On the Zeta-Functions of Some Simple Shimura Varieties*

by

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1. Introduction.

In an earlier paper [14] I have adumbrated a method for establishing that the zero-function of a Shimura variety associated to a quaternion algebra over a totally real field can be expressed as a product of L-functions associated to automorphic forms. Now I want to add some body to that sketch. The representation-theoretic and combinatorial aspects of the proof will be given in detail, but it will simply be assumed that the set of geometric points has the structure suggested in [13]. This is so at least when the algebra is totally indefinite, but it is proved by algebraic-geometric methods that are somewhat provisional in the context of Shimura varieties. However, contrary to the suggestion in [13] the general moduli problem has yet to be treated fully. There are unresolved difficulties, but they do not arise for the problem attached to a totally indefinite quaternion algebra, which is discussed in detail in [17].

It does not add to the essential difficulties if we enlarge our perspective a little and consider not only the zeta-function defined by the constant sheaf but also that defined by the sheaves associated to finite dimensional representatives of the group defining the variety, and we might even dissipate some of the current misconceptions about the nature of these sheaves. Their existence is a formal consequence of Shimura's conjecture. We should moreover not confine ourselves to the multiplicative group of the quaternion algebra, but should in addition consider subgroups lying between the full multiplicative group and the kernel of the norm, for then we can see the effect of L-indistinguishability [7] in the place where it was first noticed.

In this introduction the results of [7], to which [22] is meant to serve as a kind of exegesis, are used in conjunction with facts about continuous cohomology to arrive at an assertion about the zeta-function which the remainder of the paper is devoted to proving. Some readers will find that I have given too free rein to a lamentable tendency to argue from the general to the particular, and have obfuscated them by interjecting unfamiliar concepts of representation theory into what could be a purely geometric discussion. My intention is not that, but rather to equip myself, and perhaps them as well, for a serious study of the Shimura varieties in higher dimensions. We are in a forest whose trees will not fall with a few timid hatchet blows. We have to take up the double-bitted axe and the cross-cut saw, and hope that our muscles are equal to them.

The method of proof has already been described in [14]. It is ultimately combinatoric. The Bruhat-Tits buildings, which arise naturally in the study of orbital integrals and Shimura varieties, are used systematically. However the automorphic L-functions used to express the zeta-functions of the varieties are unusual and most of §2 is taken up with the attempt to understand them and express their coefficients in manageable, elementary terms. The appearance of L-indistinguishability complicates the task considerably.

The meaning of the conjectures of [13] is also obscure, even to their author, and considerable effort is necessary before it is revealed sufficiently that a concrete expression for the coefficients of the zeta-functions is obtained. Once this is done, in §3 and the appendix, the equality to be proved is reduced to elementary assertions which are proved by combinatorial arguments in §4.

A connected reductive group G over $\mathbf{Q}$ and a weight μ of the associate group ${}^LG^0$ are the principal data specifying a Shimura variety. The conditions they must satisfy are described in [4]. If $\mathbf{A}_f$ is the group of finite adèles one needs an open compact subgroup K of $G(\mathbf{A}_f)$ as well. μ is the weight of ${}^LG^0$ defined by the co-weight h_0 of [13]. The primary datum is h_0, rather than μ. To completely define $S(K)$ one needs h_0. The variety will be denoted by $S(K)$ and only K will appear explicitly, for G and μ are usually fixed. The group ${}^LG^0$ comes provided with a Borel subgroup ${}^LB^0$ and a Cartan subgroup ${}^LT^0$ in ${}^LB^0$. We may suppose that μ is a positive

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weight of ${}^LT^0$. Moreover if L is a large Galois extension of $\mathbf{Q}$ then $\mathfrak{G}(L/\mathbf{Q})$ acts on ${}^LG^0$, fixing the subgroups ${}^LB^0$ and ${}^LT^0$. If $\mathfrak{G}(L/E)$ is the stabilizer of μ then the Shimura conjecture, which has been proved for the groups we shall consider, states that $S(K)$ has a model over E characterized by the arithmetic structure of its special points [4]. We will always use this model. The set of complex points on $S(K)$ is a double coset space

$$G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K \ . \tag{1.1}$$

Here $K_\infty \subseteq G(\mathbf{R})$ and $G(\mathbf{R})/K_\infty$ is a finite union of Hermitian symmetric spaces.

Let Z_0 be the intersection of the kernels of the rational characters over $\mathbf{Q}$ of the center Z of G and let ξ be a representation of G on the vector space V which is trivial on Z_0, both ξ and V being defined over $\mathbf{Q}$. If K is sufficiently small, as we assume, then

$$V(Z) \times_{(G(\mathbf{Q}),\xi)} G(\mathbf{A})/K_\infty K \to S(K)$$

defines a locally constant sheaf F_ξ^K or F_ξ on $S(K)$. Using the étale coverings $S(K') \to S(K)$, $K \subseteq K'$, which are defined over E, and the formalism of [12], we may defined the sheaves F_ξ as l-adic sheaves in the étale topology.

For the groups that occur in this paper the quotient (1.1) is compact and the varieties $S(K)$ are proper, and we introduce the cohomology groups

$$H^i(S(K), F_\xi) \ .$$

They can be taken over $\mathbf{Q}$ or over $\mathbf{Q}_l$, according to the exigencies of the context. If $g \in G(\mathbf{A}_f)$ the formalism of [12], which is the usual formalism, associates to g a linear transformation

$$T^i(g)\colon H^i(S(K), F_\xi) \to H^i(S(K), F_\xi) \ .$$

The $T^i(g)$ act to the right and commute with $\mathfrak{G}(\overline{E}/E)$ which acts to the left. I recall that in the theory of Shimura varieties E is given as a subfield of $\mathbf{C}$.

We shall be concerned with the zeta-function of F_ξ as a formal rather than as an analytic object, and so we shall only be interested in the individual local factors, and these only at the primes $\mathfrak{p}$ of E for which the suggestions of [13] apply. If $\Phi_\mathfrak{p}$ is the Frobenius at $\mathfrak{p}$ and τ^i the representation of $\mathfrak{G}(\overline{E}/E)$ on $H^i(S(K), F_\xi)$ then the logarithm of the zeta-function is given by

$$\log\, Z_\mathfrak{p}(s, S(K), F_\xi) = \sum_{n=1}^{\infty} n^{-1} \sum_i (-1)^t \mathrm{trace}\, \tau^i(\Phi_\mathfrak{p}^n) |\varpi_\mathfrak{p}|^{ns} \ .$$

If q is the number of elements in the residue field then

$$|\varpi_\mathfrak{p}| = q^{-1} \ .$$

We shall be more interested in

$$Z_\mathfrak{p}(s, S(K), F_\xi) = \prod_{\mathfrak{p}|p} Z_\mathfrak{p}(s, S(K), F_\xi) \ .$$

We want to show that the zeta-function can be expressed in terms of the L-functions associated to automorphic forms. Considerations that will be explained shortly suggest an elegant conjecture. It is false and, in general, meaningless, but it is meaningless for interesting reasons, stemming from L-indistinguishability, and for the groups treated in this paper we will be able to modify and correct it, by taking the results of [7] into account. I will present it in a slicker form than it at first appeared, even though its genesis is thereby somewhat obscured.

Some auxiliary objects must be introduced. To simplify our considerations we suppose that the restriction of ξ to the center Z of G is of the form

$$\xi(z) = \nu(z) I$$

where ν is a rational character. This does not affect the generality, since ξ can be decomposed into a direct sum of representations satisfying this condition.

If T is a Cartan subalgebra of G over $\mathbf{R}$, γ is a regular element of $T(\mathbf{R})$, δ belongs to $\mathfrak{D}(T/R)$ (the explanation of this and other notation used below will be found in [16]), and f is a smooth function in $G(\mathbf{R})$ which is compactly supported modulo $Z(\mathbf{R})$ and satisfies

$$f(zg) = \nu(z)f(g) \qquad z \in Z(\mathbf{R})$$

we may introduce

$$\Phi^{\delta}(\gamma, f) = \int_{T^h(\mathbf{R})\backslash G(\mathbf{R})} f(g^{-1}h^{-1}\gamma h g)dg$$

as in [7]. Here h in $\mathfrak{A}(T)$ represents δ. We may also introduce the stable orbital integrals [7]

$$\Phi^{T/1}(\gamma, f) = \sum_{\mathfrak{D}(T/\mathbf{R})} \Phi^{\delta}(\gamma, f) \ .$$

Choose $f = f_\xi$ so that

$$\Phi^{T/1}(\gamma, f_\xi) = 0$$

unless $T(\mathbf{R})$ is fundamental, that is, unless $Z(\mathbf{R})\backslash T(\mathbf{R})$ is compact, and so that

$$\Phi^{T/1}(\gamma, f_\xi) = \frac{\operatorname{trace}\xi(\gamma)}{\operatorname{meas} Z(\mathbf{R})\backslash T(\mathbf{R})}$$

if $T(\mathbf{R})$ is fundamental. It has to be shown that f_ξ exists, but for the groups we shall ultimately consider this has been done (cf. [15], §4).

If π_∞ is a representation of $G(\mathbf{R})$ set $\mathfrak{m}(\pi_\infty) = \mathfrak{m}(\pi_\infty, \xi)$ equal to 0 unless

$$\pi_\infty(z) = \nu^{-1}(z)I \qquad z \in Z(\mathbf{R}) \ .$$

Otherwise let $\mathfrak{m}(\pi_\infty)$ be the trace of

$$\pi_\infty(f_\xi) = \int_{Z(\mathbf{R})\backslash G(\mathbf{R})} f_\xi(g)\pi_\infty(g)dg \ .$$

We temporarily disregard the circumstance that $\mathfrak{m}(\pi_\infty)$ is in fact not well-defined. If π_f is a representation of $G(\mathbf{A}_f)$ let $\mathfrak{m}(\pi_f) = \mathfrak{m}(\pi_f, K)$ be the multiplicity with which the trivial representation of K occurs in π_f.

The numbers $\mathfrak{m}(\pi_\infty)$ and $\mathfrak{m}(\pi_f)$ will occur as exponents in the expression of $Z(s, S(K), F_\xi)$ as a product of L-functions associated to automorphic representations $\pi = \pi_\infty \otimes \pi_f$, but to specify an L-function one needs a representation of ${}^LG$ as well. Let r^0 be the representation of ${}^LG^0$ with highest weight μ. The element μ, or rather its restriction to the derived group, is a *poids minuscule* in the terminology of Bourbaki. Thus the weights of r^0 are the $\omega\mu$ with ω in the Weyl group $\Omega({}^LT^0, {}^LG^0)$ of ${}^LT^0$ in ${}^LG^0$. Let ${}^LM^0$ be the group generated by ${}^LT^0$ and the coroots $\alpha^\vee$ orthogonal to μ. The stabilizer of μ is $\Omega({}^LT^0, {}^LM^0)$ and the dimension of r^0 is

$$[\Omega({}^LT^0, {}^LG^0) \colon \Omega({}^LT^0, {}^LM^0)] \ .$$

Let x be a non-zero vector transforming according to the weight μ. There is exactly one way on extending r^0 to a representation, again denoted r^0, of

$${}^LG^0 \times \mathfrak{G}(L/E)$$

on the same space so that

$$r^0(\sigma)x = x \qquad \sigma \in \mathfrak{G}(L/E) \ .$$

L is here just some large Galois extension of $\mathbf{Q}$, and could be taken to be $\overline{\mathbf{Q}}$. Let

$$r = \mathrm{Ind}({}^LG, {}^LG^0 \times \mathfrak{G}(L/E), r^0) \ .$$

The group ${}^LG$ is a semi-direct product

$${}^LG^0 \times \mathfrak{G}(L/\mathbf{Q}) \ .$$

Let q be the dimension of $S(K)$. If the phenomenon of L-indistinguishability did not manifest itself, one might suspect that

$$Z(s, S(K), \xi) = \prod L(s - q/2, \pi, r)^{\mathfrak{m}(\pi)\mathfrak{m}(\pi_\infty)\mathfrak{m}(\pi_f)} \ . \tag{1.2}$$

The zeta-function on the left is the product of the local zeta-functions, including a factor from the infinite places. The product on the right is over all automorphic representations of $G(A)$ and $\mathfrak{m}(\pi)$ is the multiplicity with which π occurs in the space of automorphic forms. The grounds for the suspicion are initially flimsy, but I shall try to explain them. If the conjectures of Weil and Ramanujan are compatible the shift in the variable form from s to $s - q/2$ must be present.

Let N be the number of absolutely irreducible components of ξ, counted with multiplicity, and let $\lambda^1, \dots, \lambda^k$ be the highest weights of the distinct components $\tilde{\xi}^1, \dots, \tilde{\xi}^k$ of the contragredient representation $\tilde{\xi}$ with respect to some order on the roots of a fundamental Cartan subgroup $T(\mathbf{R})$ in $G(\mathbf{R})$. Let g be one-half the sum of the positive roots with respect to this order, and for every ω in the complex Weil group $\Omega(T(\mathbf{C}), G(\mathbf{C}))$ set

$$\Lambda^i_\omega = \omega(\lambda^i + g) \ .$$

For each Λ^i_ω there is a discrete series representation $\pi(\Lambda^i_\omega)$. We set

$$\Pi(\xi) = \{\pi(\Lambda^i_\omega) | 1 \le i \le k,\ \omega \in \Omega(T(\mathbf{C}), G(\mathbf{C}))\} \ .$$

It is a union of the L-indistinguishable classes [11]

$$\Pi(\xi^i) = \{\pi(\Lambda^i_\omega) | \omega \in \Omega(T(\mathbf{C}), G(\mathbf{C}))\} \ .$$

f_ξ has been so chosen that

$$\sum_{\pi_\infty \in \Pi(\xi)} \mathfrak{m}(\pi_\infty) \ ,$$

which is well-defined, is equal to $(-1)^1 dN$ if

$$d = \text{ dimension } r^0 \ .$$

In the notation of [2]

$$\oplus_{\pi_\infty \in \Pi(\xi)} H^i(\mathfrak{g}, \mathfrak{k}_\infty, \pi_\infty \otimes \xi)$$

is 0 unless $i = q$ when its dimension is dN, for by the results of those notes

$$\oplus_{\pi_\infty \in \Pi(\xi^j)} H^i(\mathfrak{g}, \mathfrak{k}_\infty, \pi_\infty \otimes \xi^{j\prime})$$

is 0 unless $j = j'$ and $i = q$ when its dimension is d. To see this one has to observe, among other things, that $\Omega(T(\mathbf{C}), G(\mathbf{C}))$ is isomorphic to $\Omega({}^LT^0, {}^LG^0)$, that $\Omega({}^LT^0, {}^LM^0)$ is isomorphic to $\Omega(T(\mathbf{R}), K_\infty)$, and that the restriction of

$$\oplus_{\pi_\infty \in \Pi(\xi^j)} \pi_\infty$$

to the connected component of $G(\mathbf{R})$ is therefore the direct sum of d irreducible representations, namely the discrete series representations with the same infinitesimal character as $\tilde{\xi}^j$.

If, as occasionally happens ([12], [14]), $\Pi(\xi)$ consisted of a single element π_∞, then each time that

$$\pi = \pi_\infty \otimes \pi_f$$

occurred in the space of automorphic forms it would contribute a subspace to the cohomology group $H^q(S(K), F_\xi)$ over $\mathbf{C}$ of dimension $d\mathfrak{m}(\pi_f)$. A corresponding L-function should appear as a factor of the zeta-function. The degree of the factors appearing in its expression as an Euler product should be

$$[E\colon \mathbf{Q}]d\mathfrak{m}(\pi_f) = \mathfrak{m}(\pi_f) \text{ dimension } r$$

for almost all $\mathfrak{p}$. The L-function should appear in the numerator or denominator according as q is even or odd. We are led to guess that it is

$$L(s - q/2, \pi, r)^{(-1)^q m(\pi_f)} = L(s - 1/2, \pi, r)^{\mathfrak{m}(\pi_\infty)\mathfrak{m}(\pi_f)} \ .$$

Occam's razor and the ordinary Eichler-Shimura theory then suggest (1.2). Unfortunately $\prod(\xi)$ generally consists of several elements.

We might still be able to hold onto (1.2) if whenever π_∞ and π'_∞ were two L-indistinguishable representations of $G(\mathbf{R})$ the representations

$$\pi = \pi_\infty \otimes \pi_f \text{ and } \pi' = \pi'_\infty \otimes \pi_f$$

occurred in the space of automorphic forms with the same multiplicity. We would just have to choose a representative from each L-indistinguishable class and agree that the product in (1.2) was to be taken over those $\pi = \pi_\infty \otimes \pi_f$ for which π_∞ belonged to our set of representatives. But we would have to search for another definition of the exponent $\mathfrak{m}(\pi_\infty)$, because, as it stands, different choices of f_ξ lead to different values for trace $\pi_\infty(f_\xi)$. It does not matter, for π and π' do not always occur with the same multiplicity [7]. One may occur while the other does not. This clearly means that the degrees of the Euler products $L(s - q/2, \pi, r)$ are then too large. We must seek Euler products of smaller degree.

L-indistinguishability appears when the sets $\mathfrak{D}(T)$of [16] have more than one element. If, as we may assume, the center of ${}^LG^0$ is connected then we may use the definitions of [16] to introduce groups H over $\mathbf{Q}$ and homomorphisms $\psi\colon {}^LH \to {}^LG$. Suppose the principle of functoriality applies and the L-indistinguishable class of π is the image under ψ_* of that of π'. Then

$$L(s, \pi, r) = L(s, \pi', r \circ \psi) \ .$$

In general r is irreducible but $r \circ \psi$ is often reducible

$$r \circ \psi = \oplus r_i \ .$$

We must expect that the functions $L(s - q/2, \pi', r_i)$ will appear in the ultimate, correct form of (1.2). The definition of H leads naturally to such a decomposition of $r \circ \psi$. The constituents r_i might not be irreducible, but they seem nonetheless to yield the L-functions necessary for an analysis of $Z(s, S(K), F_\xi)$.

Without repeating the definition of the groups H, we recall that each of them is attached to a triple (T, κ, g_1). If T_{sc} is the Cartan subgroup of the simply connected form of G defined by T the term κ is a homomorphism of its lattice of coweights $X_*(T_{\text{sc}})$ into $\mathbf{C}^\times$. Moreover g_1 allows an identification of $X_*(T_{\text{sc}})$ with the lattice $X^*({}^LT^0_{\text{sc}})$ of rational characters of the Cartan subgroup of the L-group ${}^LG^0_{\text{sc}}$ and κ can therefore be transported ot a homomorphism κ' of ${}^*({}^LT^0_{\text{sc}})$ into $\mathbf{C}^\times$. We may extend κ' to a $\mathfrak{G}(K/\mathbf{Q})$-invariant homomorphism

$$\epsilon\colon X^*({}^LT^0) \to \mathbf{C}^\times \ .$$

Since

$${}^LT^0 = \operatorname{Hom}(X^*({}^LT^0), \mathbf{C}^\times) \ ,$$

ϵ is an element of ${}^LT^0$ and lies in the center of $\psi({}^LH)$. The representation $r \circ \psi$ is the direct sum of its restrictions r_i to the eigenspaces of $r(\epsilon)$, but these may not be irreducible. Since any two choices of the extension ϵ differ by a central element in ${}^LG$, these subspaces are well-defined.

We now try to modify (1.2) by including the $L(s - q/2, \pi', r_i)$ in such a way that at least the local factors at infinity of the hypothetically equal Euler products, $Z(s, S(K), F_\xi)$ on the left and some combination of the $L(s, \pi', r_i)$ on the right, are likely to be the same. It seems to be sufficient to consider H defined by a T for which $T(\mathbf{R})$ is fundamental.

The representation r^0 may be regarded as a subrepresentation of the restriction of r to ${}^LG^0 \rtimes \mathfrak{G}(\overline{\mathbf{Q}}/E)$. It is clear that the eigenspaces of $r(\epsilon)$ also decompose r^0 into

$$\oplus r_i^0$$

and that

$$r_i = \mathrm{Ind}({}^LG, {}^LG^0 \rtimes \mathfrak{G}(\overline{\mathbf{Q}}/E), r_i^0) \ .$$

To define $L(s, \pi_\infty, r_i)$ we need only know the restriction of r_i to the local associate group at infinity, ${}^LG_\infty = {}^LG^0 \times \mathfrak{G}(\mathbf{C}/\mathbf{R})$.

Implicit in the definition of E is an imbedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, and hence E is a subfield of $\mathbf{C}$ and $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ is a subgroup of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. The double cosets

$$\mathfrak{G}(\mathbf{C}/\mathbf{R})\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathfrak{G}(\overline{\mathbf{Q}}/E)$$

parametrize the infinite places v of E, the coset represented by r defining the valuation

$$x \to |\tau(x)| \ .$$

The decomposition group $\mathfrak{G}_v$ at $v = v(\tau)$ is

$$\mathfrak{G}(\overline{\mathbf{Q}}/E) \cap \tau^{-1}\mathfrak{G}(\mathbf{C}/\mathbf{R})\tau$$

and the restriction of r_i to ${}^LG_\infty$ is

$$\oplus_v \mathrm{Ind}({}^LG_\infty, {}^LG^0 \times \mathfrak{g}_v, r_i^0|\tau\mathfrak{G}_v\tau^{-1}) = \oplus r_i(v) \ .$$

Consequently

$$L(s, \pi'_\infty, r_i) = \prod_v L(s, \pi'_\infty, r_i(v)) \ ,$$

and we attempt to arrange that the contributions from the $L(s - q/2, \pi'_\infty, r_i(v))$ for a given v yield the local factor of $Z(s, S(K), F_\xi)$ at the same place.

However, our primary interest in this paper is not with equality of the Euler factors at infinity, but with equality at almost all finite places, and we are only using the infinite places as a guide to the correct statement. The one place given by the imbedding $E \subseteq \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ provided by the definition of E will serve.

The weights of r^0 on ${}^LT^0$ are the elements of

$$\{\omega\mu | \omega \in \Omega({}^LT^0, {}^LG^0)\}$$

and the differences of any two μ_1, μ_2 of these weights is an integral linear combination of roots of ${}^LT^0$. Thus

$$\mu_1(\epsilon)/\mu_2(\epsilon) = \kappa'(\mu_1 - \mu_2) = \pm 1 \ ,$$

because $T(\mathbf{R})$ is fundamental and κ' therefore of order one or two. Thus there are one or two r_i^0 and, at the cost of adding a second of dimension zero, we suppose there are two, r_1^0 and r_2^0. We are also going to decompose the set $\Pi(\xi)$ into two subsets $\Pi^1(\xi)$ and $\Pi^2(\xi)$, with r_i^0 and $\Pi^i(\xi)$ matched. We first see how to distinguish between r_1^0 and r_2^0.

The L-indistinguishable class $\Pi(\xi^j)$ is equal to Π_{φ_j}, where

$$\varphi_j\colon W_{\mathbf{C}/\mathbf{R}} \to {}^LG \ .$$

The notation is that of [11]. Suppose $\varphi_j = \psi \circ \varphi'_j$ where

$$\varphi'_j\colon W_{\mathbf{C}/\mathbf{R}} \to {}^LH$$

and that π'_∞ lies in $\Pi_{\varphi'_j}$. Then

$$L(s, r\circ\varphi_j) = L(s,\pi_\infty, r) = L(s,\pi'_\infty, r\circ\psi) = L(s, r\circ\psi\circ\varphi'_j) \ .$$

If $T(\mathbf{R})$ is fundamental we may suppose [11] that φ'_j takes $\mathbf{C}^\times \subseteq W_{\mathbf{C}/\mathbf{R}}$ to ${}^LT^0$ and then

$$\varphi'_j(z) = z^{\Lambda'}\overline{z}^{\sigma\Lambda'}, \qquad z\in\mathbf{C}^\times \ .$$

Here Λ' is one of the Λ^j_ω, at least if we use the identification of $X^*(T)$ with $X_*({}^LT^0)$ provided by g_1, and σ is the non-trivial element of $\mathfrak{G}(\mathbf{C}/\mathbf{R})$. Λ' is non-singular and there is exactly one weight μ' of r^0 which lies in the closure of the Weil chamber opposite to that containing Λ'. Since any other weight μ'' is of the form $\omega\mu'$,

$$\langle\Lambda',\mu''\rangle = \langle\Lambda',\mu''-\mu'\rangle + \langle\Lambda',\mu'\rangle > \langle\Lambda',\mu'\rangle$$

if $\mu''\neq\mu'$. Given π'_∞ we take r^0_1 to be the representation with μ' as a weight, and r^0_2 to be the other. Observe that the labeling depends on the L-indistinguishability class of π'_∞, and hence on the class in $\Phi(H)$ represented by φ'_j, but not directly on Λ'.

We now decompose $\prod(\xi)$ into two subsets $\prod^1(\xi)$, $\prod^2(\xi)$, matching $\prod^i(\xi)$ with r_i. Since $T(\mathbf{R})$ is taken to be fundamental μ is defined by a coweight

$$\mu^\vee = h'_0$$

of T. Here h'_0 is conjugate under $G(\mathbf{R})$ to h_0, and if we use g_1 as in [16] to introduce an isomorphism

$$X_*(T)\xrightarrow{\sim}X^*({}^LT^0)$$

then μ lies in the orbit of $\mu^\vee$ under the Weyl group. For each j let Λ^j be an element of $\{\Lambda^j_\omega|\omega\in\Omega(T(\mathbf{C}),G(\mathbf{C}))\}$ which is such that it and $\mu^\vee$ lie in opposing closed Weyl chambers. Since any two choices of Λ^j lie in the same orbit under $\Omega(T(\mathbf{R}),G(\mathbf{R}))$, the representation $\pi(\Lambda^j)=\pi^j(\mu^\vee)$ is well-defined and independent of the choice of Λ^j. Every element w of the normalizer of $T(\mathbf{C})$ in $G(\mathbf{C})$ lies in $\mathfrak{A}(T/\mathbf{R})$ [21] and

$$w\to\{a_\tau=\tau(w)w^{-1}|\tau\in\mathfrak{G}(\mathbf{C}/\mathbf{R})\}$$

yields an injection

$$\Omega(T(\mathbf{C}),G(\mathbf{C}))/\Omega(T(\mathbf{R}),G(\mathbf{R}))\hookrightarrow\mathfrak{E}(T/\mathbf{R}) \ .$$

The image is $\mathfrak{D}(T/\mathbf{R})$, but that does not matter. If ω is represented by w we put $\pi(\omega^{-1}\Lambda^j)$ in $\prod^1(\xi)$ or in $\prod^2(\xi)$ according as $\kappa(\{a_\tau\})$ is 1 or -1. The assignment does depend on the choice of Λ^j, but that may be inevitable. I observe that it is not difficult to see [21] that under the isomorphism

$$H^{-1}(\mathfrak{G}(\mathbf{C}/\mathbf{R}),X_*(T))\xrightarrow{\sim}H^1(\mathfrak{G}(\mathbf{C}/\mathbf{R}),T(\mathbf{C}))$$

the cocycle $\{a_\tau\}$ corresponds to $\omega\mu^\vee-\mu^\vee$.

One point on which we have insisted when assigning the elements of $\prod(\xi)$ to the two sets $\prod^1(\xi)$ and $\prod^2(\xi)$ is that $\pi(\Lambda^j)$ lie in $\prod^1(\xi)$ for each j. In order to justify this we recall the way in which the complex structure on $S(K)$ is introduced as well as the form suggested by Serre [20] for the local factors $Z_v(s,S(K),F_\xi)$ in the case of trivial ξ.

If K'_∞ is the centralizer of h'_0 then

$$G(\mathbf{R})/K'_\infty \simeq G(\mathbf{R})/K_\infty$$

and the complex structure defining that on $S(K)$ is obtained from an imbedding

$$G(\mathbf{R})/K'_\infty \hookrightarrow G(\mathbf{C})/P(\mathbf{C})$$

if P is the parabolic subgroup whose Lie algebra is spanned by those X for which

$$\mu^\vee(z)X \equiv X \quad \text{or} \quad \mu^\vee(z)X = z^{-1}X$$

for $z \in \mathrm{GL}(1)$.

We choose the order on the roots of T with respect to which Λ^j lies in the positive Weyl chamber. Suppose U is a subspace of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A})/K)$ transforming under $G(\mathbf{R})$ according to $\pi(\Lambda^j)$. It is explained in [12] (cf. also [2]) how to pass from an element of

$$\mathrm{Hom}_{K'_\infty}(\Lambda^q\mathfrak{g}/\mathfrak{k}'_\infty, U \otimes \xi) = \oplus_j \mathrm{Hom}_{K'_\infty}(\Lambda^q\mathfrak{g}/\mathfrak{k}'_\infty, U \otimes \xi^j)^{N_j}$$

to a q-form on $S(K)$ with values in F_ξ. Here N_j is the multiplicity with which ξ^j occurs in ξ. It is easier to work with

$$\mathrm{Hom}_{K'_\infty}(\Lambda^q\mathfrak{g}/\mathfrak{k}'_\infty \otimes \tilde{\xi}^j, U)\ .$$

Clearly

$$\mathfrak{g}/\mathfrak{k}'_\infty = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+$ is spanned by the image of the root vectors associated to noncompact positive roots. The elements of $\mathfrak{p}^-$ yield holomorphic tangent vectors; those of $\mathfrak{p}^+$ yield anti-holomorphic tangent vectors. We have

$$\mathrm{Hom}_{K'_\infty}(\Lambda^q\mathfrak{p}^+ \otimes \tilde{\xi}^j, U) \subseteq \mathrm{Hom}_{K'_\infty}(\Lambda^q\mathfrak{g}/\mathfrak{k}'_\infty \otimes \tilde{\xi}^j, U)\ ,$$

and it is the elements of the first space which yield forms of Hodge type $(q, 0)$.

Let ρ_P be one-half the sum of the non-compact positive roots and ρ_K one-half the sum of the compact positive roots. The space $\Lambda^q\mathfrak{p}^+$ is one-dimensional and transforms under K'_∞ according to the weight $2\rho_P$. The highest weight of $\tilde{\xi}^j$ is $\Lambda^j - \rho_P - \rho_K$ and that of $\Lambda^q\mathfrak{p}^+ \otimes \tilde{\xi}^j$ is therefore $\Lambda^j + \rho_P - \rho_K$. However, it is a fundamental fact [5] that the restriction of $\pi(\Lambda^j)$ to K'_∞ has an irreducible component with highest weight $\Lambda^j + \rho_P - \rho_K$. Thus

$$\mathrm{Hom}(\Lambda^q\mathfrak{p}^+ \otimes \tilde{\xi}^j, U) \neq 0$$

and U or $\pi(\Lambda^j)$ contributes cohomology of type $(0, q)$. No other element of $\prod(\xi^j)$ does so.

On the other hand, we have written the restriction of φ'_j to $\mathbf{C}^\times \subseteq W_{\mathbf{C}/\mathbf{R}}$ as

$$z \to z^{\Lambda'} z^{\sigma\Lambda'}$$

and we have taken μ' and Λ' to lie in opposing Weyl chambers. If ξ is trivial and g' is one-half the sum of the roots α for which $\langle\Lambda', \alpha\rangle < 0$, then $\Lambda' = -g'$ and

$$\mu'(\varphi'_j(z)) = z^{-\langle g',\mu'\rangle}\overline{z}^{-\langle\sigma g',\mu'\rangle} = \overline{z}^{2\langle g',\mu'\rangle}(z\overline{z})^{-\langle g',\mu'\rangle}\ .$$

If q is the dimension of $S(K)$ then

$$\langle g', \mu'\rangle = q/2\ .$$

[Added in proof (November, 1979). It appears that the local zeta-functions at $\mathfrak{p}$ calculated in this paper are those associated to $\tilde{\xi}$ and not to ξ. The correct definitions would entail replacing r by its contragredient $\tilde{r}$ and $r(v)$ by

$$\tilde{r}(v) = \tilde{r}_1(v) + \tilde{r}_2(v)\ ,$$

if v is the place of E defined by $E \subseteq \mathbf{Q} \subseteq \mathbf{C}$. If α is the composite of

$$W_{\mathbf{C}/\mathbf{R}} \to W_{\mathbf{R}/\mathbf{R}} = \mathbf{R}^{\times}$$

with the absolute value, then

$$L(s - q/2, \pi'_{\infty}, \tilde{r}(v) \circ \varphi) = L(s, \alpha^{-q/2} \otimes (\tilde{r}(v) \circ \varphi_j)) \ .$$

The one-dimensional subspace corresponding to the weight $-\mu'$ transforms under the restriction of $\alpha^{-q/2} \otimes (\hat{r}(v) \circ \varphi_j))$ to $\mathbf{C}^{\times}$, ξ being trivial, according to the character

$$\overline{z}^{-2\langle g', \mu' \rangle} = \overline{z}^{-1} \ \mathbf{]}$$

We have been led to our labeling by the principle that the $\pi(\Lambda^j)$, as the representations giving rise to forms of type $(0, q)$, should be matched with the weight $-\mu'$, which yields, when ξ is trivial, the character $z \to \overline{z}^{-q}$. One is led to this principle by a suggestion of Serre [20].

If S is a variety over the number field E the factor at infinity of its zeta-function can of course be expressed in terms of Γ-functions and is a product over the infinite places of E of local zeta-functions. For each place v we take a corresponding imbedding $E \hookrightarrow \mathbf{C}$ and introduce the set $S(\mathbf{C})$ of complex points on

$$S \otimes_E \mathbf{C} \ .$$

The local zeta-function at the place v may be expressed as

$$\textstyle\prod_i L(s, \rho_i)^{(-1)i}$$

where ρ_i is a representation of the Weil group $W_{\mathbf{C}/E_v}$ on the cohomology group

$$H^i(S(\mathbf{C})) = \oplus_{p+q=i} H^{p,q}(S(\mathbf{C})) \ .$$

Here, in order to conform to custom, q loses temporarily its significance as the dimension of $S(K)$. The restriction of ρ_i to $\mathbf{C}^{\times}$ is defined by demanding that $\rho_i(z)$ act on $H^{p,q}(S(\mathbf{C}))$ as $z^{-p}\,\overline{z}^{-q}$. If E_v is complex there is nothing more to be said. If it is real we have to define $\rho_i(w)$ if w is an element of $W_{\mathbf{C}/E_v}$ which maps to the complex conjugation in $\mathfrak{G}(\mathbf{C}/E_v)$ and has square -1. When E_v is real, complex conjugation defines an involution ι of $S(\mathbf{C})$ and an associated map ι^* on cohomology. We let $\rho_i(w)$ act on $H^{p,q}(S(\mathbf{C}))$ as $(-1)^p \iota^*$.

Now we must bring these puzzling divagations to bear upon some simple examples. Suppose F is a totally real field and $\widetilde{G}$ is the multiplicative group of a quaternion algebra D over F. The split algebra is excluded. Let A be a connected subgroup of $G_1 = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}(1)$ which is defined over $\mathbf{Q}$ and let G be the inverse image of A in $\widetilde{G}_1 = \mathrm{Res}_{F/\mathbf{Q}} \widetilde{G}$ with respect to the norm.

The group ${}^L G$ is a quotient of

$$\left(\prod\nolimits_{\mathfrak{G}(\overline{\mathbf{Q}}/F) \backslash \mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})} \mathrm{GL}(2, \mathbf{C}) \right) \times \mathfrak{G}(K/\mathbf{Q}) = {}^L G^0 \times \mathfrak{G}(K/\mathbf{Q})$$

by a subgroup of the center of ${}^L G^0$, namely by the set of (z_σ) for which

$$\prod\nolimits_{\mathfrak{G}(K/F) \backslash \mathfrak{G}(K/\mathbf{Q})} \lambda_\sigma(z_\sigma) = 1, \qquad (\lambda_\sigma) \in Y_*,$$

with Y_* defined as in §6 of [7]. ${}^L T^0$ is the image of the diagonal matrices and $X^*({}^L T^0)$ is

$$\{(a_\sigma, b_\sigma) | a_\sigma, b_\sigma \in \mathbf{Z}\,, \quad (\lambda_\sigma) \in Y_* \text{ if } \lambda_\sigma = a_\sigma + b_\sigma\} \ .$$

We have supposed that $F \subseteq \overline{\mathbf{Q}} \subseteq \mathbf{C}$. Then μ will be (μ_σ) and (μ_σ) will be 0 unless the imbedding $x \to \sigma(x)$ of F in $\mathbf{C}$ splits D, when it is $(1,0)$. Observe that (λ_σ) with λ_σ equal to 1 if the quaternion algebra splits at σ and 0 otherwise must belong to Y_*. This is to be treated as a condition on A, because μ is determined by D alone.

For such a G and μ we want to present a correct and verifiable expression for $Z(s, S(K), F_\xi)$ as a product of L-functions associated to automorphic forms. Two representations $\pi = \otimes \pi_w$ and $\pi' = \otimes \pi'_w$ will be said to be L-indistinguishable if π_w and π'_w are L-indistinguishable for all w and equivalent for almost all w. Our expression for $Z(s, S(K), F_\xi)$ will be given as a product over L-indistinguishable classes. We must describe the contribution from each class.

Suppose that $\mathfrak{m}(\pi')$, the multiplicity with which π' occurs in the space of automorphic forms, is constant within the L-indistinguishability class Π of π. If Π_w is the L-indistinguishability class of π_w then

$$\varpi = \otimes_w \left(\oplus_{\Pi_w} \pi'_w\right) = \otimes_w \varpi_w = \varpi_\infty \otimes \varpi_f$$

is a representation which contains each π' in Π exactly once. We set

$$\mathfrak{m}\left(\Pi_\infty\right) = \sum_{\pi_\infty \in \Pi_\infty} \text{trace}\ \pi_\infty(f_\xi)$$

provided

$$\pi_\infty(z) = \nu^{-1}(z) I, \qquad z \in Z(\mathbf{R}),$$

for one, and hence all elements of Π_∞. Otherwise $\mathfrak{m}(\Pi_\infty)$ is to be 0. $\mathfrak{m}(\Pi_\infty)$ is well-defined, and is easily seen to be 0, or -1 when ξ is absolutely irreducible. Let $\mathfrak{m}(\Pi_f)$ be the multiplicity with which the trivial representation of K occurs in ϖ_f; and let $\mathfrak{m}(\Pi)$ be $\mathfrak{m}(\pi)$. The contribution of the class Π to the zeta-function is

$$L(s - q/2, \pi, r)^{\mathfrak{m}(\Pi)\mathfrak{m}(\Pi_\infty)\mathfrak{m}(\Pi_f)} \ .$$

If $\mathfrak{m}(\pi')$ is not constant within Π there is a Cartan subgroup T of G with

$$[\mathfrak{E}(T/\mathbf{A}), \mathfrak{E}(T/F)] = 2 \tag{1.3}$$

and a character θ of $T(\mathbf{Q})\backslash T(\mathbf{A})$ such that $\Pi = \Pi(\theta)$ [7]. Let $\gamma \to \overline{\gamma}$ be the automorphism of $T(\mathbf{A})$ deduced from conjugation on the corresponding quadratic field, and define $\bar\theta$ by

$$\bar\theta(\overline{\gamma}) = \theta(\gamma) \ .$$

We suppose not only that T satisfies (1.3) but also that $T(R)$ is fundamental and that $\theta \neq \bar\theta$ for otherwise $\mathfrak{m}(\Pi_\infty) = 0$ and the L-indistinguishability class Π does not contribute to the zeta-function.

We introduce $S^0 \subseteq S$ as in §8 of [7], or as in [22]. Then $S^0\backslash S$ is of order two. Let ϵ represent the non-trivial element. As we know, ϵ is associated to

$$\kappa' \colon X_*(T_{\mathrm{sc}}) \to \mathbf{C}^\times \ .$$

Let $\langle \epsilon, \pi_\infty \rangle$ be the pairing of [7]. It is easily seen that there is a constant $\eta = \pm 1$ such that

$$(-1)^{i-1} = \eta \left\langle \epsilon, \pi_\infty \right\rangle, \qquad \pi_\infty \in \prod^{i}(\xi) \ .$$

The reason can be briefly given. In $\mathrm{SL}(2)$ over $\mathbf{R}$ we take

$$T = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \, a^2 + b^2 = 1 \right\} \ .$$

The non-trivial element of the normalizer of T is represented by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbf{C})$ and by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $\mathrm{GL}(2, \mathbf{R})$. The cocycle associated to the first matrix is

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the determinant of the second is -1. Thus $\langle \epsilon, \pi_\infty \rangle$ is constant on each Π^i_∞.

