Shimura varieties and the Selberg trace formula *

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This paper is a report on work in progress rather than a description of theorems which have attained their final form. The results I shall describe are part of an attempt to continue to higher dimensions the study of the relation between the Hasse-Weil zeta-functions of Shimura varieties and the Euler products associated to automorphic forms, which was initiated by Eichler, and extensively developed by Shimura for the varieties of dimension one bearing his name. The method used has its origins in an idea of Sato, which was exploited by Ihara for the Shimura varieties associated to $GL(2)$.

To define a Shimura variety S one needs an algebraic group G over $\mathbf{Q}$ together with certain supplementary data [2]; the set $S(\mathbf{C})$ of complex points on the variety then appears as a double coset space

$$S(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K. \tag{1}$$

Here $\mathbf{A} = \mathbf{R} \times \mathbf{A}_f$ is the ring of adèles. K_∞ is a subgroup of $G(\mathbf{R})$ for which $G(\mathbf{R})/K_\infty$ is a bounded symmetric domain and K is a compact open subgroup of $G(\mathbf{A}_f)$ which intervenes in the definition of S. The suggestion of Sato, as modified by Ihara, is simple to describe. The Euler products associated to automorphic forms are Dirichlet series defined by group theoretical data, the traces of the Hecke operators, and if the Selberg trace formula is used to obtain an explicit formula for the coefficients, this formula will involve only the internal structure of the group G. On the other hand the zeta-function is defined in terms of the number of points of S with coefficients from various fields, extensions of the residue field κ at a prime $\mathfrak{p}$ of the field over which S is defined; so these numbers too must be computed in terms of G if a comparison is to be made. To find the number of points N_n in $S(\kappa_n)$, κ_n the extension of κ of degree n, it is in principle sufficient to describe the set $S(\bar\kappa)$, $\bar\kappa$ the algebraic closure of κ, together with the action of the Frobenius $\Phi_\mathfrak{p}$ over κ on it, for N_n would be the number of fixed points of $\Phi_\mathfrak{p}^n$. In analogy with $S(\mathbf{C})$ we might expect to describe $S(\bar\kappa)$ in terms of double coset spaces. Once armed with explicit expressions for the coefficients of the two Dirichlet series, we could set out to prove they are equal.

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* This material was presented at the 1975 U.S.-Japan Seminar on Number Theory in Ann Arbor, Michigan. The paper appeared in Can. J. Math. XXIX (1977)

If we follow this suggestion, we might divide the problem into three parts.

(a) If a reductive group H over $\mathbf{Q}$ is given, together with an automorphic form π on it, or rather what has recently been called an *automorphic representation*, as well as a representation r of the associate group ${}^LH$, then one may introduce an L-function $L(s,\pi,r)$, which is defined as an Euler product. We want to show that the Hasse-Weil zeta-function of S is equal to a product

$$\Pi L(s-a,\pi,r)^m.$$

It is first necessary to decide which H,π, and r should intervene in this product, and with what exponent m and what translation a each L-function should appear. Then it is necessary to derive an explicit expression for its coefficients, or rather those of its logarithm, by means of the trace formula.

(b) If E is the field over which S is defined, in those cases in which Shimura's conjecture is verified, and if one is content, as seems appropriate for the moment, to verify the equality of the factors appearing in the two Euler products for almost all p, then for all primes $\mathfrak{p}$ of E dividing such a p one must describe $S(\bar{\kappa})$ together with the action of the Frobenius on it. If S is not complete, it is also necessary to analyze its structure near infinity, but this is a complication which it is best to avoid for now.

(c) The final step is to compare the results obtained from (a) and (b). Here the Harish-Chandra transform on reductive groups over local fields comes into play. Since our knowledge of this transform is pretty meagre, a real obstacle has presented itself. It does not seem possible to treat (c), or (a), with any generality until it has been removed.

