{\bar{G}_1}) = \mathrm{Ind}(\mathrm{Gal}(\bar{E}/E),\ \mathrm{Gal}(\bar{E}/F),\ X_*(T^{G_1}))\ .$$

Both these modules consist of functions on $\mathrm{Gal}(\bar{E}/E)$, and if $\bar{\psi}'$ takes λ to λ_1, then

$$\lambda_1(\tau) = \psi'(\lambda(\tau)), \qquad \tau \in I\ .$$

Shapiro's Lemma shows that

$$\lambda \to \sum_{\tau \in I} \lambda(\tau)$$

yields an isomorphism $\mathfrak{E}(T^{\bar{G}}) \xrightarrow{\sim} \mathfrak{E}(T^G)$. Thus κ pulls back to

$$\bar{\kappa}\colon\ \lambda \to \sum_{\tau \in I} \kappa(\lambda(\tau))$$

and κ' to

$$\bar{\kappa}'\colon\ \lambda \to \sum_{\tau \in I} \kappa'(\lambda(\tau))\ .$$

${}^L\bar{G}^0$ consists of the functions φ on $\mathrm{Gal}(\bar{E}/E)$ with values in ${}^LG^0$ satisfying

$$\varphi(\sigma\tau) = \sigma(\varphi(\tau)), \qquad \sigma \in \mathrm{Gal}(\bar{E}/F)\ .$$

It is clear that ${}^L\bar{H}^0$ consists of those φ for which

$$\varphi(\tau) \in {}^LH^0 \tag{2}$$

for all $\tau \in I$.

If

$$\tau\sigma = \alpha_\tau(\sigma)\tau'$$

with $\alpha_\tau(\sigma) \in \mathrm{Gal}(\bar{E}/F)$ and $\tau' \in I$ and $\bar{T} = T^{\bar{G}}$, then $\omega_{\bar{T}/\bar{G}}(\sigma)$ takes λ to λ' with

$$\lambda'(\tau) = \omega_{T/G}(\alpha_\tau(\sigma))\lambda(\tau'), \qquad \tau \in I \ .$$

If

$$\omega_{\bar{T}/\bar{G}}(\sigma) = \bar{\omega}^1(\sigma)\bar{\omega}^2(\sigma) \ ,$$

where $\bar{\omega}^2(\sigma)$ lies in the Weyl group of ${}^L\bar{T}^0$ in ${}^L\bar{H}^0$ and $\bar{\omega}^1(\sigma)$ takes positive roots in ${}^L\bar{H}^0$ to positive roots, then $\bar{\omega}^1(\sigma)$ takes λ to λ' with

$$\lambda'(\tau) = \omega^1(\alpha_\tau(\sigma))\lambda(\tau'), \qquad \tau \in I \ .$$

Thus ${}^L\bar{H}$ is the associate group attached to ${}^LH$ by the functor $\mathfrak{G}^\vee(F) \to \mathfrak{G}^\vee(E)$ of [10]. Consulting the definitions of [9], we see that if ξ: ${}^LH \to {}^LG$ extends ${}^LH^0 \hookrightarrow {}^LG^0$ then the homomorphism $\bar{\xi}$: ${}^LH \to {}^LG$ associated to ξ by the functorial process of [10] extends ${}^LH^0 \hookrightarrow {}^LG^0$.

The conclusion is that the constructions of this paper behave simply under restriction of scalars, as one expects. It should also be noticed that the functorial constructions of [10] also allow one to construct the homomorphism of Proposition 1 even in situations that do not strictly arise from restriction of scalars. They can sometimes be used for connected subgroups of $\bar{G} = \mathrm{Res}_{F/E}G$ with abelian quotients. We will meet an example of this in another paper.

References

[1] A. Borel, *Formes automorphes et séries de Dirichlet*, Sem. Bourbaki (1975).

[2] N. Bourbaki, *Groupes et algèbres de Lie*, chap. 4, 5, et 6, Hermann (Paris).

[3] E. Dynkin, *Semi-simple subalgebras of semi-simple Lie algebras*, A.M.S. Trans. Ser. 2, 6 (157).

[4] D. Flath, Thesis, Harvard (1977).

[5] H. Freudenthal and H. de Vries, *Linear Lie groups*, Academic Press (New York).

[6] H. Jacquet and R. P. Langlands, *Automorphic forms on* $\mathrm{GL}(2)$, Springer Lecture Notes 114 (1970).

[7] J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* $\mathrm{SL}(2)$, to appear in Can. J. Math.

[8] R. P. Langlands, *Representations of abelian algebraic groups*, preprint (1968).

[9] ———, *Problems in the theory of automorphic forms*, Springer Lecture Notes 170.

[10] ———, *On the classification of irreducible representations of real algebraic groups*, preprint (1973).

[11] R. Steinberg, *Regular elements of semi-simple algebraic groups*, Publ. Math. I.H.E.S. (1965).

[12] D. Shelstad, *Characters and inner forms of a quasi-split group over* **R**, to appear in Comp. Math.

[13] ———, *Orbital integrals and a family of groups attached to a real reductive group*, to appear.

[14] ———, *Notes on L-indistinguishability* (based on a lecture of R. P. Langlands), to appear in Proc. of A.M.S. Summer Institute (Corvallis).