Π_f is of course the set of π_f for which $\pi_\infty \otimes \pi_f$ lies in Π for some π_∞. If $\pi_f = \otimes \pi_w$ set

$$\langle \epsilon, \pi_f \rangle = \prod_w \langle \epsilon, \pi_w \rangle \ .$$

Let

$$\Pi^i_f = \{\pi_f \in \Pi_f \,|\, \langle \epsilon, \pi_\infty \rangle \, \langle \epsilon, \pi_f \rangle = 1 \quad \text{for} \quad \pi_\infty \in \Pi^i_\infty\}$$

and let $\mathfrak{m}^i(\Pi_f)$ be the multiplicity with which the trivial representation of K occurs in

$$\varpi^i_f = \oplus_{\Pi_f \in \Pi^i_f} \pi_f \ .$$

Let $\mathfrak{m}'(\theta)$ be 0 unless $\Pi(\theta_\infty)$ is contained in a $\Pi(\xi^j)$, when it is to be $(-1)^q N^j$. Recall that N^j is the multiplicity with which ξ^j occurs in ξ. On the set of $\pi \in \Pi$ for which $\mathfrak{m}(\pi) > 0$, $\mathfrak{m}(\pi)$ is constant. Denote this constant value by $\mathfrak{m}(\Pi)$. When $\mathfrak{m}(\Pi)$ is not constant on all of Π the contribution of $\Pi = \Pi(\theta)$ to the zeta-function $Z(s, S(K), F_\xi)$ is

$$\prod\nolimits_\pi L(s - q/2, \theta, r_i)^{\mathfrak{m}'(\theta_\infty)\mathfrak{m}^i(\Pi_f)\mathfrak{m}(\Pi)} \ . \tag{1.4}$$

We should recall that although the collection $\{r_i\}$ is the same for all θ_∞ the labeling may vary. Moreover

$$\mathfrak{m}'(\theta_\infty) = \mathfrak{m}(\Pi_\infty) \ .$$

If $i\mathfrak{m}(\pi)$ is not constant on Π then the stable multiplicity $\mathfrak{n}(\pi)$, which we also write as $\mathfrak{n}(\Pi)$, introduced in [7] is $\mathfrak{m}(\Pi)/2$ and (1.4) may be written as

$$L(s - q/2, \pi, r)^{\mathfrak{n}(\Pi)\mathfrak{m}(\Pi_\infty)\mathfrak{m}(\Pi_f)} \left\{ \frac{L(s - q/2, \theta, r_1)}{L(s - q/2, \theta, r_2)} \right\}^{(\mathfrak{m}(\Pi_\infty)/2)(\mathfrak{m}^1(\Pi_f) - \mathfrak{m}^2(\Pi_f))\mathfrak{m}(\Pi)}$$

if

$$\mathfrak{m}(\Pi_f) = \mathfrak{m}^1(\Pi_f) + \mathfrak{m}^2(\Pi_f)$$

If, perchance, there is only one r_i we must as above introduce a second of dimension 0 in order to employ this notation. When $\mathfrak{m}(\pi)$ is constant on Π we define

$$\mathfrak{m}^1(\Pi_f) - \mathfrak{m}^2(\Pi_2) = 0 \ .$$

If $\mathfrak{m}(\pi)$ is constant on Π then $\mathfrak{m}(\Pi) = \mathfrak{n}(\Pi)$. Thus we are asserting that the zeta-function $Z(s, S(K), F_\xi)$ can be presented as the product of a stable part

$$\prod_{\Pi} L(s - q/2, \pi, r)^{\mathfrak{n}(\Pi)\mathfrak{n}(\Pi_\infty)\mathfrak{m}(\Pi_f)} \tag{1.5}$$

and a labile part. The labile part is itself a product over the stable conjugacy classes of Cartan subgroups T with $[\mathfrak{E}(T/\mathbf{A})\colon \mathrm{Im}\,\mathfrak{E}(T/F)] = 2$. We want to represent the labile contribution from T as a product over characters of $T(\mathbf{Q})\backslash T(\mathbf{A})$, but we must rememvber that two different $\bar{\theta}$ can yield the same L-indistinguishable class $\Pi(\theta)$. Since $\mathfrak{m}(\Pi_\infty)$ will be 0 if θ is not of type (a) in the sense of [7] and since $\mathfrak{m}^1(\Pi_f) - \mathfrak{m}^2(\Pi_f)$ will be 0 if $\theta_v = \bar{\theta}_v$ for some finite v, we may apply Lemmas 6.7 and 7.1 of [7]. They allow us to write, in the notation of that paper,

$$\mathfrak{m}(\Pi) = \frac{e(\pi)}{2}\,\mu(T)\ .$$

Since $e(\pi)$ is hte number of different θ yielding the same $\Pi(\theta)$, the contribution from T is

$$\Pi_\theta \left(\frac{L(s-q/2,\theta,r_1)}{L(s-1/2,\theta,r_2)}\right)^{(\mathfrak{m}(\Pi_\infty)/4)(\mathfrak{m}^1(\Pi_f(\theta))-\mathfrak{m}^2(\Pi_f(\theta)))\mu(T)} \tag{1.6}$$

To prove the assertion one proves, in particular, that if we substitute $L_\mathfrak{p}(s - q/2, \pi, r)$ for $L(s - q/2, \pi, r)$ in (1.5) and $L_\mathfrak{p}(s - q/2, \theta, r_i)$ for $L(s - q/2, \theta, r_i)$ in (1.6) and take the product with the same exponents then the result is $Z_\mathfrak{p}(s, S(K), F_\xi)$. We shall take K sufficiently small, and prove that this is so for almost all $\mathfrak{p}$. The restriction on K is ultimately of no consequence.

Before beginning I take this opportunity to mention that while studying the problems arising from Shimura varieties I have frequently been instructed by Deligne's conversation and correspondence., His comments on the structure of the set of geometric points over $\overline{F}_\mathfrak{p}$ on a Shimura variety associated to a quaternion algebra over a real quadratic field were invaluable.

2. The trace formula.

The procedure to be followed is that of [12] and [14]. We apply the trace formula to calculate the coefficients of the logarithms of the products in (1.5) and (1.6), and then compare with the coefficients of the logarithm of the zeta-function, which are obtained from the explicit description of the sets $S(\bar{\kappa}_\mathfrak{p})$.

We first turn our attention to the problem of expressing the logarithms of (1.5) and (1.6) in a form suitable for the final comparison. The expressions both are defined for all ξ, whether defined over $\mathbf{Q}$ or not. Since they are multiplicative in ξ, we may as well suppose that ξ is absolutely reducible, for that will simplify some of our considerations. The logarithm of (1.6) demands the most modification, and we begin with it. We are really interested in the logarithm of the product of the local L-functions at $\mathfrak{p}$, and it is

$$\mu(T)\sum_\theta \frac{\mathfrak{m}(\prod_\infty(\theta))}{4}\left(\mathfrak{m}^1(\Pi_f(\theta)) - \mathfrak{m}^2(\Pi_f(\theta))\right) \times \log\, L_p(s-q/2,\theta,r_1-r_2)\ . \tag{2.1}$$

The first step in the transformation of this expression leads to a clumsy, intermediate result, which we will be able to put in a useful form only after applying the trace formula to the group T.

In its role as carrier of θ, T is appearing as one of the groups H of [16]. This means in particular that the element g_1 of that note, which allows us to identify $X^*(T)$ with $X_*({}^LT^0)$, is fixed. Moreover it is fixed in a manner consonant with the remarks at the end of that paper. Then, as in the discussion at the end of §6 of [7], the given g_1 leads to

$$\psi\colon {}^LT \to {}^LG\ .$$

The map ψ was denoted ξ in [16].

The following lemma has been implicit in the earlier discussion, and may seem to be a matter of definition, but so far as I can see it needs a proof.

Lemma 2.1. *Suppose* $\varphi'\colon W_{\mathbf{C}/\mathbf{R}} \to {}^LT$, φ *is* $\psi \circ \varphi'$, *and* $\theta_\infty \in \Pi_{\varphi'}$. *Then* $\Pi_{\varphi'} = \{\theta_\infty\}$ *and*

$$\Pi_\varphi = \Pi(\theta_\infty) \ .$$

That $\Pi_{\varphi'}$ consists of the single element θ_∞ does indeed follow from the definitions of [11], which show in addition that is enough to verify the equality $\Pi_\varphi = \Pi(\theta_\infty)$ when T is not split and $G = \mathrm{GL}(2)$. In this case, one must compare the definitions of [5] and [11].

In general the explicit description of ${}^LG$ in [7] gives ${}^LT^0$ as a quotient of the group matrices

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \times \ldots \times \begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}, \qquad \alpha_i, \beta_i \in \mathbf{C}^\times \ .$$

The co-weights are given by n pairs of integers, (a_i, b_i), and the co-weight is positive if and only if $a_i \geq b_i$ for all i.

Taking T to be non-split and G to be $\mathrm{GL}(2)$, we let

$$\varphi'\colon z \to \begin{pmatrix} z^a\,\overline{z}^b & 0 \\ 0 & z^b\,\overline{z}^a \end{pmatrix}, \qquad z \in \mathbf{C}^\times \ .$$

It is easily verified that the class of φ' is determined by its restriction to $\mathbf{C}^\times$. Let γ_1 and γ_2 be the co-weights $(1, 0)$ and $(0, 1)$ of ${}^LT^0$. They are also weights of T. The definition of [11] gives

$$\theta_\infty\colon t \to \gamma_1(t)^a \overline{\gamma_1(t)}^b, \qquad t \in T(\mathbf{R}),$$

as the unique element of $\Pi_{\varphi'}$. Note that $\gamma_1(t) = \overline{\gamma}_2(t)$ when $t \in T(\mathbf{R})$.

Π_φ also consists of a single element. If $a = b$ then φ factors through $W_{\mathbf{C}/\mathbf{R}} \to W_{\mathbf{R}/\mathbf{R}} \to \mathbf{R}^\times$ and, in terms of $\mathbf{R}^\times$,

$$\varphi\colon x \to \begin{pmatrix} |x|^a & 0 \\ 0 & (\operatorname{sgn} x)|x|^a \end{pmatrix} \ .$$

The unique element of Π_φ is the representation of the principal series corresponding to the two characters of $\mathbf{R}^\times$ appearing here. According to the remark preceding Corollary 5.14 of [6], this is also $\pi(\theta_\infty)$.

If $a \neq b$ then Π_φ consists of a representation π in the discrete series. It does not change if a and b are interchanged. It will be simpler to be explicit if we assume $a > b$. According to the definitions of §3 of [11], the character of π on $T(\mathbf{R})$, viewed now as a subset of $G(\mathbf{R})$, is

$$-\frac{\{\gamma_1(t)^{a-1/2}\gamma_1(t)^{b+1/2} - \gamma_2(t)^{a-1/2}\overline{\gamma}_2(t)^{b+1/2}\gamma_2(t)\gamma_1^{-1}(t)\}}{1 - \gamma_2(t)\gamma_2^{-1}(t)}.$$

This may be written as

$$-\frac{\gamma_2(t)}{|\gamma_2(t)|}\,\frac{\sum_{\omega \in \Omega(T(\mathbf{C}),G(\mathbf{C}))} \operatorname{sgn} \omega \theta_\infty^0(\omega(t))}{1 - \gamma_2(t)\gamma_1^{-1}(t)} \tag{2.2}$$

It follows from Corollary 5.14 of [6] that if μ_1 and μ_2 are the two characters of $\mathbf{R}^\times$ defined by

$$\mu_1(x) = |x|^a, \qquad \mu_2(x) = |x|^b (\operatorname{sgn} x)^{a-b+1} \ ,$$

then the character of $\pi(\theta_\infty)$ on $T(\mathbf{R})$ is the negative of the character of the finite-dimensional representation $\pi(\mu_1, \pi_2)$. Lemma 5.7 of [6] allows one to compute easily the character of $\pi(\mu_1, \mu_2)$ on $T(\mathbf{R})$, and it is seen to equal the negative of (2.2).

For the purposes of the following corollary and lemma we choose an order on roots of T with respect to which $\mu^\vee$ lies in the closed negative Weyl chamber. Otherwise we use the customary parameters to represent roots and weights. For example, let ζ be the weight, dominant with respect to this order, represented by

$$\zeta = (1, 0) \times \ldots \times (1, 0) \ .$$

Let λ be the highest weight of ξ and set

$$\theta_\infty^0(t) = \lambda(t)(\zeta(t)/|\zeta(t)|) \ .$$

Corollary 2.2. *$\Pi(\theta_\infty)$ is $\Pi(\xi)$ if and only if θ_∞ is conjugate under $\Omega(T(\mathbf{C}), G(\mathbf{C}))$ to θ_∞^0.*

Once again it is enough to verify the assertion for $G = \mathrm{GL}(2)$. Then $\prod(\xi)$ consists of a single element π and on $T(\mathbf{R})$ the character of π is

$$t \to -\mathrm{trace}\,\tilde{\xi}(t) \ .$$

If $\lambda = (a, b)$, $a > b$, this is

$$-\frac{\gamma_1(t)^a\gamma_2(t)^b - \gamma_1(t)^b\gamma_2(t)^a\gamma_2(t)\gamma_1^{-1}(t)}{1-\gamma_2(t)\gamma_1^{-1}(t)} \ .$$

Since $\gamma_1(t) = \overline{\gamma_2}(t)$, this is equal to

$$-\frac{\gamma_2(t)}{|\gamma_2(t)|}\ \frac{\sum_\omega \mathrm{sgn}\,\omega\theta_\infty^0(\omega(t))}{1-\gamma_2(t)\gamma_1^{-1}(t)} \ .$$

Comparing this with (2.2), we obtain the corollary.

It will be useful to have the following lemma on record.

Lemma 2.3. *Let $\Lambda \in \{\Lambda_\omega\}$ be $\lambda + g$, wehre g is one-half the sum of the positive roots. If φ': $W_{\mathbf{C}/\mathbf{R}} \to {}^LT$ and its restriction to $\mathbf{C}^\times$ is*

$$z \to z^\Lambda\,\overline{z}^{\sigma\Lambda}$$

then

$$\prod\nolimits_{\varphi'} = \{\theta_\infty^0\} \ .$$

We may write an element of $T(\mathbf{R})$ as $t = e^H$ with $\sigma(H) = H$. The unique element of $\Pi_{\varphi'}$ takes t to

$$e^{\Lambda(H)} = e^{\lambda(H)}e^{g(H)} \ .$$

Since

$$\lambda(t) = e^{\lambda(H)}$$

we need only verify that

$$e^{g(H)} = \zeta(t)/|\zeta(t)| \ .$$

It is enough to do this when $G = \mathrm{GL}(2)$. If

$$\gamma_1(H) = z\,, \quad \gamma_2(H) = \overline{z}$$

then

$$\gamma_1(t) = e^z\,, \quad \gamma_2(t) = e^{\overline{z}}$$

and

$$e^{g(H)} = e^{z/2-\overline{z}/2} = e^z/|e^z| = \zeta(t)/|\zeta(t)| \ .$$

We know that T is associated to a quadratic extension L of F. if L is not totally imaginary then $\mathfrak{m}(\Pi(\theta_\infty))$ is 0 for all characters of $T(\mathbf{Q})\backslash T(\mathbf{A})$; hence the product (1.6) is equal to 1, and of no interest. Suppose that L is totally imaginary, and let κ be the associated character of I_F. It is also a character of $Z(\mathbf{A})$ and $Z(\mathbf{R})$. We set $\mathfrak{m}(\theta_\infty)$ equal to 0 unless the restriction of θ_∞ to $Z(\mathbf{R})$ is

$$z \to \nu^{-1}(z)\kappa(z) \ ,$$

and then we take

$$\mathfrak{m}(\theta_\infty) = \int_{Z(\mathbf{R})/T(\mathbf{R})} \theta_\infty(t) f_\xi^T(t)dt$$

with

$$f_\xi^T(t) = (-1)^q \frac{\sum_{\Omega(T(\mathbf{C}),G(\mathbf{C}))} \operatorname{sgn}\omega\theta_\infty^0(\omega(t^{-1}))}{\operatorname{meas} Z(\mathbf{R})\backslash T(\mathbf{R})} .$$

Lemma 2.4. *The number* $\mathfrak{m}(\theta_\infty)$ *is* 0 *if and only if* $\mathfrak{m}(\Pi(\theta_\infty))$ *is* 0*, and* $\mathfrak{m}(\Pi(\theta_\infty))$ *is not* 0 *if and only if* $\Pi(\theta_\infty)$ *is* $\Pi(\xi)$*. If* $\theta_\infty(t) \equiv \theta_\infty^0(\omega(t))$ *then*

$$\mathfrak{m}(\theta_\infty) = \operatorname{sgn}\omega\mathfrak{m}(\Pi(\theta_\infty)) .$$

The first assertion is a consequence of the Weyl integration formula, and the explicit formula for the restriction of

$$\sum_{\pi\in\prod(\theta_\infty)} \chi_\pi$$

to $T(\mathbf{R})$. The same formulae also show that $\mathfrak{m}(\Pi(\xi)) = (-1)^q$. It is clear that $\mathfrak{m}(\theta_\infty)$ has been so defined that

$$\mathfrak{m}(\theta_\infty) = (-1)^q \operatorname{sgn}\omega$$

when

$$\theta_\infty(t) \equiv \theta_\infty^0(\omega(t)) .$$

If $\mathfrak{m}(\Pi(\theta_\infty))$ is 0 the corresponding factor of (1.6) is 1, and there is nothing to be said. Suppose therefore that $\Pi(\theta_\infty) = \Pi(\xi)$. Then $\Pi_\infty^i = \prod^i(\xi)$ may be introduced. We have agreed that Π_∞^1 will contain the representation $\pi(\Lambda_1)$. Let ϕ be the characteristic function of $K \subseteq G(\mathbf{A}_f)$ divided by the measure of $Z_K\backslash K$ with

$$Z_K = Z(\mathbf{A}_f) \cap K .$$

We are supposing that

$$[\mathfrak{E}(T/\mathbf{A})\mathrm{Im}\,\mathfrak{E}(T/\mathbf{Q})] = 2 ,$$

and we regard κ as the non-trivial character of this quotient. Following §2 of [7], we set $\Phi^{T/\kappa}(t,\phi)$ equal to

$$\left\{\prod\nolimits_v \Big(\lambda(L_v/F_v,\psi_v)\kappa_v\Big(\frac{t_1-t_2}{t_1^0-t_2^0}\Big)\frac{|(t_1-t_2)^2|v^{1/2}}{|t_1t_2|_v^{1/2}}\Big)\right\} \times \left\{\sum_{\mathfrak{E}(T/\mathbf{A}_f)} \kappa(\delta)\Phi^\delta(t,\phi)\right\} .$$

Here the product is over all finite places of F. If δ is represented by h in $\mathfrak{A}(T/\mathbf{A}_f) = \prod_w \mathfrak{A}(T/\mathbf{A}_w)$, the product now being taken over all finite places of $\mathbf{Q}$, then

$$\Phi^\delta(t,\phi) = \int_{T^h(\mathbf{A}_f)G\backslash(\mathbf{A}_f)} \phi(g^{-1}t^hg)dg .$$

It is a consequence of the definitions and principles of [7] that if $\pi_\infty \in \Pi_\infty^1$ then

$$\langle\epsilon,\pi_\infty\rangle \{\mathfrak{m}^1(\Pi_f(\theta)) - \mathfrak{m}^2(\Pi_f(\theta))\}$$

is equal to

$$\int_{Z_K\backslash T(\mathbf{A}_f)} \theta_f(t)\Phi^{T/\kappa}(t,\phi)dt = \Big\langle \theta_f, \Phi^{T/\kappa}(\,\cdot\,,\phi)\Big\rangle .$$

It should not be forgotten that the pairing $\langle\epsilon,\pi_\infty\rangle$ depends on θ_∞. The representations r_1 and r_2 do also. In order to have a pair of representations that do not depend on θ_∞, we let r^+ be that r_i for which X_i contains $\mu^\vee$, and r^- the other. If $\theta_\infty = \theta_\infty^0$, then Lemma 2.3 implies that

$$r_1 - r_2 = r^+ - r^- .$$

In general if $\theta_\infty(t) = \theta^0_\infty(\omega^{-1}(t))$ then $\{\theta_\infty\} = \Pi_{\varphi'}$ if φ' restricted to $\mathbf{C}^\times$ is

$$z \to z^{\omega\Lambda}\,\overline{z}^{\sigma\omega\Lambda} \ .$$

Since

$$\langle \omega\Lambda, \omega\mu^\vee \rangle = \langle \Lambda, \mu^\vee \rangle \ ,$$

the labeling must be such that r^0_1 contains the weight $\omega\mu^\vee$. Thus

$$r_1 - r_2 = \kappa'(\omega\mu^\vee - \mu^\vee)(r^+ - r^-) \ .$$

In order to stress its dependence on θ_∞, rather than on ω, we denote the coefficient $\kappa'(\omega\mu - \mu)$ appearing here by $\eta(\theta_\infty)$.

Lemma 2.5. *If we take* $\theta_\infty(t) = \theta^0_\infty(\omega(t))$ *then*

$$\alpha = \langle \epsilon, \pi_\infty \rangle\, \eta(\theta_\infty) \operatorname{sgn} \omega \ ,$$

with $\pi_\infty \in \prod^1_\infty$, *is independent of* ω.

Before verifying the lemma, we observe that the pairing $\langle \epsilon, \pi_\infty \rangle$ depends not only on θ_∞ but also on the choice of an additive character and a regular element in $T(\mathbf{R})$. Eventually we will be forced to recognize this, but not yet.

To prove the lemma we show that α does not change if ω is replaced by $\omega'\omega$, where ω' is the reflection defined by a simple root. For each real place of F there is one such reflection.

a) If the division algebra defining G is not split at the place then, according to the definitions of [7], replacing ω by $\omega'\omega$ changes the sign of $\langle \epsilon, \pi_\infty \rangle$. It does not affect $\eta(\theta_\infty)$.

b) If the division algebra splits at the place then $\langle \epsilon, \pi_\infty \rangle$ remains the same but $\eta(\theta_\infty)$ changes sign.

Putting all these lemmas together, we conclude that the sum (2.1) is equal to

$$\frac{1}{4}\,\alpha\mu(T) \sum_\theta \mathfrak{m}(\theta_\infty) \left\langle \theta_f, \Phi^{T/\kappa}(\ \cdot\ ,\phi) \right\rangle \log\ L_{\mathfrak{p}}(s - q/2, \theta, r^+ - r^-) \ . \tag{2.3}$$

The sum here is taken over those θ for which

$$\begin{aligned} \theta(z) &= \nu^{-1}(z)\kappa(z), \quad & z \in Z(\mathbf{R}),\\ \theta(z) &= \kappa(z), & z \in Z_K \ . \end{aligned}$$

If we are to put (2.3) in a form to which the trace formula can be applied, we must view the Hecke algebra in the manner of [9]. We are now going to assume that $K = K^{\mathfrak{p}} K_{\mathfrak{p}}$, where $K^{\mathfrak{p}} \subseteq G(\mathbf{A}^{\mathfrak{p}}_f)$ and $K_{\mathfrak{p}}$ is a special maximal compact of $G(\mathbf{A}_{\mathfrak{p}})$. According to §2 of [7]

$$\Phi^{T/\kappa}(\phi) = 0$$

unless L is unramified at every place of F dividing $\mathfrak{p}$. This we may as well assume. We also assume that F is itself unramified over $\mathfrak{p}$. Then

$$\left\langle \theta_f, \Phi^{T/\kappa}(\ \cdot\ ,\phi) \right\rangle = 0$$

unless $\theta_{\mathfrak{p}}$, the restriction of θ to $T(\mathbf{Q}_{\mathfrak{p}})$, is unramified. To such a $\theta_{\mathfrak{p}}$ is associated a conjugacy class $\{t(\theta_{\mathfrak{p}})\}$ in ${}^LT$, and

$$\log\ L_{\mathfrak{p}}(s - q/2, \theta, r^+ - r^-) = \sum_{j=1}^\infty (|\varpi^j|^{s-q/2}/j) \left\{ \operatorname{trace}\ r^+(t^j(\theta_{\mathfrak{p}})) - \operatorname{trace}\ r^-(t^j(\theta_{\mathfrak{p}})) \right\}$$

if ϖ is a uniformizing parameter for $\mathbf{Q}_{\mathfrak{p}}$. In addition there is an element $f_{\mathfrak{p}}^j$ of the Hecke algebra of $T(\mathbf{Q}_{\mathfrak{p}})$ such that

$$|\varpi^j|^{-q/2}\left\{\text{trace } r^+(t^j(\theta_{\mathfrak{p}}))-\text{trace } r^-(t^j(\theta_{\mathfrak{p}}))\right\}=\int_{Z_{\mathfrak{p}}T\backslash(\mathbf{Q}_{\mathfrak{p}})}\theta_{\mathfrak{p}}(t)f_{\mathfrak{p}}^j(t)dt$$

for all unramified $\theta_{\mathfrak{p}}$. Here

$$Z_{\mathfrak{p}}=Z(\mathbf{Q}_{\mathfrak{p}})\cap K_{\mathfrak{p}}\ .$$

If $t=(t^{\mathfrak{p}},t_{\mathfrak{p}})\in T(\mathbf{A}_f)=T(\mathbf{A}_f^{\mathfrak{p}})T(\mathbf{Q}_{\mathfrak{p}})$, we write

$$\phi(t)=\phi^{\mathfrak{p}}(t^{\mathfrak{p}})\phi_{\mathfrak{p}}(t_{\mathfrak{p}})\ ,$$

where, for example, $\phi_{\mathfrak{p}}$ is the characteristic function of $K_{\mathfrak{p}}$ divided by the measure of $Z_{\mathfrak{p}}\backslash K_{\mathfrak{p}}$. We may also write

$$\Phi^{T/\kappa}(t,\phi)=\Phi^{T/\kappa}(t^{\mathfrak{p}},\phi^{\mathfrak{p}})\Phi^{T/\kappa}(t_{\mathfrak{p}},\phi_{\mathfrak{p}})\ .$$

For brevity we denote the second factor by $\varphi_{\mathfrak{p}}(t_{\mathfrak{p}})$. It is $+1$ times the characteristic function of the maximal compact $U_{\mathfrak{p}}$ of $T(\mathbf{Q}_{\mathfrak{p}})$ divided by the measure of $Z_{\mathfrak{p}}\backslash U_{\mathfrak{p}}$.

Applying the trace formula on $Z(\mathbf{R})Z_KT(\mathbf{Q})\backslash T(\mathbf{A})$ to the function

$$t=(t_\infty,t^{\mathfrak{p}},t_{\mathfrak{p}})\to f_\xi^T(t_\infty)\phi^{T/\kappa}(t^{\mathfrak{p}},\phi^{\mathfrak{p}})\varphi_{\mathfrak{p}}^*f_{\mathfrak{p}}^j(t_{\mathfrak{p}})\ ,$$

we see that the coefficient of $|\varpi^j|^s/j$ in the expansion of (2.3) is

$$\frac{\alpha\mu(T)}{2}\operatorname{meas}(Z(\mathbf{R})Z_KT(\mathbf{Q})\backslash T(\mathbf{A}))\sum f_\xi^T(t)\Phi^{T/\kappa}(t,\phi^{\mathfrak{p}})\varphi_{\mathfrak{p}}^*f_{\mathfrak{p}}^j(t)\ . \tag{2.4}$$

The sum is over $T(\mathbf{Q})\cap Z(\mathbf{R})Z_K\backslash T(\mathbf{Q})$.

The first thing to observe is that the term corresponding to a t in $Z(\mathbf{Q})$ is 0 because f_ξ^T vanishes on $Z(\mathbf{R})$. If t is not central we write $\Phi^{T/\kappa}(T,\phi^{\mathfrak{p}})$ as the product of

$$\Phi_0^{T/\kappa}(t,\phi^{\mathfrak{p}})=\sum_{\mathfrak{E}(\mathbf{A}_f^{\mathfrak{p}})}\kappa(\delta)\Phi^\delta(t,\phi)$$

and

$$\prod\lambda(L_v/F_v,\phi_v)\kappa_v\left(\frac{t_1-t_2}{t_1^0-t_2^0}\right)\frac{|(t_1-t_2)^2|_v^{1/2}}{|t_1t_2|_v^{1/2}}\ .$$

The product is over all finite places of F which do not divide $\mathfrak{p}$. Using various product formulae, we may replace it by the inverse of the same product taken over the infinite primes and the primes dividing $\mathfrak{p}$.

The expression

$$\alpha\left\{\prod\lambda(L_v/F_v,\psi_v)\kappa_v\left(\frac{t_1-t_2}{t_1^0-t_2^0}\right)\frac{|(t_1-t_2)^2|_v^{1/2}}{|t_1t_2|_v^{1/2}}\right\}^{-1}f_\xi^T(t)\ , \tag{2.5}$$

in which the product is taken over the infinite places, depends on t and on $\mu^\vee$, but it does not depend on the choice of t^0 or of the ψ_v.

Lemma 2.6. *If t lies in $T(\mathbf{R})$ and is regular then the expression (2.5) is equal to*

$$\frac{\operatorname{trace} \xi(t)}{\operatorname{meas} Z(\mathbf{R})\backslash T(\mathbf{R})} \ .$$

It is clear that we could define the expression (2.5) for any group lying between

$$\prod\nolimits_v S_v(\mathbf{R}) \ ,$$

and

$$\{(g_v) \in \prod\nolimits_v G_v(\mathbf{R}) | \prod\nolimits_v \operatorname{Nm} g_v > 0\} \ .$$

Here the product is over the infinite places of F. $G_v(\mathbf{R})$ is $\mathrm{GL}(2, \mathbf{R})$ if the quaternion algebra defining G splits at v and the multiplicative group of a quaternion algebra of it does not. $S_v(\mathbf{R})$ consists of the elements of norm 1 in $G_v(\mathbf{R})$. It is certainly enough to prove the lemma for the largest of these groups. Since the group

$$\Big\{(g_v) \in \prod\nolimits_v G_v(\mathbf{R}) | \det g_v > 0 \quad \text{for all} \quad v\Big\}$$

contains $T(\mathbf{R})$ and supports the character of ψ_∞, we may work with it instead. This yields a situation that factors, and we may finally suppose that there is only one place.

We take the additive character ψ_v to be $x \to e^{2\pi i x}$ and define t^0 by

$$t_1^0 = \gamma_1(t^0) = i \ .$$

We choose γ_1, γ_2 so that $\gamma_1 - \gamma_2$ and $\mu^\vee$ lie in opposite Weyl chambers, and set

$$t_1 = \gamma_1(t) = re^{i\theta} \ .$$

According to [10],

$$\lambda(E_v/F_v, \psi_v) = i \ .$$

Thus

$$\lambda(E_v/F_v, \psi_v)\kappa\left(\frac{t_1 - t_2}{t_1^0 - t_2^0}\right)\frac{|(t_1 - t_2)^2|_v^{1/2}}{|t_1 t_2|^{1/2}} = e^{i\theta} - e^{-t\theta} \ .$$

If D does not split and ω is the non-trivial element of the Weyl group,

$$\frac{\theta_\infty^0(t) - \theta_\infty^0(\omega(t))}{e^{i\theta} - e^{-i\theta}} = \frac{\lambda(t)e^{i\theta} - \lambda(\omega(t))e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} = \operatorname{trace} \pi_\infty(t) \ .$$

Consequently the value of $\langle \epsilon, \pi_\infty \rangle$, defined with respect to θ_∞^0, is -1. Since $\eta(\theta_\infty)$ is clearly also 1, the value of α is 1. Since π_∞ is $\tilde{\xi}$ in this case, the assertion of the lemma now follows from the definition of f_ξ^T.

Suppose D splits at v. There is a unique element π_∞ in $\Pi^1(\xi)$ and it contains a lowest weight of $T(\mathbf{R})$ with respect to the order making $\gamma_1 - \gamma_2$ positive. The character of π_∞ is easily calculated and is found to be

$$-\theta_\infty^0(t)/(e^{i\theta} - e^{-i\theta}) \ .$$

If $\widetilde{\pi}_\infty$ is the corresponding element of the holomorphic discrete series, then

$$\chi_{\pi_\infty}(t) - \chi_{\widetilde{\pi}_\infty}(t) = -\frac{\theta_\infty^0(t) + \theta_\infty^0(\omega(t))}{e^{i\theta} - e^{-i\theta}} \ .$$

As a consequence

$$\langle \epsilon, \pi_\infty \rangle = 1 \ .$$

Since $\eta(\theta_\infty^0)=1$, the number α is -1. Once again, the assertion of the lemma follows from the definition of f_ξ^T.

The expression

$$\left\{\prod\nolimits_{v|\mathfrak{p}}\lambda(L_v/F_v,\psi_v)\kappa_v\left(\frac{t_1-t_2}{t_1^0-t_2^0}\right)\frac{|(t_1-t_2)^2|_v^{1/2}}{|t_1t_2|_v^{1/2}}\right\}^{-1}\varphi_{\mathfrak{p}}^* f_{\mathfrak{p}}^j(t) \tag{2.6}$$

depends on the regular element t in $T(\mathbf{Q}_\mathfrak{p})$ and on the order on the roots of T provided by the identification of $X_*(T)$ and $X^*({}^LT^0)$. It is easily seen that r^0, the restriction to ${}^LT^0\times\mathfrak{G}(\overline{\mathbf{Q}}/E)$ of the representation of ${}^LG^0\times\mathfrak{G}(\overline{\mathbf{Q}})/E)$ that appeared in the construction of r, is the direct sum of two representations r_0^+ and r_0^- such that

$$r^+=\mathrm{Ind}({}^LT,{}^LT^0\times\mathfrak{G}(\overline{\mathbf{Q}}/E),r_0^+)$$
$$r^-=\mathrm{Ind}({}^LT,{}^LT^0\times\mathfrak{G}(\overline{\mathbf{Q}}/E),r_0^-)\ .$$

The restriction of r^+ or r^- to the local associate group ${}^LT_\mathfrak{p}={}^LT^0\times\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$ is therefore the direct sum of induced representations parametrized by the double coset space

$$\mathfrak{G}(\overline{\mathbf{Q}}/E)\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})\ .$$

We had fixed $E\subseteq\overline{\mathbf{Q}}\subseteq\mathbf{C}$ and we have now fixed $\overline{\mathbf{Q}}\to\overline{\mathbf{Q}}_\mathfrak{p}$ as well. This double coset space also indexes the primes $\mathfrak{P}$ of E dividing $\mathfrak{p}$, the coset containing σ yielding the valuation defined by the imbedding $x\to\sigma^{-1}(x)$ of E in $\overline{\mathbf{Q}}_\mathfrak{p}$. We write

$$r^+=\oplus r_\mathfrak{P}^+\quad r^-=\oplus r_\mathfrak{P}^-$$

and define $f_\mathfrak{P}^j$ in the Hecke algebra by

$$|\varpi^j|^{-q/2}\{\mathrm{trace}\ r_\mathfrak{P}^+(t^j(\theta_\mathfrak{p}))-\mathrm{trace}\ r_\mathfrak{P}^-(t^j(\theta_\mathfrak{p}))\}=\int_{Z_\mathfrak{p}/T(\mathbf{Q}_\mathfrak{p})}\theta_\mathfrak{p}(t)f_\mathfrak{P}^j(t)dt\ .$$

Then

$$f_\mathfrak{p}^j=\sum\nolimits_\mathfrak{P} f_\mathfrak{P}^j\ .$$

The representation of $f_\mathfrak{p}^j$ as a sum yields a representation of (2.6) as a sum, the terms being obtained by replacing $f_\mathfrak{p}^j$ with $f_\mathfrak{P}^j$. Although we should consider each of them, there is no loss of generality in fixing our attention on the prime $\mathfrak{P}$ defined by the imbedding $E\to\overline{\mathbf{Q}}\to\overline{\mathbf{Q}}_\mathfrak{p}$.

The group $G(F)$ is a subgroup of $\widetilde{G}_1(F)$, if $\widetilde{G}_1=\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(2)$, and ${}^LG^0$ is a quotient of ${}^L\widetilde{G}_1^0$. Let $\widetilde{T}_1$ be the centralizer of T in $\widetilde{G}_1$, and then ${}^LT^0$ is a quotient of ${}^L\widetilde{T}_1^0$. If $\widetilde{U}_\mathfrak{p}$ and $U_\mathfrak{p}$ are the maximal compact subgroups of $\widetilde{T}_1(\mathbf{Q}_\mathfrak{p})$ and of $T(\mathbf{Q}_\mathfrak{p})$ we may define an imbedding $f\to f'$ of the Hecke algebra $\mathcal{H}_\mathfrak{p}(T)$ of $T(\mathbf{Q}_\mathfrak{p})$ into $\mathcal{H}_\mathfrak{p}(\widetilde{T}_1)$. The value of f' at t is 0 unless $t=su$ with $u\in\widetilde{U}_\mathfrak{p}$ and $s\in T(\mathbf{Q}_\mathfrak{p})$ and then

$$f'(t)=\frac{\mathrm{meas}\,U_\mathfrak{p}}{\mathrm{meas}\,\widetilde{U}_\mathfrak{p}}f(s)\ .$$

If we take ${}^LT$ and ${}^L\widetilde{T}_1$ to be ${}^LT^0\times\mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$ and ${}^L\widetilde{T}_1^0\times\mathfrak{G}\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/(\mathbf{Q}_\mathfrak{p})$, then $\mathcal{H}_\mathfrak{p}(T)$ and $\mathcal{H}_\mathfrak{p}(\widetilde{T}_1)$ may be regarded as algebras of funcitons on ${}^LT^0\times\Phi_\mathfrak{p}$ and ${}^L\widetilde{T}_1^0\times\Phi_\mathfrak{p}$, with $\Phi_\mathfrak{p}$ being the Frobenius. The map $f\to f'$ may also be obtained by pulling back functions by means of

$${}^L\widetilde{T}_1^0\times\Phi_\mathfrak{p}\to{}^LT^0\times\Phi_\mathfrak{p}\ .$$

The representations $r_\mathfrak{P}^+$ and $r_\mathfrak{P}^-$ may be lifted to ${}^LT$, and it will be advantageous for us to regard $f_\mathfrak{P}^j$ as an element of $\mathcal{H}_\mathfrak{p}(\widetilde{T}_1)$.

We have supposed that the totally real field used to define G is imbedded in $\overline{\mathbf{Q}}$ and hence in $\mathbf{C}$ and $\overline{\mathbf{Q}}_{\mathfrak{p}}$. The set

$$Q = \mathfrak{G}(\overline{\mathbf{Q}}/F)\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$$

parametrizes the imbeddings of F in $\mathbf{R} \subseteq \mathbf{C}$ and in $\overline{\mathbf{Q}}_{\mathfrak{p}}$. We represent it as a set of crosses and circles, the crosses denoting the infinite places at which the quaternion algebra defining G splits.

$$\bigcirc \times \ldots \times \bigcirc \ldots \times$$

We decompose the set into orbits under $\Phi_{\mathfrak{p}}$, and suppose that the action of $\Phi_{\mathfrak{p}}$ on each orbit is by a cyclic shift to the right

$$\underbrace{\times \ldots \bigcirc}_{Q_{v_1}} \quad \underbrace{\bigcirc \times \ldots \times}_{Q_{v_2}} \quad \ldots$$

Let $\mathfrak{n}_v$ be the number of elements in the v^{th} orbit. Then

$$\sum \mathfrak{n}_v = \mathfrak{n} = [F : \mathbf{Q}] \ .$$

We let $\mathfrak{m}_v$ be the number of points in the v^{th} orbit which are marked by a cross. The orbits also parametrize the places of F dividing ρ, and so we may label an orbit by the place it defines. That is why we have chosen the symbol v.

Over $\mathbf{Q}_{\mathfrak{p}}$

$$\widetilde{T}_1 \simeq \prod_{v|\mathfrak{p}} T_v$$

with

$$T_v(\mathbf{Q}_{\mathfrak{p}}) = L_v^{\times}$$

if

$$L_v = L \otimes_F F_v \ .$$

Thus $\mathcal{H}_{\mathfrak{p}}(\widetilde{T}_1) \simeq \otimes_v \mathcal{H}_{\mathfrak{p}}(T_v)$.