In [5] I described in conjectural form a solution of (b), indicating that, by applying the techniques of modern algebraic geometry, it was possible to verify this conjecture in sufficiently many cases to make it worthwhile to carry out (a) and (c), which are a matter of harmonic analysis rather than of algebraic geometry. In this paper, I want to describe the solution of (a) and (c) when G is defined by the multiplicative group of a quaternion algebra over a totally real field. Here (a) is easy, as is the harmonic analysis required by (c), but the comparison is difficult enough to be worth describing. It may also be worthwhile to see the form taken by the solution of (b) described in [5] for some specific groups. However, the methods mentioned there only yield a solution to (b), which is real rather than conjectural, when the algebra is totally indefinite; so that this is the only case at present in which the method can yield a full solution of the problem.

I observe that there are two oversights in [5]. One is mathematical and will be rectified below. The other is more regrettable. It has been pointed out to me that I might have said more about the origins of the problems posed and that, in particular, I might have drawn attention to Section 5 of Ihara's notes in [4]. It would be unfortunate if this omission caused any misunderstanding for I am certainly much indebted to Ihara's ideas, but rather through [3] and indirectly [1], than through [4].

I begin by recapitulating, in as simple a manner as possible, some formal aspects of the associate groups. Suppose G is the group over $\mathbf{Q}$ defined by the multiplicative group of a quaternion algebra D over F, a totally real field of finite degree over $\mathbf{Q}$. Then $G(\mathbf{Q}) = D^{\times}$ and for almost all primes p,

$$G(\mathbf{Q}_p) \simeq \prod_{\mathfrak{p}|p} GL(2, F_{\mathfrak{p}}). \tag{2}$$

If $F' \subseteq \bar{\mathbf{Q}}$ is a Galois extension of $\mathbf{Q}$ containing F then $\mathfrak{G}(F'/\mathbf{Q})$ acts on the set I of imbeddings of F into $\bar{\mathbf{Q}} \subseteq \mathbf{C}$. Set

$$^{L}G^{0} = \prod_{\iota \in I} GL(2, \mathbf{C}).$$

We let $\mathfrak{G}(F'/\mathbf{Q})$ act on $^{L}G^{0}$ by

$$\sigma : (g_{\iota}) \to (g_{\sigma^{-1}\iota})$$

and then form the semi-direct product

$$^{L}G =^{L} G^{0} \times \mathfrak{G}(F'/\mathbf{Q}).$$

^{L}G will be referred to as the *associate group*.

Suppose p is a prime at which (2) is satisfied and $G(\mathbf{Z}_p) \subseteq G(\mathbf{Q}_p)$ corresponds to

$$\prod_{\mathfrak{p}|p} GL(2, O_{\mathfrak{p}}).$$

$O_{\mathfrak{p}}$ is the ring of integers in $F_{\mathfrak{p}}$. Suppose moreover that $F'/\mathbf{Q}_p$ is unramified. Fix a Frobenius element Φ_p in $\mathfrak{G}(F/\mathbf{Q})$. The associate group has been so defined that the Hecke algebra of $G(\mathbf{Q}_p)$ with respect to $G(\mathbf{Z}_p)$ is isomorphic to the algebra of functions on $^{L}G^{0} \times \Phi_p$ obtained by taking linear combinations over $\mathbf{C}$ of restrictions of the characters of finite-dimensional complex analytic representations of $^{L}G^{0} \times \Phi_p^{\mathbf{Z}}$. Suppose the function φ corresponds to f_{φ} in the Hecke algebra. To each irreducible admissible representation π_p of $G(\mathbf{Q}_p)$ which contains the trivial representation of $G(\mathbf{Z}_p)$ there corresponds a semi-simple element $g(\pi_p)$ in $^{L}G^{0} \times \Phi_p$ which satisfies