Since $t = t(\theta_{\mathfrak{p}}) \in {}^LT^0 \times \Phi_{\mathfrak{p}}$ and $r_{\mathfrak{p}}^+$ and $r_{\mathfrak{p}}^-$ are induced,

$$\text{trace } r_{\mathfrak{p}}^+(t^j) - \text{trace } r_{\mathfrak{p}}^-(t^j) = 0$$

if j is not divisible by $k = [E_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]$. Consequently $f_{\mathfrak{p}}^j$ is then to be 0. Let $t = a \times \Phi_{\mathfrak{p}}$ with $a \in {}^LT^0$. Then

$$t^k = a\Phi_{\mathfrak{p}}(a) \ldots \Phi_{\mathfrak{p}}^{k-1}(a) \times \Phi_{\mathfrak{p}}^k$$

and

$$\Phi_{\mathfrak{p}} t^k \Phi_{\mathfrak{p}}^{-1} = a^{-1} t^k a$$

is conjugate under ${}^LG^0$ to g^k. Hence if $k|j$

$$\text{trace } r_{\mathfrak{p}}^+(t^j) - \text{trace } r_{\mathfrak{p}}^-(t^j) = [E_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]\{\text{trace } r_0^+(t^j) - \text{trace } r_0^-(t^j)\} \ .$$

The representations r_0^+ and r_0^- are, for the present purposes, to be treated as representations of ${}^LT^0 \times \mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}^{\text{un}}/E_{\mathfrak{p}})$ or of ${}^L\widetilde{T}_1^0 \times \mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}^{\text{un}}/E_{\mathfrak{p}})$. For each i in $\mathbf{Q}$, let γ_1^i, γ_2^i be the weights of ${}^L\widetilde{T}_1^0$ or of ${}^LT$ given by

$$\gamma_1^i = (0,0) \times \ldots \times (0,0) \times (1,0) \times (0,0) \times \ldots \times (0,0),$$
$$\gamma_2^i = (0,0) \times \ldots \times (0,0) \times (0,1) \times (0,0) \times \ldots \times (0,0) \ .$$

There is one non-zero factor and it is at the i^{th} place. For the moment we will not be too concerned about which order makes the roots $\gamma_1^i - \gamma_2^i$ positive. We take it to be that coming from the identification of $X_*(T)$ and $X^*({}^LT^0)$.

Let $\overline{Q} \subseteq Q$ be the set of marked spots. It is easily seen that

$$r_0^+ - r_0^- = + \otimes_{i\in\overline{Q}} (\gamma_1^i - \gamma_2^i) \tag{2.7}$$

on ${}^L\widetilde{T}_1^0$. To examine this difference on ${}^L\widetilde{T}_1^0 \times \mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$ we must not only describe r_0 on ${}^LG^0 \times \mathfrak{G}(\mathbf{Q}/E)$ but also explicitly describe the lifting of ψ: ${}^LT \to {}^LG$ to $\bar{\psi}_1$: ${}^L\widetilde{T}_1 \to {}^L\widetilde{G}_1$.

We lift r_0 and regard it as a representation of ${}^L\widetilde{G}_1^0 \times \mathfrak{G}(\mathbf{Q}/E)$. The group ${}^L\widetilde{G}_1^0$ is a product

$$\prod\nolimits_{i\in Q} \mathrm{GL}(2,\mathbf{C}) \ .$$

Let r^i be the representation obtained by projecting on the i^{th} factor and then taking the standard representation of $\mathrm{GL}(2,\mathbf{C})$ on the space X^i of column vectors of length two. The restriction of r_0 to ${}^L\widetilde{G}_1^0$ is

$$\otimes_{i\in Q}\, r^i \ ,$$

acting on

$$\otimes_{i\in Q}\, X^i \ .$$

Recall that Q is a homogeneous space on which $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts to the right. By its definition $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ consists of those elements of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ that leave E invariant. If $\sigma \in \mathfrak{G}(\overline{\mathbf{Q}}/E)$ then

$$r_0(\sigma)\colon \otimes_{i\in\overline{Q}}\, x^i \to \otimes_{i\in\overline{Q}}\, x^{i^\sigma} \ .$$

On ${}^L\widetilde{T}_1^0$ the homomorphism $\bar{\psi}_1$ is easily described. It takes

$$t \to \prod\nolimits_{i\in Q} \begin{pmatrix} \gamma_1^i(t) & 0 \\ 0 & \gamma_2^i(t) \end{pmatrix} \ .$$

To define it explicitly on ${}^L\widetilde{T}_1$ we need to choose a set of representatives z_i for the cosets in Q. If we examine the constructions in [16], we see that this entails choosing g_1 correctly, but we are in fact allowed to choose g_1 anew continually. We let

$$\tau_i\sigma = d_{\tau_i}(\sigma)\tau_j, \quad d_{\tau_i}(\sigma) \in \mathfrak{G}(\overline{\mathbf{Q}}/F) \ .$$

Set

$$a(\rho) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho \in \mathfrak{G}(\overline{\mathbf{Q}}/L), \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \rho \in \mathfrak{G}(\overline{\mathbf{Q}}/F),\ \rho \notin \mathfrak{G}(\overline{\mathbf{Q}}/L) \ . \end{cases}$$

Then

$$\bar{\psi}_1(\sigma) = \prod\nolimits_i a(d_i(\sigma)) \times \sigma \ .$$

Lemma 2.7. *The function $f_\mathfrak{p}^j$ is 0 if for some v_0 the algebra $L_{v_0} = L \otimes_F F_{v_0}$ is a field and Q_{v_0} has marked points.*

We may suppose that $[E_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]$ divides j. Let l_v be the greatest common divisor of $\mathfrak{n}_v$ and j. Up to equivalence the representation r_0 and the representations r_0^+ and r_0^-, as representations of ${}^L\widetilde{T}_1$, do not depend on the choice of coset representatives. In an orbit under $\rho = \Phi_\mathfrak{p}^j$ we take the representatives to be

$$\tau, \tau\rho, \tau\rho^2, \ldots, \tau\rho^{e_v-1}, \quad e_v = \mathfrak{n}_v/l_v \ .$$

Then

$$d_{\tau\rho^i}(\rho) = \begin{cases} 1, & 0 \leqq i < e_v - 1, \\ \tau\rho^{e_v}\tau^{-1}, & i = e_v - 1\ . \end{cases}$$

Certainly

$$\tau\rho^{e_v}\tau^{-1} = \tau\Phi_{\mathfrak{p}}^{n_v j/l_v}\tau^{-1}$$

is the Frobenius over F_v to the power j/l_v. If L_v is a field, then it is quadratic over F_v and, with the assumption under which we are working at present, unramified. Thus $\tau\rho^{e_v}\tau^{-1}$ lies in $\mathfrak{G}(\overline{\mathbf{Q}}/L)$ if and only if $2|j/l_v$.

Let $\{x_1^i, x_2^i\}$ be the standard basis of X^i. The collection

$$\otimes_{i\in\overline{Q}}\ x^i_{j(i)}, \qquad j(i) = 1, 2$$

is a basis for the space in which r_0 acts. The representation r_0^+ acts on the span of those elements for which

$$(-1)^{\sum_{i\in\overline{Q}} j(i)} = \pm(-1)^{|\overline{Q}|}\ ,$$

and r_0^- on the span of the other elements. The sign is determined by (2.7). It is clear that $r_0(\Phi_{\mathfrak{p}}^j)$ permutes the basis elements amongst themselves, and that the element indexed by $\{j(i)\}$ is fixed if and only if $j(i)$ is constant on orbits of $\Phi_{\mathfrak{p}}^j$ and

$$a(\tau\rho^{e_v}\tau^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1\ . \end{pmatrix}$$

Since each basis element is an eigenvector for ${}^L\widetilde{T}_1^0$, we have

$$\text{trace } r_0^+(t^j) = \text{trace } r_0^-(t^j) = 0,$$

and

$$\text{trace } r_0^+(t^j) - \text{trace } r_0^-(t^j) = 0,$$

if $2l_{v_0}$ does not divide j and $t = a \times \Phi_{\mathfrak{p}}$, $a \in {}^L\widetilde{T}_1^0$. Thus in this case at least, $f_{\mathfrak{p}}^j = 0$.

We now suppose that $2l_{v_0}$ divides j, and make a different choice of coset representatives. We take the representatives of the cosets in Q_v to be of the form

$$\tau, \tau\Phi_{\mathfrak{p}}, \ldots, \tau\Phi_{\mathfrak{p}}^{n_v-1}.$$

If $\rho = \Phi_{\mathfrak{p}}$ then

$$d_{\tau\rho^i}(\rho) = \begin{cases} 1, & 0 \leqq i < \mathfrak{n}_v - 1, \\ \tau\rho^{\mathfrak{n}_v}\tau^{-1}, & i = \mathfrak{n}_v - 1\ . \end{cases}$$

It helps to picture Q_v as

$$\overbrace{\times\times\bigcirc\ldots\times}^{l_v}\ \times\times\bigcirc\ldots\times\bigcirc\ldots$$

There are l_v orbits under $\Phi_{\mathfrak{p}}^j$ and each orbit consists entirely of marked or entirely of unmarked points. If k_v is the number of marked orbits then

$$\mathfrak{m}_v/k_v = \mathfrak{n}_v/l_v\ .$$

We may write

$$r_0^+(t^j) - r_0^-(t^j) = \pm\otimes_z (r_z^+(t^j) - r_z^-(t^j))\ .$$

The tensor product is taken over the marked orbits and the meaning of $r_z^+(t^j)$ and $r_z^-(t^j)$ is, I hope, clear. We show that

$$\text{trace } r_z^+(t^j) = \text{trace } r_z^-(t^j)$$

if $z \in Q_{v_0}$.

All our calculations will be within Q_{v_0}. So we may as well take the indices of the points in this set to be $1, \dots, \mathfrak{n}_{v_0}$, agreeing that indices in the formulae to follow are to be read modulo $\mathfrak{n}_{v_0}$. If $t^j = b \times \Phi_{\mathfrak{p}}^j$, then

$$\gamma_e^i(b) = \gamma_e^i(a)\gamma_e^{i+1}(a)\dots\gamma_e^{\mathfrak{n}_{v_0}}(a)\gamma_{e'}^{\mathfrak{n}_{v_0}+1}(a)\dots\gamma_{e'}^{2\mathfrak{n}_{v_0}}(a)\gamma_e^{2\mathfrak{n}_{v_0}+1}(a)\dots$$

Here $e' \neq e$, and the subscripts, except perhaps at the beginning and end, appear in blocks of length $\mathfrak{n}_{v_0}$. There are exactly two vectors of the form

$$\otimes_{i\in z} x^i_{j(i)}$$

fixed by $\Phi_{\mathfrak{p}}^j$, but since we have chosen a new set of coset representations the subscripts $j(i)$ are no longer constant. At all events,

$$\text{trace } r_z^+(t^j) - \text{trace } r_z^-(t^j) = \left(\prod\nolimits_{i\in z}\gamma^i_{j(i)}(b) - \prod\nolimits_{i\in z}\gamma^i_{j'(i)}(b)\right) .$$

Since

$$\Phi_{\mathfrak{p}}(b) = a^{-1} b \Phi_{\mathfrak{p}}^j(a),$$

the character

$$a \to \prod\nolimits_{i\in z}\gamma^i_{j(i)}(b)$$

is invariant under $\Phi_{\mathfrak{p}}$. It is therefore a power of

$$a \to \gamma_1^1(a)\dots\gamma_1^{n_{v_0}}(a)\gamma_2^1(a)\dots\gamma_2^{n_{v_0}}(a) ,$$

and the power must clearly be

$$\frac{j\mathfrak{n}_{v_0}}{2\mathfrak{n}_{v_0} l_{v_0}} = \frac{j}{2 l_{v_0}} .$$

We conclude that

$$\prod\nolimits_{i\in z}\gamma^i_{j(i)}(b) = \prod\nolimits_{i\in z}\gamma^i_{j'(t)}(b) .$$

The lemma follows.

Our purpose has been to find an explicit expression for (2.6), with $f_{\mathfrak{p}}^j$ replacing $f_{\mathfrak{p}}^j$, when every prime of F dividing $\mathfrak{p}$ is unramified in L. We are still not finished, but we have shown that it is 0 unless v splits in L whenever Q_v contains a marked point. If we pass from $\mathcal{H}_{\mathfrak{p}}(T)$ to $\mathcal{H}_{\mathfrak{p}}(\widetilde{T}_1)$ we replace $\varphi_{\mathfrak{p}}$ by $\varphi'_{\mathfrak{p}}$ and $\varphi'_{\mathfrak{p}}$ is a product

$$\varphi'_{\mathfrak{p}}(t) = \prod\nolimits_v \varphi_v(t^v)$$

if $t = (t^v)$, $t^v \in T_v(\mathbf{Q}_{\mathfrak{p}})$. The expression

$$\lambda(L_v/F_v, \psi_v)\kappa_v\left(\frac{t_1^v - t_2^v}{t_1^0 - t_2^0}\right)\varphi_v(t^v) \tag{2.8}$$

does not depend on the choice of t^0 or of ψ_v. For lack of space, the image of the global element t^0 in $T_v(\mathbf{Q}_{\mathfrak{p}})$ is also denoted t^0.

Lemma 2.8. *Let U_v be the maximal compact subgroup of $T_v(\mathbf{Q}_{\mathfrak{p}})$ and let δ_v be the characteristic function of U_v divided by its measure. If v splits in L then (2.8) is equal to $\delta_v(t^v)$. If v does not split and U_v and $K_{\mathfrak{p}}$ are contained in a common maximal subgroup of $\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})$, then (2.8) is equal to $\delta_v(t^v)$ if the order of $t_1^v - t_2^v$ in L_v is even and to $-\delta_v(t^v)$ if the order of $t_1^v - t_2^v$ in L_v is odd.*

The assertion pertaining to split v is clear. If v is not split we take ψ_v to be such that the largest ideal on which it is trivial is the ring of integers of F_v and we take t^0 to be such that $t_1^0 - t_2^0$ is a unit in L_v. Then

$$\lambda(L_v/F, \psi_v) = 1$$

and

$$\kappa_v\left(\frac{t_1^v - t_2^v}{t_1^0 - t_2^0}\right)$$

is 1 when the order of $t_1^v - t_2^v$ is even and -1 when it is odd. However, it is observed after the proof of Lemma 2.2 of [7] that with these choices of ψ_v and t^0, and the assumption that U_v and $K_{\mathfrak{p}}$ are contained in a common maximal compact subgroup $\widetilde{K}_{\mathfrak{p}}$ of $\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})$, the function

$$\varphi_v(t) = \Phi^{T_v/\kappa_v}(t, \phi_v)$$

is the characteristic function of U_v divided by its measure. Here we write

$$\widetilde{K}_{\mathfrak{p}} = \prod\nolimits_v K_v$$

and take ϕ_v to be the characteristic function of K_v divided by its measure.

We return to the functions $f_{\mathfrak{p}}^j$, which we regard as elements of

$$\mathcal{H}_{\mathfrak{p}}(\widetilde{T}_1) \simeq \otimes_v \mathcal{H}_{\mathfrak{p}}(T_v)\ .$$

We suppose now that Q_v contains no marked points if v does not split in L. If we regard r_0^+ and r_0^- as representations of the associate group of $\widetilde{T}_1$ over $\mathbf{Q}_{\mathfrak{p}}$ then we may factor r_0^+ and r_0^- as a tensor product

$$\otimes_{v|\mathfrak{p}}(r_v^+ - r_v^-)\ .$$

In order to specify r_v^+ and r_v^- conveniently we choose the γ_1^i and γ_2^i in such a manner that all of the $\gamma_1^i - \gamma_2^i$ are positive with respect to an order that puts $\mu^\vee$ in the closed negative Weyl chamber. This is a choice that refers only to T and not to the identification of $X_*(\widetilde{T}_1)$ and $X^*({}^L\widetilde{T}_1)$.

If Q_v contains no marked points then r_v^+ is the trivial one-dimensional representation and r_v^- is zero-dimensional. If v splits in L then $r_v^+ \oplus r_v^-$ acts on the span of

$$\otimes_{i\in\overline{Q}_v} x_{j(i)}^i\ .$$

The action of $\Phi_{\mathfrak{p}} = \sigma$ sends

$$\otimes_{i\in\overline{Q}_v} x_{j(i)}^i \to \otimes_{i\in\overline{Q}_v} x_{j(i^\sigma)}^i$$

and $a \in {}^L\widetilde{T}_1^0$ acts as

$$\otimes x_{j(i)}^i \to \left(\prod\nolimits_{i\in\overline{Q}_v} \gamma_{j(i)}^i(a)\right)\left(\otimes x_{j(i)}^i\right)\ .$$

If we write

$$\mu^\vee = \sum \mu_v^\vee,$$

then

$$\mu_v = \sum\nolimits_{i\in\overline{Q}_v} \gamma_2^i\ .$$

The vector $\otimes x_{j(i)}^i$ lies in the space of r_v^+ or r_v^- according as $(-1)^{\sum j(i)}$ is 1 or -1.

The following lemma is in any case clear.

Lemma 2.9. *The function $f_{\mathfrak{p}}^j$ is a product $\prod f_{\mathfrak{p},v}^j$ with $f_{\mathfrak{p},v}^j$ in $\mathcal{H}_{\mathfrak{p}}(T_v)$.*

In order to save on subscripts we suppose for the purposes of describing the $f_{\mathfrak{p},v}^j$ that $\mathfrak{p}$ remains prime in F and that F splits in L. This allows us to drop the v when we choose.

Suppose μ is a weight of $r_0^+ + r_0^-$. Then there is a partition of $\overline{Q}$ into two disjoint subsets $\overline{Q}\,'$ and $\overline{Q}\,''$ with $\mathfrak{m}'$ and $\mathfrak{m}''$ elements respectively, and

$$\mu = \sum\nolimits_{i\in\overline{Q}\,'} \gamma_2^i + \sum\nolimits_{i\in\overline{Q}\,''} \gamma_1^i \ .$$

Moreover $\mathfrak{m}' + \mathfrak{m}'' = \mathfrak{m}$ ($= \mathfrak{m}_v$). Let

$$\delta_e = \sum\nolimits_{i\in Q} \gamma_e^i$$

and set

$$\nu = \mathfrak{m}'\delta_2 + \mathfrak{m}''\delta_1 \ .$$

If $k_{\mathfrak{p}}$ is a large unramified Galois extension of $\mathbf{Q}_{\mathfrak{p}}$ then

$$\mathrm{Nm}_{k_{\mathfrak{p}}/Q_{\mathfrak{p}}}\,\mu = [k_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]\nu/\mathfrak{n} \ .$$

We set

$$\beta(\nu) = \kappa(\mu - \mu^\vee) = (-1)^{k''} \ .$$

Lemma 2.10. *Suppose there is only one prime v of F dividing $\mathfrak{p}$ and that it splits in L. Let $\mathfrak{n} = \mathfrak{n}_v$, $l = l_v$, $k = k_v$. If $j|[E_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]$ and $t = a \times \Phi_{\mathfrak{p}}$ then*

$$\text{trace } r_0^+(t^j) - \text{trace } r_0^-(t^j)$$

is equal to

$$\sum\nolimits_{\mathfrak{m}/k|\mathfrak{m}'} \frac{k'}{k'!k''!}\,\beta(\nu)\nu(a)^{j/\mathfrak{n}} \ .$$

Here

$$k' = \frac{\mathfrak{m}'}{\mathfrak{m}}\,k \quad k'' = \frac{\mathfrak{m}''}{\mathfrak{m}}\,k \ .$$

Notice that

$$\frac{j}{\mathfrak{n}}\,\nu = \frac{j}{l}\,\left(\frac{\mathfrak{n}}{l}\,\nu\right) = \frac{j}{l}\,\left(\frac{k}{\mathfrak{m}}\,\nu\right)$$

is a weight if $\mathfrak{m}/k|\mathfrak{m}'$, and so the terms of the sum appearing in the lemma are well-defined.

The argument used to prove Lemma 2.7 shows that

$$\text{trace } r_0^+(t^j) - \text{trace } r_0^-(t^j) = \prod_z (\text{trace } r_z^+(t^j) - \text{trace } r_z^-(t^j)) \ .$$

The basis vectors $\otimes_{i\in z} x_{j(i)}^i$ for $z_z^+ \oplus r_z^-$ are permuted amongst themselves by $\Phi_{\mathfrak{p}}^j$. The only fixed vectors are

$$\otimes_{i\in z}\, x_j^i, \quad j = 1, 2 \ .$$

Thus

$$\text{trace } r_z^+(t^j) - \text{trace } r_z^-(t^j) = \delta_2(a)^{j/l} + (-1)^{\mathfrak{n}/l}\delta_1(a)^{j/l}$$

Since there are k orbits, we must raise the right side t the k^{th} power. Expanding by the binomial theorem and recalling that

$$\mathfrak{m}/k = \mathfrak{n}/l \ ,$$

we obtain the lemma.

If ϖ is a local uniformizing parameter for $\mathbf{Q}_\mathfrak{p}$ then the element of the Hecke algebra corresponding to the function

$$a \times \Phi_\mathfrak{p} \to \nu(a)^{j/\mathfrak{n}}$$

is the characteristic function of

$$\varpi^{j\nu/\mathfrak{n}} \widetilde{U}_\mathfrak{p}$$

divided by its measure. We denote this function by $\theta_{j\nu/\mathfrak{n}}$.

Before summarizing our results on the sum (2.1) we observe that since $Z(\mathbf{R})\backslash T(\mathbf{R})$ is compact the image of $T(\mathbf{Q})$ in

$$T(R)Z_K\backslash T(\mathbf{A}) \simeq Z_K\backslash T(\mathbf{A}_f)$$

is discrete. Clearly

$$\mathrm{meas}(Z_K T(\mathbf{Q})\backslash T(\mathbf{A}_f)) = \frac{\mathrm{meas}(Z(\mathbf{R})Z_K T(\mathbf{Q})\backslash T(\mathbf{A}))}{\mathrm{meas}\ Z(\mathbf{R})\backslash T(\mathbf{R})}\ .$$

Summary of the discussion of the sum (2.1).

(a) It can be expressed as a sum over the primes $\mathfrak{p}$ of E dividing $\mathfrak{p}$.

(b) If $\mathfrak{p}$ is such a prime let $\varpi_\mathfrak{p}$ be a uniformizing parameter for $E_\mathfrak{p}$. The term of the sum corresponding to $\mathfrak{p}$, which is now our only object of interest, may be expanded in powers of

$$|\varpi_\mathfrak{p}|^s = |\varpi|^{se}$$

if $e = [E_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]$.

(c) The coefficient of $j^{-1}|\varpi_\mathfrak{p}|^{js}$ is 0 if, for some $v|\mathfrak{p}$, v does not split in L and there is a marked place in Q_v. It contains the factor

$$\frac{\mu(T)}{2[\mathfrak{E}(T/\mathbf{A}) : \mathrm{Im}\,\mathfrak{E}(T/F)]}\,\mathrm{meas}(Z_K T(\mathbf{Q})\backslash T(\mathbf{A}_f))\ .$$

(d) To obtain the coefficient this factor has to be multiplied by a sum over the non-central elements of

$$T(\mathbf{Q}) \cap Z(\mathbf{R})Z_K\backslash T(\mathbf{Q}) = T(\mathbf{Q}) \cap Z_K\backslash T(\mathbf{Q})$$

and over the possible ν arising from collections $\{(\mathfrak{m}'_v, \mathfrak{m}''_v) | \mathfrak{m}'_v + \mathfrak{m}''_v = \mathfrak{m}_v\}$. In the quotient on the right, $T(\mathbf{Q})$ is regarded as a subgroup of $T(\mathbf{A}_f)$. The terms of the sum are themselves the product of three factors. The first is trace $\xi(t)$, and depends only on ξ and t. The second is $\Phi_0 T/K(t, \phi^\mathfrak{p})$, and depends only on the image of t in $T(\mathbf{A}_f^\mathfrak{p})$.

(e) The third factor may be represented by $\mathrm{meas}\,\widetilde{U}_\mathfrak{p}/\mathrm{meas}\,U_\mathfrak{p}$ times a product over the places v of F dividing $\mathfrak{p}$ of further factors, each depending only on the image of t in $T_v(\mathbf{Q}_\mathfrak{p})$, times $\beta(\nu) = \kappa(\mu - \mu^\vee)$. Here μ is any weight of r_0 such that, for some large Galois extension $k_\mathfrak{p}$ of $\mathbf{Q}_\mathfrak{p}$,

$$\mathrm{Nm}_{k_\mathfrak{p}/Q_\mathfrak{p}}\,\mu$$

is a multiple of ν.

(f) The factor corresponding to a given v is 0 unless v splits in F or there are no marked places in Q_v.

(g) If there are no marked places in Q_v and v does not split in F then the corresponding factor is

$$\frac{\pm 1}{\mathrm{meas}\,U_v} \cdot \frac{|t_1 t_2|_v^{1/2}}{|(t_1 - t_2)^2|_v^{1/2}}\ .$$

The sign ± 1 may be expressed as a product of two factors. The first is $(-1)^{\mathrm{ord}(t_1-t_2)}$. To obtain the second we write

$$\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}}) = \prod_{v|\mathfrak{p}} G_v(\mathbf{Q}_{\mathfrak{p}}) \ ,$$

with

$$G_v = \mathrm{Res}_{F_v/Q}\widetilde{G} \ ,$$

and let $\widetilde{K}_{\mathfrak{p}} = \prod K_v$ be a maximal compact subgroup of $\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})$ containing $K_{\mathfrak{p}}$. If $g \in G_v(\mathbf{Q}_{\mathfrak{p}})$ and $g^{-1}U_v g \subseteq K_v$ then the second factor is $(-1)^{\mathrm{ord}\,\mathrm{Nm}_g}$.

(h) We write $\nu = \sum \nu_v$. If v splits in F, the corresponding factor is 0 unless $k_v\nu_v/\mathfrak{m}_v$ is a weight. If it is, the corresponding factor is to be

$$\frac{k_v!}{k_v'!k_v''!}\, p^{je\mathfrak{m}_v/2}\theta_{\nu_v'}(t)\, \frac{|t_1t_2|_v^{1/2}}{|(t_1-t_2)^2|_v^{1/2}} \ .$$

Here

$$k_v' = \mathfrak{m}_v'k_v/\mathfrak{m}_v \qquad k_v'' = \mathfrak{m}_v''k_v/\mathfrak{m}_v \qquad \nu_v' = je\nu_v/\mathfrak{n}_v \ .$$

Notice that l_v must now be taken to be the greatest common divisor of $\mathfrak{n}_v$ and je. Moreover if $\mathfrak{m}_v' + \mathfrak{m}_v''$, $\theta_{\nu_v'}(t) \neq 0$, and q_v is the smallest of $\mathfrak{m}_v'$ and $\mathfrak{m}_v''$ then

$$\mathfrak{p}^{jem_v/2}\, \frac{|t_1t_2|_v^{1/2}}{|(t_1-t_2)^2|_v^{1/2}} = p^{jeq_v} \ .$$

The logarithm of (1.5) has to be subjected to a similar treatment. Fortunately, we can handle it with more dispatch, aprtly because we can rely to some extent on our discussion of (1.6), and partly because (1.5) has been so set up that the trace formula, in its stabilized form [7], is immediately applicable and quickly leads to the expressions needed for the comparison.

The logarithm of the local factor at $\mathfrak{p}$ of (1.5) is

$$\sum\nolimits_{\prod} \mathfrak{n}(\Pi)\mathfrak{m}(\Pi_\infty)\mathfrak{m}(\Pi_f)\log\, L_{\mathfrak{p}}(s-q/2,\pi,r) \ , \tag{2.9}$$

where $\pi \in \Pi$. The number $\mathfrak{n}(\Pi)$ is the common value of $\mathfrak{n}(\pi)$, $\pi \in \Pi$. Moreover we may as well agree that the sum is to be taken only over those Π such that

$$\begin{aligned} &\pi_\infty(z) = \nu^{-1}(z)I, && z \in Z(\mathbf{R}), \\ &\pi_f(z) = I, && z \in Z_K, \end{aligned}$$

if $\pi = \pi_\infty \otimes \pi_f \in \prod$, for otherwise either $\mathfrak{m}(\Pi_\infty)$ or $\mathfrak{m}(\Pi_f)$ is 0. This said, we may replace $\mathfrak{m}(\Pi_\infty)$ by its value

$$\sum\nolimits_{\prod_\infty \in \Pi_\infty} \mathrm{trace}\ \pi_\infty(f_\xi) \ .$$

We also may write

$$\mathfrak{m}(\Pi_f) = \sum\nolimits_{\pi_f \in \Pi_f} \mathfrak{m}(\pi_f) \ .$$

Then $\mathfrak{m}(\pi_f)$ is non-zero only if $\pi_{\mathfrak{p}}$ contains the trivial representation of $K_{\mathfrak{p}}$. Thus we may replace

$$\log\, L_{\mathfrak{p}}(s-q/2,\pi,r)$$

by

$$\sum\nolimits_{\mathfrak{p}|\mathfrak{p}} \sum_{j=1}^{\infty} \frac{|\varpi|^{js}}{j} \text{trace } \pi_{\mathfrak{p}}(h_{\mathfrak{p}}^j) \ ,$$

where $h_{\mathfrak{p}}^j k$ is an appropriate element of the Hecke algebra of $G(\mathbf{Q}_{\mathfrak{p}})$. It is defined by

$$\text{trace } \pi_{\mathfrak{p}}(h_{\mathfrak{p}}^j) = |\varpi|^{-jq/2} \text{trace } r_{\mathfrak{p}}(g(\pi_{\mathfrak{p}})) \ ,$$

whenever $\pi_{\mathfrak{p}}$ is an irreducible representation of $G(\mathbf{Q}_{\mathfrak{p}})$ containing the trivial representation of $K_{\mathfrak{p}}$. Here $r_{\mathfrak{p}} = r_{\mathfrak{p}}^+ \oplus r_{\mathfrak{p}}^-$ and $g(\pi_{\mathfrak{p}})$ is an element of the conjugacy class in ${}^L G^0 \times \Phi_{\mathfrak{p}}$ associated to $\pi_{\mathfrak{p}}$. We conclude that (2.9) may be written as the sum over $\mathfrak{p}|\mathfrak{p}$ and the sum of j from 1 to ∞ of $|\varpi|^{js}/j$ times

$$\sum\nolimits_{\pi} \mathfrak{n}(\pi) \text{trace } \pi_{\infty}(f_{\xi}) \text{trace } \pi^{\mathfrak{p}}(\phi^{\mathfrak{p}}) \text{trace } \pi_{\mathfrak{p}}(h_{\mathfrak{p}}^j) \ . \tag{2.10}$$

Recall that $\phi^{\mathfrak{p}}$ is the characteristic function of $K^{\mathfrak{p}} \subseteq G(\mathbf{A}_f^{\mathfrak{p}})$ divided by its measure.

We may apply the stabilized trace formula for $G(\mathbf{Q})Z(\mathbf{R})Z_K \backslash G(\mathbf{A})$ to the function

$$g \to f_{\xi}(g_{\infty}) \phi^{\mathfrak{p}}(g^{\mathfrak{p}}) h_{\mathfrak{p}}^j(g_{\mathfrak{p}})$$

to obtain the form for (2.10) that we need.

In order to write down the contribution from the scalars we need to draw on our knowledge of harmonic analysis on real groups, namely on the limit formula of Harish-Chandra, to see that

$$f_{\xi}(z) = (-1)^q \text{trace } \xi(z) / \text{meas } Z(\mathbf{R}) \backslash G'(\mathbf{R}), \qquad z \in Z(\mathbf{R}) \ .$$

Here G' is a form of G over $\mathbf{R}$, the one for which $Z(\mathbf{R}) \backslash G'(\mathbf{R})$ is compact, and the measure on $Z \backslash G'$ is obtained from that on $Z \backslash G$ by transporting invariant forms of highest degree as in §15 of [6]. To tell the truth, I am unable to supply a reference to the appropriate computation. The method to be used is described in [8], and the reader can verify for himself that the constant is correct. Appealing to §7 of [7], we see that the contribution of the scalar matrices to (2.10) is

$$\sum\nolimits_{Z(\mathbf{Q}) \cap Z(\mathbf{R}) Z_K \backslash Z(\mathbf{Q})} \frac{\text{meas}(Z(\mathbf{R}) Z_K G(\mathbf{Q}) \backslash G(\mathbf{A}))}{\text{meas}(Z(\mathbf{R}) \backslash G(\mathbf{R})} \times (-1)^q \text{trace } \xi(z) \phi^{\mathfrak{p}}(z) h_{\mathfrak{p}}^j(z) \ . \tag{2.11}$$

It will remove some complication from the discussion of the remaining terms of the trace formula if we take advantage of the possibility we have allowed ourselves of only working with sufficiently small K. It is convenient so to arrange matters that the equation

$$r^{-1} t g = z t$$

with $g \in G(\mathbf{Q})$, $t \in G(\mathbf{Q})$, and $z \in Z(\mathbf{Q}) \cap Z(\mathbf{R}) Z_K$ implies that $z = 1$. The equation certainly implies that z lies in the center of the derived group. Since this is finite, its intersection with $Z(\mathbf{R}) Z_K$ will be $\{1\}$ when K is small.

The remaining contribution to the stabilized trace formula is a sum over a set of representatives T for the stable conjugacy classes of Cartan subgroups of

$$\frac{1}{2} \sum\nolimits'_{T(\mathbf{Q}) \cap Z(\mathbf{R}) Z_K \backslash T(\mathbf{Q})} \frac{\text{meas}(T(\mathbf{Q}) Z(\mathbf{R}) Z_K \backslash T(\mathbf{A})) \mu(T)}{[\mathfrak{E}(T/\mathbf{A}) : \text{Im } \mathfrak{E}(T/F)]} \times \Phi^{T/1}(t, f_{\xi}) \Phi^{T/1}(t, \phi^{\mathfrak{p}}) \Phi^{T/1}(t, h_{\mathfrak{p}}^j) \ .$$

The prime indicates that scalars are excluded from the sum. Since $\Phi^{T/1}(t, f_{\xi})$ is 0 unless $T(\mathbf{R})$ is fundamental, we agree to sum only over such T. Then

$$\Phi^{T/1}(t, f_{\xi}) = \text{trace } \xi(t) / (Z(\mathbf{R}) \backslash T(\mathbf{R})) \ .$$

By appealing to our earlier discussion and to the formalism of Hecke algebras, we easily see that the following assertions are valid.

(i) $h^j_{\mathfrak{p}} = 0$ if $e = [E_{\mathfrak{p}} \cdot \mathbf{Q}_{\mathfrak{p}}]$ does not divide j.

(ii) If

$$G_v = \mathrm{Res}_{F_v/\mathbf{Q}_{\mathfrak{p}}} \widetilde{G}$$

there is a natural imbedding of $\mathcal{H}_{\mathfrak{p}}$ into $\otimes_{v/\mathfrak{p}} \mathcal{H}_{\mathfrak{p}}(G_v)$.

(iii) The image of $h^j_{\mathfrak{p}}$ under this imbedding is a product of $\otimes h^j_{\mathfrak{p},v}$.

(iv) If T_v is a split Cartan subgroup of G_v and t_1, t_2 the eigenvalues of $t \in T_v(\mathbf{Q}_{\mathfrak{p}}) \simeq L_v^\times$, then, when $e|j$,

$$\int_{T_v(\mathbf{Q}_{\mathfrak{p}})\backslash G_v(\mathbf{Q}_c urp)} h^j_{\mathfrak{p},v}(g^{-1}tg)dg$$

is equal to

$$\frac{|t_1t_2|_v^{1/2}}{|t_1-t_2|_v} \sum\nolimits_{\mathfrak{m}_v/k_v|\nu_v} \frac{k_v!}{k_v'!k_v''!}\,\theta_{\nu_v'}(t)\left(\nu_v' = \frac{j}{\mathfrak{n}_v}\,\nu_v\right) \ .$$

The last assertion is, of course, a consequence of the definition of the Satake homomorphism, and the relation of the Satake homomorphism with the formalism of the associate group. Here ν_v is defined by a pair of non-negative integers $\mathfrak{m}_v'$, $\mathfrak{m}_v''$ with $\mathfrak{m}_v' + \mathfrak{m}_v'' = \mathfrak{m}_v$, and the sum is over those for which

$$k_v' = k_v\mathfrak{m}_v'/\mathfrak{m}_v \qquad k_v'' = k_v\mathfrak{m}_v''/\mathfrak{m}_v$$

are integral.

For any T let $U_{\mathfrak{p}}$ be, as before, the maximal compact subgroup of $T(\mathbf{Q}_{\mathfrak{p}})$, and $\widetilde{U}_{\mathfrak{p}} = \prod U_v$ the maximal compact subgroup of $\widetilde{T}_1(\mathbf{Q}_{\mathfrak{p}})$. Let $\widetilde{K}_{\mathfrak{p}} = \prod K_v$ be the maximal compact subgroup of $\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})$ containing $K_{\mathfrak{p}}$. Replacing $\widetilde{T}_1$ by a conjugate if necessary, suppose $\widetilde{U}_{\mathfrak{p}} \subseteq \widetilde{K}_{\mathfrak{p}}$ and define

$$\mathfrak{u}(T) = [\widetilde{K}_{\mathfrak{p}} : K_{\mathfrak{p}}\widetilde{U}_{\mathfrak{p}}] \ .$$

Then

$$\mathfrak{u}(T)\mathrm{meas}\ U_{\mathfrak{p}}\Phi^{T/1}(t, h^j_{\mathfrak{p}})$$

is equal to

$$\prod\nolimits_v \mathrm{meas}\ U_v \int_{T_v(\mathbf{Q}_{\mathfrak{p}})\backslash G_v(\mathbf{Q}_{\mathfrak{p}})} h^j_{\mathfrak{p},v}(g^{-1}tg)dg \ . \tag{2.12}$$

To verify this we observe that both sides are independent of the measures chosen, and that it therefore suffices to work with one convenient choice of measure. Let h be $(\mathrm{meas}\ K_{\mathfrak{p}})h^j_{\mathfrak{p}}$, where $h^j_{\mathfrak{p}}$ is taken in $\mathcal{H}_{\mathfrak{p}}(G)$ and let h' be $(\mathrm{meas}\ \widetilde{K}_{\mathfrak{p}})h^j_{\mathfrak{p}}$, with $h^j_{\mathfrak{p}}$ now regarded as an element of $\mathcal{H}_{\mathfrak{p}}(\widetilde{G}_1) = \otimes_\iota \mathcal{H}_{\mathfrak{p}}(G)$. Then h is the restriction of h' to $G(\mathbf{Q}_{\mathfrak{p}})$.

$$\Phi^{T/1}(t,h) = \sum\nolimits_{\widetilde{T}(\mathbf{Q}_{\mathfrak{p}})G(\mathbf{Q}_{\mathfrak{p}})\backslash \widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})} \int_{T(\mathbf{Q}_{\mathfrak{p}})\backslash G(\mathbf{Q}_{\mathfrak{p}})} h(g^{-1}tg)dg \ .$$

If we take the measure on $T(\mathbf{Q}_{\mathfrak{p}})\backslash G(\mathbf{Q}_{\mathfrak{p}})$ to be the same as that on its image $\widetilde{T}_1(\mathbf{Q}_{\mathfrak{p}})\backslash \widetilde{T}_1(\mathbf{Q}_{\mathfrak{p}})G(\mathbf{Q}_{\mathfrak{p}})$ in $\widetilde{T}_1(\mathbf{Q}_{\mathfrak{p}})\backslash\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})$, the right side is equal to

$$\int_{\widetilde{T}_1(\mathbf{Q}_{\mathfrak{p}})/\widetilde{G}_1(\mathbf{Q}_{\mathfrak{p}})} h'(g^{-1}tg)dg \ .$$

To check the assertion completely we need to show that with this choice

$$\mathfrak{u}(T)\mathrm{meas}\ J_{\mathfrak{p}}\backslash K_{\mathfrak{p}} = \mathrm{meas}\ \widetilde{U}_{\mathfrak{p}}\backslash\widetilde{K}_{\mathfrak{p}} \ .$$

However

$$\text{meas } U_{\mathfrak{p}} \backslash K_{\mathfrak{p}} = \text{meas } \widetilde{U}_{\mathfrak{p}} \backslash \widetilde{U}_{\mathfrak{p}} K_{\mathfrak{p}}$$

and

$$\text{meas } \widetilde{U}_{\mathfrak{p}} \backslash \widetilde{K}_{\mathfrak{p}} = \mathfrak{u}(T) \text{meas } \widetilde{U}_{\mathfrak{p}} \backslash \widetilde{U}_{\mathfrak{p}} K_{\mathfrak{p}} \ .$$

We have only to evaluate the individual integrals on the right of (2.12). They have been given in §6 of [12], albeit for the group $\mathrm{GL}(2, \mathbf{Q}_{\mathfrak{p}})$ and not for $G_v(\mathbf{Q}_{\mathfrak{p}}) = \mathrm{GL}(2, F_v)$, but the formulae and the proofs are valid without change. In order to disencumber ourselves of the subscript v when stating the results, we suppose that $\mathfrak{p}$ remains prime in F.

We start from $G_v(\mathbf{Q}_{\mathfrak{p}}) = \mathrm{GL}(2, F_v)$ and a

$$\nu = a'\delta_2 + a''\delta_1 \ .$$

Let

$$\tilde{\nu} = a'\delta_1 + a''\delta_2$$

and let h be a spherical function on $\mathrm{GL}(2, F_v)$ for which

$$(|t_1 - t_2|_v / |t_1 t_2|_v^{1/2}) \int_{A(F_v)\backslash \mathrm{GL}(2,F_v)} h(g^{-1}tg)dg = \frac{1}{2}\{\theta_\nu(t) + \theta_{\tilde{\nu}}(t)\}$$

if A is a split torus in $\mathrm{GL}(2)$. If $a' = a''$ and K_v the maximal compact subgroup containing $K_{\mathfrak{p}}$, then h is the characteristic function of

$$\begin{pmatrix} \varpi_v^{a'} & 0 \\ 0 & \varpi_v^{a''} \end{pmatrix} K_v$$

divided by the measure of K_v. There will be no need for an explicit evaluation of the other orbital integrals in this case.

In general set

$$\Phi^{T_v}(t,h) = \int_{T_v(\mathbf{Q}_{\mathfrak{p}})\backslash G_v(\mathbf{Q}_{\mathfrak{p}})} h(g^{-1}th) \ .$$

If $a' \neq a''$ these integrals can be computed directly, along the lines of §3 of [15] or of §7 of [6], but I prefer to apply the method of §6 of [12]. Lemma 6.4 of that paper and the Weyl integration formula imply that

$$\frac{1}{2} {\sum}'_{T_v} \int_{T_v(\mathbf{Q}_{\mathfrak{p}})} \chi_\pi(t) \Phi^{T_v}(t,h) \Delta^2(t) dt = 0$$

if π is an absolutely cuspidal representation of $G_v(\mathbf{Q}_{\mathfrak{p}})$. The sum is over a set of representatives for the non-split Cartan subgroups and

$$\Delta(t) = |(t_1 - t_2)^2|_v^{1/2} / |t_1 t_2|_v^{1/2} \ .$$

In addition $\Phi^{T_v}(t,h)$ is 0 unless

$$|\det t|_v = |\varpi_v|^a \qquad a = a' + a'' \ .$$

Let $G'_v(\mathbf{Q}_{\mathfrak{p}})$ be the multiplicative group of the quaternion algebra over F_v. Exploiting the relation between characters of $G_v(\mathbf{Q}_{\mathfrak{p}})$ and characters of $G'_v(\mathbf{Q}_{\mathfrak{p}})$ provided in [6], and using as well the Weyl integration formula for $G'_v(\mathbf{Q}_{\mathfrak{p}})$ and orthogonality of characters, we deduce that there is a function $\varphi(x)$ on $F^\times$ such that

$$\Phi^{T_v}(t,h) = \text{meas}(T_v(\mathbf{Q}_{\mathfrak{p}})\backslash G'_v(\mathbf{Q}_{\mathfrak{p}}))\varphi(\det t)$$

if T_v is not split. The measure on $G'_v(\mathbf{Q}_{\mathfrak{p}})$ may be assumed to be obtained by transferring that of $G_v(\mathbf{Q}_{\mathfrak{p}})$.

Let χ be the representation $g \to \chi(\det g)$. If we use the Satake homomorphism to compute the trace of $\chi(h)$ we obtain

$$\begin{array}{ll} 0 & \chi \text{ ramified} \\ \frac{1}{2}\left(|\varpi_v|^{a'-a''/2} + |\varpi_v|^{a''-a'/2}\right)\chi(\varpi_v)^a & \chi \text{ unramified} \ . \end{array}$$

We may also compute it by using the Weyl integration formula. This yields the sum of

$$\frac{1}{2}\sum\nolimits_{T_v}' \operatorname{meas}(T_v(\mathbf{Q}_\mathfrak{p})\backslash G_v'(\mathbf{Q}_\mathfrak{p}))\int_{T_v(\mathbf{Q}_\mathfrak{p})}\chi(\det t)\varphi(\det t)\Delta^2(t)dt$$

and

$$\frac{1}{2}\int_{A_v(\mathbf{Q}_\mathfrak{p})}\chi(\det t)\theta_\nu(t)\Delta(t)dt\ .$$

The second expression is 0 unless χ is unramified, but then it equals

$$\frac{1}{2}\,|\varpi_v|^{-|a'-a''|/2}\chi(\varpi_v)^a\ .$$

This yields a value for the first expression. Applying the orthogonality relations and the Weyl integration formula to the characters of the one-dimensional representations of $G_v'(\mathbf{Q}_\mathfrak{p})$, we conclude that $\varphi(x)$ is 0 unless $|x|=|\varpi_v|^a$ and that then

$$\varphi(x)=\frac{|\varpi_v|^{|a'-a''|/2}}{2\operatorname{meas}K_v'}$$

if K_v' is the maximal compact subgroup of $G_v'(\mathbf{Q}_\mathfrak{p})$. In order to have a convenient notation we let Ξ_a be 0 unless $|\det t|=|\varpi|^a$ when $\Xi_a(t)$ is to be 1.

Finally we deduce from Lemma 4.1 of [15] that if z is a scalar then $h(z)=0$ unless $|\det z|=|\varpi_v|^a$ but then

$$h(z)=-\frac{|\varpi_v|^{|a'-a''|/2}}{2\operatorname{meas}K_v'}\ .$$

All the information we need is now at our disposal. It will be a help to review it once again, and display it in a form convenient for reference.

Summary of the discussion of the sum (2.9).

(a) It can be expressed as a sum over the primes $\mathfrak{p}$ of E dividing $\mathfrak{p}$ and over the possible ν, defined by $\{(\mathfrak{m}_v',\mathfrak{m}_v'')|\mathfrak{m}_v'+\mathfrak{m}_v''=\mathfrak{m}_v\}$. The $\mathfrak{m}_v'$, $\mathfrak{m}_v''$ are to be integral except perhaps when $\mathfrak{m}_v'=\mathfrak{m}_v''=\mathfrak{m}_v/2$, and then we allow them to be half-integral.

(b) The term corresponding to $\mathfrak{p}$ and ν may itself be expanded in powers of $|\varpi_\mathfrak{p}|^s=|\varpi_\mathfrak{p}|^{se}$, $e=[E_\mathfrak{p}:\mathbf{Q}_\mathfrak{p}]$. We now fix a $\mathfrak{p}$ and a ν and consider the coefficients of $|\varpi_\mathfrak{p}|^j/j$.

(c) We treat the case that $\mathfrak{m}_v'\neq\mathfrak{m}_v''$ for at least one v first. Then the coefficient is a double sum, over the stable conjugacy classes of Cartan subgroups T fundamental at infinity and split at every v for which $\mathfrak{m}_v'\neq\mathfrak{m}_v''$, and over non-scalar t in $T(\mathbf{Q})\cap Z(\mathbf{R})Z_K\backslash T(\mathbf{Q})$. The individual terms are given as products. One factor is

$$\frac{\mu(T)}{2[\mathfrak{E}(T/\mathbf{A}):\operatorname{Im}\mathfrak{E}(T/F)]}\operatorname{meas}(T(\mathbf{Q})Z_K\backslash T(\mathbf{A}_f))\ .$$

A second is

$$\Phi^{T/1}(t,\phi^\mathfrak{p})\text{trace }\xi(t)\ \cdot\ u(T)\,\frac{\operatorname{meas}\widetilde{U}_\mathfrak{p}}{\operatorname{meas}U_\mathfrak{p}}\ .$$

The third may again be written as the product over the places of v dividing $\mathfrak{p}$ of factors depending only on the image of t in $T_v(\mathbf{Q}_\mathfrak{p})$. If T_v is split, the factor is 0 unless $\mathfrak{m}_v/k_v$ divides $\mathfrak{m}_v'$ and $\mathfrak{m}_v''$. Otherwise it is

$$\mathfrak{m}^{je\mathfrak{m}_v/2}\,\frac{k_v!}{k_v'!k_v''!}\,\theta_{\nu_v'}(t)\,\frac{|t_1t_2|_v^{1/2}}{|t_1-t_2|_v}\ .$$

If T_v is not split the expression is more complicated. It is the sum of two terms. The first is

$$\left\{\sum\nolimits_{0 \leqq i < k_v/2} p^{je\mathfrak{m}_v i/k_v} \binom{k_v}{i}\right\} \frac{\operatorname{meas} T_v(\mathbf{Q}_\mathfrak{p})\backslash G'_v(\mathbf{Q}_\mathfrak{p})}{\operatorname{meas} K'_v} \Xi_{je\mathfrak{m}_v/\mathfrak{n}_v}(t) \ .$$

The second is 0 unless k_v is even, but it is then

$$p^{je\mathfrak{m}_v/2} \binom{k_v}{k_v/2} \frac{1}{\operatorname{meas} K_v} \int_{T_v(\mathbf{Q}_\mathfrak{p})\backslash G_v(\mathbf{Q}_\mathfrak{p})} \varphi_{\mathfrak{m}_v}(g^{-1}tg)dg$$

if $\varphi_{\mathfrak{m}_v}$ is the characteristic function of

$$\begin{pmatrix} \varpi^{\mathfrak{m}_v/2} & 0 \\ 0 & \varpi^{\mathfrak{m}_v/2} \end{pmatrix} K_v \ .$$

The point is that for split T_v there are several possible ν_v, and we can decompose the contributions from the orbital integrals of $h^j_\mathfrak{p}$ into parts labeled by them; but for a T_v that is not split there is only one reasonable ν_v and we have to take the orbital integral in one piece.

(d) Whatever is not included in the terms gathered in (c) must now be put together and credited to that ν for which $\mathfrak{m}'_v = \mathfrak{m}''_v$ for all v. There are two contributions; the first is a sum over z in $Z(\mathbf{Q}) \cap Z(\mathbf{R})Z_K\backslash Z(\mathbf{Q})$. Each term of the sum is a product, the first factor being

$$\frac{\operatorname{meas} Z(\mathbf{R})Z_K G(\mathbf{Q})\backslash G(\mathbf{A})}{\operatorname{meas} Z(\mathbf{R})\backslash G'(\mathbf{R})} \operatorname{trace} \xi(z)\phi^\mathfrak{p}(z) \frac{\operatorname{meas} \widetilde{K}_\mathfrak{p}}{\operatorname{meas} K_\mathfrak{p}} \ .$$

The second factor is itself a product over the places v dividing $\mathfrak{p}$ of terms depending only on the image of z in the group $G_v(\mathbf{Q}_\mathfrak{p})$. It will be easier to write them down if we first observe that a simple calculation, which can be left to the reade4r (cf. §15 of [6]), shows that

$$\operatorname{meas} K_v = \left(|\varpi_v|^{-1} - 1\right) \operatorname{meas} K'_v \ .$$

The factor at v is itself a sum of two terms. The first is

$$-(-1)^{\mathfrak{m}_v} \frac{\left(|\varpi_v|^{-1} - 1\right)}{\operatorname{meas} K_v} \Xi_{je\mathfrak{m}_v/\mathfrak{n}_v}(z) \left\{\sum\nolimits_{0 \leqq i < k_v/2} \binom{k_v}{i} \mathfrak{p}^{je\mathfrak{m}_v i/k_v}\right\} \ .$$

The second is 0 unless k_v is even, when it is

$$\frac{(-1)^{\mathfrak{m}_v}}{\operatorname{meas} K_v} \Xi_{je\mathfrak{m}_v/\mathfrak{n}_v}(z) \binom{k_v}{k_v/2} \mathfrak{p}^{je\mathfrak{m}_v/2} \ .$$

The second contribution is a sum over the stable conjugacy classes of Cartan subgroups T with $T(\mathbf{R})$ fundamental of a sum over the non-scalar elements in $T(\mathbf{Q}) \cap Z(\mathbf{R})Z_K\backslash T(\mathbf{Q})$. The terms of the sum are themselves products. The first factor of the product is

$$\frac{\mu(T)}{2[\mathfrak{E}(T/\mathbf{A}) : \operatorname{Im} \mathfrak{E}(T/F)]} \operatorname{meas}(T(\mathbf{Q})Z_K\backslash T(\mathbf{A}_f))\Phi^{T/1}(t, \phi^\mathfrak{p}) \times \operatorname{trace} \xi(t) \cdot u(T) \frac{\operatorname{meas} \widetilde{U}_\mathfrak{p}}{\operatorname{meas} U_\mathfrak{p}} \ .$$

The second is a product over v dividing $\mathfrak{p}$ of terms that once again depend only on the image of t in $T_v(\mathbf{Q}_\mathfrak{p})$. If T_v is split the first factor is 0 unless k_v is even, but then it is

$$\mathfrak{p}^{je\mathfrak{m}_v/2} \binom{k_v}{k_v/2} \Theta_{\nu'_v}(t) \frac{|t_1 t_2|_v^{1/2}}{|t_1 - t_2|_v}, \qquad \nu'_v = \frac{je}{\mathfrak{n}_v} \nu_v \ .$$

If T_v is not split, the factor is the same as in (c).

3. The zeta-function.

The next step in the proof is to use the structure of the set of geometric oints on $S(K)$ over the algebraic closure $\tilde{\kappa}_{\mathfrak{p}}$ of the residue field $\kappa_{\mathfrak{p}}$ of E at $\mathfrak{p}$ to obtain a formula for

$$\log\, Z_{\mathfrak{p}}(s, S(K), F_{\xi})$$

which can be compared with the formulae of the preceding paragraph. This logarithm has a power series expansion

$$\sum\nolimits_{j=1}^{\infty} M(j) \frac{|\varpi_{\mathfrak{p}}|^{js}}{j}$$

with

$$M(j) = \sum\nolimits_{i} (-1)^i \operatorname{trace} \tau^i(\Phi)\ ,$$

and we shall be concerned only with the coefficients $M(j)$.

According to [13] the set $S(K, \tilde{\kappa}_{\mathfrak{p}})$ is a union of subsets indexed by equivalence classes of Frobenius pairs (γ, h^0). These must be described. Recall that we have supposed that the totally real field F used to define G is imbedded in $\overline{\mathbf{Q}} \subseteq \mathbf{C}$. The set of imbeddings of F in $\overline{\mathbf{Q}}$ or $\mathbf{C}$ is then represented by the finite homogeneous space

$$Q = \mathfrak{G}(\overline{\mathbf{Q}}/F)\backslash \mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})\ .$$

To nourish our intuition we wrote this set as

$$\times \times \ldots \times \bigcirc \ldots \bigcirc \times\ .$$

The infinite places at which the algebra splits are marked by a cross; the others are not marked, but represented by circles. We assumed that F is unramified at $\mathfrak{p}$. Hence the Frobenius $\Phi_{\mathfrak{p}}$ acts, and we decomposed the set into its orbits.

$$\underbrace{\times \ldots \bigcirc \ldots \times}_{Q_{v_1}} ; \underbrace{\bigcirc \ldots \times \ldots \bigcirc}_{Q_{v_2}} ;$$

let $\mathfrak{n}_v$ be the number of elements in the orbit labelled by v. Then

$$\sum \mathfrak{n}_v = \mathfrak{n} = [F : \mathbf{Q}]\ .$$

To be definite we viewed the Frobenius as acting on each orbit by a cyclic shift to the right. We also let $\mathfrak{m}_v$ be the number of marked points in the orbit, and observed that the orbits or the indices v also label the primes of F dividing $\mathfrak{p}$.

There are two kinds of Frobenius pairs, or rather of equivalence classes of such pairs. To describe one of the first type we start from a totally imaginary quadratic extension L of F which splits at least one prime of F dividing $\mathfrak{p}$. We choose a set of representatives for the isomorphism classes over F of such fields. Each L determines a stable conjugacy class of Cartan subgroups of G over $\mathbf{Q}$. Choose a representative T_L for each such class. T_L is contained in a unique Cartan subgroup $\widetilde{T}_L$ of $\widetilde{G}_1$. it will be less taxing on my powers of abstraction if I fix an isomorphism of $\widetilde{T}_L$ with the algebraic group over $\mathbf{Q}$ defined by the multiplicative group of L. In particular $\widetilde{T}_L(\mathbf{Q})$ will be identified with $L^{\times}$, and every imbedding of L in $\overline{\mathbf{Q}}$ defines a rational character of $\widetilde{T}_L$ and as such it is a basis of the lattice of all rational characters. The dual basis of the lattice of coweights is also indexed by P.

We fix, to have a point of reference, h_L: $R \to T_L$ which is conjugate under $G(\mathbf{R})$ to the h defining the Shimura variety. Suppose $g \in \mathfrak{A}(T/\mathbf{Q})$. Let

$$T = T_L^g = g^{-1} T g\ .$$

An $h^0\colon R \to T$ must be of the form

$$h_L^x\colon r \to x^{-1}h_L(r)x, \qquad x \in G(\mathbf{R}) \ .$$

The diagram

$$\begin{tikzcd} & T \\ R \arrow[ur, "h^0"] \arrow[dr, "h_L^{xg^{-1}}"'] & \\ & T_L \arrow[uu, "g"'] \end{tikzcd}$$

is commutative. Since $\mathfrak{D}(T/F) \to \mathfrak{D}(T/\mathbf{R})$ is surjective we may by a suitable choice of x and g arrange that $h_L^{xg^{-1}}$ is any element in the orbit of h_L under the Weyl group of $T_L(\mathbf{C})$ in $G(\mathbf{C})$. However, h_L itself can only vary within an orbit of the Weyl group of $T(\mathbf{R})$ in $G(\mathbf{R})$.

As in [13] h^0 defines a coweight $\mu^\vee$ of T. Since g has been fixed for now, we may pull $\mu^\vee$ back to a coweight of T_L and hence of $\widetilde{T}_L$, which I again denote by $\mu^\vee$. It may be written in terms of the dual basis we have chosen. The coefficients will be 0 or 1. It will help to have a pictorial way of representing these coefficients.

Let P_v be the inverse image of Q_v in P. The group $\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$ acts on P_v. If v splits in L then P_v falls into two orbits and we represent the map $P_v \to Q_v$ by the diagram:

$$\begin{array}{cl} P_v & \left\{\begin{array}{l} \times\bigcirc\ldots\bigcirc\ldots\times \\ \bigcirc\times\ldots\bigcirc\ldots\times \end{array}\right. \\ \big\downarrow & \\ Q_v & \quad\times\times\ldots\bigcirc\ldots\times\ . \end{array}$$

Each horizontal line represents an orbit. A point is marked by an $\times$ if the coefficient of the corresponding element of the dual basis is 1 and is left unmarked if the corresponding coefficient is 0. Above a marked point of Q_v there is one marked and one unmarked point, and above an unmarked point there is no marked point. let $\mathfrak{m}'_v$ and $\mathfrak{m}''_v$ be the number of marked points in the two orbits. Then

$$\mathfrak{m}'_v + \mathfrak{m}''_v = \mathfrak{m}_v \ .$$

If v does not split in L we may still represent $P_v \to Q_v$ as

$$\begin{array}{c} \left\{\begin{array}{l} \bigcirc\ldots\times\ldots\bigcirc \\ \bigcirc\ldots\bigcirc\ldots\times \end{array}\right. \\ \big\downarrow \\ \bigcirc\ldots\times\ldots\times\ . \end{array}$$

However there is only one orbit in P_v and no significance is to be attached to the two horizontal rows inits pictorial representation.

Starting from L, g, and h^0 we set about defining Frobenius pairs (γ, h^0) of the first kind. We must suppose that $\mathfrak{m}'_v \neq \mathfrak{m}''_v$ for at least one v with v split. If $k_\mathfrak{p}$ is a sufficiently large but finite Galois extension of $\mathbf{Q}_\mathfrak{p}$ we set

$$\mathrm{Nm}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}\mu^\vee = \nu^\vee$$

and write $\nu^\vee$ as an integer linear combination of the elements of the dual basis. If $l = [k_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]$ the coefficient of a coweight in P_v is $l\mathfrak{m}_v/2\mathfrak{n}_v$ if v does not split in L. If v splits the coefficient is $l\mathfrak{m}'_v/\mathfrak{n}_v$ in the first row and $l\mathfrak{m}''_v/\mathfrak{n}_v$ in the second.

We first construct an appropriate γ_1 in $T_L(\mathbf{Q}) \subseteq L^\times$ and then take $\gamma = g^{-1}\gamma_1 g$. If v splits let $\mathfrak{a}_v$ and $\mathfrak{b}_v$ be the two prime ideals in L dividing it. If it does not let $\mathfrak{g}_v$ be the ideal of L generated by the prime ideal in F defined by the valuation v. Set

$$\mathfrak{a} = \left(\prod \mathfrak{g}_v^{\mathfrak{m}_v}\right)\left(\prod \mathfrak{a}_v^{2\mathfrak{m}'_v}\mathfrak{b}_w^{2\mathfrak{m}''_v}\right) .$$

Some power $\mathfrak{a}^s$ of $\mathfrak{a}$ is principal. Let

$$\mathfrak{a}^s = (\delta_1) .$$

δ_1 lies in $\widetilde{T}_L(\mathbf{Q})$. Recall that G was defined as the inverse image of A in $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(1)$. If B is the quotient of these two groups we have injections

$$T_L(\mathbf{Q})\backslash\widetilde{T}_L(\mathbf{Q}) \hookrightarrow B(\mathbf{Q}) \qquad A(\mathbf{Q})\backslash F^\times \hookrightarrow B(\mathbf{Q}) .$$

Let $\bar{\delta}_1$ be the image of δ_1. Since $\mu^\vee$ and $\nu^\vee$ are coweights of T_L, $\chi(\bar{\delta}_1)$ is a unit for all weights of B and some power $\bar{\delta}_1^t$ of $\bar{\delta}_1$ is the image of a unit β in $F^\times$. Set

$$\gamma_1 = \delta_1^{2t}/\beta .$$

The pair (γ, h^0) is then of Frobenius type.

Suppose $\overline{\gamma}$ lies in T and $(\overline{\gamma}, h^0)$ is also of Frobenius type. Let

$$\overline{\gamma} = g^{-1}\overline{\gamma}_1 g .$$

By the definition of pairs of Frobenius type there are positive integers c and d for which

$$|\overline{\gamma}_1^d|_v = |\gamma_1^c|_v$$

at every finite place v of L. Thus $\gamma_1^{-c}\,\overline{\gamma}_1^d$ is a unit. Since L is a totally imaginary extension of F we may even arrange, by multiplying c and d by a common factor, that $\gamma_1^{-c}\,\overline{\gamma}_1^d$ is a unit in F. Consequently (γ, h^0) and $(\overline{\gamma}, h^0)$ are equivalent, and the critical data for the construction of a pair of the first kind are L, g, and h^0, or, if one prefers,$h_L^{xg^{-1}}$. Actually, $h_L^{xg^{-1}}$ is determined by $\mu^\vee$, regarded as a coweight of T_L, and we use $\mu^\vee$ rather than h^0. It is also clear that the class of (γ, h^0) depend sonly on the image δ of g in $\mathfrak{D}(T/\mathbf{Q})$, and it is finally most convenient to take L, δ, and $\mu^\vee$ as the fundamental data. I note in passing that, because $\mathfrak{m}'_v \neq \mathfrak{m}''_v$ for some i, no power of γ is central and the group H^0 of [13], now denoted by I^0, is T.

The use of the symbol H in [13] conflicts with its use in [7]. Since the construction of those two papers appear simultaneously in the study of Shimura varieties, it will be best to use I^0 and I for the groups denoted H^0 and H in [13]. It will also be best to denote the groups $\overline{G}^0$ and $\overline{G}$ of that paper by J^0 and J, and not overburden the letter G.

Not every pair δ, $\mu^\vee$ can arise, and it may save us some confusion if we describe now the relation between δ and $\mu^\vee$ that must be satisfied. Observe first that $\mu^\vee$ must lie in the orbit of ${\mu_L}^\vee$ under the Weyl group if ${\mu_L}^\vee$ is the coweight associated to h_L. Since the image of T_L is anisotropic, ${\mu_L}^\vee - \mu^\vee$ defines an element of $H^{-1}(\mathfrak{G}(\mathbf{C}/\mathbf{R})$, $X_*(T_L))$ and hence, by the Tate-Nakayama theory, an element $\alpha_\infty(\mu^\vee)$ of $H^1(\mathbf{R}, T_L)$. On the other hand δ lies in $\mathfrak{D}(T_L/\mathbf{Q})$ which may be mapped to $\mathfrak{D}(T_L/\mathbf{R}) \subseteq H^1(\mathbf{R}, T_L)$. The condition is that the image of δ is $\alpha_\infty(\mu^\vee)$. To verify this we observe that the image of δ is the class of $H^1(\mathbf{R}, T_L)$ corresponding to

$${\mu_L}^\vee - gx^{-1}({\mu_L}^\vee) = {\mu_L}^\vee - \mu^\vee .$$

Lemma 3.1. *Suppose L, T_L, and h_L are given. Then L, δ, $\mu^\vee$ are possible data for the construction of a Frobenius pair if and only if $\alpha_\infty(\mu^\vee)$ is the image of δ in $\mathfrak{D}(T/\mathbf{R})$.*

The necessity has just been verified. On the other hand if g represents δ, the image of δ is $\alpha_\infty(\mu^\vee)$, and $\mu^\vee = \omega({\mu_L}^\vee)$, with ω in the Weyl group over $\mathbf{C}$, then there is an x in $G(\mathbf{R})$ such that gx^{-1} normalizes T and represents ω. We then define h^0 by

$$h^0(r) = x^{-1}h_L(r)x .$$

We next describe sufficient conditions for the class of pairs associated to L, δ, and $\mu^\vee$ to be the same as that associated to L, $\bar{\delta}$, and $\bar{\mu}^\vee$. We continue to assume that the pairs are of the first kind. Let $\Omega(T_L, G; \mathbf{Q})$ be the quotient of the intersection of $\mathfrak{A}(T/\mathbf{Q})$ with the normalizer of T by $T(\overline{F})$. It is a group and may be properly larger than $\Omega(T_L(\mathbf{Q}), G(\mathbf{Q}))$, the Weyl group over $\mathbf{Q}$. It acts on $\mathfrak{D}(T_L/\mathbf{Q})$ to the right, and in the present circumstances consists of two elements, corresponding to the two automorphisms of L over F.

Lemma 3.2. *Suppose δ, $\mu^\vee$ and $\bar\delta$, $\bar\mu^\vee$ are given, and yield classes of Frobenius pairs of the first kind. These classes are the same if the following conditions are satisfied:*

(i) *There is an ω in $\Omega(T_L, G; \mathbf{Q})$ such that*

$$\mathrm{Nm}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}(\bar\mu^\vee - \omega(\mu^\vee)) = 0$$

if $k_\mathfrak{p}$ is any sufficiently large Galois extension of $\mathbf{Q}_\mathfrak{p}$.

(ii) *If* (i) *is satisfied then $\bar\mu^\vee - \omega(\mu^\vee)$ defines an element of $H^{-1}(\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), X_*(T_L))$ and hence, by the Tate-Nakayama theory, an element $\alpha_\mathfrak{p}(\bar\mu^\vee, \omega(\mu^\vee))$ of $H^1(\mathbf{Q}_\mathfrak{p}, T)$. The equation*

$$\omega\delta\alpha_\mathfrak{p}(\bar\mu^\vee, \omega(\mu^\vee)) = \bar\delta$$

is to obtain.

(iii) *If l is a finite prime different from $\mathfrak{p}$ then $\omega\delta$ and $\bar\delta$ have the same image in $\mathfrak{D}(T/\mathbf{Q}_l)$.*

Let δ and $\bar\delta$ be represented by g and $\overline{g}$. Because of the first condition we may take

$$\gamma = g^{-1}\gamma_1 g, \quad \overline{g}^{-1}\omega(\gamma_1)\overline{g},$$

with a common γ_1 in $T_L(\mathbf{Q})$. If ω is represented by w the third condition implies that for $l \neq \mathfrak{p}$

$$\overline{g} = twgu, \quad t \in T_L(\overline{\mathbf{Q}}_l), \quad u \in G(\mathbf{Q}_l),$$

and, consequently, that

$$\overline{\gamma} = u^{-1}\gamma u \ .$$

There is one more condition to be verified if equivalence is to be established. It is a condition on b, the element associated to (γ, h^0) in [13] and discussed at length in the appendix of this paper, and on $\bar b$, the element associated in the same way to $(\overline{\gamma}, \bar h^0)$. The elements γ and $\overline{\gamma}$ are conjugate over $G(\overline{\mathbf{Q}})$ and hence, by the corollary on p. 170 of [18] over $G(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}})$. Let

$$\overline{\gamma} = u^{-1}\gamma u \qquad u \in G(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}) \ .$$

In order to establish equivalence we must show that there is a t in $T(\mathfrak{k})$ for which

$$\bar b = t^{-1}u^{-1}b\sigma(u)\sigma(t)$$

if σ is the Frobenius on $\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}$ or $\mathfrak{k}$.

Since no power of γ_1 is central

$$\overline{g} = swgu, \qquad s \in T_L(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}) \ .$$

Thus $\bar\delta$ is represented by the cocycle

$$\{\tau(w)\tau(g)\tau(u)u^{-1}g^{-1}w^{-1}\}, \qquad \tau \in \mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}),$$

and

$$\bar\delta = \omega\delta\beta$$

in $\mathfrak{D}(T/\mathbf{Q}_\mathfrak{p})$ if

$$\beta = wg\tau(u)u^{-1}g^{-1}w^{-1} \ .$$

Notice that β is represented by the inflation of a cocycle of $\mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$ and is completely determined by

$$wg\sigma(u)u^{-1}g^{-1}w^{-1}a_\sigma \ .$$

It is a consequence of the definitions and the corollary on p. 170 of [18] that, after multiplying b and $\overline{b}$ by $y\sigma(y^{-1})$ and $\overline{y}\sigma(\overline{y}^{-1})$, $y \in T(\mathfrak{k})$, $\overline{y} \in \overline{T}(\mathfrak{k})$, we may suppose that $\alpha_{\mathfrak{p}}(\bar{\mu}^{\vee}, \omega(\mu^{\vee}))$ is represented by the inflation of a cocycle $\{\alpha_{\tau}\}$ of $\mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}/\mathbf{Q}_{\mathfrak{p}})$ and that

$$\overline{g}\overline{b}\overline{g}^{-1} = wgbg^{-1}w^{-1}a_{\sigma} \ .$$

Thus

$$u\overline{b}u^{-1} = bg^{-1}w^{-1}a_{\sigma}wg \ .$$

The second assumption is that

$$\beta \sim \alpha_{\mathfrak{p}}(\bar{\mu}^{\vee}, \omega(\mu^{\vee})) \ .$$

Thus we may even assume that

$$g^{-1}w^{-1}a_{\sigma}wg = \sigma(u)u^{-1} \ .$$

Then

$$\overline{b} = u^{-1}b\sigma(u) \ .$$

Lemma 3.3. *Suppose the classes of Frobenius pairs associated to L, δ, $\mu^{\vee}$ and $\bar{L}$, $\bar{\delta}$, $\bar{\mu}^{\vee}$ are of the first kind and the same. Then $L = \bar{L}$ and the conditions of the previous lemma are fulfilled.*

If the classes are the same then $L \otimes F_v$ and $\bar{L} \otimes F_v$ are isomorphic for all finite places v of F which do not divide $\mathfrak{p}$. Consequently L and $\bar{L}$ are isomorphic. Since we are working within a set of representatives for the isomorphism classes of quadratic extensions, $L = \bar{L}$ and $T_L = T_{\bar{L}}$.

Let δ and $\bar{\delta}$ be represented by g and $\overline{g}$ and let (γ, h^0) and $(\overline{\gamma}, h^0)$ be corresponding Frobenius pairs. We suppose they are equivalent. Replacing γ and $\overline{\gamma}$ by appropriate powers of themselves and perhaps multiplying by an appropriate central element as well, we may suppose they are conjugate in $G(\mathbf{Q}_l)$ for $l \neq \mathfrak{p}$ and hence in $G(\mathbf{Q})$. As usual let

$$\gamma = g^{-1}\gamma_1 g \qquad \overline{\gamma} = \overline{g}^{-1}\overline{\gamma}_1\overline{g} \ .$$

Then γ_1 and $\overline{\gamma}_1$ are also conjugate in $G(\mathbf{Q})$ and there is an ω in $\Omega(T_L, G, \mathbf{Q})$ such that

$$\overline{\gamma}_1 = \omega(\gamma_1) \ .$$

Recall that if λ is a rational character of T_L then

$$|\lambda(\gamma_1)| = |\varpi|^{\tau\langle\lambda,\nu^{\vee}\rangle}$$

and

$$|\lambda(\gamma_2)| = |\varpi|^{\tau\langle\lambda,\bar{\nu}^{\vee}\rangle} \ .$$

Here r and $\overline{r}$ are two positive rational numbers and

$$\nu^{\vee} = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\mu^{\vee}, \qquad \bar{\nu}^{\vee} = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\bar{\mu}^{\vee} \ .$$

Consequently

$$r\bar{\nu}^{\vee} = \overline{r}\omega(\nu^{\vee}) \ .$$

Since the sum of the coefficients in the expression of $\nu^{\vee}$ as a linear combination of the elements in the dual basis is the same as the sum of the coefficients for $\bar{\nu}^{\vee}$, $r = \overline{r}$ and

$$\mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\bar{\mu}^{\vee} = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\omega(\mu^{\vee}) \ .$$

This is the first condition of the previous lemma. To verify the second we have only to observe that if ω is represented by w then

$$\gamma = g^{-1}\gamma_1 g = g^{-1}w^{-1}\overline{\gamma}_1 wg$$

and

$$\overline{\gamma} = \overline{g}^{-1}\overline{\gamma}_1\overline{g}$$

are conjugate in $G(\mathbf{Q}_l)$, and hence

$$wg = t\overline{g}u, \qquad t \in T(\overline{\mathbf{Q}}_l), \quad u \in G(\mathbf{Q}_l)\ .$$

As in the proof of the previous lemma

$$\overline{g} = swgu, \qquad s \in T_L(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}), \quad u \in G(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}),$$

and

$$\overline{\gamma} = u^{-1}\gamma u\ .$$

Since γ is not central, u is determined modulo $T(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}})$ and the class of the cocycle in T_L

$$\{wg\tau(u)u^{-1}g^{-1}w^{-1} \mid \tau \in \mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}/\mathbf{Q}_{\mathfrak{p}})\}$$

is well-defined. As we observed before

$$\overline{\delta} = \omega\delta\beta\ .$$

Moreover we may once again suppose that $\alpha_{\mathfrak{p}}(\bar{\mu}^\vee, \omega(\mu^\vee))$ is obtained by inflating a cocycle $\{\alpha_\tau\}$ of $\mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}/\mathbf{Q}_{\mathfrak{p}})$ and that

$$u\overline{b}u^{-1} = bg^{-1}w^{-1}a_\sigma wg\ .$$

Since we are assuming the two Frobenius pairs equivalent, there is a t in $\overline{T}(\mathfrak{k})$ for which

$$\overline{b} = t^{-1}u^{-1}b\sigma(u)\sigma(t)$$

and

$$(ut^{-1}u^{-1})b\sigma(u)u^{-1}(u\sigma(t)u^{-1}) = bg^{-1}w^{-1}a_\sigma wg\ .$$

Canceling b and taking

$$z = wgutu^{-1}g^{-1}w^{-1},$$

we obtain

$$zwg\sigma(u)u^{-1}g^{-1}w^{-1}\sigma(z^{-1}) = a_\sigma\ .$$

Since $\{a_\tau\}$ and $\{\tau(u)u^{-1}\}$ are both continuous cocycles, it follows readily from this equation that $z \in T_L(\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}})$ and that

$$\beta = \alpha_{\mathfrak{p}}(\bar{\mu}^\vee, \omega(\mu^\vee))\ .$$

We introduce next another type of Frobenius pair, which we will say is of the second kind. We again start from the totally imaginary quadratic extension L of F, the Cartan subgroup T_L, an element g in $\mathfrak{A}(T/\mathbf{Q})$, and an h^0, but suppose that $\mathfrak{m}'_v = \mathfrak{m}''_v$ for every place v of F dividing $\mathfrak{p}$ and splitting in L. Such data exist, for we may so choose L that no place dividing $\mathfrak{p}$ splits in it. We construct γ as before. The ideal $\mathfrak{a}$ is now an ideal in F, and so some power of γ is central.

Lemma 3.4. *Any two Frobenius pairs of the second kind are equivalent.*

Suppose (γ, h^0) and $(\overline{\gamma}, \bar{h}^0)$ are two pairs of the second kind. We take γ and $\overline{\gamma}$ central. Choosing $k_\mathfrak{p}$ so large that it splits both T_L and $T_{\bar{L}}$ and noting that

$$\mathfrak{m}'_v = \mathfrak{m}''_v = \bar{\mathfrak{m}}'_v = \bar{\mathfrak{m}}''_v = \mathfrak{m}_v/2$$

we see easily that there are two positive integers c and d and a unit ξ in $F^\times$ for which

$$\xi\gamma^c = \overline{\gamma}^d \ .$$

Replacing γ by $\xi\gamma^c$ and $\overline{\gamma}$ by $\overline{\gamma}^d$ we may suppose $\gamma = \overline{\gamma}$. The equivalence now follows from the fact that b is well-defined (cf. Lemma A.2).

It is clear that a pair of the second kind cannot be equivalent to a pair of the first kind. With the following lemma, which is really a matter of definition, our classification of Frobenius pairs is complete.

Lemma 3.5. *Every Frobenius pair is equivalent to one of the first or second kind.*

The definition of (γ, h^0) involves the introduction of a subgroup I^0 of G [13], and the choice of a Cartan subgroup T of I^0 which is defined over $\mathbf{Q}$ and through which h^0 factors. Replacing γ by a conjugate over $\mathbf{Q}$ if necessary, we may suppose that, for one of the representatives L and some g in $\mathfrak{A}(T/\mathbf{Q})$,

$$T = g^{-1}T_L g \ .$$

If we also suppose, as we may, that the image of T in I^0_{ad} is anisotropic over $\mathbf{Q}_\mathfrak{p}$ then, by the very definitions, (γ, h^0) must be a Frobenius pair associated to L, g, and h^0.

In addition to the group I^0 there is a group I over $\mathbf{Q}$ and groups J^0 and J over $\mathbf{Q}_\mathfrak{p}$ attached to a Frobenius pair (recall that J^0 and J are denoted by $\overline{G}^0$ and $\overline{G}$ in [13]). I is an inner twisting of I^0. For a pair of the first kind, I^0 is T and the twisting is trivial. For a pair of the second kind I^0 is G. An inner twisting of G is obtained from an inner twisting of $\mathrm{Res}_{F/\mathbf{Q}}\widetilde{G}$ or a twisting of the quaternion algebra D defining $\widetilde{G}$. Thus for pairs of the second kind, I will be defined by the same subgroup A of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(1)$ and a new quaternion algebra D'. According to the prescription for passing from I^0 to I we are not to twist away from infinity and $\mathfrak{p}$, but D' must be ramified at every infinite place. Since the number of infinite places at which D splits is

$$\mathfrak{m} = \sum\nolimits_v \mathfrak{m}_v$$

the invariant will be changed at $\mathfrak{m}$ infinite places. Once we decipher the prescription given in [13] for the twisting at $\mathfrak{p}$, we will see that the invariant of D at v is to be changed if and only if $\mathfrak{m}_v$ is odd. In particular, the total number of places at which the invariant is to be changed is even, and the prescription can actually be carried out.

Over $\mathbf{Q}_\mathfrak{p}$

$$\mathrm{Res}_{F/\mathbf{Q}}\widetilde{G} = \prod\nolimits_v \mathrm{Res}_{F_v/\mathbf{Q}_\mathfrak{p}}\widetilde{G} = \prod\nolimits_v G^v \ .$$

The centralizer of T in this group is a product $\prod T^v$, and if we regard $\mu^\vee$ as a coweight of the product it may be factored as

$$\prod \mu_v^\vee \quad \text{(multiplicative notation)} \ .$$

The cocycle which defines the twisting is also a product and the v^{th} factor lifts to the cochain

$$\sigma \to a_\sigma^v = \prod\nolimits_{\tau\in\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})} a_{\sigma,\tau}^{\sigma\tau\mu^\vee}$$

in $G^\vee$. Here $\{a_{\sigma,\tau}\}$ is a representative of the fundamental class of the extension $k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}$.