$$\varphi(g(\pi_p)) = \mathrm{trace}\ \ \pi_p(f_{\varphi}).$$

The elements of I may also be regarded as imbeddings of F into $\mathbf{R}$. Suppose $J \subseteq I$ is the set of such imbeddings which split D. If $K \subset G(\mathbf{A}_f)$ then the canonical model of the Shimura variety $S = S_K$ corresponding to K is defined over the fixed field E of $\mathfrak{G}(F'/E)$, consisting of those elements of $\mathfrak{G}(F'/\mathbf{Q})$ which leave the set J invariant. Suppose r_ι is the representation of ${}^LG^0$ obtained by projection on the ι-th factor. We extend the representation $\otimes_{\iota \in J} r_\iota$ to ${}^LG^0 \times \mathfrak{G}(F'/E)$ by letting $\sigma \in \mathfrak{G}(F'/E)$ send the vector $\otimes \nu_\iota$ to $\otimes \nu_{\sigma^{-1}\iota}$. We then induce to ${}^LG$ to obtain the representation r which will play a role in this paper. It is one of the simplifying features of the groups G we are considering that the zeta-functions of its Shimura varieties can be expressed entirely in terms of L-functions attached to automorphic forms on G itself; no auxiliary groups H need be introduced. The simplification is to some extent spurious. It would not occur if we attempted to study the zeta-function of the connected components, or even of the parts irreducible over E.

The zeta-function $Z(s, S_K)$ is an Euler product

$$\prod_p Z_p(S, S_K)$$

with

$$Z_p(S, S_k) = \prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}(s, S_K),$$

where $\mathfrak{p}$ denotes a prime of E. If $\pi = \pi_\infty \otimes \pi_f = \pi_\infty \otimes \pi_2 \otimes \pi_3 \dots$ is an automorphic representation almost all factors of the Euler product $L(s, \pi, r)$ are defined by

$$L_p(S, \pi, r) = \frac{1}{\det(1 - \frac{r(g(\pi_p))}{p^s})}.$$

Suppose Z_0 is an open subgroup of the centre $Z(\mathbf{A})$ of $G(\mathbf{A})$ containing $Z(\mathbf{R})$ and with $Z_0 \cap G(\mathbf{A}_f) \subseteq K$. The representation of $G(\mathbf{A})$ on $L^2(Z_0 G(\mathbf{Q})\backslash G(\mathbf{A}))$ is a direct sum of irreducible representations. Let Π be the set of representations occurring. If $\pi \in \Pi$ then π_∞ is a representation of

$$G(\mathbf{R}) \simeq \left(\prod_{\iota \in J} GL(2, \mathbf{R}) \right) \times \left(\prod_{\iota \notin J} G'(\mathbf{R}) \right)$$

where $G'(\mathbf{R})$ is the multiplicative group of the Hamilton quaternion algebra. We may therefore write $\pi_\infty = \otimes \pi_\iota$. If $\iota \notin J$ we set $m(\pi_\iota)$ equal to 1 or 0 according as π_ι is or is not trivial; if $\iota \in J$ then $m(\pi_\iota)$

is to be -1 if π_ι is trivial, 1 if it is the first member of the discrete series, and 0 otherwise. Observe that if π_ι is trivial for one $i \in J$ then it is trivial for all i. Let

$$m(\pi_\infty) = \prod_{\iota \in I} m(\pi_\iota).$$

Finally, $m(\pi, K)$ is the product of $m(\pi_\infty)$ with $m(\pi_f, K)$, the multiplicity with which the trivial representation of K occurs in π_f; it is 0 for all but finitely many π. We shall show that if the solution of (b) is granted, then

$$Z_p(s, S_K) = \prod_{\pi \in \Pi} L_p(s - \frac{q}{2}, \pi, r)^{m(\pi,K)} \tag{3}$$

for almost all p. Here q is the number of elements in the set J.