A straightforward calculation shows that

$$H^1(\mathbf{Q}_\mathfrak{p}, G^v_{\mathrm{ad}}) \simeq H^1(F_v, \widetilde{G}_{\mathrm{ad}}) \ .$$

The isomorphism is obtained by writing G^v_{ad} over $\overline{\mathbf{Q}}_{\mathfrak{p}}$ as a product

$$\prod\nolimits_{\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})/\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/F_v)} {}^{\sigma}\widetilde{G}_{\mathrm{ad}},$$

and then restricting a cocycle of $\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$ with values in G^v_{ad} to $\mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}/F_v)$ and projecting to the factor ${}^1\widetilde{G}_{\mathrm{ad}}$. We also have the familiar imbedding

$$H^1(F_v, \widetilde{G}_{\mathrm{ad}}) \hookrightarrow H^2(\mathfrak{G}(\mathbf{Q}_{\mathfrak{p}}/F_v), \mathbf{Q}_{\mathfrak{p}}) \ .$$

We have to show that, starting from the given element in $H^1(\mathbf{Q}_{\mathfrak{p}}, G^v_{\mathrm{ad}})$, we finish with the element of $H^2(\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/F_v), \overline{\mathbf{Q}}_{\mathfrak{p}})$ with invariant $\mathfrak{m}_v/2$. We may restrict and project before or after taking the co-boundary of $\{a^v_\sigma\}$, and it is convenient to take the co-boundary first. We obtain

$$\left\{\prod\nolimits_{\tau} a_{\rho,\tau}^{\rho\tau\mu_v^\vee}\right\}\left\{\prod\nolimits_{\tau} \rho(a_{\sigma,\tau})^{\rho\sigma\tau\mu_v^\vee}\prod\nolimits_{\tau} a_{\rho\sigma,\tau}^{-\rho\sigma\tau\mu_v^\vee}\right\} \ .$$

The products run over $\tau \in \mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$. substituting $\sigma\tau$ for τ in the first factor and using the co-cycle relation

$$a_{\rho,\sigma} = \rho(a_{\sigma,\tau})a_{\rho,\sigma\tau}a^{-1}_{\rho\sigma,\tau}$$

we see that this co-boundary equals

$$\{a_{\rho,\sigma}^{\nu_v^\vee}\}$$

with

$$\nu_v^\vee = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\mu_v^\vee \ .$$

The invariant of the restriction of $\{a_{\rho,\sigma}\}$ to $\mathfrak{G}(k_{\mathfrak{p}}/F_v)$ is

$$\mathfrak{n}_v/[k_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}] \ ,$$

and the invariant of the image of the composite homomorphism is therefore

$$\frac{\mathfrak{m}_v[k_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]}{2\mathfrak{n}_v} \cdot \frac{\mathfrak{n}_v}{[k_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}]} = \frac{\mathfrak{m}_v}{2} \ .$$

For the pairs of the second kind $J^0 = I^0$ and $J = I$. For pairs of the first kind J^0, a group over $\mathbf{Q}_{\mathfrak{p}}$ is the inverse image of A in

$$\prod J^v$$

where J^v is T^v if v splits in L and $\mathfrak{m}'_v \neq \mathfrak{m}''_v$ and is G^v otherwise. J is obtained in the same manner with D' replacing D.

There is also a space X and a multiplicity d to associate to a pair (γ, h^0) or, rather, to a class of such pairs. For the moment we ignore the multiplicity and forego a detailed analysis of the space X. The group K is taken to be a product $K^{\mathfrak{p}}K_{\mathfrak{p}}$ with $K^{\mathfrak{p}} \subseteq G(\mathbf{A}^{\mathfrak{p}}_f)$ and $K_{\mathfrak{p}}$ a special compact subgroup of $G(\mathbf{Q}_{\mathfrak{p}})$, that is, the stabilizer of a special vertex in the Bruhat-Tits building. X depends on $K_{\mathfrak{p}}$. The set of points in $S(\bar{\kappa}_{\mathfrak{p}})$ corresponding to the class of (γ, h^0) is formally d copies of

$$Y_K = H(\mathbf{Q})\backslash G(\mathbf{A}^{\mathfrak{p}}_v) \times X/K^{\mathfrak{p}} \ .$$

The group $K^{\mathfrak{p}}$ acts on the right through its action on $G(\mathbf{A}^{\mathfrak{p}}_f)$. $I(\mathbf{Q})$ acts on both factors, and the Frobenius $\Phi_{\mathfrak{p}}$ acts on Y through its action on X.

As in [12] we use the Lefschetz fixed point formula to compute the alternating sum of the traces of $\Phi^j_{\mathfrak{p}}$, $j > 0$, on the cohomology of F_ξ. We take the sum over the fixed points of the traces on the fibers. The fibers over $\mathbf{Q}_l$ are obtained by first taking the fibers of the sheaves over $\mathbf{Z}/l^k\mathbf{Z}$, then letting $k \to \infty$ to obtain fibers over $\mathbf{Z}_l$, and then tensoring with $\mathbf{Q}_l$. Since the fixed points lie in $S(\bar\kappa_{\mathfrak p}) = S_K(\bar\kappa_{\mathfrak p})$, the only thing that really matters is the resulting sheaf over $S(\bar\kappa_{\mathfrak p})$, a set with the discrete topology, and the action of $\Phi_{\mathfrak p}$ on it.

We first look at the points in Y_K, and find a formula for their contribution to the alternating sum. If $\overline{K} = \overline{K}^{\mathfrak p} K_{\mathfrak p}$ with $\overline{K}^{\mathfrak p} \subseteq K^{\mathfrak p}$ then the inverse image of Y_K in $S_{\overline{K}}(\bar\kappa_{\mathfrak p})$ is

$$Y_{\overline{K}} = H(\mathbf{Q})\backslash G(\mathbf{A}_v^{\mathfrak p}) \times X/\,\overline{K}^{\mathfrak p} \ .$$

The map is the obvious one. Consequently [12] the sheaf over Y_K is

$$(H(\mathbf{Q})\backslash G(\mathbf{A}_f^{\mathfrak p}) \times X) \times_{K^{\mathfrak p}} V(\mathbf{Q}_l) \ .$$

Here $K^{\mathfrak p}$ acts on $V(\mathbf{Q}_l)$ through its projection on $G(\mathbf{Q}_l)$. The action of $\Phi_{\mathfrak p}$ on the sheaf is obtained by letting it act on X.

The point $y \in Y_K$ represented by (g,x) is fixed by $\Phi_{\mathfrak p}^j$ if

$$(g, \Phi_{\mathfrak p}^j x) = (hgk, hx), \qquad k \in K^{\mathfrak p},\ h \in I(\mathbf{Q}) \ .$$

There are really two equations here

$$\Phi_{\mathfrak p}^j x = hx \qquad \text{and} \qquad k = g^{-1}h^{-1}g \ .$$

The map from the fiber at $\Phi_{\mathfrak p}^j$ to the fiber at y is

$$(g, \Phi_{\mathfrak p}^j x) \times v \to (g,x) \times v \ .$$

At a fixed point,

$$(g, \Phi_{\mathfrak p}^j x) \times v = (hgk, hx) \times v = (g,x) \times kv$$

and the trace is

$$\operatorname{trace} \xi(k^{-1}) = \operatorname{trace} \xi(h) \ .$$

There is a lemma to be proved before we can find an expression for the contribution of the points on Y_K to the alternating sum of the traces. If h_1, h_2 lie in $I(\mathbf{Q})$ and k_1, k_2 lie in $K^{\mathfrak p}$, the equation

$$(h_1gk_1, h_1x) = (h_2gk_2, h_2x)$$

is equivalent to the two equations

$$h_1^{-1}h_2 = gk_1k_2^{-1}g^{-1} \qquad \text{and} \qquad h_1^{-1}h_2x = x \ .$$

Since the center Z of G is contained in I^0 and I is obtained from I^0 by an inner twisting, Z is also a subgroup of I.

Lemma 3.6 *There is an open compact subgroup K_0 of $G(\mathbf{A}_f)$ such that if $K \subseteq K_0$ then for any Frobenius pair the equations*

$$h = gkg^{-1}, \qquad hx = x,$$

with $h \in I(\mathbf{Q})$, $g \in G(\mathbf{A}_f^{\mathfrak p})$, $k \in K^{\mathfrak p}$, $x \in X$ imply that h lies in Z.

We may as well divide by Z, and consequently suppose that Z is $\{1\}$. Since $I(\mathbf{R})$ is compact, h is semi-simple. Let it lie in the torus T over $\mathbf{Q}$. I claim that if λ is a rational character of T and v any valuation of $\overline{\mathbf{Q}}$, then

$$|\lambda(h)|_v = 1 \ .$$

If v is archimedean, this is a consequence of the compactness of $I(\mathbf{R})$. If v is non-archimedean but prime to $\mathfrak p$, it is a consequence of the first of the assumed equations. If v divides $\mathfrak p$, it is a consequence of the second assumed equation and the definition of X [13]. We conclude that $\lambda(h)$ is a root of unity. Since $\lambda(h)$ lies in a Galois extension whose degree is at most the product of the order of the Weyl group of G with the order of the group of automorphisms of the Dynkin diagram, it is one of a fixed finite set of roots of unity. We have merely to take K_0 sufficiently small that the ensuing congruence conditions force it to be 1.

We assume henceforth that $K \subseteq K_0$. There is another vexatious possibility with which I would prefer not to have to deal, simply because it encumbers the notation. Suppose $h \in I(\mathbf{Q})$, $g \in I(\mathbf{Q})$ and $g^{-1}hg = zh$ with $z \in (\mathbf{Q}) \cap K$. Then z certainly must lie in the center of the derived group which is finite. I take K_0 so small that this equation implies $z = 1$.

If $z \in Z(\mathbf{Q}) \cap K$ then $(zg, zx) = (gz, x)$. If

$$(g, \Phi_{\mathfrak{p}}^j x) = (hgk, hx)$$

and $h_1 \in I(\mathbf{Q})$, $k_1 \in K''$, then

$$(h_1 g k_1, \Phi_{\mathfrak{p}}^j h_1 x) = ((h_1 h h_1^{-1}) h_1 g k_1 (k_1^{-1} k k_1), (h_1 h h_1^{-1}) h_1 x) \ .$$

Thus to each fixed point in Y_K is associated a conjugacy class $\{h\}$ in $Z(\mathbf{Q}) \cap K \backslash I(\mathbf{Q})$, and if $N^j(h)$ is the number of fixed points yielding the conjugacy class $\{h\}$ then the total contribution of Y_K to the alternating sum of the traces is

$$\sum\nolimits_{(h)} N^j(h) \mathrm{trace}\, \xi(h) \ .$$

If $x \in X$ and $j > 0$, set

$$T_x^j = \{g \in J(\mathbf{Q}_{\mathfrak{p}}) \mid \Phi_{\mathfrak{p}}^j x = gx\} \ .$$

Lemma 3.7. *If h lies in $I(\mathbf{Q})$ and in T_x^j, $j > 0$, then the centralizer $I(h, \mathbf{Q}_{\mathfrak{p}})$ of h in $I(\mathbf{Q}_{\mathfrak{p}})$ is the same as its centralizer $J(h, \mathbf{Q})_{\mathfrak{p}})$ in $J(\mathbf{Q}_{\mathfrak{p}})$.*

The proof of this lemma will have to be postponed until we have examined the sets T_z^j more closely.

Let ψ_x^j be the characteristic function of T_z^j, and if $\{x\}$ is a set of representatives for the orbits of $J(\mathbf{Q}_{\mathfrak{p}})$ in X, set

$$\varphi^j(h) = \sum\nolimits_{\{x\}} \frac{1}{\mathrm{meas}\, J(x)} \int_{J(h,\mathbf{Q}_{\mathfrak{p}})\backslash J(\mathbf{Q}_{\mathfrak{p}})} \psi_x^j(g^{-1}hg)dg \ .$$

Here h is an arbitrary element of $J(\mathbf{Q}_{\mathfrak{p}})$ and $J(x)$ is the stabilizer of x in $J(\mathbf{Q}_{\mathfrak{p}})$. We shall eventually see that the integrals are finite and that, for each j, all but finitely many of the ψ_x^j are identically zero.

Lemma 3.8. *Suppose $\phi^{\mathfrak{p}}$ is the characteristic function of $K^{\mathfrak{p}}$ divided by its measure, and let*

$$Z_K = Z(\mathbf{A}_f) \cap K \ .$$

Then $N^j(h)$ is equal to

$$\frac{\mathrm{meas}(Z_K I(h, \mathbf{Q})\backslash I(h, \mathbf{A}_f))\varphi^j(h)}{\mathrm{meas}\, Z_K} \int_{I(h,\mathbf{A}_f^{\mathfrak{p}})\backslash G(\mathbf{A}_f^{\mathfrak{p}})} \phi^{\mathfrak{p}}(g^{-1}hg)dg \ .$$

This lemma is a more general form of some of the lemmas in §5 of [12]. The number $N^j(h)$ is equal to the sum over i of

$$\sum\nolimits_{h_1} \sum\nolimits_{(g,\overline{g})} \mathrm{meas}\, K^{\mathfrak{p}} \phi(g^{-1}h_1 g) \psi_{x_i}^j(\overline{g}^{-1} h_1 \overline{g}) \ .$$

Here h_1 runs over the conjugates of h in $I(\mathbf{Q})$ modulo $Z(\mathbf{Q}) \cap K$ and $(g, \overline{g})$ runs over

$$I(\mathbf{Q})\backslash G(\mathbf{A}_f^{\mathfrak{p}}) \times J(\mathbf{Q}_{\mathfrak{p}})/K^{\mathfrak{p}} \times J(x_i) \ .$$

We may drop the sum of h_1 if we divide on the left, not by $I(\mathbf{Q})$, but by $I(h, \mathbf{Q})$. Since

$$Z_K I(\mathbf{Q}) \cap (gK^{\mathfrak{p}} g^{-1} \times \overline{g} J(x_i) \overline{g}^{-1}) = Z_K,$$

we may also replace the sum by an integral over $Z_K I(h,\mathbf{Q})\backslash G(\mathbf{A}_f^{\mathfrak{p}}) \times J(\mathbf{Q}_{\mathfrak{p}})$ provided we multiply by

$$\operatorname{meas} Z_K / (\operatorname{meas} K^{\mathfrak{p}})(\operatorname{meas} J(x_i)) \ .$$

The integration may be taken first over $Z_K I(h,\mathbf{Q})\backslash I(h,\mathbf{A}_f)$ and then over

$$I(h,\mathbf{A}_f^{\mathfrak{p}})\backslash G(\mathbf{A}_f^{\mathfrak{p}}) \times I(h,\mathbf{Q}_{\mathfrak{p}})\backslash J(\mathbf{Q}_{\mathfrak{p}}) \ .$$

Appealing to Lemma 3.7, we replace the denominator in the second factor by $J(h,\mathbf{Q}_{\mathfrak{p}})$. The first integration simply yields a factor

$$\operatorname{meas}(Z_K I(h,\mathbf{Q})\backslash I(h,\mathbf{A}_f)) \ .$$

The lemma follows.

We must next consider the multiplicity d attached to a Frobenius pair. I have first to confess that the multiplicity suggested in [13] is not quite correct. It was suggested that it could be incorrect because insufficiently many examples had been studied, and that is still a possibility, but the error to be mentioned now arises from a difference source, a misinterpretation of my calculations for the special cases. Such mistakes — another is correction [14] — must be annoying to anyone who is seriously attempting to understand this sequence of papers. I can only apologize and assure him that they do not seem serious. I hope to have a fairly thorough discussion of the conjectures and the examples available sooner or later. It appears to be safe for now to take d to be the number of elements in $H^1(\mathbf{Q},I)$ which become trivial in $H^1(\mathbf{Q}_v,I)$ for all places v of $\mathbf{Q}$ and which have trivial image in $H^1(\mathbf{Q},G_{\mathrm{der}}\backslash G)$. In [13] triviality in $H^1(\mathbf{Q}_{\mathfrak{p}},I)$ was not demanded. I observe as well that in [13] a rather eccentric notation was employed. The set $H^1(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}),I(\overline{\mathbf{Q}}))$ was denoted by $H^1(\overline{\mathbf{Q}},I)$ and not by $H^1(\mathbf{Q},I)$.

With this definition of d, we have the following simple lemma.

Lemma 3.9 *(a) For Frobenius pairs of the second kind $d=1$.*

(b) For Frobenius pairs of the first kind d is equal to $\mu(T)$, if $T=I^0$, a Cartan subgroup, and $\mu(T)$ is the order of the kernel of $\mathfrak{E}(T/\mathbf{Q}) \to \mathfrak{E}(T/\mathbf{A})$.

For a pair of the second kind,

$$G_{\mathrm{der}}\backslash G \simeq I_{\mathrm{der}}\backslash I$$

and I_{der} is a simply-connected group. Thus it follows from the Hasse principle or, more directly, from the fact that $H^1(\mathbf{Q},I_{\mathrm{der}})=1$ that $d=1$. For pairs of the first kind, $T=I^0=I$ and the kernel of

$$H^1(\mathbf{Q},T) \to H^1(\mathbf{Q},G_{\mathrm{der}}\backslash G)$$

is $\mathfrak{E}(T/\mathbf{Q})$. The lemma is verified.

In order to compare the alternating sum of the traces with the results of §2, we need to express it as the sum of a stable and labile part. We begin with the contribution from the Frobenius pairs of the first kind attached to a given totally imaginary quadratic extension of F.

Earlier we fixed T_L and $\mu_L^\vee$ and, when $g \in \mathfrak{A}(T/\mathbf{Q})$ and h^0 were given, regarded $\mu^\vee$ as a coweight of T_L. There is more than one possibility for

$$\nu^\vee = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}} \mu^\vee \ .$$

Let $\nu_1^\vee, \nu_2^\vee, \ldots$ be the finitely many possibilities. For each of them we choose a $\mu_j^\vee$ with norm $\nu_j^\vee$ and a $g_j \in \mathfrak{A}(T_L/\mathbf{Q})$ with image δ_j in $\mathfrak{D}(T_L/\mathbf{Q})$ so that δ_j and $\mu_j^\vee$ satisfy the condition of Lemma 3.1. It follows from the density of $D(\mathbf{Q})$ in $D(\mathbf{R})$ that g_j exists. Set $T_j = g_j^{-1} T_L g_j$ and regard $\mu_j^\vee$ now as a coweight of T_j. Rather than working with T_L and $\mu_L^\vee$ we prefer now to work with T_j,$g \in \mathfrak{A}(T_j/\mathbf{Q})$, and those $\mu^\vee$ for which

$$\mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}} \mu^\vee = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}} \mu_j^\vee \ .$$

However, when we come to assemble the contributions from the various T_j, we must divide by 2, the order of the group $\Omega(T_L,G;\mathbf{Q})$, because, by Lemma 3.2, $\nu_i^\vee$ and $\nu_j^\vee$ yield the same classes of Frobenius pairs if

$$\nu_j^\vee = \omega(\nu_i^\vee), \qquad \omega \in \Omega(T_j,G;\mathbf{Q}) \ .$$

If

$$\mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\mu^{\vee} = \mathrm{Nm}_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}\mu_i^{\vee}$$

then by the Tate-Nakayama theory $\mu^{\vee} - \mu_i^{\vee}$ defines elements $\alpha_{\mathfrak{p}}(\mu^{\vee}, \mu_i^{\vee})$ and $\alpha_{\infty}(\mu^{\vee}, \mu_i^{\vee})$ in $H^1(\mathbf{Q}_{\mathfrak{p}}, T_i)$ and $H^1(\mathbf{Q}_{\infty}, T_i)$. By Lemma A.9 there is an element $\alpha(\mu^{\vee}, \mu_i^{\vee})$ in $H^1(\mathbf{Q}, T_i)$ whose image in $H^1(\mathbf{Q}_v, T_i)$ is $\alpha_{\infty}^{-1}(\mu^{\vee}, \mu_i^{\vee})$ if $v = \infty$, $\alpha_{\mathfrak{p}}(\mu^{\vee}, \mu_i^{\vee})$ if $v = \mathfrak{p}$, and 1 otherwise. If g and $\mu^{\vee}$ actually define a Frobenius pair then $\delta\alpha^{-1}(\mu^{\vee}, \mu_i^{\vee})$ is trivial at ∞, and, by Lemma 3.2, the class of the Frobenius pair is determined by its local behavior at the finite places. Conversely Lemma 3.3 shows that, for a given i, the cohomology class of $\delta\alpha^{-1}(\mu^{\vee}, \mu_i^{\vee})$ is determined locally by the Frobenius pair. To free ourselves of any ambiguity we first of all agree to take $\mu^{\vee} = \mu_i^{\vee}$, and then to choose g from a set of representatives $\{g\}$ for those elements $\mathfrak{A}(T_i/\mathbf{Q})$ which are trivial at ∞ modulo those which are trivial everywhere.

For such g set $T_i^g = g^{-1}T_i g$. A datum such as a measure may be transported from T_i toT_i^g. The part of the contribution to the alternating sum of the traces corresponding to T_i is the product of

$$(d/2)\,\mathrm{meas}(Z_K T_i(\mathbf{Q})\backslash T_i(\mathbf{A}_f))/\mathrm{meas}\, Z_K$$

and

$$\sum_{\{g\}} \sum_{h\in T_i(\mathbf{Q})} \varphi_g^j(h) \int_{T^g(\mathbf{A}_f^{\mathfrak{p}})\backslash G(\mathbf{A}_f^{\mathfrak{p}})} \phi(x^{-1}h^g x)dx \ . \tag{3.1}$$

Here φ_g^j is φ^j, but for the Frobenius pair attached to g. I have not stressed this before, but it is understood that any term of this sum is zero if the first factor is zero, even when the second factor is infinite. It will eventually be clear that if the first factor is not zero, then the second factor is finite.

If $\mathfrak{E}(T_i/\mathbf{A}_f)$ is the set of elements in $\mathfrak{E}(T_i/\mathbf{A})$ which are trivial at ∞, then

$$\mathfrak{E}(T_i/\mathbf{A}_f)/\mathfrak{E}(T_i/\mathbf{A}_f)\cap \mathrm{Im}\,\mathfrak{E}(T_i/\mathbf{Q}) \simeq \mathfrak{E}(T_k/\mathbf{A})/\mathrm{Im}\,\mathfrak{E}(T_i/\mathbf{Q}) \ .$$

Thus it should be possible to write (3.1) as a sum over the characters κ of $\mathfrak{E}(T/\mathbf{A})/\mathrm{Im}\,\mathfrak{E}(T/\mathbf{Q})$ of

$$[\mathfrak{E}(T_i/\mathbf{A}) : \mathrm{Im}\,\mathfrak{E}(T_i/\mathbf{Q})]^{-1} \sum_h \sum_{\{g\}} \kappa(\delta)\varphi_g^j(h^g) \times \int_{T_i^g(\mathbf{A}_f^{\mathfrak{p}})\backslash G(\mathbf{A}_f^{\mathfrak{p}})} \phi(x^{-1}h^g x)dx \ .$$

Here g runs over a set of representatives for $\mathfrak{E}(T_i/\mathbf{A}_f)$ and δ in $\mathfrak{E}(T_i/\mathbf{A}_f)$ is the image of g. However we do have to observe that in the definition of X and J it was not essential that g lie in $\mathfrak{A}(T_i/\mathbf{Q})$. It need only lie in $\mathfrak{A}(T_i/\mathbf{Q}_{\mathfrak{p}})$. Thus φ_g^j is defined for $g \in \prod_v \mathfrak{A}(T/\mathbf{Q}_v)$ by its coordinate in $\mathfrak{A}(T_i/\mathbf{Q}_{\mathfrak{p}})$. We set

$$\varphi_g^j(h^g) = \varphi_\delta^j(h) \ ,$$

sometimes taking δ in $\mathfrak{D}(T_i/\mathbf{Q})$ and sometimes in $\mathfrak{D}(T_i/\mathbf{Q}_{\mathfrak{p}})$.

The inner sum may be written as the product of

$$\Phi^{T_i/\kappa}(h) = \sum_{\mathfrak{E}(T_i/\mathbf{A}_f^{\mathfrak{p}})} \kappa(\delta)\Phi^{\delta}(h,f)$$

and

$$\sum_{\mathfrak{E}(T_i/\mathbf{Q}_{\mathfrak{p}})} \kappa(\delta)\varphi_\delta^j(h) \ .$$

The first of these two factors we have met before, and there is little to be said about it. It is the second which must be studied carefully.

Let $\mathfrak{k}$ be the completion of the maximal unramified extension of $\mathbf{Q}_{\mathfrak{p}}$. If b_i is the element of $T_i(\mathfrak{k})$ associated to $\mu_i^{\vee}$ by the procedure of the appendix, then the element b of $T^g(\mathfrak{k})$ associated to $\mu_i^{\vee}$ by the same procedure is $b_g = g^{-1}b_t g$. We recall the manner in which b_g is used to construct the space X [13]. To stress that it depends on g, I write X_g instead of X.

The group $G(\mathfrak{k})$ is contained in $\widetilde{G}_1(\mathfrak{k})$, and if $Q = \mathfrak{G}(\overline{\mathbf{Q}}/F)\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ then

$$\widetilde{G}_1(\mathfrak{k}) = \prod\nolimits_Q \mathrm{GL}(2,\mathfrak{k}) \ .$$

Since F is unramified at $\mathfrak{p}$ the action of $\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$ on Q factors through $\mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$, which is also the group of continuous automorphisms of $\mathfrak{k}$ and contains σ as $\Phi_\mathfrak{p}$. To obtain the action of σ on $\widetilde{G}_1(\mathfrak{k})$ we choose a set of representatives $\{\tau\}$ for the cosets of Q and write $\tau\sigma = d_\tau(\sigma)\tau'$ with $d_\tau(\sigma) \in \mathfrak{G}(\overline{\mathbf{Q}}/F)$. Then

$$\sigma : (g_\tau) \to (g'_\tau)$$

with $g'_\tau = d_\tau(\sigma)(g'_\tau)$. Observe that in fact $d_\tau(\sigma)$ is determined only modulo the inertial group of $F \subseteq \overline{\mathbf{Q}}_\mathfrak{p}$, but that does not matter, for $d_\tau(\sigma)$ is acting on $g'_\tau \in \mathrm{GL}(2,\mathfrak{k})$ in the usual way.

Let $\mathfrak{o}_\mathfrak{k}$ be the ring of integers in $\mathfrak{k}$. The element σ acts in the same way on the collection of (M_i), $i \in Q$, where, for each i, M_i is an $\mathfrak{o}_\mathfrak{k}$-lattice in the space of column vectors of length two over $\mathfrak{k}$. Let (M_i^0) be a fixed point of σ and let $\widetilde{K}_\mathfrak{p}(\mathfrak{k})$ be the stabilizer of (M_i^0) in $\widetilde{G}_1(\mathfrak{k})$. We may so choose (M_i^0) that if

$$\widetilde{K}_\mathfrak{p} = \widetilde{K}_\mathfrak{p}(\mathfrak{k}) \cap \widetilde{G}_1(\mathbf{Q}_\mathfrak{p})$$

then

$$K_\mathfrak{p} = \widetilde{K}_\mathfrak{p} \cap G(\mathbf{Q}_\mathfrak{p}) \ .$$

Observe that $\widetilde{G}_1(\mathbf{Q}_\mathfrak{p})$ is taken to be the set of points in $\widetilde{G}_1(\mathfrak{k})$ fixed by σ.

We introduce the set

$$\mathfrak{X} = G(\mathfrak{k})/K_\mathfrak{p}(\mathfrak{k}) \subseteq \widetilde{\mathfrak{X}} = \widetilde{G}_1(\mathfrak{k})/\widetilde{K}_\mathfrak{p}(\mathfrak{k})$$

with

$$K_\mathfrak{p}(\mathfrak{k}) = \widetilde{K}_\mathfrak{p}(\mathfrak{k}) \cap G(\mathfrak{k}) \ .$$

$\widetilde{\mathfrak{X}}$ is just the set of (M_i). the action of σ on $\mathfrak{X}$ is given by its action on $G(\mathfrak{k})$ or by its action on the sequences (M_i). We introduce the transformation $\mathbf{F}_v$ of $\mathfrak{X}$ which sends $\mathfrak{c}$ to $b_g\sigma(\mathfrak{c})$.

Let $\mathfrak{x}$ in $\mathfrak{X}$ be (M_i) and let $\mathfrak{Y} = \mathbf{F}_g\mathfrak{x}$ be (N_i). According to the definition of [13], supplemented by the correction in [14], the point $\mathfrak{x}$ lies in X_g if and only if the following two conditions are satisfied.

(i) $M_i = N_i$ if i is an unmarked point.

(ii) $M_i \supsetneq N_i \supsetneq \mathfrak{p}M_i$ if i is a marked point.

It will be easier to make the comparison of the following paragraph if we can express

$$\sum\nolimits_{\mathfrak{E}(T_i/\mathbf{Q}_\mathfrak{p})} \kappa(\delta)\varphi_\delta^j(h) \tag{3.2}$$

entirely in terms of the set $\widetilde{\mathfrak{X}}$. By the corollary on p. 170 of [18]

$$H^1(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}, T_i) = 1 \ .$$

Thus we may choose the g to lie in $G(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}})$ and then write

$$g = tu, \qquad t \in \widetilde{T}_i(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}), \qquad u \in \widetilde{G}_1(\mathbf{Q}_\mathfrak{p}) \ .$$

Here $\widetilde{T}_i$ is the centralizer of T_i in $\widetilde{G}_1$. Thus

$$T_i^g = u^{-1}T_iu$$

and

$$b_g = u^{-1} b_i u \ .$$

We let $\mathbf{F}_i$ be the operator on $\widetilde{\mathfrak{X}}$ which takes the point represented by x in $\widetilde{G}_1(\mathfrak{k})$ to the point represented by $b_i\sigma(x)$, and define $\widetilde{X}_i P$ in the same way that we defined X_g except that $\mathbf{F}_i$ replaces $\mathbf{F}_g$. Then

$$uX_g = u\mathfrak{X} \cap \widetilde{X}_i \ .$$

As in [7], $\mathfrak{E}(T_i/\mathbf{Q}_\mathfrak{p})$ may be identified with

$$\widetilde{G}_1(\mathbf{Q}_\mathfrak{p})/\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})G(\mathbf{Q}_\mathfrak{p}) \simeq \prod_{v|\mathfrak{p}} F_v^\times A(\mathbf{Q}_\mathfrak{p}) \prod_{v|\mathfrak{p}} \mathrm{Nm}\, L_v^\times$$

with

$$L_v = L \otimes_F F_v \ .$$

We may therefore regard κ as a character of $\widetilde{G}_1(\mathbf{Q}_\mathfrak{p})$. If

$$J(\mathbf{Q}_\mathfrak{p}) = \{\overline{g} \in G(\mathbf{Q}_\mathfrak{p}) \mid b_\sigma(\overline{g})b^{-1} = \overline{g}\}$$

the sum (3.2) is equal to

$$\sum\nolimits_{u\in \widetilde{G}_1(\mathbf{Q}_\mathfrak{p})/T_i(\mathbf{Q}_\mathfrak{p})G(\mathbf{Q}_\mathfrak{p})} \sum\nolimits_{\{x\}} \frac{\kappa(u)}{\text{meas}\, J(X)} \int_{T_i(\mathbf{Q}_\mathfrak{p})\backslash(\mathbf{Q}_\mathfrak{p})} \psi_x^j(g^{-1}hg)dg \ . \tag{3.3}$$

Here $\{x\}$ is a set of representatives for the orbits of $J(\mathbf{Q}_\mathfrak{p})$ in $u\mathfrak{X} \cap \widetilde{X}_i$ and ψ_x^j is the characteristic function of

$$\{g \in J(\mathbf{Q}_\mathfrak{p}) \mid \Phi_\mathfrak{p}^j x = gx\}$$

with $\Phi_\mathfrak{p} = \mathbf{F}_i^e$ if $e = [E_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]$. I observe that, by the definition of E, the operator $\Phi_\mathfrak{p}$ must take the set of marked points to itself, and does in fact operate on $\widetilde{X}_i$.

It is manifest that $\widetilde{G}_1(\mathbf{Q}_\mathfrak{p}) \cap \widetilde{K}_\mathfrak{p}(\mathfrak{k})$ takes $u\mathfrak{X}$ to itself for all u, and hence that (3.3) is 0 unless κ is unramified. We assume then that κ is unramified. For a given T_i there are at most two possibilities for κ. If it is not trivial, it can be unramified only if L is unramified at every place of F dividing $\mathfrak{p}$.

We want to transform the expression (3.3), and in order to do so we need the following lemma.

Lemma 3.10. *The set $\widetilde{X}_i$ is contained in $\widetilde{G}_1(\mathbf{Q}_\mathfrak{p})\mathfrak{X}$.*

If $G_1 = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(1)$ then

$$\mathrm{Nm} : \widetilde{G}_1(\mathbf{Q}_\mathfrak{p}) \to G_1(\mathbf{Q}_\mathfrak{p})$$

is surjective. Let $g \in \widetilde{G}_1(\mathfrak{k})$ and let $a = \mathrm{Nm}\, g$. We want to show that if the image of g in $\widetilde{\mathfrak{X}}$ lies in $\widetilde{X}$ then

$$g \in \widetilde{G}_1(Q_\mathfrak{p})G(\mathfrak{k})\widetilde{K}_\mathfrak{p}(\mathfrak{k}) \ .$$

Since G is the inverse image of A in $\widetilde{G}_1$, all we need do is show that

$$a = a_1 a_2 a_3$$

with $a_1 \in G_1(\mathbf{Q}_\mathfrak{p})$, $a_2 \in A(\mathfrak{k})$, $a_3 \in G_1(\mathfrak{k})$, and $|\lambda(a_3)| = 1$ for every rational character of G_1.

The composition of $\mu_i^\vee$ with $T_i \to A$ is a coweight $\bar{\mu}_i^\vee$ of A and thus of G_1. Both A and G_1 are split over $\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}$ because F is assumed to be unramified at $\mathfrak{p}$. Consequently $\varpi^{\mu_i'}$ lies in $A(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}})$ if ϖ is a uniformizing parameter of $\mathbf{Q}_\mathfrak{p}$. Let $\overline{b} = \mathrm{Nm}\, b_i$. The condition for the image of g to lie in $\widetilde{X}$ which was added in [14] and used to deduce conditions (i) and (ii) above is that

$$a\sigma(a^{-1}) = \overline{b}\,\varpi^{-\mu_i^\vee} b_3$$

with $|\lambda(b_3)| = 1$ for every rational character of G_1. A simple variant of Lemma A.7 allows us to establish that $b_3 = a_3\sigma(a_3^{-1})$ with a_3 of the desired form. For simplicity we replace a by aa_3^{-1} and suppose that $b_3 = 1$.

The considerations of the appendix apply to the group A and the coweight $\bar{\mu}_i^\vee$. There are two ways to construct an element b_A, that is, the element b of the appendix when A replaces G. On one hand, we can take $\overline{b}$. On the other, we can observe that A splits over an unramified extension $l_\mathfrak{p}$ of $\mathbf{Q}_\mathfrak{p}$ with Galois group generated by σ, and that the fundamental cocycle of $l_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}$ may be taken to be

$$a_{\sigma^j,\sigma^k} = \begin{cases} 1, & 0 \leqq j,k < [l_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}], j+k < [l_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}], \\ \varpi, & 0 \leqq j,k < [l_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}], j+k \geqq [l_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}] \end{cases}$$

This leads to $\varpi^{\bar{\mu}_i^\vee}$. The considerations of the appendix show that

$$\overline{b}\varpi^{-\mu_i^\vee} = a_2\sigma(a_2^{-1}), \qquad a_2 \in A(\mathfrak{k}) \ .$$

Thus

$$a_1 = aa_2^{-1} \in G_1(\mathbf{Q}_\mathfrak{p}) \ .$$

We may now regard x as a function on $\widetilde{X}_i$, defined by

$$\kappa(x) = \kappa(u)$$

if $x \in u\mathfrak{X}$. It is constant on orbits of $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$. Let

$$\widetilde{J}_1(\mathbf{Q}_\mathfrak{p}) = \{g \in \widetilde{G}_1(\mathbf{Q}_\mathfrak{p}) \mid b\sigma(g)b^{-1} = g\} \ .$$

This group contains $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$ and the usual bijection

$$T_i(\mathbf{Q}_\mathfrak{p})\backslash J(\mathbf{Q}_\mathfrak{p}) \simeq \widetilde{T}_i(\mathbf{Q}_\mathfrak{p})\backslash(\mathbf{Q}_)J(\mathbf{Q}_\mathfrak{p}) \subseteq \widetilde{T}_i(\mathbf{Q}_\mathfrak{p})\backslash\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$$

is defined. We choose measures on $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$ and $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ in such a way that the restriction of the quotient measure to the image of the arrow corresponds by transport of structure to the measure on $T_i(\mathbf{Q}_\mathfrak{p})\backslash J(\mathbf{Q}_\mathfrak{p})$ appearing in (3.3).

Recall the definition of the number $u(T_i)$ (p. 1156). Replacing T_i by a conjugate if necessary, we suppose that $K_\mathfrak{p} \subseteq \widetilde{K}_\mathfrak{p} \subseteq \widetilde{U}_\mathfrak{p}$ where $\widetilde{K}_\mathfrak{p}$ is a maximal compact subgroup of $\widetilde{G}_1(\mathbf{Q}_\mathfrak{p})$ and $\widetilde{U}_\mathfrak{p}$ the maximal compact subgroup of $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$. Let $\widetilde{J}_1(x)$ be the stabilizer of x in $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$. We want to show now that the expression (3.3) is equal to

$$u(T_i)\,\frac{\operatorname{meas}\widetilde{U}_\mathfrak{p}}{\operatorname{meas}U_\mathfrak{p}}$$

times

$$\sum \frac{\kappa(x)}{\operatorname{meas}J_1(x)} \int_{\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})\backslash\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})} \psi_x^j(g^{-1}hg)\kappa(g)dg \ . \tag{3.4}$$

The sum is over a set of representatives for the orbits of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ in $\widetilde{X}_i$, and $U_\mathfrak{p}$, $\widetilde{U}_\mathfrak{p}$ are the maximal compact subgroups of $T_i(\mathbf{Q}_\mathfrak{p})$ and $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$.

We described $J(\mathbf{Q}_\mathfrak{p})$ explicitly above. It is clear that $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ admits a similar description. It follows from these descriptions that

$$\begin{aligned} \operatorname{Norm}\ J(\mathbf{Q}_\mathfrak{p}) &= \operatorname{Norm}\ G(\mathbf{Q}_\mathfrak{p}), \\ \operatorname{Norm}\ \widetilde{J}_1(\mathbf{Q}_\mathfrak{p}) &= \operatorname{Norm}\ \widetilde{G}_1(\mathbf{Q}_\mathfrak{p}) \ . \end{aligned}$$

Consequently the outer sum in (3.3) may be taken over

$$\widetilde{T}_1(\mathbf{Q}_\mathfrak{p})J(\mathbf{Q}_\mathfrak{p})\backslash\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$$

and the index in the inner sum may be taken to be ux where x runs over a set of representatives for the orbits of $J(\mathbf{Q}_\mathfrak{p})$ in $\mathfrak{X}\cap\widetilde{X}_i$. Changing the order of summation and combining the new inner sum and the integral into a single integral, we obtain

$$\sum\nolimits_{\{x\}} \frac{\kappa(x)}{\mathrm{meas}\, J(x)} \int_{\widetilde{T}_1(\mathbf{Q}_\mathfrak{p})\backslash\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})} \psi_x^j(g^{-1}hg)\kappa(g)dg\ .$$

The sum is over a set of representatives for the orbits.

Every orbit of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ in $\widetilde{X}$ meets $\mathfrak{X}$, and so the set $\{x\}$ does meet every orbit of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$. The difficulty with which we have to contend is that it may contain several points from the same orbit. Given x in $\{x\}$, choose a maximal compact subgroup $\widetilde{C}$ of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ containing $\widetilde{J}_1(x)$ and let $C=\widetilde{C}\cap J(\mathbf{Q}_\mathfrak{p})$. If $g\in\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ then $gx\in\mathfrak{X}$ if and only if $g\in J(\mathbf{Q}_\mathfrak{p})\widetilde{C}$. The number of orbits of $J(\mathbf{Q}_\mathfrak{p})$ in

$$\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})x\cap\mathfrak{X}$$

is

$$[J(\mathbf{Q}_\mathfrak{p})\widetilde{C} : J(\mathbf{Q}_\mathfrak{p})\widetilde{J}_1(x)] = [\widetilde{C} : C\widetilde{J}_1(x)]\ .$$

Thus we are free to sum over a set of representatives for the orbits of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$, provided we incorporate the factor

$$[\widetilde{C} : C\widetilde{J}_1(x)]/\mathrm{meas}\, J(x)\ .$$

What we must do is show that this equals

$$u(T_i)\mathrm{meas}\, \widetilde{U}_\mathfrak{p}/\mathrm{meas}\, U_\mathfrak{p}\mathrm{meas}\, \widetilde{J}_1(x)\ .$$

Let $\widetilde{T}_i(x)$ and $T_i(x)$ be the stabilizers of x in $\widetilde{T}_i(\mathbf{Q}_\mathfrak{p})$ and $T_i(\mathbf{Q}_\mathfrak{p})$. Then

$$\frac{[\widetilde{C} : C\widetilde{J}_1(x)]}{\mathrm{meas}\, J(x)} = \frac{[\widetilde{C} : C\widetilde{T}_i(x)]}{[\widetilde{J}_1(x) : J(x)\widetilde{T}_i(x)]} \cdot \frac{1}{\mathrm{meas}\, T_i(x)\backslash J(x)} \cdot \frac{1}{\mathrm{meas}\, T_i(x)}\ .$$

The middle factor on the right combines with the denominator of the first factor to give

$$(\mathrm{meas}\, \widetilde{T}_i(x)\backslash\widetilde{J}_1(x))^{-1} = \mathrm{meas}\, \widetilde{T}_i(x)/\mathrm{meas}\, \widetilde{J}_1(x)\ .$$

On the other hand,

$$[\widetilde{C} : C\widetilde{T}_i(x)] = u(T_i)[\widetilde{U}_\mathfrak{p} : U_\mathfrak{p}\widetilde{T}_i(x)]$$

and

$$[\widetilde{U}_\mathfrak{p} : U_\mathfrak{p}\widetilde{T}_i(x)] = \frac{[\widetilde{U}_\mathfrak{p} : \widetilde{T}_i(x)]}{[U_\mathfrak{p} : T_i(x)]} = \frac{\mathrm{meas}\, \widetilde{U}_\mathfrak{p}}{\mathrm{meas}\, U_\mathfrak{p}} \cdot \frac{\mathrm{meas}\, T_i(x)}{\mathrm{meas}\, \widetilde{T}_i(x)}\ .$$

The desired equality follows.

We started with a particular quadratic extension L, chose T_L, and then $\nu_1^\vee, \nu_2^\vee, \dots$, and $g_1, g_2, \dots$, as well as $T_1, T_2, \dots$. On the other hand, in the previous paragraph we expressed at leaast part of the contribution to the sums (2.1) and (2.9) as a sum over T and ν. We may as well assume the T are the T_L. Moreover the $\nu_i^\vee$ and the ν are essentially the same. If

$$\nu_i^\vee = \sum\nolimits_v \nu_v^\vee \qquad \text{and} \qquad \nu = \sum \nu_v$$

then

$$\nu_v^\vee = \frac{\mathfrak{n}_v}{[k_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]}\,\nu_v\ .$$

Since we are dealing with Frobenius pairs of the first kind, $\mathfrak{m}_v' \neq \mathfrak{m}_v''$ for at least one v.

Let δ_i be the image of g_i in $\mathfrak{D}(T/\mathbf{Q})(T = T_L)$ and δ_∞^i and $\delta_\mathfrak{p}^i$ the images of δ_i in $\mathfrak{D}(T/\mathbf{R})$ and $\mathfrak{D}(T/\mathbf{Q}_\mathfrak{p})$. Then

$$\Phi^{T/\kappa}(t, \phi^\mathfrak{p}) = \kappa(\delta_\infty^i)\kappa(\delta_\mathfrak{p}^i)\Phi^{T_i/\kappa}(t, \phi^\mathfrak{p})\ .$$

The term $\kappa(\delta_\infty^i)$ is 1 if κ is trivial and cancels the term $\beta(\nu)$ if κ is not trivial. The term $\kappa(\delta_\mathfrak{p}^i)$ if 1 if κ is trivial. Otherwise it is just what is needed to allow us to refer the definition of the sign appearing in part (g) of our summary of the discussion of (2.1) to T_i rather than T.

Observe that $u(T)$ is 1 when every place v of F dividing $\mathfrak{p}$ is unramified, that the space $\widetilde{X}_i$, together with the action of $\Phi_\mathfrak{p}$, may be represented as a Cartesian product over the v dividing $\mathfrak{p}$, and that the expression is then also a product. A comparison of the summaries in the previous paragraph with our results for Frobenius pairs of the first kind shows that, in order to have perfect cancellation of the contributions of these pairs with the contributions from the Selberg trace formula parametrized by the ν with $\mathfrak{m}_v' \neq \mathfrak{m}_v''$ for some v, we need only verify the combinatorial facts to be reviewed below.

Combinatorial facts to be proved. Before describing these facts, we recapitulate the relevant definitions in the form and with the notation that is now appropriate. $F_v \subseteq \mathbf{Q}_\mathfrak{p}^{\mathrm{un}} \subseteq \overline{\mathbf{Q}}_\mathfrak{p}$ is an unramified extension of $\mathbf{Q}_\mathfrak{p}$ and we take G to be $\mathrm{Res}_{F_v/\mathbf{Q}_\mathfrak{p}}\mathrm{GL}(2)$. The imbeddings of F_v in $\overline{\mathbf{Q}}_\mathfrak{p}$ are indexed by

$$\mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/F_v)\backslash\mathfrak{G}(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q})$$

and the imbedding $x \to \Phi_\mathfrak{p}^{-i}(x)$ will also be labelled by the integer i. Let M^0 be a lattice in the space of column vectors of length two over O_v invariant under $\mathfrak{G}(F_v/\mathbf{Q}_\mathfrak{p})$ and let $K \subseteq \mathrm{GL}(2, F_v) = G(\mathbf{Q}_\mathfrak{p})$ be the stabilizer of M^0. K is a maximal compact subgroup. Let $\mathfrak{k}$ be be completion of $\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}$. Then

$$G(\mathfrak{k}) = \{(g_i) \mid 1 \leqq i \leqq n, g_1 \in \mathrm{GL}(2, \mathfrak{k})\}\ .$$

Let M^0 also denote $M^0 \otimes \mathfrak{o}_\mathfrak{k}$ and let

$$K(\mathfrak{k}) = \{(g_i) \in G(\mathfrak{k}) \mid g_i M^0 = M^0 \quad \text{for all} \quad i\}\ .$$

A point $\mathfrak{x}$ in

$$\mathfrak{X} = G(\mathfrak{k})/K(\mathfrak{k})$$

is a sequence (M_i), $1 \leqq i \leqq n$, where M_i is an $\mathfrak{o}_\mathfrak{k}$-lattice in the space of column vectors of length two over $\mathfrak{k}$. The action of σ, the Frobenius of $\mathfrak{k}$, on (M_i) is $\sigma : (M_i) \to (M_i')$ with $M_i' = M_{i=1}$, $2 \leqq i \leqq n$, and $M_1' = \sigma^n(M^n)$, if in the last equation σ^n denotes the usual action of σ^n on $\mathfrak{o}_\mathfrak{k}$-lattices.

Suppose J^0 is either a split Cartan subgroup of G or G itself, and T is a Cartan subgroup of J^0 whose image in J_{ad}^0 is anisotropic. Let μ be a coweight of T of the form

$$\sum_{i=1}^n a_i \gamma_{j(i)}^i\ ,$$

with $j(i)$ equal to 1 or 2 and a_i equal to 0 or 1. The set $\{\gamma_j^i\}$ is the standard basis of the lattice of coweights. We suppose that

$$\mathrm{Nm}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}\mu$$

factors through the center of J^0, but if $J^0 \neq G$ not through the enter of G, and use μ as in the appendix to define b, and then let **F** be the operator $\mathfrak{x} \to b_\sigma(\mathfrak{x})$.

The set X is a subset of $\mathfrak{X}$. If $\mathfrak{x} = (M_i)$ and $\mathfrak{Y} = \mathbf{F}\mathfrak{x} = (N_i)$, then $\mathfrak{x}$ lies in X if and only if the following two conditions are satisfied:

(i) $M_8 = N_i$ if i is an unmarked point.

(ii) $M_i \supsetneq N_i \supsetneq \mathfrak{p}M_i$ if i is a marked point.

Observe in particular that if (M_i) and (M_i') with $M_i' = g_iM_i$ lie in X then $\mathrm{ord}(\det g_i)$ is independent of i.

If we fix an index i_0, $1 \leqq i_0 \leqq \mathfrak{n}$, we may define a function κ on X by

$$\kappa(\mathfrak{x}) = \epsilon^{\mathrm{ord}(\det g_{i_0})},$$

if $\mathfrak{x} = (M_i)$ and $M_i = g_iM_i^0$. Here ϵ is 1 if J^0 is a split Cartan subgroup and ± 1 if J^0 is G. If $\mathfrak{g} = g\mathfrak{x}$ with $g = (g_i)$ in $J(\mathbf{Q}_\mathfrak{p})$ and $u = \mathrm{ord}(\det g_i)$, then

$$\kappa(\mathfrak{g}) = \epsilon^u \kappa(\mathfrak{x}) \ .$$

If there are no marked places κ is independent of i_0 even when $\epsilon = -1$. If there are marked places a different choice of i_0 may change its sign, but that does not matter.

If $e = [E_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]$ then $\Phi_\mathfrak{p} = \mathbf{F}^e$ maps X to itself and we set

$$T_x^j = \{g \in J(\mathbf{Q}_\mathfrak{p}) \mid \Phi_\mathfrak{p}^j x = gx\} \ .$$

J is obtained from J^0 by the usual inner twisting. Finally, if $\{x\}$ is a set of representatives for the orbits of $J(\mathbf{Q}_\mathfrak{p})$ in X and ψ_x^j is the characteristic function of T_x^j, we set

$$\varphi_\kappa^j(h) = \sum\nolimits_{\{x\}} \frac{\kappa(x)}{\mathrm{meas}\, J(x)} \int_{J(h,\mathbf{Q}_\mathfrak{p})\backslash J(\mathbf{Q}_\mathfrak{p})} \psi_x^j(g^{-1}hg)\kappa(g)dg$$

with $\kappa(g) = \epsilon^{\mathrm{ord}(\mathrm{Norm}\, g)}$. We must assume that $g \to x(g)$ is trivial on $J(h, \mathbf{Q}_\mathfrak{p})$.

Here is what we must establish.

(a) Suppose J^0 is a split Cartan subgroup and $\epsilon = 1$. Let

$$\mathrm{Nm}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} \mu = \frac{[k_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}]}{\mathfrak{n}} (\mathfrak{m}''\delta_1 + \mathfrak{m}'\delta_2)$$

with

$$\delta_c = \sum_{i=1}^n \gamma_c^i, \ \ c = 1, 2 \ ,$$

$\mathfrak{m}' \neq \mathfrak{m}''$, and $\mathfrak{m}' + \mathfrak{m}'' = \mathfrak{m}$, the number of marked places. Let l be the number of orbits under $\Phi_\mathfrak{p}^j$ and k the number of marked orbits. Then $\varphi_\kappa^j(h)$ is 0 unless $\mathfrak{m}/k$ divides $\mathfrak{m}'$ and $\mathfrak{m}''$. However, if $k' = \mathfrak{m}'/k\mathfrak{m}$ and $k'' = \mathfrak{m}''k/\mathfrak{m}$ are integral, then

$$\varphi_\kappa^j(h) = \frac{k!}{k'!k''!} \mathfrak{p}^{jeq} \theta_v(h)$$

with q equal to the smallest of $\mathfrak{m}'$ and $\mathfrak{m}''$ and

$$\nu = jcn^{-1}(\mathfrak{m}''\delta_1 + \mathfrak{m}'\delta_2)$$

(b) Suppose $\mathfrak{m}' = \mathfrak{m}''$ and they are both integral. Then $J^0 = J = G$. Let t lie in the split Cartan subgroup T and have distinct eigenvalues t_1, t_2. Let ϵ be 1. If $k' = \mathfrak{m}'k/\mathfrak{m} = k/2$ and $k'' = \mathfrak{m}''/\mathfrak{m} = k/2$ are not integral, then $\varphi_\kappa^j(t)$ is 0. Otherwise it is

$$\binom{k}{k/2}^{p^{jem}/2} \frac{|t_1t_2|^{1/2}}{|t_1 - t_2|^{1/2}} \theta_\nu(t)$$

with

$$\nu = jen^{-1}(\mathfrak{m}'\delta_2 + \mathfrak{m}''\delta_1) \ .$$

Suppose t lies in a Cartan subgroup T that is not split. If $\epsilon = -1$, only the case of a T associated to an unramified quadratic extension matters. If $\mathfrak{m} \neq 0$, then $\varphi^j_\kappa(t)$ must be 0. If $\mathfrak{m} = 0$, then $b = 1$ and

$$\varphi^j_\kappa(t) = \frac{\pm 1}{\operatorname{meas} U}\ \frac{|t_1 t_2|^{1/2}}{|(t_1 - t_2)^2|^{1/2}} \ .$$

Here U is the maximal compact subgroup of $T(\mathbf{Q}_\mathfrak{p})$, and the sign is to be determined by the rule given in part (g) of the summary of our discussion of (2.1).

If $\epsilon = 1$ but the eigenvalues are still distinct, then $\varphi^j_\kappa(t)$ is a sum of two terms. The first is

$$\left\{\sum\nolimits_{0 \leqq i < k/2} p^{jem/k} \binom{k}{i}\right\} \frac{\operatorname{meas} T(\mathbf{Q}_\mathfrak{p})\backslash G'(\mathbf{Q}_\mathfrak{p})}{\operatorname{meas} K'}\, \Xi_{jem/\mathfrak{n}}(t) \ .$$

Here G' is the group obtained from the quaternion algebra over F_v, and K' is a maximal compact subgroup of $G'(\mathbf{Q}_\mathfrak{p})$. The second term is 0 unless $k/2$ is integral, when it is

$$\mathfrak{p}^{jem/2}\binom{k}{k/2}\frac{1}{\operatorname{meas} K}\int_{T(\mathbf{Q}_\mathfrak{p})\backslash G(\mathbf{Q}_\mathfrak{p})} \varphi_{ejm/\mathfrak{n}}(g^{-1}tg)dg$$

if $\varphi_{ejm/\mathfrak{n}}$ is the characteristic function of

$$\begin{pmatrix} \varpi^{ejm/2\mathfrak{n}} & 0 \\ 0 & \varpi^{ejm/2\mathfrak{n}} \end{pmatrix} K \ .$$

(c) The final case to consider is that $\mathfrak{m}' = \mathfrak{m}''$, and they are both half-integral. The formulae are to be the same as before. Notice, however, that split T no longer come into question, and that $k/2$ can no longer be integral.

Notice also that Lemma 3.7 is a consequence of part (a).

Finally, we have to examine the contribution of Frobenius pairs of the second kind to the alternating sum of the traces and compare it with the results of §2. We take the sum over conjugacy classes $\{h\}$ in $I(\mathbf{Q})$ of the contributions $N^j(h)$ given by Lemma 3.8 and decompose it into stable and labile parts. The group I is now G', defined in the same manner as G, but in terms of a different quaternion algebra D'. The Hasse invariants of D' are the same as those of D at all finite places that do not divide $\mathfrak{p}$, but they are $\frac{1}{2}$ at every infinite place, and the invariant at a place v dividing $\mathfrak{p}$ is $\mathfrak{m}_v/2$.

The contribution of the scalar matrices is already stable. It is a sum over $Z(\mathbf{Q}) \cap K\backslash Z(\mathbf{Q})$. The terms of the sum may be represented as products. The first factor is

$$\frac{\operatorname{meas}(Z_K G'(\mathbf{Q})\backslash G'(\mathbf{A}_f))}{\operatorname{meas} Z_k}\, \phi^\mathfrak{p}(h) \ .$$

In order to make the comparison with the results of §2, we must recall a simple property of Tamagawa numbers, viz.,

$$\operatorname{meas}(Z_K Z(\mathbf{R}) G(\mathbf{Q})\backslash G(\mathbf{A})) = \operatorname{meas}(Z_K Z(\mathbf{R}) G'(\mathbf{Q})\backslash G'(\mathbf{A})) \ .$$

The measure on the right is equal to

$$\operatorname{meas}(Z(\mathbf{R})\backslash G'(\mathbf{R}))\operatorname{meas}(Z_K G'(\mathbf{Q})\backslash G'(\mathbf{A}_f))$$

The second factor is $\varphi^j(h)$, defined with respect to the space X associated to G. Since it is $\widetilde{X}$, the space associated to $\widetilde{G}_1$, that factors as a Cartesian product, we want to replace $\varphi^j(h)$ by $\tilde{\varphi}^j(h)$. Lemma 3.10 is still valid, and hence every orbit of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p}) = \widetilde{G}'_1(\mathbf{Q}_\mathfrak{p})$ in $\widetilde{X}$ meets X. The number of orbits of $J(\mathbf{Q}_\mathfrak{p}) = G'(\mathbf{Q}_\mathfrak{p})$ contained in the $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$-orbit of x is

$$[\widetilde{C} : C\widetilde{J}_1(x)] \ .$$

Here $\widetilde{C}$ is again a maximal compact subgroup of $\widetilde{J}_1(\mathbf{Q}_\mathfrak{p})$ containing $\widetilde{J}_1(x)$. Since

$$[\widetilde{C} : C\widetilde{J}_1(x)] = \frac{[\widetilde{C} : \widetilde{J}_1(x)]}{[C : J_1(x)]} = \frac{\operatorname{meas} \widetilde{C}}{\operatorname{meas} C}\ \frac{\operatorname{meas} J_1(x)}{\operatorname{meas} \widetilde{J}_1(x)}$$

and

$$\frac{\operatorname{meas} \widetilde{C}}{\operatorname{meas} C} = \frac{\operatorname{meas} \widetilde{K}_\mathfrak{p}}{\operatorname{meas} K_\mathfrak{p}}\ ,$$

we may replace $\varphi^j(h)$ by $\tilde{\varphi}^j(h)$, provided that we multiply by the quotient

$$\operatorname{meas} \widetilde{K}_\mathfrak{p} / \operatorname{meas} K_\mathfrak{p}\ .$$

If we observe that $\sum \mathfrak{m}_v$ is the dimension of the Shimura variety, we see that, to establish that the contribution of the scalar matrices is equal to their contribution to (2.9), we need only the following:

Additional combinatorial facts. We must revert to the notations ued when describing the other combinatorial facts to be proved.

(d) Let κ be identically 1 and let z lie in the center of $G'(\mathbf{Q}_\mathfrak{p})$. If k is odd, then $\varphi^j_\kappa(h)$ is equal to

$$-\frac{(|\varpi_v|^{-1} - 1)}{\operatorname{meas} K}\, \Xi_{jem/\mathfrak{n}}(z) \left\{ \sum\nolimits_{0 \leqq i < k/2} \binom{k}{i} p^{jemi/k} \right\}\ .$$

If k is even, it is the sum of this and

$$\frac{1}{\operatorname{meas} K}\, \Xi_{jem/\mathfrak{n}}(z) \binom{k}{k/2} p^{jem/2}\ .$$

The contribution of the non-scalar elements in $G'(\mathbf{Q})$ remains to be treated, but for it no new combinatorial facts are needed. The treatment is by now routine. We obtain a double sum, over a set of representatives T for the stable conjugacy classes of Cartan subgroups, and over the characters κ of

$$\mathfrak{E}(T/\mathbf{A})/\operatorname{Im} \mathfrak{E}(T/\mathbf{Q}) = \mathfrak{E}(T/\mathbf{A}_f)/\backslash \mathfrak{E}(T/\mathbf{A}_f) \cap \operatorname{Im} \mathfrak{E}(T/\mathbf{Q})\ .$$

Each term is itself a sum over h in $Z_K T(\mathbf{Q})\backslash T(\mathbf{Q})$ of

$$\frac{\mu(T)}{2[\mathfrak{E}(T/\mathbf{A}) : \operatorname{Im} \mathfrak{E}(T/F)]}\ \operatorname{trace} \xi(y) \Phi^{T/\kappa}(t, \phi^\mathfrak{p})$$

times

$$\sum\nolimits_{\delta \in \mathfrak{E}(T/\mathbf{Q}_\mathfrak{p})} \kappa(\delta) \varphi^j(h^\delta)\ .$$

This sum is basically the same as (3.2), except that $\varphi^j(h^\delta)$ is here defined with respect to a fixed X rather than with respect to varying X_g. It may, however, be treated in exactly the same way, with the same conclusions.

This is the reason that no additional combinatorial facts are needed. Now $\mathfrak{m}'_v = \mathfrak{m}''_v$ for all v, whereas earlier this happened only for some v. Since we factored the set into a product over v, every possibility for the individual factors has had to be taken into account already.

4. Combinatorics.

The preparation over, we come now, with sighs of relief from reader and author, to the amusing part of the paper. The combinatorial facts to be verified turn out to be statements about a simple type of tree, the Bruhats-Tits buildings for $\mathrm{SL}(2)$. They may well be familiar to combinatorialists, but a cursory glance at the standard texts yielded nothing of help.

The notation will now be that used when stating the facts to be proved. Once a few preliminary remarks are out of the way, we will be able to dismiss most of the preceding discussion from our minds, and indulge ourselves in a little elementary mathematics.

We have agreed to take $b = 1$ when $\mu = 0$. Otherwise, we have not made any particular choice of the element b used to define X. It will be convenient to do so for the calculations of this paragraph. b lies in $J^0(\mathfrak{k})$, but we are free to modify it to $cb\sigma(c^{-1})$ with c in $J^0(\mathfrak{k})$. Thus we can suppose it is of the form $b = (b_i)$ with

$$b_2 = \ldots = b_\mathfrak{n} = 1 \qquad \text{and} \qquad b_1 = B\ .$$

If J^0 is a split torus, which we take to be the group of diagonal matrices, we may take

$$B = \begin{pmatrix} p^{\mathfrak{m}''} & 0 \\ 0 & p^{\mathfrak{m}'} \end{pmatrix}\ .$$

If J^0 is G we may take

$$B = \begin{pmatrix} p^{\mathfrak{m}/2} & 0 \\ 0 & p^{\mathfrak{m}/2} \end{pmatrix}$$

when $\mathfrak{m}$ is even and

$$B = p^{\mathfrak{m}-1/2} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

when $\mathfrak{m}$ is odd. There is no real need for a specific choice of B, but it does no harm.

A point $\mathfrak{x} \in \mathfrak{X}$ is represented by a sequence $\{M_i \mid 1 \leqq i \leqq n\}$. Define M_i for all $i \in \mathbf{Z}$ by the periodicity condition

$$B\sigma^n(M_{i+n}) = M_i\ .$$

In addition, extend the notion of marked or unmarked point by periodicity. Then $\{M_i\}$ defines a point of X if and only if the following conditions are satisfied.

(o) $B\sigma^n(M_{i+n}) = M_i$;

(i) At an unmarked point i, $M_i = M_{i-1}$;

(ii) At a marked point i, $M_i \supsetneqq M_{i-1} \supsetneqq \mathfrak{p}M_i$.

The supplementary condition is absorbed into (i) and (ii). Because of the special form of b, it states simply that, if $M_i = g_i M_i^0$, then

$$\mathrm{ord}(\det(g_i g_{i-1}^{-1})) = \begin{cases} \ \ 0, & i \text{ unmarked}, \\ -1, & i \text{ marked}\ . \end{cases}$$

The operation of $\Phi_\mathfrak{p}^j$ takes (M_i) to (M_i') with

$$M_i' = M_{i-ej}\ .$$

The point $h = (h_i)$, $1 \leqq i \leqq n$, lies in $J(\mathbf{Q}_\mathfrak{p})$ if and only if h_i is independent of i and

$$B\sigma^n(h_i)B^{1} = h_i$$

for all i. It will be convenient to change the notation slightly and to write $h = (h, \ldots, h)$, that is, we identify h with any one of its coordinates. If $x = (M_i)$, then $h \in T_x^j$ if and only if

$$M_{i-ej} = hM_i$$

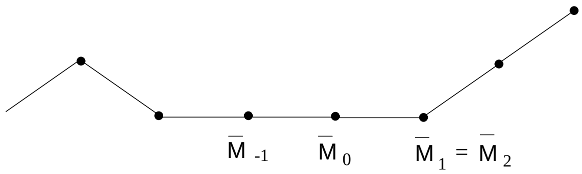
for all i.

If M and M' are two-dimensional lattices over $\mathfrak{o}_{\mathfrak{k}}$, they are said to be *homothetic* if $M' = \lambda M$ with $\lambda \in \mathfrak{k}^x$. Let $\overline{M}$ be the class of lattices homothetic to M. The Bruhat-Tits building for $\mathrm{SL}(2, \mathfrak{k})$ is a tree whose vertices consist of the homothety classes of lattices, the classes $\overline{M}$ and $\overline{N}$ being joined by an edge if, for some $\lambda \in \mathfrak{k}^x$,

$$M \supsetneq \lambda N \supsetneq \mathfrak{p}M \ .$$

σ acts on this building and the fixed point set of σ^j is the Bruhat-Tits building for $\mathrm{SL}(2, k)$ if k is the unramified extension of $\mathbf{Q}_{\mathsf{p}}$ of degree j.

The point x in X determines an infinite path in the building



with vertices $\overline{M}_i$. If x and $x' = (M'_i)$ determine the same path then, because of conditions (i) and (ii), there is a $z \in Z(\mathbf{Q}_{\mathsf{p}})$ for which $x' = zx$, that is, $M'_i = zM_i$ for all i. Since $J(\mathbf{Q}_{\mathsf{p}})$ contains $Z(\mathbf{Q}_{\mathsf{p}})$, x and x' lie in the same orbit. Observe also that if we have a sequence $(\overline{M}_i)$ satisfying the following three conditions, then it may be lifted to a point of X.

(o) $B\overline{M}_{i+n} = \sigma^{-n}(\overline{M}_i)$;

(i) $\overline{M}_i = \overline{M}_{i-1}$ if i is unmarked;

(ii) $\overline{M}_i$ and $\overline{M}_{i-1}$ are joined by an edge if i is marked.

We may as well dispose immediately of the case that $\mu = 0$ and $B = 1$. Then $J(\mathbf{Q}_{\mathsf{p}}) = G(\mathbf{Q}_{\mathsf{p}}) = \mathrm{GL}(2, F_v)$, and $M_i = M$ is independent of i. Since $BM_{i+n} = \sigma^{-n}(M_i)$, L is actually a lattice over F_v. There is only one orbit and we may take it to be the point with $M = M^0$ or, if we were being extremely precise, $M^0 \otimes \mathfrak{o}_{\mathfrak{k}}$, but at present it is best not to distinguish between a lattice over $\mathfrak{o}_v$ and the lattice over $\mathfrak{o}_{\mathfrak{k}}$ it determines. If ψ is the characteristic function of K, then

$$\varphi^j_\kappa(h) = (\operatorname{meas} K)^{-1} \int_{G(h,\mathbf{Q}_{\mathsf{p}})\backslash G(\mathbf{Q}_{\mathsf{p}})} \kappa(g)\psi(g^{-1}hg)dg \ .$$

That $\varphi^j_\kappa(h)$ has the desired value when h is regular and lies in a split torus follows form the properties of the Satake homomorphism. That it has the desired value otherwise is immediate for κ trivial, and follows from an observation in §2 of [7] when κ is not trivial.

It is also easy to show that, if there are marked points and κ is not trivial, then $\varphi^j_\kappa(h)$ is 0. Suppose $x = (M_i)$ is a point of X. We may define another point $x' = (M'_i)$ by demanding that $M'_i = M_{i'}$ whenever i is marked and i' is the first marked point following i. If $h \in J(\mathbf{Q}_{\mathsf{p}})$ and $y = hx$, then $y' = hx'$. Moreover

$$\kappa(x') = -\kappa(x) \ .$$

Since $\varphi^j_\kappa(h)$ can be calculated by a sum over $\{x'\}$ rather than a sum over $\{x\}$, we conclude that $\varphi^j_\kappa(h) = -\varphi^j_\kappa(h)$.

We suppose henceforth not only that there are marked points, but also that κ is trivial, and fix our attention on a specific h in $J(\mathbf{Q}_{\mathsf{p}})$. We observe first of all that $\varphi^j_\kappa(h)$ is the sum over the orbits of $J(h, \mathbf{Q}_{\mathsf{p}})$, the centralizer of h in $J(\mathbf{Q}_{\mathsf{p}})$, of

$$\kappa(x)\psi^j_x(h)/\operatorname{meas} J(h,x) \ .$$

Here $J(h,x)$ is the stabilizer of x in $J(h,\mathbf{Q}_\mathfrak{p})$. To see this start from the definition of $\varphi_\kappa^j(h)$ as

$$\sum\nolimits_{\{x\}} \frac{\kappa(x)}{\operatorname{meas} J(x)} \int_{J(h,\mathbf{Q}_\mathfrak{p})\backslash J(\mathbf{Q}_\mathfrak{p})} \kappa(g)\psi_x^j(g^{-1}hg)dg \ .$$

This expression is equal to

$$\sum\nolimits_{\{x\}} \frac{\kappa(x)}{\operatorname{meas} J(x)} \sum\nolimits_{J(h,\mathbf{Q}_\mathfrak{p})\backslash J(\mathbf{Q}_\mathfrak{p})/J(x)} \kappa(g)\psi_{gx}^j(h) \times \operatorname{meas}(J(h,\mathbf{Q}_\mathfrak{p})\backslash J(h,\mathbf{Q}_\mathfrak{p})gJ(x)) \ .$$

Since

$$\kappa(g)\kappa(x) = \kappa(gx)$$

and

$$\frac{\operatorname{meas}(J(h,\mathbf{Q}_\mathfrak{p})\backslash J(h,\mathbf{Q}_\mathfrak{p})gJ(x))}{\operatorname{meas} J(x)} = \frac{1}{\operatorname{meas} J(h,gx)} \ ,$$

the assertion follows.

We are therefore interested only in the set U of those x for which $h \in T_x^j$. Suppose we have a subset U' of U and an open subgroup J_0 of $J(h,\mathbf{Q}_\mathfrak{p})$ satisfying the following three conditions.

(i) Every orbit of $J(h,\mathbf{Q}_\mathfrak{p})$ in U meets U'.

(ii) If x and y in U' lie in the same orbit of $J(h,\mathbf{Q}_\mathfrak{p})$, then $x = gy$ with $g \in J_0$.

(iii) For all $x \in U'$, $J(h,x)$ is a subgroup of J_0.

It is then clear that

$$\varphi_\kappa^j(h) = \operatorname{meas} J_0 \sum\nolimits_{x\in U'} \kappa(x) \ . \tag{4.1}$$

We also want to reformulate the two conditions of periodicity:

(a) $\sigma^n(BM_i) = M_{i-n}$;

(b) $hM_i = M_{i-ej}$.

Let $uej + vn = l$. I claim that (a) and (b) are equivalent to the conditions

(c) $\sigma^{ejn/l}(M_i) = B^{-ej/l}h^{n/l}M_i$;

(d) $h^uB^v\sigma^{vn}(M_i) = M_{i-l}$.

From (b)

$$h^{n/l}M_i = M_{i-enj/l} \ .$$

Since $\sigma(B) = (B)$, the relation (a) implies that

$$M_{i-enj/l} = \sigma^{enj/l}B^{ej/l}M_i \ ,$$

and we deduce (c). Applying (a) and (b) again, we have

$$M_{i-l} = M_{i-vn-uej} = h^uM_{i-vn} = h^uB^ve^{vn}(M_i) \ .$$

Conversely, if we assume (d), we may write

$$M_{i-n} = M_{i-ln/l} = h^{un/l}B^{vn/l}\sigma^{vn^2/l}(M_i)$$

because $h \in J(\mathbf{Q}_{\mathfrak{p}})$ and

$$B\sigma^n(h) = hB \ .$$

We then apply (c) to infer that the right side of

$$h^{un/l} B^{vn/l} \sigma^{vn^2/l}(M_i) = h^{un/l} B B^{-uej/l} \sigma^{n-uejn/l}(M_i)$$

is equal to

$$B\sigma^n(M_i) = \sigma^n(BM_i) \ .$$

This is (a). To deduce (b) from (c) and (d), we write

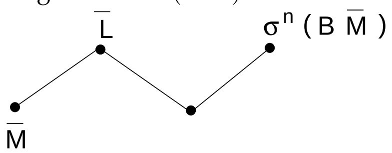
$$M_{i-ej} = M_{i-lej/l} = h^{uej/l} B^{vej/l} \sigma^{vejn/l}(M_i) \ ,$$

and replace the right side by

$$h^{-vn/l} B^{vej/l} \sigma^{vejn/l}(M_i) = M_i \ .$$

The conditions (c) and (d) will be more useful than (a) and (b), but there is still one useful consequence to be drawn from (a). Consider the set $\bar{A}$ of points $\overline{M}$ in the Bruhat-Tits building for which $\mathrm{dist}(\overline{M}, \sigma^n(B\overline{M}))$ is a minimum. It is clear that $\bar{A}$ is invariant under $\overline{M} \to \sigma^n(B\overline{M})$. Moreover, $\bar{A}$ is convex, in the sense that every vertex on the path joining two points of $\bar{A}$ is again in $\bar{A}$. To prove this statement, we proceed by induction on the length of the path. Thus it is enough to show that if $\overline{M}$ and $\overline{N}$ lie in $\bar{A}$ but no other point on the path joining $\overline{M}$ and $\overline{N}$ does, then $\overline{M}$ and $\overline{N}$ are adjacent.

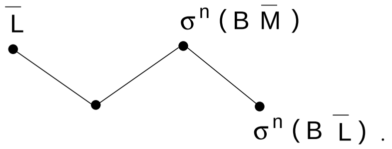
We examine first the path connecting $\overline{M}$ and $\sigma^n(B\overline{M})$



if $\overline{M} \neq \sigma^n(B\overline{M})$ and $\bar{L}$ is the point in the path succeeding $\overline{M}$ then

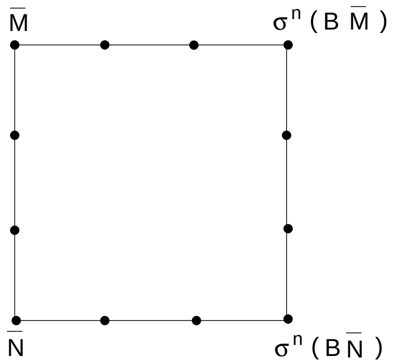
$$\mathrm{dist}(\bar{L}, \sigma^n(B\bar{L})) \leqq \mathrm{dist}(\overline{M}, \sigma^n(B\overline{M})) \ .$$

Since we must have equality, $\sigma^n(B, \overline{M})$ is on the path joining $\bar{L}$ to $\sigma^n(BI)$



Observe that $\bar{L}$ lies in $\bar{A}$.

Returning to $\overline{M}$ and $\overline{N}$, we notice that we can construct a cycle of the form:

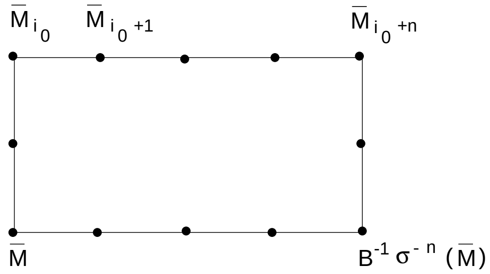


If $\overline{M}$ and $\overline{N}$ are not adjacent, the sides can have no edge in common with the top or bottom, and the cycle is not trivial. This is impossible, because the building is a tree.

Suppose $x = (M_i)$ lies in X. Let $\overline{M}_{i_0}$ be such that

$$\mathrm{dist}(\overline{M}_{i_0}, \bar{A}) \leqq \mathrm{dist}(\overline{M}_i, \bar{A})$$

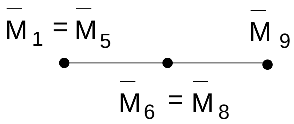
for all i. I claim that $\overline{M}_{i_0} \in \bar{A}$. Otherwise, we could again construct a nontrivial cycle



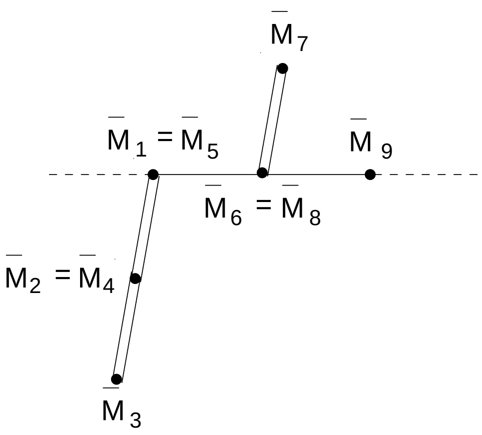
Here $\overline{M}$ is the point in $\bar{A}$ closest to $\overline{M}_{i_0}$.

The skeleton $S(x)$ of $x = (M_i)$ in X will either be the set of integers i for which $\overline{M}_i \in \bar{A}$ or the path formed by joining these vertices in succession. Suppose, for example, that $\bar{A}$ is

and $\overline{M} \to \sigma^n(B\overline{M})$ is a shift to the left by two. Let $\mathfrak{n}$ be 8. Then the skeleton could be



To form the full path we have to add subpaths which issue from $\bar{A}$. Thus the full path, or at least a representative part, could be



The sets A are easily described explicitly. If $J(\mathbf{Q}_\mathfrak{p})$ is the split Cartan subgroup, then A consists of the lattices $M(u', u'')$ formed by the set of

$$\begin{pmatrix} \varpi^{u'} x \\ \varpi^{u''} y \end{pmatrix}$$

with x, y in $\mathfrak{o}_{\mathfrak{k}}$. Here u' and u'' are any two integers. The set $\bar{A}$ is an infinite line

$$\cdots\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\cdots$$

and $\overline{M} \to \sigma^n(B\overline{M})$ is a shift of size $|\mathfrak{m}' - \mathfrak{m}''|$. We take it to be to the right.

If $J(\mathbf{Q}_{\mathfrak{p}})$ is $\mathrm{GL}(2, F_v)$, then A is the set of lattices defined over F_v and $\overline{m} \to \sigma^n(B\overline{M})$ acts trivially on $\bar{A}$. If $J(\mathbf{Q}_{\mathfrak{p}})$ is $G'(F_v)$, then $\bar{A}$ consists of two points, the image of the lattice formed by

$$\begin{pmatrix} x \\ y \end{pmatrix}, \quad x, y \in \mathfrak{o}_{\mathfrak{k}},$$

together with the image of the lattice formed by

$$\begin{pmatrix} x \\ \varpi y \end{pmatrix}, , \quad x, y \in \mathfrak{o}_{\mathfrak{k}} \ .$$

The map $\overline{M} \to \sigma^n(B\overline{M})$ interchanges the two points in $\bar{A}$.

The sets A and $\bar{A}$ are invariant under $J(\mathbf{Q}_{\mathfrak{p}})$, and $J(\mathbf{Q}_{\mathfrak{p}})$ acts transitively on $\bar{A}$. I now introduce a convex subset $\bar{D}$ of $\bar{A}$, as well as D, the set of lattices M for which $\overline{M} \subseteq \bar{D}$. $\bar{D}$ will be invariant under $J(h, \mathbf{Q}_{\mathfrak{p}})$. If h is central in $J(\mathbf{Q}_{\mathfrak{p}})$, then $\bar{D}$ will be $\bar{A}$. Otherwise, $J(h, \mathbf{Q}_{\mathfrak{p}})$ is a Cartan subgroup of $J(\mathbf{Q}_{\mathfrak{p}})$. We can pass to an extension b_v, at most quadratic, of F_v that splits $J(h)$. The Bruhat-Tits building over F_v becomes, perhaps after taking the first barycentric subdivision, a subset of the Bruhat-Tits building over L_v (cf. §3 of [15]). Over L_v we may associate an apartment to $J(h, \mathbf{Q}_{\mathfrak{p}})$. $\bar{D}$ is to consist of the points in $\bar{A}$ at a minimum distance from this apartment. It is geometrically clear that $\bar{D}$ can contain at most two points.

It is also easily seen (cf. §3 of [15]) that in all cases $\bar{D}$ is a connected tree and that the same number $q_0 + 1$ of edges issue from each vertex. We tabulate the possibilities and give, in addition, the number f of orbits in $\bar{D}$ or D under the action of $J(h, \mathbf{Q}_{\mathfrak{p}})$.

(a) $J^0 = J$ is a split torus. Then $q_0 = 1$ and $f = 1$.

(b) $J^0 = G$ and h is central. Then $q_0 = \mathfrak{p}^n$ and $f = 1$.

(c) $J^0 = G$ and the centralizer of h is a split torus. Then $q_0 = 1$ and $f = 1$.

(d) $J^0 = J = G$ and the centralizer of h is not a split torus. If the extension splitting $J(h)$ is unramified, then $q_0 = -1$ and $f = 1$. If it is ramified, then $q_0 = 0$ and $f = 1$.

(e) $J^0 = G$ but $J = G'$, and the centralizer of h is not a split torus. If the extension splitting $J(h)$ is unramified, then $q_0 = 0$ and $f = 2$. If it is ramified, then $q_0 = 0$ and $f = 1$.

The reduced skeleton $\mathrm{RS}(x)$ will either be the set of i on which $\mathrm{dist}(\bar{L}_i, \bar{D})$ attains its minimum or the path obtained by joining these points in order. The reduced skeleton is contained in the skeleton. We choose a set of representatives D_1 for the orbits of $J(h, \mathbf{Q}_{\mathfrak{p}})$ in D. Then $\bar{D}_1$ is a set of representatives for the orbits of $J(h, \mathbf{Q}_{\mathfrak{p}})$ in $\bar{D}$. In all cases but one, D_1 consists of a single element. All points of D_1 have the same stabilizer J_0 and J_0 is a maximal compact subgroup of $J(h, \mathbf{Q}_{\mathfrak{p}})$.