Let A_f^p be the ring of adèles which are 0 at infinity and at p. For almost all $p, K = K^p K_p$ with $K^p \subseteq G(\mathbf{A}_f^p)$ and with K_p equal to $G(\mathbf{Z}_p)$. We calculate the logarithm of the right side of (3) for such p. We write $\pi_f = \pi_f^p \otimes \pi_p$ and let $m(\pi_f^p, K^p)$ be the multiplicity with which the trivial representation of K^p occurs in π_f^p. The integer $m(\pi, K)$ is 0 unless π_p contains the trivial representation of K_p, when it equals $m(\pi_\infty) m(\pi_f^p, K^p)$. There is a smooth function f_∞, with support which is compact modulo $Z(\mathbf{R})$, so that if π_∞ is trivial on $Z(\mathbf{R})$ and

$$\pi_\infty(f_\infty) = \int_{Z(\mathbf{R})\backslash G(\mathbf{R})} f_\infty(g) \pi_\infty(g) dg$$

then

$$m(\pi_\infty) = \text{trace } \pi_\infty(f_\infty).$$

Otherwise $m(\pi_\infty)$ is 0. If f^p is the characteristic function of K^p divided by its measure then

$$m(\pi_f^p, K^p) = \text{ trace } \pi_f^p(f^p).$$

If π_p contains the trivial representation of K_p then

$$\log L_p(s - \frac{q}{2}, \pi, r) = \sum_{n=1}^{\infty} \frac{1}{np^{ns}} p^{nq/2} \text{ trace } r(g(\pi_p))^n.$$

According to our introductory remarks there is for each n an element $f_p^{(n)}$ of the Hecke algebra such that

$$p^{nq/2} \text{ trace}\, r(g(\pi_p))^n = \text{ trace}\, \pi_p(f_p^{(n)}).$$

If π_p does not contain the trivial representation of K_p the right hand side is 0. Thus the coefficient of $1/np^{ns}$ in

$$\sum_{\pi} m(\pi, K) \log L_p(s - \frac{q}{2}, \pi, r)$$

is

$$\sum_{\pi \in \Pi} \text{trace } \pi(f^{(n)}) \tag{4}$$

if

$$f^{(n)}(g) = f^{(n)}(g_\infty, g^p, g_p) = f_\infty(g_\infty) f^p(g^p) f_p^{(n)}(g_p)$$

for $g \in G(\mathbf{A})$.

The Selberg trace formula immediately yields a formula for (4). Exploiting our knowledge of the harmonic analysis on $G(\mathbf{R})$ we obtain the sum over conjugacy classes $\{\gamma\}$ in the set $G(\mathbf{Q}) \cap \mathbf{Z}_0 \backslash G(\mathbf{Q})$ which are elliptic at infinity of the product

$$\frac{(\text{meas } Z(\mathbf{R})\backslash Z_0)(\text{meas } Z_0 G_\gamma(\mathbf{Q})\backslash G_\gamma(\mathbf{A}))}{\text{meas } Z(\mathbf{R})\backslash G'_\gamma(\mathbf{R})} \tag{5}$$

and

$$\left\{ \int_{G_\gamma(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f^p(g^{-1}\gamma g) dg \right\} \left\{ \epsilon(\gamma) \int_{G_\gamma(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} f_p^{(n)}(g^{-1}\gamma g) dg \right\}. \tag{6}$$

G_γ is the centralizer of γ and G'_γ is the twisted form of G_γ over $\mathbf{R}$ for which $Z(\mathbf{R})\backslash G'_\gamma(\mathbf{R})$ is compact. $\epsilon(\gamma)$ is 1 if γ is not central, otherwise it is $(-1)^q$ if q is the number of elements in J. The important term in (5) and (6) is the orbital integral of $f_p^{(n)}$, which defines its Harish-Chandra transform.