We may now define the set U'. For each possible reduced skeleton RS, we choose an integer $i(\mathrm{RS})$ in it. Then U' consists of those $x = (M_i)$ which are such that if $\mathrm{RS} = \mathrm{RS}(x)$ and $i = i(\mathrm{RS})$, then the minimum distance from the path $(\ldots, \overline{M}_{-1}, \overline{M}_0, \overline{M}_1, \ldots)$ to $\bar{D}$ is equal to $\mathrm{dist}(\overline{M}_i, \overline{N})$ with $N \in D_1$ and $M_i = gN$ with

$$\mathrm{order}(\det g) = \mathrm{dist}(\overline{M}_i, \overline{N}) \ .$$

To verify that U' satisfies the three conditions imposed, we have only to observe that if g is any element of $J(h, \mathbf{Q}_{\mathfrak{p}})$ and, for the same M_i and N,

$$\mathrm{dist}(g\overline{M}_i, \overline{N}) = \mathrm{dist}(\overline{M}_i, \overline{N}) \ ,$$

then $g\overline{N} = \overline{N}$.

There are some remarks to be made about the periodicity condition (c) before we examine the set U' more closely. Since that condition is to be valid for all i, it implies in particular that if $\varphi_\kappa^j(h) \neq 0$, the transformation

$$M \to h^{-n/l} B^{ej/l} \sigma^{ejn/l}(M)$$

has a fixed point in D, because it fixes a point in the reduced skeleton and therefore the closest point in $\bar{D}$ to the reduced skeleton. Recalling that in all but a single case there is only one orbit in D, and that in the exceptional case $\bar{D}$ consists of two points, we conclude that the transformation fixes every point of D. So does $\sigma^{ejn/l}$. Thus, $h^{-n/l} B^{ej/l}$ fixes every point of D.

Examining the various cases, we come to the following conclusions about the nature of those h for which $\varphi_\kappa^j(h) \neq 0$.

(a) If $J^0 = J$ is a split torus, then

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

with

$$|\alpha| = |\varpi^{ej\mathfrak{m}''/\mathfrak{n}}|, \qquad |\beta| = |\varpi^{ej\mathfrak{m}'/\mathfrak{n}}| \ .$$

In particular, $\mathfrak{n}$ must divide $ej\mathfrak{m}'$ and $ej\mathfrak{m}''$, and consequently $\mathfrak{n}/l = \mathfrak{m}/k$ must divide $\mathfrak{m}'$ and $\mathfrak{m}''$.

(b) If $J = G$ and the centralizer of h is a split Cartan subgroup, then the eigenvalues α and β of h have equal absolute values and

$$|\alpha| = |\beta| = |\varpi|^{ej\mathfrak{m}/2\mathfrak{n}} \ .$$

(d) $J^0 = G$ and the centralizer of h is not split, then

$$|\mathrm{Norm}\, h| = |\varpi|^{ej\mathfrak{m}/\mathfrak{n}} \ .$$

In all cases, the order of the determinant of $h^u B^v$ is

$$uej\mathfrak{m}/\mathfrak{n} + v\mathfrak{m} = \mathfrak{m}l/\mathfrak{n} = k \ .$$

The conditions described here are also sufficient for the transformation to fix every element of D. Moreover, that $\varphi_\kappa^j(h)$ is 0 when they are nto satisfied is a part of the combinatorial facts to be proved. We suppose henceforth that the conditions are satisfied.

Suppose

$$M = h^{-\mathfrak{n}/l} B^{ej/l} \sigma^{ej\mathfrak{n}/l}(M) \ .$$

The set of points in the Bruhat-Tits building over $\mathfrak{k}$ that can be joined to $\overline{M}$ by an edge may be viewed as the projective line over the algebraic closure of the finite field $\mathbf{F}_\mathfrak{p}$. The transformation

$$\overline{N} \to h^{-\mathfrak{n}/l} B^{ej/l} \sigma^{ej\mathfrak{n}/l}(\overline{N})$$

allows us to put on it the structure of a projective line over $\mathbf{F}_{\mathfrak{p}^d}$, $d = ej\mathfrak{n}/l$. According to Lang's theorem, there is only one such structure. Thus the set of $\overline{N}$ that are fixed by the transformation and can be joined to $\overline{M}$ by an edge contains $\mathfrak{p}^{ej\mathfrak{n}/l} + 1$ elements.

If the minimum distance from $\{\ldots, \overline{M}_{-1}, \overline{M}_0, \overline{M}_1, \ldots\}$ to $\bar{D}$ is positive, then the path of the reduced skeleton consists of a single point, say $\overline{M}_i$. Since $\bar{D}$ is invariant under $B\sigma^n$ and under h, the periodicity condition

$$\overline{M}_{i-1} = h^u B^v \sigma^{v\mathfrak{n}}(\overline{M}_i) = h^u \overline{M}_i$$

implies that $\overline{M}_{i-1}$ also lies in the reduced skeleton. Hence $\overline{M}_{i-1} = \overline{M}_i$ when the minimum distance is positive, as we now assume. Since u and $\mathfrak{n}/l$ are relatively prime, we deduce from the two equations

$$h^u \overline{M}_i = \overline{M}_i \qquad \text{and} \qquad h^{-\mathfrak{n}/l}\overline{M}_i = \overline{M}_i$$

that

$$h\overline{M}_i = \overline{M}_i \ .$$

Since there are k marked points in an interval of length l, the path from $\overline{M}_i$ to $\overline{M}_{i-1} = \overline{M}_i$ must have k edges. Consequently, points of the above type can exist only for k even.

It is now an easy matter to construct the points of U' for which the minimum distance to $\bar{D}$ is positive. Given a possible reduced skeleton RS and $i = i(\mathrm{RS})$, we choose any $\overline{M}_i$ and $\bar{A}$ which is fixed by h, and construct a path $\overline{M}_i, \overline{M}_{i-1}, \ldots, \overline{M}_{i-l}$, from $\overline{M}_i$ to $\overline{M}_{i-l}$. To construct it we must suppose that k is even. Then $\overline{M}_{i'-1} = \overline{M}_{i'}$ if i' is unmarked and one of the $\mathfrak{p}^{ej\mathfrak{n}/l} + 1$ elements that can be joined to $\overline{M}_{i'}$ by an edge and satisfy

$$h^{\mathfrak{n}/l}\overline{M}_{i'-1} = b^{ej/l}\sigma^{ej\mathfrak{n}/l}(\overline{M}_{i'-1})$$

if i' is marked. We say that the edge from $\overline{M}_{i'}$ to $\overline{M}_{i'-1}$ is progressive or retrogressive according as $\overline{M}_{i'-1}$ is at a greater or a lesser distance from $\bar{D}$ than $\overline{M}_{i'}$. We must be careful that at any stage we have added at least as many progressive as retrogressive edges, for otherwise we would approach too closely to $\bar{D}$, and we must ultimately take as many retrogressive as progressive steps, in order that we arrive back at $\overline{M}_i = \overline{M}_{i-l}$. Indeed we must be careful to return to the initial $\overline{M}_i$ at any i' that lies in the given rational skeleton. Finally, we have to choose $\overline{M}_i$ so that the closest point to it in $\bar{D}$ is an $\overline{N}$ with N in D_1. We define $\overline{M}_{i'}$ in general by the periodicity condition (d), and we lift the full path to a point $x = (M_{i'})$ in X with $M_i = gN$ and

$$\mathrm{order}(\det g) = \mathrm{dist}(\overline{M}_i, \overline{N}) \ .$$

Then x lies in U'.

There are some observations to be made. First of all, the number W of points that are yielded by a given choice of $\overline{M}_i$ and all possible choices of the reduced skeleton and all possible paths from $\overline{M}_i$ to $\overline{M}_{i-l}$ is independent of $\overline{M}_i$. We shall show that

$$W = \binom{k}{k/2}\mathfrak{p}^{ej\mathfrak{m}/2} \ . \tag{4.2}$$

The minimum distance can be positive only when $D \neq A$, that is, only when $J(\mathbf{Q}_\mathfrak{p}) = G(\mathbf{Q}_\mathfrak{p})$ and $J(h, \mathbf{Q}_\mathfrak{p})$ is a Cartan subgroup. Then A is just the Bruhat-Tits building over F_v. The argument that led to (4.1) shows that if $T(\mathbf{Q}_\mathfrak{p}) = J(h, \mathbf{Q}_\mathfrak{p})$ and $|\det h| = |\varpi|^{ej\mathfrak{m}/\mathfrak{n}}$ then

$$(\mathrm{meas}\, K_\mathfrak{p})^{-1} \int_{T(\mathbf{Q}_\mathfrak{p})\backslash G(\mathbf{Q}_\mathfrak{p})} \varphi_{ej\mathfrak{m}/\mathfrak{n}}(g^{-1}hg)dg \tag{4.3}$$

is $(\mathrm{meas}\, U_\mathfrak{p})^{-1}$ times the number of points $\overline{M}$ in the Bruhat-Tits building over F_v that are fixed by h and satisfy

$$\mathrm{dist}(\overline{M}, \bar{D}_1) \leqq \mathrm{dist}(\overline{M}, \bar{D}) \ .$$

However, $\binom{k}{k/2}\mathfrak{p}^{ej\mathfrak{m}/2}$ times (4.3) is the second term of the desired formula for $\varphi^j_\kappa(h)$ when κ is trivial. Therefore, the contribution to (4.3) of those $\overline{M}$ for which

$$0 \leqq \mathrm{dist}(\overline{M}, \bar{D}_1) \leqq \mathrm{dist}(\overline{M}, \bar{D})$$

is yielded by those x for which the minimum distance from$\{\ldots, \overline{M}_{-1}, \overline{M}_0, \overline{M}_1, \ldots\}$ to $\bar{D}$ is positive.

The equality (4.2) is a consequence of the next lemma, which will be proved towards the end of the paragraph. Let T_1 be a connected tree and suppose that from every vertex of T_1 there issue $q_1 + 1$ edges, with $q_1 \geqq 0$. Let $k \geqq 0$ be an even integer, and for each nonempty subset S of $\mathbf{Z}$ with period l choose $i = i(S)$ in S. Let P be a point of T_1 and L an edge containing it.

Lemma 4.1. *Let $\mathfrak{k}$ be the set of pairs (x,S) where $x=(P_i,P_{i-1},\dots,P_{i-k})$, with $i=i(S)$, is a path from P to P in T_1 with no edge in L and with $P_{i'}=P$ if and only if $i'\in S$. The number of pairs in $\mathfrak{k}$ is then*

$$\binom{k}{k/2}q_1^{k/2}\ .$$

The equality (4.2) follows upon taking $q_1=\mathfrak{p}^{ej\mathfrak{n}/l}$. This lemma will be proved at the same time as another lemma from which we can deduce the combinatorial facts needed but not yet proved. Suppose $T_0\subseteq T_1$ is another connected tree and that there issue q_0+1 points from every vertex of T_0. Let P and P' be two points in T_0 a distance d apart and let $r=(k-d)/2$ be a non-negative integer.

Lemma 4.2. *Let $\mathfrak{Q}$ be the set of pairs (x,S) where $x=(P_i,P_{i-1},\dots,P_{i-k})$ is a path from $P_i=P$ to $P_{i-l}=P'$ and $P_{i'}\in T_0$ if and only if $i'\in S$. The number of points in $\mathfrak{Q}$ is*

$$-\sum\nolimits_{0\,\leqq\,a<r}(q_0-1)\binom{k}{a}q_1^a+\binom{k}{r}q_1^r\ .$$

This lemma will enable us to count the number of paths in U' for which the minimum distance to $\bar D$ is 0. The reduced skeleton is then a path in $\bar D$. The distance

$$d=\operatorname{dist}(\bar L,h^uB^v\sigma^{v\mathfrak{n}}(\bar L))$$

is constant on $\bar D$. If $J^0=G$ then d is 0 if k is even and is 1 if k is odd. If J^0 is a split Cartan subgroup, then

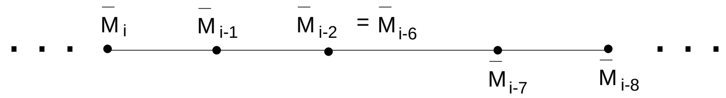
$$h^uB^v=\begin{pmatrix}\alpha^u\varpi^{v\mathfrak{m}''} & 0\\ 0 & \beta^u\varpi^{v\mathfrak{m}'}\end{pmatrix}$$

and

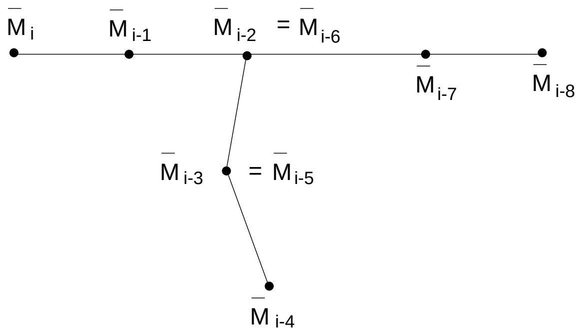
$$\left|\frac{\alpha^u\varpi^{v\mathfrak{m}''}}{\beta^u\varpi^{v\mathfrak{m}'}}\right|=|\varpi|^{v(\mathfrak{m}''-\mathfrak{m}')+uej(\mathfrak{m}''-\mathfrak{m}')/\mathfrak{n}}=|\varpi|^{(k''-k')}\ .$$

Consequently, $d=[k''-k']=k-2r$ if r is the minimum of k' and k''.

To construct the set U_0' of points of U' whose reduced skeleton lies in $\bar D$, we have merely to construct the associated path in the Bruhat-Tits building over $\mathfrak{k}$, for we can uniquely lift the path to a point in U'. If RS is the reduced skeleton, we have only to construct that part of the path lying between $i(\mathrm{RS})$ and $i(\mathrm{RS})-l$, for we may complete the part of the path to the full infinite path by invoking the periodicity condition (d). Moreover, the unmarked points are irrelevant, for at them we just mark time, and so we might as well discard them and obtain the new period k. The reduced skeleton is also a path in $\bar D$ labeled by the points of RS. That part of it between $i(\mathrm{RS})$ and $i(\mathrm{RS})-k$ joins $\overline{M}_{i(\mathrm{RS})}=\overline{M}$ in $\bar D_1$ to $\overline{M}_{i(\mathrm{RS})-k}=h^uB^v\sigma^{v\mathfrak{n}}(\overline{M})$. We complete the reduced skeleton to the complete path segment from $\overline{M}_{i(\mathrm{RS})}$ to $\overline{M}_{i(\mathrm{RS})-k}$ by adding flagella which project into the ambient Bruhat-Tits building. Thus, if $n=8$ and $i=i(\mathrm{RS})$, the reduced skeleton could be



and the full path



Applying the lemma with $q_1 = \mathfrak{p}^{ej\mathfrak{n}/l}$, we see that the number of points in U' for which the reduced skeleton lies in $\bar{D}$ is

$$f\left\{\binom{k}{r}\mathfrak{p}^{ej\mathfrak{n}r/l} - (q_0 - 1)\sum\nolimits_{0 \leqq a < r}\binom{k}{a}\mathfrak{p}^{ej\mathfrak{n}a/l}\right\} , \tag{4.4}$$

if f is the number of orbits of $J(h, \mathbf{Q}_\mathfrak{p})$ in $\bar{D}$.

To see that this gives us all the combinatorial facts we needed, we just have to run through the various possibilities, adding the occasional simple comment. If $J(h, \mathbf{Q}_\mathfrak{p})$is a split Cartan subgroup, then $q_0 = 1, q_0 - 1 = 0$, and $f = 1$. Moreover, $\mathfrak{n}r/l = q$ is the smallest of $\mathfrak{m}'$ and $\mathfrak{m}''$. Thus

$$\frac{|U'_0|}{\operatorname{meas} J_0} = \frac{1}{\operatorname{meas} U_\mathfrak{p}}\binom{k}{k'}\mathfrak{p}^{ejq}$$

is the anticipated value of $\varphi^j_\kappa(h)$ if $k' \neq k''$. If $k' = k''$, it is exactly what is required to supplement the contribution to $\varphi^j_\kappa(h)$ from $U' - U'_0$.

Suppose $J(h, \mathbf{Q}_\mathfrak{p})$ is a Cartan subgroup of $J(\mathbf{Q}_\mathfrak{p})$ but is not split. Then $-f(q_0 - 1)$ is 2 if the corresponding extension is unramified and 1 if it is ramified. If $T(\mathbf{Q}_\mathfrak{p}) = J(h, \mathbf{Q}_\mathfrak{p})$, then

$$\frac{\operatorname{meas} T(\mathbf{Q}_\mathfrak{p})\backslash G'(\mathbf{Q}_\mathfrak{p})}{\operatorname{meas} K'} = \frac{-f(q_0 - 1)}{\operatorname{meas} U_\mathfrak{p}} .$$

$U_\mathfrak{p}$ is the maximal compact subgroup of $T(\mathbf{Q}_\mathfrak{p})$.

If k is odd, then $U' = U'_0$. Moreover, the extension can be unramified only if $\mathfrak{m}$ is odd; but then $J(\mathbf{Q}_\mathfrak{p}) = G'(\mathbf{Q}_\mathfrak{p})$ and $f = 2$. The value of $\varphi^j_\kappa(h)$ is seen to be exactly that stated in the list of combinatorial facts to be proved. If k is even, the value of $\varphi^j_\kappa(h)$ is given there as a sum of two terms, the first a sum over $0 \leqq i < k/2$. This term is yielded by the second part of (4.4), a sum over $0 \leqq a < r = k/2$. The second part of $\varphi^j_\kappa(h)$ was expressed in terms of the integral (4.3). Most of it was accounted for by the contribution from $U' - U'_0$. The remainder is taken care of by the first term of (4.4). Notice that f must be 1 if k is even.

Suppose, finally, that h is central. Then $U' = U'_0$ and $q_0 = |\varpi_v|^{-1}$. It is consequently manifest that $\varphi^j_\kappa(h)$ has the anticipated value.

We have still to prove Lemmas 4.1 and 4.2. Since there is nothing else to be done, all symbols apart from those entering the statements of these two lemmas are free. The skeleton of a pair (x, S) in $\mathfrak{P}$ or $\mathfrak{Q}$ is S. If (x, S) is in $\mathfrak{P}$, an edge will be called *progressive* if it is moving away from P. If it is in $\mathfrak{Q}$, an edge will be called *progressive* if it is moving away from T_0. An edge that is not progressive will be called *retrogressive*. The sense of the path is from P_i to P_{i-l}.

We represent that part of the skeleton lying between $i = i(S)$ and $i - l$ as

$$\underset{i}{\times} \bigcirc \times \times \bigcirc \bigcirc \bigcirc \times \ldots \underset{i-l}{\times}$$

The points in S are unmarked. If there is to be any pair with S as skeleton, the gaps must all be of even length. Let there be $\mathfrak{m} = \mathfrak{m}(S)$ gaps of length $2s_1, \ldots, 2s_\mathfrak{m}$. Given a pair (x, S), we add to S the integers j' for which the edge from $L_{j'}$ to $L_{j'-1}$ is progressive if $i \geqq j \geqq i - l$ and $j \equiv j' (\operatorname{mod} k)$. The result will be called the *frame* F. We can recover S from F. To do this, let ϵ_j be $+1$ or -1, according as j is or is not in F. If $j_2 \leqq j_1$, set

$$N_{j_1,j_2} = \sum_{j=j_2}^{j_1} \epsilon_j .$$

Then $j_1 \in S$ if and only if $N_{j_1,j_2} \geqq 0$ for all j_2. There are $k - s$ points in a period of F and $k - s \geqq k/2$.

Conversely, suppose we start from a subset F of $\mathbf{Z}$ which is periodic of period l and contains $k - s \geqq k/2$ points in each period. Define ϵ_j and N_{j_1,j_2} as before, and let S be the set of j_1 for which $N_{j_1,j_2} \geqq 0$ whenever $j_2 \leqq j_1$. We verify by induction that S is not empty. Choose $j_1 \in F$. If $j_1 \in S$, there is nothing to prove.

Otherwise, choose the largest $j_2 \leqq j_1$ for which $N_{j_1,j_2} < 0$. Then $j_2 < j_1$ because $j_1 \in F$ and $N_{j_1,j_2} = 1$. Moreover, $j_2 > j_1 - k$ because

$$N_{j_1,j_2-l} = (k-2s) + N_{j_1,j_2} \ .$$

Finally, the set $\{j_2+1,\dots,j_1\}$ must contain an even number of elements because $N_{j_1,j_2+1} = 0$. Discard this set and all sets congruent to it modulo k, and pull the remaining points together to obtain a new set F' of period $k-(j_1-j_2)$. Since exactly half of the points $\{j_2+1,\dots,j_1\}$ lie in F, the set F' contains

$$k-s-\frac{1}{2}(j_1-j_2)$$

points within a period and

$$k-s-\frac{1}{2}(j_1-j_2) \geqq \frac{1}{2}(k-(j_1-j_2)) \ .$$

It is clear that F and F' have the same skeleton. The induction assumption guarantees that the skeleton of F' is not empty.

The points in S immediately preceding the gaps, as one moves toward smaller integers, will be called *extremities*. The integer j_1 is an extremity if and only if $N_{j_1,j_2} = 0$ for some $j_2 \leqq j_1$. To a frame we can attach:

(i) the skeleton;

(ii) the number $\mathfrak{m}$ of extremities within a period;

(iii) to each extremity i_α, $1 \leqq \alpha \leqq \mathfrak{m}$, $i-k < i_\alpha \leqq i$ the length $2s_\alpha$ of the succeeding gap;

(iv) the spine, which is obtained from the skeleton by discarding the extremities. If $s = \sum s_\alpha$, the spine has $k-2s$ elements in a period and may therefore be empty.

We want to treat the two lemmas in a uniform manner, and to this purpose we introduce, when dealing with Lemma 4.1, the tree T_0 consisting of P alone. We take P' to be P, and let $q=0$, $q_0=-1$. When dealing with Lemma 4.2, we take q to be q_0. Let $N(c)$ be the number of paths of length c in T_0 joining P to P'. The number of elements in $\mathfrak{P}$ or $\mathfrak{Q}$ is

$$\sum (1-q/q_1)^{\mathfrak{m}} q_1^s N(k-2s) \ , \tag{4.5}$$

the sum being taken over all possible frames. To see this we observe that to construct a path $x = (P_i,\dots,P_{i-k})$, we first take one of the $N(k-2s)$ paths from P to P' in T_0, with points labeled by the integers between i and $i-l$ lying in the spine, and then at each extremity add one of the $q_1 - q$ possible edges from T_0 into T_1, and finally, at all other points of the frame, add one of the q_1 possible progressive edges.

The expression (4.5) is equal to

$$\sum_{l=0}^{r} \left\{ \sum\nolimits_{\mathfrak{m} \geqq l} \binom{m}{l} (-q)^l N(k-2s) \right\} q_1^{s-1} \ . \tag{4.6}$$

The inner sum is taken over all frames with $\mathfrak{m} \geqq l$. It can be put in a more manageable form.

If we have a frame F with $k-s$ elements and $\mathfrak{m} \geqq l$, then $s \geqq l$. We construct $\binom{\mathfrak{m}}{l}$ new frames with $k-(s-l)$ elements. Take any subset E of the extremities with l elements (within a period) and, for each element of the subset, add to F the last element of the gap in the skeleton following it. Since the added elements do not lie in F, the result is a frame with $k-(s-l)$ elements. The added elements will not be extremities of F', because the skeleton S of F is contained in the skeleton S' of F'. The extremities of F' are the extremities of F that do not lie in E.

The procedure yields not only F' but also a subset E' of its spine. E' consists of the added elements, and any two elements of E' are separated by a point in the spine of F'. Conversely, suppose we start with F' and a set E' of l separated points in its spine. Remove these l points from F'. The result is still a frame, and the skeleton S of F is contained in the skeleton S' of F'. The points of E' lie in the gaps of S. I claim that they are the last points of the gaps in which they lie. Since we may argue by induction, we have only to show that if j is a point of the spine, j' is the smallest point of the spine with $j < j'$, $j' < j + l$, and we remove j from F' to obtain F, then the skeleton of F is obtained by removing all j_1 with $j \leqq j_1 < j'$ from the skeleton of F'.

We must certainly remove j. Let j_1 with $j < j_1 < j + l$ lie in the skeleton of F'. Suppose first that $0 < j_1 < j'$. Then j_1 must be an extremity, and so $N'_{j_1,j_2} = 0$ for some j_2. If $j_2 \leqq j$, then

$$N'_{j_1,j_2} = N'_{j_1,j+1} + N'_{j,j_2} \geqq N'_{j,j_2} > 0 \ ,$$

because j lies in the spine. This is impossible and $j < j_2 < j_1$. If $j_3 < j_2$, then

$$0 \leqq N'_{j_1,j_3} = N'_{j_1,j_2} + N'_{j_2-1,j_3} = N'_{j_2-1,j_3} \ .$$

Thus, $j_2 - 1$ also lies in the skeleton. Iterating, we conclude that

$$N'_{j_1,j+1} = 0 \ .$$

Hence, $N'_{j_1,j} = -1$, and j_1 is not in the skeleton of F.

Now suppose that $j' \leqq j_1 < j + l$. If $j_1 \geqq j_2 > j$, then

$$N_{j_1,j_2} = N'_{j_1,j_2} \geqq 0 \ .$$

If $j - l < j_2 \leqq j$, then

$$N_{j_1,j_2} = N'_{j_1,j_2} - 1 = N'_{j_1,j'+1} + (N_{j',j+1} - 1) + (N_{j,j_2} - 1) \ .$$

Since j and j' lie in the spine, all summands on the right are positive or zero.

Let $S_{\mathfrak{n},l}$ be the number of ways of choosing l separated points from a cyclic set with $\mathfrak{n}$ elements. We now regard the inner sum in (4.6) as taken over all F' with $k - (s - l)$ elements, and all possible choices of E'. Since the spine of F' has $k - 2(s - l)$ elements and there are $\binom{k}{s-l}$ possible choices for F', the sum (4.6) is equal to

$$\sum_{l=0}^{r} \sum_{s=l}^{r} \binom{k}{s-l} (-q)^l N(k-2s) S_{k-2(s-l),l} q_1^{s-l} \ .$$

We reverse the order of summation and replace $s - l$ by l to obtain

$$\sum_{s=0}^{r} \sum_{l=0}^{s} \binom{k}{l} (-q)^{s-l} N(k-2s) S_{k-2l,s-l} q_1^{l} \ .$$

We then change the order of summation once again, and consider

$$\sum_{s=l}^{r} (-q)^{s-l} N(k-2s) S_{k-2l,s-l} \ .$$

Here we substitute s for $s - l$ to obtain

$$\sum_{s=0}^{r'} (-q)^{s} N(k'-2s) S_{k',s} \tag{4.7}$$

with $r' = r - l$, $k' = k - 2l$.

To establish the lemmas, we must show that the sum (4.7) has the following values.

(i) When $q = 0$ and $q_0 = -1$, it is 0 unless $r' = 0$, and then it is 1.

(ii) When $q = q_0$, it is 1 if $r' = 0$ and $-(q_0 - 1)$ if $r' > 0$.

Observe that in the first case $d = 0$ and $k' = 2r'$. If $q = 0$, the sum reduces to

$$N(k')S_{k',0} \ .$$

If $q_0 = -1$, then $N(k') = 0$ unless $k' = 0$ and $N(0) = S_{0,0} = 1$. If $q = q_0 = -1$, the $k' = 2r'$ and the sum reduces to

$$N(0)S_{2r',r'} = S_{2r',r'} \ .$$

Since $S_{2r',\cap'}$ is 2 if $r' > 0$, the value of the sum is again correct. If $q = q_0 = 0$, the sum reduces to

$$N(k')S_{k',0} \ .$$

Since $N(k')$ is always 1 when $q_0 = 0$ and since $S_{k',0} = 1$ for all $k' \geqq 0$, the value is again correct.

Suppose that $q = q_0 > 0$. We regard $S_{k',s}$ as the number of ways of choosing a set of s points from $1, \ldots, k'$ which is cyclically separated. Let $S^0_{k',s}$ be the number of ways of choosing such X which do not contain k'. We first show that

$$\sum\nolimits_{s=0}^{r'} (-q_0)^s N(k' - 2s) S^0_{k',s} = 1 \ . \tag{4.8}$$

Suppose that X doe snot contain k'. Remove from $\{1, \ldots, k'\}$ the points of X and their immediate successors. This leaves $k' - 2s$ points which can be used ot label the edges of a path of length $k' - 2s$ from P to P'. Starting from this path and the labeling, we construct q_0^s new paths of length k'. Choose any edge emanating from P and cal it the exceptional edge. The exceptional edge at any other point will be the edge leading toward P. If $i \in X$ and there is no $i' < i$ not in X, then we add to the path an edge issuing from P and then the same edge in the opposite direction. We are allowed to take any but the exceptional edge. These new edges are labeled by i and $i + 1$. If there is such an i we take the largest and add an adge and its opposite in just the same way, except that it must issue form the final point of the edge labeled by $i - 1$, and it must not be the exceptional edge. Carrying this out for each $i \in X$, we obtain a path of length k' from P to P' labeled by $1, \ldots, k'$.

The sum (4.8) is a sum over all paths from P to P' of length k' of the sum over s of $(-1)^s$ times the multiplicity with which it is obtained by the above construction. The sum over s is easily evaluated. Given a path of length k', let $\mathfrak{n}$ be the number of subpaths of length two which consist of a move out from a point along an edge which is not exceptional and a return. Then the sum is

$$\sum\nolimits_{s=0}^{\mathfrak{n}} (-1)^s \binom{\mathfrak{n}}{s} \ .$$

It is 0 unless there are no such subpaths, and then it is 1. A little reflection convinces one that there are no such subpaths in only one case, that of the path which moves out from P along the exceptional edge and returns r' times, and then proceeds directly to P'. This establishes (4.8).

Let $S^1_{k',s}$ be the number of ways of choosing X so that it does contain k'. To complete the proof of the lemma, we have only to show that

$$\sum\nolimits_{s=1}^{r'} (-q)^{s-1} N(k' - 2s) S^1_{k',s} = 1$$

if $r' \geqq 1$ and $k' \geqq 2$. A separated subset of $\{1, \ldots, k'\}$ that contains k' yields, upon removal of k', a separated subset of $\{2, \ldots, k' - 1\}$ that does not contain $k' - 1$. Conversely, a separated subset of $\{2, \ldots, k' - 1\}$ that does not contain $k' - 1$ yields, upon addition of k', a separated subset of $\{1, \ldots, k'\}$. Thus

$$S^1_{k',s} = S^0_{k'-2,s-1} \ ,$$

and our sum is equal to

$$\sum\nolimits_{s=0}^{r'-1} (-q_0)^s N(k' - 2 - 2s) S^0_{k'-2,s-1} \ ,$$

which is have already seen to equal 1.

Appendix.

Suppose (γ, h^0) is a Frobenius pair [13]. We choose a Cartan subgroup T over $\mathbf{Q}$ such that T_{ad}, the image of T in I^0_{ad}, is anisotropic at infinity and $\mathfrak{p}$, and let $k_\mathfrak{p}$ be a finite Galois extension of $\mathbf{Q}_\mathfrak{p}$ which splits T. We suppose h^0 factors through T and let $\mu^\vee$ be the coweight h^0_0 of T. We set

$$\nu^\vee = \sum\nolimits_{\tau \in \mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})} \tau\mu^\vee \ .$$

Let $\{a_{\sigma,\tau}\}$ be a fundamental 2-cocycle of the extension $k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}$. We define the Weil group $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ as the set of pairs (x, σ) with $x \in k_\mathfrak{p}^\times$, and $\sigma \in \mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$, multiplication being defined by

$$(x, \sigma)(y, \tau) = (x\sigma(y)a_{\sigma,\tau}, \sigma\tau) \ .$$

If σ belongs to $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$, we set

$$a_\sigma = \prod\nolimits_{\tau \in \mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})} a_{\sigma,\tau}^{\sigma\tau\mu^\vee} \ .$$

It lies in

$$T(k_\mathfrak{p}) \simeq X_*(T) \otimes k_\mathfrak{p}^\times \ .$$

If $w = (x, \sigma)$, we set

$$b_w = x^{\nu^\vee} a_\sigma \ .$$

Lemma A.1. *The* 1*-cochain* $w \to b_w$ *is a cocycle.*

It must be verified that

$$a_\rho \rho(a_\sigma) a_{\rho\sigma}^{-1} = a_{\rho,\sigma}^{\nu^\vee} \ .$$

The left side is

$$\left\{\prod\nolimits_\tau a_{\rho,\tau}^{\rho\tau\mu^\vee}\right\}\left\{\prod\nolimits_\tau \rho(a_{\sigma,\tau})^{\rho\sigma\tau\mu^\vee}\right\}\left\{\prod a_{\rho\sigma,\tau}^{-\rho\sigma\tau\mu^\vee}\right\} \ .$$

Replace τ by $\sigma\tau$ in the first product and use the relation

$$a_{\rho\sigma,\tau}\rho(a_{\sigma,.\tau})a_{\rho\sigma,\tau} = a_{\rho,\sigma}$$

to obtain

$$\prod\nolimits_\tau a_{\rho,\sigma}^{\rho\sigma\tau\mu^\vee} = a_{\rho,\sigma}^{\nu^\vee} \ .$$

Suppose we replace $a_{\rho,\sigma}$ by

$$\overline{a}_{\rho,\sigma} = c_\rho \rho(c_\sigma) c_{\rho\sigma}^{-1} a_{\rho,\sigma} \ .$$

Then $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ is replaced by $\overline{W}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$, but

$$(x, \sigma) \to (xc_\sigma^{-1}, \sigma)$$

is an isomorphism from $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ to $\overline{W}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$. The 1-cochain $\{a_\sigma\}$ is replaced by

$$\begin{aligned}\sigma \to \overline{a}_\sigma &= a_\sigma \prod\nolimits_\tau (c_\sigma^{\sigma\tau\mu^\vee} \sigma(c_\tau)^{\sigma\tau\mu^\vee} c_{\sigma\tau}^{-\sigma\tau\mu^\vee}) \\ &= a_\sigma c_\sigma^{\nu^\vee} \sigma(d) d^{-1}\end{aligned}$$

if

$$d = \prod\nolimits_\tau c_\tau^{\tau\mu^\vee} \ .$$

If we pull back the cocycle $\{\overline{b}_w\}$ from $\overline{W}_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ to $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$, we obtain

$$w \to \sigma(d) b_w d^{-1} = w(d) b_w d^{-1} \ ,$$

because w acts on $T(k_\mathfrak{p})$ through its projection to σ. At all events, we obtain a cocycle in the same class, and so our constructions do not depend on the choice of a fundamental 2-cocycle.

We choose another Cartan subgroup $\overline{T}$ for which $\overline{T}_{\mathrm{ad}}$ is still anisotropic at infinity and $\mathfrak{p}$ and which still splits over $k_\mathfrak{p}$. We suppose that $\bar{h}^0 : R \to I^0$ is conjugate under $I(\mathbf{R})$ to h^0 and factors through $\overline{T}$. Let $\bar{\mu}^\vee$ be the coweight $\bar{h}^0_0$ of $\overline{T}$. We use it to define the 1-cocycle $\{\overline{b}_w\}$.

Lemma A.2. *There is a c in $I^0(k_\mathfrak{p})$, such that*

$$\overline{b}_w = cb_w w(c^{-1})$$

for all w.

Since T_{ad} is anisotropic at $\mathfrak{p}$, the coweight $\nu^\vee$ is actually a coweight of the center of I^0, and hence b_w and a_σ have the same image a'_σ in $I^0_{\text{ad}}(k_\mathfrak{p})$. We use the cocycle $\{a'_\sigma\}$ to twist I^0 and obtain a group I over $\mathbf{Q}_\mathfrak{p}$. Then

$$d_w = \overline{b}_w b_w^{-1}$$

is a cocycle of $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ with values in $I(k_\mathfrak{p})$ and we have to show that it bounds.

The first step is to verify that it takes values in the derived group I_{der} of I and factors through $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$. The difference between $\mu^\vee$ and $\bar{\mu}^\vee$, and hence that between $\nu^\vee$ and $\bar{\nu}^\vee$, is a sum of coroots. Since $\nu^\vee$ and $\bar{\nu}^\vee$ are coweights of the center, $\nu^\vee = \bar{\nu}^\vee$. Thus

$$d_w = a_\sigma \overline{a}_\sigma^{-1}$$

and factors through $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$. I write d_σ instead of d_w. If λ is a rational character of I, then λ is orthogonal to coroots and

$$\langle \sigma\tau\mu^\vee, \lambda\rangle = \langle \sigma\tau\bar{\mu}^\vee, \lambda\rangle \ .$$

Consequently, $\lambda(d_\sigma) = 1$ and d_σ lies in I_{der}.

If I_{sc} is the simply-connected covering group of I_{der}, then

$$H^1\mathfrak{G}((k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), I_{\text{sc}}(k_\mathfrak{p})) = H^1(\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), I_{\text{sc}}(\overline{\mathbf{Q}}_\mathfrak{p})) = 1 \ .$$

If C is the kernel of

$$I_{\text{sc}} \to I_{\text{der}} \ ,$$

then the composition

$$H^1(\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), I_{\text{der}}(k_\mathfrak{p})) \to H^1(\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), I_{\text{der}}(\overline{\mathbf{Q}}_\mathfrak{p})) \to H^2(\mathfrak{G}(\overline{\mathbf{Q}}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}), C(\overline{\mathbf{Q}}_\mathfrak{p}))$$

is injective. We show that the image of d_σ is trivial.

Choose an integer m so that

$$m\mu^\vee = \mu_1^\vee + \mu_2^\vee$$

where $\mu_1^\vee$ is a coweight of T_{sc} and $\mu_2^\vee$ is a coweight of the center Z. Then

$$m\bar{\mu}^\vee = \bar{\mu}_1^\vee + \bar{\mu}_2^\vee \ .$$

For each ρ, σ let $b_{\rho,\sigma}$ be an m^{th} root of $a_{\rho,\sigma}$. Then

(1) $$\prod\nolimits_\tau b_{\sigma,\tau}^{\sigma\tau(\mu_1^\vee+\mu_2^\vee)}$$

and

(2) $$\prod\nolimits_\tau b_{\sigma,\tau}^{\sigma\tau(\bar{\mu}_1^\vee+\mu_2^\vee)}$$

are liftings of a_σ and $\overline{a}_\sigma$ to $I_{\text{sc}}(\overline{\mathbf{Q}}_\mathfrak{p}) \times Z(\overline{\mathbf{Q}}_\mathfrak{p})$. Moreover

(3) $$\left\{\prod\nolimits_\tau b_{\sigma,\tau}^{\rho\tau(\bar{\mu}_1^\vee+\mu_2^\vee)}\right\}\left\{\prod\nolimits_\tau b_{\sigma,\tau}^{\sigma\tau(\mu_1^\vee+\mu_2^\vee)}\right\}^{-1} = \left\{\prod\nolimits_\tau b_{\sigma,\tau}^{\sigma\tau\bar{\mu}_1^\vee}\right\}\left\{\prod\nolimits_\tau b_{\sigma,\tau}^{\sigma\tau\mu_1^\vee}\right\}^{-1}$$

is a lifting of d_σ to $I_{\text{dc}}(\overline{\mathbf{Q}}_\mathfrak{p})$.

Let $\{c_{\rho,\sigma,\tau}\}$ be the boundary of $\{b_{\rho,\sigma}\}$. The boundary of (1) is

$$\left\{\prod\nolimits_{\tau} b_{\rho,\tau}^{\rho\tau(\mu_1^\vee+\mu_2^\vee)}\right\}\left\{\prod\nolimits_{\tau} \rho b_{\sigma,\tau}^{\rho\sigma\tau(\mu_1^\vee+\mu_2^\vee)}\right\}\left\{\prod\nolimits_{\tau} b_{\rho\sigma,\tau}^{\rho\sigma,\tau(\mu_1^\vee+\mu_2^\vee)}\right\}^{-1} ,$$

which equals

$$b_{\rho,\sigma}^{m\nu^\vee}\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\rho\sigma\tau(\mu_1^\vee+\mu_2^\vee)}$$

It must of course lie in the center of $I_{\rm sc}\times Z$. There is a similar formula for the boundary of (2). Taking account of the Galois action on $I(\overline{\mathbf{Q}}_{\mathfrak{p}})$, one readily concludes that the boundary of (3) is

$$\rho_{\rho,\sigma}=\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\rho\sigma\tau(\bar\mu_1^\vee+\mu_2^\vee)}\right\}\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\rho\sigma\tau(\mu_1^\vee+\mu_2^\vee)}\right\}^{-1} .$$

We should perhaps remind ourselves that $\mu_1^\vee$ and $\bar\mu_1^\vee$ are coweights of different groups, namely, $T_{\rm sc}$ and $\overline{T}_{\rm sc}$. We may also write this boundary as

$$e_{\rho,\sigma}=\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\rho\sigma\tau\bar\mu_1^\vee}\right\}\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\rho\sigma\tau\mu_1^\vee}\right\}^{-1} .$$

Let $X_*(T_{\rm der})$ and $X_*(T_{\rm sc})$ be the lattices of coweights of $T_{\rm der}$ and $T_{\rm sc}$ and let

$$Y_*=X_*(T_{\rm der})/X_*(T_{\rm sc}) .$$

If $X^*(T_{\rm der})$ and $X^*(T_{\rm sc})$ are the lattices of weights and

$$Y^*=X^*(T_{\rm sc})/X^*(T_{\rm der}),$$

then

$$Y^*=\mathrm{Hom}(Y_*,\mathbf{Q}/\mathbf{Z})$$

and

$$C(\overline{\mathbf{Q}}_{\mathfrak{p}})=\mathrm{Hom}(Y^*,\overline{\mathbf{Q}}_{\mathfrak{p}}^\times) .$$

Replacing T by $\overline{T}$, we obtain $\overline{Y}^*$ and $\overline{Y}_*$, but

$$\overline{Y}_*\simeq Y_*$$

and

$$\overline{Y}^*\simeq Y^* .$$

The isomorphisms are canonical.

If we apply the local duality of Tate for finite $\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$ modules, we have only to check that the cup product of $\{e_{\rho,\sigma}\}$ with any element of

$$H^0(\mathfrak{G}(\overline{\mathbf{Q}}_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}),Y^*)$$

is trivial. An element of this group is represented by a $\lambda\in X^*(T_{\rm sc})$ with $\sigma\lambda-\lambda\in X^*(T_{\rm der})$ for all σ or by a $\bar\lambda\in X^*(\overline{T}_{\rm sc})$ with $\sigma\bar\lambda-\bar\lambda\in X^*(\overline{T}_{\rm der})$. The cup product is

$$f_{\rho,\sigma}=\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\langle\bar\lambda,\rho\sigma\tau\bar\mu_1^\vee\rangle}\right\}\left\{\prod\nolimits_{\tau} c_{\rho,\sigma,\tau}^{\langle\lambda,\rho\sigma\tau\mu_1^\vee\rangle}\right\}^{-1} .$$

To be definite, we take $\bar\lambda$ to be $h(\lambda)$ where h is an element of $I(\overline{\mathbf{Q}}_{\mathfrak{p}})$ taking T to $\overline{T}$ and

$$h(\lambda)(hth^{-1})=\lambda(t) .$$

We write

$$\langle \lambda, \rho\sigma\tau\mu_1^\vee \rangle = \langle \lambda - \rho\sigma\tau\lambda, \rho\sigma\tau\mu_1^\vee \rangle + \langle \lambda, \mu_1^\vee \rangle \ .$$

Since $\lambda - \rho\sigma\tau\lambda$ is a weight of T_{der}, it is equal to the restriction to T_{der} of a weight λ' of T, and

$$\langle \lambda - \rho\sigma\tau\lambda, \rho\sigma\tau\mu_1^\vee \rangle = \langle \lambda', m\mu^\vee \rangle - \langle \lambda', \mu_2^\vee \rangle \ .$$

Each $c_{\rho,\sigma,\tau}$ is an m^{th} root of unity and

$$c_{\rho,\sigma,\tau}^{\langle \lambda', m\mu^\vee \rangle} = 1 \ .$$

We treat $\langle \bar\lambda, \rho\sigma\tau\bar\mu_1^\vee \rangle$ in the same manner, writing

$$\bar\lambda - \rho\sigma\tau\bar\lambda = h(\lambda) - \rho\sigma\tau h(\lambda) = h(\lambda') + h(\lambda'')$$

with

$$\lambda'' = (1 - h^{-1}\rho\sigma\tau(h))\rho\sigma\tau(\lambda) \ .$$

λ'' is a weight of T_{ad}, and so

$$\langle h(\lambda') + h(\lambda''), \mu_2^\vee \rangle = \langle h(\lambda'), \mu_2^\vee \rangle = \langle \lambda', \mu_2^\vee \rangle \ .$$

We conclude that

$$f_{\rho,\sigma} = \{\textstyle\prod_\tau c_{\rho,\sigma,\tau}\}^{\langle \bar\lambda, \bar\mu_1^\vee \rangle - \langle \lambda, \mu_1^\vee \rangle} \ .$$

However,

$$\textstyle\prod_\tau c_{\rho,\sigma,\tau} = \rho\left(\prod_\tau b_{\sigma,\tau}\right)\left(\prod_\tau b_{\rho,\sigma\tau}\right)\left(\prod_\tau b_{\rho\sigma,\tau}\right)^{-1}\left(\prod_\tau b_{\rho,\sigma}\right)^{-1} \ .$$

We may replace $\sigma\tau$ by τ in the second factor. The first three terms then form a boundary and $\{f_{\rho,\sigma}\}$ is cohomologous to

$$b_{\rho,\sigma}^{-[k_{\mathfrak{p}}:\mathbf{Q}_{\mathfrak{p}}]\left(\langle \bar\lambda, \bar\mu_1^\vee \rangle - \langle \lambda, \mu_1^\vee \rangle\right)} \ .$$

However,

$$\langle \bar\lambda, \bar\mu_1^\vee \rangle - \langle \lambda, \mu_1^\vee \rangle = \langle \lambda, h^{-1}(\bar\mu_1^\vee) - \mu_1^\vee \rangle = \mathfrak{m}\langle \lambda, h^{-1}(\bar\mu^\vee) - \mu^\vee \rangle \ ,$$

and

$$\langle \lambda, h^{-1}(\bar\mu^\vee) - \mu^\vee \rangle$$

is integral because

$$h^{-1}(\bar\mu^\vee) - \mu^\vee$$

is a coweight of T_{sc}. Since

$$b_{\rho,\sigma}^m = a_{\rho,\sigma}$$

and

$$\left\{ a_{\rho,\sigma}^{[k_{\mathfrak{p}}:\mathbf{Q}_{\mathfrak{p}}]} \right\}$$

is trivial, the lemma is proved.

Suppose $k'_{\mathfrak{p}} \subseteq k_{\mathfrak{p}}$ are two finite Galois extensions of $\mathbf{Q}_{\mathfrak{p}}$ that split T. There is a homomorphism $w \to w'$ from $W_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ to $W_{k'_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ and we may pull back the cocycle $\{b_{w'}\}$, but the result may not be cohomologous to $\{b_w\}$.

Lemma A.3. *There is a cocycle c_w with values in the units of $k_{\mathfrak{p}}^{\times}$ such that $w \to b_w b_{w'}^{-1}$ is cohomologous in $T(k_{\mathfrak{p}})$ to $c_w^{\nu^\vee}$.*

We begin by recalling the manner in which the homomorphism from $W_{k'_{\mathfrak{p}}/\mathbf{Q}_p}$ to $W_{k_{\mathfrak{p}}/\mathbf{Q}_p}$ is defined [1]. Let $\{a_{\rho,\sigma}\}$ be a fundamental 2-cocycle for $k_{\mathfrak{p}}/\mathbf{Q}_p$. For each ρ' in $\mathfrak{G}(k'_{\mathfrak{p}}/\mathbf{Q}_p)$ choose a representative $\bar{\rho}$ in $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_p)$ A fundamental 2-cocycle for $k'_{\mathfrak{p}}/\mathbf{Q}_p$ is then

$$a'_{\rho',\sigma'} = N_{k_{\mathfrak{p}}/k_{\mathfrak{p}'}}(a_{\bar{\rho},\bar{\sigma}} a^{-1}_{\gamma,\bar{\rho}\bar{\sigma}}) \prod\nolimits_{\beta \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})} a_{\beta,\gamma}$$

with

$$\gamma = \bar{\rho}\bar{\sigma}\rho\bar{\sigma}^{-1} \ .$$

An expression for $a'_{\rho'\sigma'}$ which is more useful to us is ([1], p. 188)

$$\prod\nolimits_{\beta \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})} a_{\rho\beta,\bar{\sigma}} a_{\beta,\bar{\rho}} a^{-1}_{\beta,\bar{\rho}\bar{\sigma}}$$

if ρ is any lifting of ρ' to $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_p)$. We apply the coboundary relation to the first factor to obtain

$$\prod\nolimits_{\beta} \rho a_{\beta,\bar{\sigma}} a_{\rho,\beta\bar{\sigma}} a^{-1}_{\rho,\beta} a_{\beta,\bar{\rho}} a^{-1}_{\beta,\bar{\rho}\bar{\sigma}} \ .$$

An element of $W_{k_{\mathfrak{p}}/\mathbf{Q}_p}$ may be written as $x\rho\bar{\sigma}$ with $x \in k_{\mathfrak{p}}^{\times}$, $\rho \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})$, and $\sigma' \in \mathfrak{G}(k'_{\mathfrak{p}}/\mathbf{Q}_p)$. It is mapped to

$$w' = \left\{\prod\nolimits_{\beta \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})} \beta x\right\} \left\{\prod\nolimits_{\beta \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})} a_{\beta,\rho}\right\} \sigma' \ .$$

Then $b_{w'}$ is the product of

$$\left\{\prod\nolimits_{\beta} \beta x^{\nu^\vee}\right\} \left\{\prod\nolimits_{\beta} a^{\nu^\vee}_{\beta,\rho}\right\}$$

and

$$\prod\nolimits_{\alpha' \in \mathfrak{G}(k'_{\mathfrak{p}}/\mathbf{Q}_p)} \prod\nolimits_{\beta \in \mathfrak{G}(k_{\mathfrak{p}}/k'_{\mathfrak{p}})} \sigma a^{\sigma\alpha\mu^\vee}_{\beta,\alpha} \, a^{\sigma\alpha\mu^\vee}_{\sigma,\beta\bar{\alpha}} \, a^{-\sigma\alpha\mu^\vee}_{\sigma,\beta} \, a^{\sigma\alpha\mu^\vee}_{\beta,\bar{\sigma}} \, a^{-\sigma\alpha\mu^\vee}_{\beta,\bar{\sigma}\bar{\alpha}}$$

Here σ is the lifting $\rho\bar{\sigma}$ of σ' and α any lifting of α' to $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_p)$. If

$$c = \prod\nolimits_{\alpha'} \prod\nolimits_{\beta} a^{\alpha\mu^\vee}_{\beta,\bar{\alpha}}$$

the first term yields $\sigma(c)$ and the last c^{-1}. Since we are only interested in the cohomology class of $w \to b_{w'}$, we may drop these two terms. The second term yields

$$\prod\nolimits_{\tau \in \mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_p)} a^{\sigma\tau\mu^\vee}_{\sigma,\tau} \ .$$

The third and fourth yield

$$\left\{\prod\nolimits_{\beta} a^{-1}_{\alpha,\beta} a_{\alpha,\bar{\sigma}}\right\}^{\nu^\vee} \ .$$

Collecting the information at our disposal, we see that the lemma is valid with

$$c_w = \prod\nolimits_{\beta} x\beta(x)^{-1} a_{\beta,\rho} a^{-1}_{\sigma,\beta} a_{\beta,\bar{\sigma}} \ .$$

It must be verified that c_w is a unit, but that is a consequence of the next lemma applied to the trivial torus $T = \mathrm{GL}(1)$ and both $k_{\mathfrak{p}}$ and $k'_{\mathfrak{p}}$.

Lemma A.4. *If λ is a rational character of T over $\mathbf{Q}_\mathfrak{p}$ and v the image of w under the homomorphism $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} \to W_{\mathbf{Q}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} = \mathbf{Q}_\mathfrak{p}^\times$, then*

$$|\lambda(b_w)| = |v|^{\langle\lambda,\mu^\vee\rangle} .$$

If $w = x \times \sigma$, the left side is equal to

$$|x|^{\langle\lambda,\nu^\vee\rangle} \prod\nolimits_\tau |a_{\sigma,\tau}|^{\langle\lambda,\mu^\vee\rangle} = \left\{\prod\nolimits_\tau |x|^{\langle\lambda,\mu^\vee\rangle}\right\} \left\{\prod\nolimits_\tau |a_{\sigma,\tau}|^{\langle\lambda,\mu^\vee\rangle}\right\} .$$

Since

$$(\sigma\tau)^{-1}\sigma = \tau^{-1}\sigma^{-1}(a_{\sigma,\tau}) \times \tau^{-1}$$

in $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$, v is equal to

$$\left\{\prod\nolimits_\tau \tau(x)\right\} \left\{\prod\nolimits_\tau \tau^{-1}\sigma^{-1}(a_{\sigma,\tau})\right\} .$$

The lemma follows.

Let $\mathfrak{k}$ be the completion of the maximal unramified extension $\mathbf{Q}_\mathfrak{p}^{\mathrm{un}}$ of $\mathbf{Q}_\mathfrak{p}$. We have still to explain in detail how the element b of $G(\mathfrak{k})$ introduced in [13] is defined. Let D_1 be the image of $\mathrm{GL}(1)$ in T under $\nu^\vee$. It is an algebraic subgroup over $\mathbf{Q}_\mathfrak{p}$. Let $W^0_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ be the inverse image of the units in $\mathbf{Q}_\mathfrak{p}^\times$ under the homomorphism $W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} \to W_{\mathbf{Q}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} = \mathbf{Q}_\mathfrak{p}^\times$.

Lemma A.5. *If $k_\mathfrak{p}$ is sufficiently large, the cocycle $\{b_w\}$ is cohomologous to a cocycle $\{b'_w\}$ with the following two properties:*

(i) *the restriction of $\{b'_w\}$ to $W^0_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ takes values in $D_1(k_\mathfrak{p})$;*

(ii) *the image of b'_w in $T/D_1(k_\mathfrak{p})$ lies in $T/D_1(\mathbf{Q}_\mathfrak{p}^{\mathrm{un}})$.*

The second property is a consequence of the first, because the first implies that b'_w is invariant under $W^0_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ modulo D_1, and the image of $W^0_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$ in $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$ is the inertial group. To obtain a cocycle with the first property, we apply results from Chap. X, §7, of [18]. We may as well suppose that D_1 is trivial, and hence that $\{b_w\} = \{b_\sigma\}$ is a cocycle of $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$.

We have a diagram of fields

$$\begin{tikzcd}[column sep=small] & k_\mathfrak{p}^{\mathrm{un}} \arrow[dl, dash] \arrow[dr, dash] & \\ k_\mathfrak{p} \arrow[dr, dash] & & \mathbf{Q}_\mathfrak{p}^{\mathrm{un}} \arrow[dl, dash] \\ & \mathbf{Q}_\mathfrak{p} & \end{tikzcd}$$

and we may regard $\{b_\sigma\}$ as a cocycle of $\mathfrak{G}(k_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$. By the corollary of Prop. 11 of [18], its restriction to $\mathfrak{G}(k_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p}^{\mathrm{un}})$ is cohomologous to the trivial cocycle, and may therefore be assumed to be trivial, for we are willing to enlarge the field $k_\mathfrak{p}$. Thus $\{b_\sigma\}$ is the lifting to $\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})$ of a cocycle of

$$\mathfrak{G}(k_\mathfrak{p} \cap \mathbf{Q}_\mathfrak{p}^{\mathrm{un}}/\mathbf{Q}_\mathfrak{p})$$

and is trivial on the inertial group. Consequently, $\{b_w\}$ is trivial on $W^0_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}}$.

Our purposes demand a strengthening of the previous lemma.

Lemma A.6. *Suppose $k_{\mathfrak{p}}$ is sufficiently large and $l_{\mathfrak{p}}$ is the maximal unramified extension of $\mathbf{Q}_{\mathfrak{p}}$ in $k_{\mathfrak{p}}$. Then $\{b_w\}$ is cohomologous to a product $\{b'_w b''_w\}$, where $\{b'_w\}$ is the lifting of a cocycle of $W_{l_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ with values in $T(l_{\mathfrak{p}})$ and $\{b''_w\}$ is of the form $b''_w = d_w^{\nu^\vee}$ where $w \to d_w$ is a cocycle of $W_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ with values in the units of $k_{\mathfrak{p}}^\times$.*

We begin with an extension $k_{\mathfrak{p}}$ over which T splits, and let $l_{\mathfrak{p}}$ be the unramified extension of $\mathbf{Q}_{\mathfrak{p}}$ with

$$[l_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}] = [k_{\mathfrak{p}} : \mathbf{Q}_{\mathfrak{p}}] = n \ .$$

We shall prove the lemma not for $k_{\mathfrak{p}}$ but for $k'_{\mathfrak{p}}$, the composition of $k_{\mathfrak{p}}$ and $l_{\mathfrak{p}}$, in which $l_{\mathfrak{p}}$ is the maximal unramified subfield. Schematically we have:

$$\begin{tikzcd}[column sep=small] & k'_{\mathfrak{p}} \arrow[dl, dash] \arrow[dr, dash] & \\ l_{\mathfrak{p}} \arrow[dr, dash] & & k_{\mathfrak{p}} \arrow[dl, dash] \\ & k_{\mathfrak{p}} \cap l_{\mathfrak{p}} \arrow[d, dash] & \\ & \mathbf{Q}_{\mathfrak{p}} & \end{tikzcd}$$

If we appeal to Lemma A.3, we see that it is sufficient to take the cocycle $\{b_w\}$ associated to the Weil group $W_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$, and then prove that its lifting to $W_{k'_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ can be factored as $\{b'_w b''_w\}$.

The Galois group $\mathfrak{G}(l_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$ is cyclic of order $\mathfrak{n}$ and is generated by the Frobenius element which we shall, during the present proof, denote by Φ. We take a uniformizing parameter ϖ for $\mathbf{Q}_{\mathfrak{p}}$, which could be $\mathfrak{p}$ itself, and take the fundamental cocycle $c_{\rho,\sigma}$ of the extension $l_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}$ to be

$$c_{\Phi^i,\Phi^j} = \begin{cases} 1, & 0 \leqq i, j < n, i + j < n, \\ \varpi, & 0 \leqq i, j < n, i + j \geqq n \ . \end{cases}$$

We may also simplify matters by supposing that the lattice of coweights of T is the free $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$-module generated by $\mu^\vee$.

Suppose we can find a chain $\{c_\sigma\}$ of $\mathfrak{G}(l_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$ with values in $T(l_{\mathfrak{p}})$ and boundary $\{c_{\rho,\sigma}^{\nu^\vee}\}$. Then we may define a cocycle $\{d_w\}$ of $W_{l_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$ with values in $T(l_{\mathfrak{p}})$ by

$$d_w = x^{\nu^\vee} c_\sigma, \quad w = x \times \sigma \ .$$

Suppose in addition that if

$$c_\Phi = \prod\nolimits_{\tau \in \mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})} c_\Phi(\tau)^{\tau\mu^\vee}$$

then

$$\prod\nolimits_\tau |c_\Phi(\tau)| = |\varpi| \ .$$

I claim that if $\{b_{w'}\}$ and $\{d_{w'}\}$ denote the liftings of $\{b_w\}$ and $\{d_w\}$ to $W_{k'_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$, then $\{b_{w'} d_{w'}^{-1}\}$ is cohomologous to a cocycle $w' \to c_{w'}^{\nu^\vee}$ where $\{c_{w'}\}$ is a cocycle with values in the group of units of $k'_{\mathfrak{p}}$. To see this we pull back $\{a_\sigma\}$ and $\{c_\sigma\}$ to cochains $\{a_{\sigma'}\}$ and $\{c_{\sigma'}\}$ of $\mathfrak{G}(k'_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$. Their boundaries are obtained by pulling back $\{a_{\rho,\sigma}^{\nu^\vee}\}$ and $\{c_{\rho,\sigma}^{\nu^\vee}\}$ to $\{a_{\rho',\sigma'}^{\nu^\vee}\}$ and $\{c_{\rho',\sigma'}^{\nu^\vee}\}$. By local class-field theory, the two cocycles $\{a_{\rho',\sigma'}\}$ and $\{c_{\rho',\sigma'}\}$ are cohomologous, and

$$a_{\rho',\sigma'} = e_{\rho'}\rho'(e_{\sigma'})e_{\rho',\sigma'}^{-1} c_{\rho',\sigma'} \ .$$

Thus $\{a_{\sigma'}\}$ and $\{e_{\sigma'}^{\nu^\vee} c_{\sigma'}\}$ have the same boundary. Because of our simplifying assumption, $T(k'_{\mathfrak{p}})$ has no cohomology in dimension 1, and

$$a_{\sigma'} = f e_{\sigma'}^{\nu^\vee} c_{\sigma'} \sigma'(f^{-1}) \ .$$

In particular,

$$w' \to u_{w'} = b_{w'} f^{-1} d_{w'}^{-1} w'(f) = b_{w'} f^{-1} d_{w'}^{-1} \sigma'(f)$$

takes values in $D_1(k'_{\mathfrak{p}})$. With the simplifying assumption that the lattice of coweights is freely generated over $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$ by $\mu^\vee$,

$$D_1(k'_{\mathfrak{p}}) = \{x^{\nu^\vee} \mid x \in k'_{\mathfrak{p}}\} \ .$$

To establish this claim, I must show that if λ is a rational character of D_1, then

$$|\lambda(u_{w'})| = 1$$

for all w'. It will be enough to show this for a rational character of T defined over $\mathbf{Q}_{\mathfrak{p}}$. Clearly,

$$|\lambda(f^{-1}\sigma'(f))| = 1$$

and, by Lemma A.4,

$$|\lambda(b_{w'})| = |v|^{\langle \lambda, \mu^\vee \rangle}$$

if v is the image of w' in $\mathbf{Q}_{\mathfrak{p}}^\times$. We check that

$$|\lambda(d_{w'})| = |\lambda(d_w)| = |v|^{\langle \lambda, \mu^\vee \rangle}$$

if w' maps to w in $W_{k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}}}$.

This is easily seen to be so if $w = x \times 1$ with $x \in l_{\mathfrak{p}}^\times$, and so the point is to verify it when $w = 1 \times \Phi$. Then $v = \varpi$ and

$$|\lambda(d_w)| = \prod\nolimits_{\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})} |c_\Phi(\tau)|^{\langle \lambda, \tau\mu^\vee \rangle} \ .$$

Since $\langle \lambda, \tau\mu^\vee \rangle = \langle \lambda, \mu^\vee \rangle$, the right side equals

$$\left\{\prod |c_\Phi(\tau)|\right\}^{\langle \lambda, \mu^\vee \rangle}$$

and is, by assumption, $|\varpi|^{\langle \lambda, \mu^\vee \rangle}$.

To completely prove the lemma we must establish the existence of the chain $\{c_\sigma\}$. We first remark that there is an a in $k_{\mathfrak{p}}^\times$ for which

$$\varpi = \mathrm{Nm}_{k'_{\mathfrak{p}}/k_{\mathfrak{p}}}\, a \ .$$

Set

$$c_\Phi = \prod\nolimits_{\tau \in \mathfrak{G}(k'_{\mathfrak{p}}/l_{\mathfrak{p}})} \tau(a)^{\tau\mu^\vee}$$

and

$$c_{\Phi^i} = c_\Phi \Phi(c_\Phi) \dots \Phi^{i-1}(c_\Phi), \quad 0 \leqq i < n \ .$$

The chain $\{c_\sigma\}$ then takes values in $T(l_{\mathfrak{p}})$ and will have the boundary $\{c_{\rho,\sigma}^{\nu^\vee}\}$ if

$$\prod\nolimits_{i=0}^{n-1} \Phi^i(c_\Phi) = \varpi^{\nu^\vee} \ .$$

The product on the left is

$$\prod\nolimits_{\mathfrak{G}(k'_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})} \tau(a)^{\tau\mu^\vee} \ .$$

If we take the product over $\mathfrak{G}(k'_{\mathfrak{p}}/k_{\mathfrak{p}})$ and then over $\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})$, we obtain

$$\prod\nolimits_{\mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})} \varpi^{\tau\nu^\vee} = \varpi^{\nu^\vee} \ .$$

Finally,

$$\prod\nolimits_\tau |c_\Phi(\tau)| = \prod\nolimits_{\mathfrak{G}(k'_{\mathfrak{p}}/l_{\mathfrak{p}})} |\tau(a)| = |a|^{[k'_{\mathfrak{p}}:l_{\mathfrak{p}}]} = |a|^{[k'_{\mathfrak{p}}:k_{\mathfrak{p}}]} = |\varpi| \ .$$

In [13] I took $\mathfrak{k}$ to be $\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}$, but I should have taken it to be the closure of $\mathbf{Q}_{\mathfrak{p}}^{\mathrm{un}}$, for I am otherwise unable to prove the next lemma. I shall denote the Frobenius automorphism of $\mathfrak{k}$ by σ.

Lemma A.7. *If $d \in \mathfrak{k}$ and $|d| = 1$, then the equation*

$$d = c\sigma(c^{-1}), \quad c \in \mathfrak{k},$$

can be solved.

Since the map $c \to c\sigma(c^{-1})$ takes $\mathbf{Q}_\mathfrak{p}$ to 1, its image is closed, and so we must only verify that it is dense in the group of units of $\mathfrak{k}^\times$. If y is a unit, we can always find a unit x such that

$$y \equiv x\sigma(x^{-1})(\text{mod } \mathfrak{p}) \ .$$

Moreover, the equation

$$(1 + ap^k)\sigma(1 + ap^k)^{-1} \equiv (a - \sigma(a))p^k(\text{mod } \mathfrak{p}^k)$$

and the simple fact that

$$a - \sigma(a) \equiv b(\text{mod } \mathfrak{p})$$

can be solved with an integer a for any given integer b allows us to approximate any $y \equiv 1(\text{mod } \mathfrak{p})$. Since $D_1(\mathfrak{k}) \simeq \mathfrak{k}^\times$, the lemma may be applied to $D_1(\mathfrak{k})$ as well.

The element b introduced in [13] can now be defined. It lies in $I^0(\mathfrak{k})$. It is not uniquely determined, but the set

$$\{cb\sigma(c^{-1}) \mid c \in I^0(\mathfrak{k})\} \tag{4}$$

is. We start from a given T, a given, but sufficiently large, $k_\mathfrak{p}$, and a given fundamental class $\{a_{\rho,\sigma}\}$ for $k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}$ and the associated cocycle $\{b_w\}$. According to Lemma A.6, $\{b_w\}$ is cohomologous to $\{b'_w b''_w\}$, where b'_w and b''_w have the properties specified there. We have the usual homomorphisms

$$W_{k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} \to W_{\mathbf{Q}_\mathfrak{p}/\mathbf{Q}_\mathfrak{p}} \to \mathbf{Z} \ .$$

We choose a w_0 that maps to $1 \in \mathbf{Z}$ and set $b = b_{w_0}$. The previous lemma shows that the collection (4) is independent of the particular w_0 chosen.

The cocycle $\{b'_w\}$ is not unique; it might be replaced by

$$b'_w x w(x^{-1}) u_w$$

with $x \in T(k_\mathfrak{p})$, $u_w \in D_1(k_\mathfrak{p})$. However, x and u_w are not arbitrary. The absolute value $|\lambda(u_w)|$ must be 1 for any rational character of D_1, and the image of x in T/D_1 must lie in $T/D_1(l_\mathfrak{p})$. By Hilbert's Theorem 90, there is a v in $T(l_\mathfrak{p})$ such that

$$x \equiv v(\text{mod } D_1(k_\mathfrak{p})) \ .$$

Let $x = vz$. Then

$$b'_w x w(x^{-1}) u_w = b'_w (v w(v^{-1}))(z w(z^{-1}) u_w) \ .$$

We apply Lemma A.7 to $z w_0(z^{-1}) u_{w_0}$ to conclude that the set (4) remains the same. To change the fundamental class $\{a_{\rho,\sigma}\}$ does not affect the class of $\{b_w\}$, and hence does not affect (4). Finally, Lemmas A.2, A.3, and A.7 show that it is not affected by the choice of T and $k_\mathfrak{p}$.

T has been so taken that its image in I^0_{ad} is anisotropic. By the definition of a Frobenius pair, there is therefore a positive rational number r such that

$$|\lambda(\gamma)| = |\varpi|^{r\langle\lambda,\nu^\vee\rangle}$$

for all rational characters of T. The element ϖ is again a uniformizing parameter of $\mathbf{Q}_\mathfrak{p}$, and absolute values are taken in $\overline{\mathbf{Q}}_\mathfrak{p}$.

In addition to the group I^0 (or H^0) over $\mathbf{Q}$, I introduced in [13] a group $\overline{J}^0$ (or $\overline{G}^0$) over $\mathbf{Q}_\mathfrak{p}$. Its definition did not involve T, but it is easily seen that it is the connected group generated by T and the one-parameter root groups corresponding to roots α for which $\langle\alpha,\nu^\vee\rangle = 0$. Consequently, D_1 lies not only in the center of I^0 but also in the center of J^0 and the image of $\{a_\sigma\}$, or what is the same, of $\{b_w\}$ in I^0_{ad} or J^0_{ad}, yields elements of $H^1(\mathbf{Q}_\mathfrak{p}, I^0_{\mathrm{ad}})$ and of $H^1(\mathbf{Q}_\mathfrak{p}, J^0_{\mathrm{ad}})$ which can be used to twist I^0 and J^0, thereby obtaining new groups I and J. The twisting of I^0 can in fact be extended to a global twisting, but that will not be discussed yet.

Changing $\{a_\sigma\}$ or $\{b_w\}$ within its cohomology class has the usual effect on $J(\mathbf{Q}_\mathfrak{p})$ and on $I(\mathbf{Q}_\mathfrak{p})$. If $\{b_w\}$ is replaced by $xb_w w(x^{-1})$, then

$$J(\mathbf{Q}_\mathfrak{p}) \to \{xgx^{-1} \mid x \in J(\mathbf{Q}_\mathfrak{p})\}\ .$$

Since it is easy to keep track of such changes, I feel free to modify $\{b_w\}$ within its class, and indeed to replace $\{b_w\}$ by $\{b'_w\}$, where $\{b'_w\}$ satisfies the conditions of Lemma A.6, for $\{b''_w\}$ commutes with $J(\overline{\mathbf{Q}}_\mathfrak{p})$. Thus

$$J(\mathbf{Q}_\mathfrak{p}) = \{g \in J^0(\mathfrak{k}) \mid b\sigma(g)b^{-1} = g\}\ .$$

I claim that

$$J(\mathbf{Q}_\mathfrak{p}) = \{g \in G(\mathfrak{k}) \mid b\sigma(g)b^{-1} = g\}\ .$$

When proving this, I may take b to be defined by the cocycle $\{b'_w\}$ constructed in the proof of Lemma A.6. Choose $k_\mathfrak{p}$, $l_\mathfrak{p}$, and $k'_\mathfrak{p}$ as in the proof of that lemma with $[k_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}] = [l_\mathfrak{p} : \mathbf{Q}_\mathfrak{p}] = \mathfrak{n}$. Then

$$b\sigma(b)\ldots\sigma^{n-1}(b) = c = x^{\nu^\vee}, \quad |x| < 1\ .$$

Iterating the relation

$$g = b\sigma(g)b^{-1}$$

we obtain

$$g = c\sigma^n(g)c^{-1}$$

and

$$g = c^{-1}\sigma^{-n}(g)c\ ,$$

or, more generally,

$$g = c^m\sigma^{mn}(g)c^{-m}$$

and

$$g = c^{-m}\sigma^{-mn}(g)c^m$$

for every positive integer m. We may choose a sequence m_i so that $\{\sigma^{m_i n}(g)\}$ and $\{\sigma^{-m_i n}(g)\}$ converge. Then $\{c^{-m_i}gc^{\mathfrak{m}_i}\}$ and $\{c^{m_i}gc^{-m_i}\}$ converge. Since G is a matrix group, we see, by passing to a larger field with respect to which T can be diagonalized and taking the form of c into account, that this is possible only if c commutes with g. Since the connected component of the centralizer of any positive power of c in G is J^0, the centralizer of c in G is connected and equals J^0 [23].

Although the groups of this paper are simple enought that the existence of the global twisting of I^0 demanded by the formalism of [13] is clear, it turns out nonetheless to be useful to say a few words about the construction of the cocycle defining this global twisting.

Recall that we started with a Cartan subgroup T of I^0 defined over $\mathbf{Q}$ such that T_{ad}, the image of T in I^0_{ad}, is anisotropic at ∞ and $\mathfrak{p}$. If $\bar\mu^\vee$ is the coweight of T_{ad} obtained by composing $\mu^\vee$ with $T \to T_{\mathrm{ad}}$, then the twisting at $\mathfrak{p}$ is given by the cocycle

$$\alpha_\mathfrak{p} = \{\overline{a}_\sigma\}$$

with

$$\overline{a}_\sigma = \textstyle\prod_{\tau\in\mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})} a_{\sigma,\tau}^{\sigma\tau\bar\mu^\vee}, \quad a \in \mathfrak{G}(k_\mathfrak{p}/\mathbf{Q}_\mathfrak{p})\ .$$

We define a twisting cocycle at ∞ in exactly the same fashion

$$\alpha_\infty = \{\overline{a}_\sigma\}$$

with

$$\overline{a}_\sigma = \textstyle\prod_{\tau\in\mathfrak{G}(\mathbf{C}/\mathbf{R})} a_{\sigma,\tau}^{-\sigma\tau\bar\mu^\vee}, \quad \sigma \in \mathfrak{G}(\mathbf{C}/\mathbf{R})\ .$$

We have changed the sign in the exponent, but that has no effect on the resultant cohomology class.

Lemma A.8. *If I is the group over $\mathbf{R}$ obtained by twisting I^0 by α_∞, then $I_{\mathrm{ad}}(\mathbf{R})$ is compact.*

Let $\mathfrak{G}(\mathbf{C}/\mathbf{R}) = \{1, \sigma\}$. We may take the fundamental cocycle to be

$$a_{1,1} = a_{1,\sigma} = a_{\sigma,1} = 1, \quad a_{\sigma,\sigma} = -1 \ .$$

Then

$$\overline{a}_1 = 1 \quad \overline{a}_\sigma = (-1)^{\bar{\mu}^\vee} \ .$$

Every root of T in I^0 or in I is imaginary. All we must do is verify that the roots of T in I are compact. Let β be a root of T in I^0 and choose root vectors $X_\beta, X_{-\beta}$ with

$$[X_\beta, X_{-\beta}] = H_\beta,$$

where

$$\lambda(H_\beta) = \langle \lambda, \beta^\vee \rangle \ .$$

Since $\sigma(\beta) = -\beta$,

$$\sigma(X_\beta) = cX_{-\beta} \quad \sigma(X_{-\beta}) = dX_\beta \ .$$

It is easily seen that c must be real and that $cd = 1$. Examining the two forms of $\mathrm{SL}(2)$ over $\mathbf{R}$, one sees that $c > 0$ if and only if β is compact. On the other hand, β is compact if and only if $\langle \beta, \mu^\vee \rangle = 0$. If β is not compact, then $\langle \beta, \mu^\vee \rangle = +1$. When we twist by $\alpha_{\mathfrak{p}}$, the new action on X_β is

$$X_\beta \to (-1)^{\langle \beta, \mu^\vee \rangle} cX_{-\beta} \ .$$

Thus c is replaced by $(-1)^{\langle \beta, \mu^\vee \rangle} c$, and compact roots remain compact while noncompact roots become compact.

The global twisting of I is by an element α of $H^1(\mathbf{Q}, T_{\mathrm{ad}})$ whose image in $H^1(\mathbf{R}, T_{\mathrm{ad}})$ is α_∞, in $H^1(\mathbf{Q}_{\mathfrak{p}}, T_{\mathrm{ad}})$ is $\alpha_{\mathfrak{p}}$, and whose image in $H^1(\mathbf{Q}_l, T_{\mathrm{ad}})$, $l \neq \mathfrak{p}$, is trivial. Its existence follows from standard results in Galois cohomology, which we will now describe. We first state the appropriate lemma formally.

Lemma A.9. *Let T be a torus over $\mathbf{Q}$ and $\mu^\vee$ a coweight of T. Suppose T is anisotropic at ∞ and $\mathfrak{p}$. Then*

$$\alpha_\infty = \left\{ \textstyle\prod_{\tau \in \mathfrak{G}(\mathbf{C}/\mathbf{R})} a_{\sigma,\tau}^{-\sigma\tau\mu^\vee} \right\},$$

$$\alpha_{\mathfrak{p}} = \left\{ \textstyle\prod_{\tau \in \mathfrak{G}(k_{\mathfrak{p}}/\mathbf{Q}_{\mathfrak{p}})} a_{\sigma,\tau}^{\sigma\tau\mu^\vee} \right\}$$

represent cohomology classes in $H^1(\mathbf{R}, T)$ and $H^1(\mathbf{Q}_{\mathfrak{p}}, T)$, respectively. There is an element α in $H^1(\mathbf{Q}, T)$ whose local components are trivial everywhere except at ∞ and $\mathfrak{p}$, where they equal α_∞ and $\alpha_{\mathfrak{p}}$.

Let K be a finite Galois extension of $\mathbf{Q}$ that splits T. Then

$$T(K) = X_*(T) \otimes K^\times,$$

$$T(\mathbf{A}_K) = X_*(T) \otimes I_K \ .$$

If $C_K = I_K/K^\times$ is the idèle-class group, set

$$T_C = X_*(T) \otimes C_K \ .$$

The exact sequence

$$1 \to T(K) \to T(\mathbf{A}_K) \to T_C \to 1$$

leads to

$$H^1(\mathfrak{G}(K/\mathbf{Q}), T(K)) \to H^1(\mathfrak{G}(K/\mathbf{Q}), T(\mathbf{A}_K)) \to H^1(\mathfrak{G}(K/\mathbf{Q}), T_C) \ .$$

Let β be the element of the middle group with component α_∞ at ∞, $\alpha_\mathfrak{p}$ at $\mathfrak{p}$, and 1 elsewhere. All we have to do is verify that its image in the final group is trivial. The middle group is

$$\oplus_v H^1(\mathfrak{G}(K_v/\mathbf{Q}_v), T(K_v))$$

and we must verify that the product of the images $\bar\alpha_\infty$, $\bar\alpha_\mathfrak{p}$ of α_∞ and $\alpha_\mathfrak{p}$ in $H^1(\mathfrak{G}(K/\mathbf{Q}), T_C)$ is trivial.

The Tate-Nakayama isomorphisms are

$$H^i(\mathfrak{G}(K_v/\mathbf{Q}_v), X_*(T) \simeq H^{i+2}(\mathfrak{G}(K_v/\mathbf{Q}_v), T(K_v))$$

and

$$H^i(\mathfrak{G}(K/\mathbf{Q}), X_*(T) \simeq H^{i+2}(\mathfrak{G}(K/\mathbf{Q}), T_C)$$

There is a diagram

$$\begin{array}{ccc} H^i(\mathfrak{G}(K_v/\mathbf{Q}_v), X_*(T) & \simeq & H^{i+2}(\mathfrak{G}(K_v/\mathbf{Q}_v), T(K_v)) \\ \Big\downarrow & & \Big\downarrow \\ H^i(\mathfrak{G}(K/\mathbf{Q}_v), X_*(T) & \simeq & H^{i+2}(\mathfrak{G}(K/\mathbf{Q}_v), T_C) \end{array}$$

The left vertical arrow is corestriction. The right vertical is the composition of

$$H^j(\mathfrak{G}_v, T(K_v)) \simeq H^j(\mathfrak{G}, T(K\otimes\mathbf{Q}_v)) \to H^j(\mathfrak{G}, T(\mathbf{A}_K)) \to H^j(\mathfrak{G}, T_c)$$

with $j = i+2$, $\mathfrak{G}_v = \mathfrak{G}(K_v/\mathbf{Q}_v)$, $\mathfrak{G} = \mathfrak{G}(K/\mathbf{Q})$. The place v of $\mathbf{Q}$ has been extended in some way, no matter which, to K. It can be verified without too much difficulty, although it is more than a mere formality, that the diagram is commutative. One examines the proof of the Tate-Nakayama theorems and recalls at the same time the relation between the local and the global fundamental classes. I forego the details, although I have no reference to furnish the reader.

Suppose in particular that $i = -1$. The corestriction takes the element of $H^{-1}((K_v/\mathbf{Q}_v), X_*(T))$ represented by λ with $\mathrm{Nm}_{K_v/\mathbf{Q}_v}\lambda = 0$ to the element of $H^{-1}(\mathfrak{G}(K/\mathbf{Q}), V_*(T))$ represented by the same λ. Therefore $\bar\alpha_\infty$ corresponds to the element of $H^{-1}(\mathfrak{G}(K/\mathbf{Q}), X_*(T))$ represented by $-\mu^\vee$ and $\bar\alpha_\mathfrak{p}$ to the element represented by $\mu^\vee$, and $\bar\alpha_\infty\cdot\bar\alpha_\mathfrak{p}$ is trivial.

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