The restriction of the representation r to ${}^L G^0 \times \Phi_p^{\mathbf{Z}}$ decomposes into a direct sum indexed by the double cosets

$$\mathfrak{G}(F'\backslash E)\backslash \mathfrak{G}(F'\backslash \mathbf{Q})\backslash \Phi_p^{\mathbf{Z}}.$$

The function $f_p^{(n)}$ is then also a sum over these double cosets, which may be identified with the primes $\mathfrak{p}$ of E dividing p; so we write

$$f_p^{(n)} = \sum f_{\mathfrak{p}}^{(n)}.$$

Moreover

$$\log Z_p(s, S_K) = \sum_{\mathfrak{p}|p} \log Z_p(s, S_k)$$

if $Z_{\mathfrak{p}}(s, S_K)$ is the zeta-function of S_K over the residue field κ of E at $\mathfrak{p}$. What we want to show is that the result of substituting $f_{\mathfrak{p}}^{(n)}$ for $f_p^{(n)}$ in the expression obtained for (3) is equal to $\log Z_{\mathfrak{p}}(s, S_K)$.

According to the conjectural solution of (b) described in [5] the set $S_K(\overline{\kappa})$ can be decomposed into a disjoint union of sets, each of which can be represented in the form

$$Y_K = H(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X \backslash K^p.$$

The groups H appearing here are not those mentioned earlier. The parameter set for these double cosets is a little difficult to describe, as is the structure of X. However, K^p acts on $G(\mathbf{A}_f^p)$ and hence on $G(\mathbf{A}_f^p) \times X$ to the right. Moreover H is a group over $\mathbf{Q}$; $H(\mathbf{A}_f^p)$ comes provided with an injection into $G(\mathbf{A}_f^p)$; and $H(\mathbf{Q}_p)$ acts on X. This yields an action of $H(\mathbf{Q})$ on $G(\mathbf{A}_f^p) \times X$. Each Y_K is left invariant by the Frobenius, whose action is obtained from a map $\Phi : X \to X$ which commutes with the action of $H(\mathbf{Q}_p)$.

The coefficient of $1/np^{ns}$ in $\log \mathbf{Z}_{\mathfrak{p}}(s, S_K)$ is the number of fixed points of $\Phi_{\mathfrak{p}}^n$ on $S_K(\overline{\kappa})$, which we can represent as the sum over Y of the number $N_n(Y_K)$ of fixed points in Y_K. It is convenient to express the formula for $N_n(Y_K)$ in terms of a group $\overline{G}(\mathbf{Q}_p)$, containing $H(\mathbf{Q}_p)$, which also acts on X.

If $n > 0$ and $\chi \in X$ set

$$T_\chi^n = \{g \in \overline{G}(\mathbf{Q}_p) \mid \Phi^n \chi = g\chi\}$$

and let δ_χ^n be the characteristic function of T_χ^n. If $\{\chi_\iota\}$ is a set of representatives for the orbits of $\overline{G}(\mathbf{Q}_p)$ in X set

$$\varphi^{(n)}(\gamma) = \sum_\iota \frac{1}{\text{meas } \overline{G}_\iota} \int_{\overline{G}_\gamma(\mathbf{Q}_p)\backslash \overline{G}(\mathbf{Q}_p)} \delta_{\chi_\iota}^n (h^{-1}\gamma h) dh.$$

Here $\overline{G}_\iota$, is the stabilizer of χ_ι in $\overline{G}(\mathbf{Q}_p)$. I hasten to reassure the reader that the structure of X is such that the right hand side makes sense. A formal set-theoretic argument yields $N_n(Y_K)$ as a sum over conjugacy classes in $H(\mathbf{Q}) \cap Z_0 \backslash H(\mathbf{Q})$, for H will contain Z, of

$$m(Z_0 \cap G(\mathbf{A}_f))(mZ_0 H_\gamma(\mathbf{Q})\backslash H_\gamma(\mathbf{A}_f))\varphi^{(n)}(\gamma) \int_{H_\gamma(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f^p(g^{-1}\gamma g) dg. \tag{7}$$

Moreover every conjugacy class $\{\gamma\}$ in $H(\mathbf{Q})$ will determine a conjugacy class $\{\gamma'\}$ in $G(\mathbf{Q})$. In $G(\mathbf{A}_f^p)$, γ and γ' will be conjugate, $\gamma' = h^{-1}\gamma h$, $G_{\gamma'}(\mathbf{A}_f^p)$ will be $h^{-1}H_\gamma(\mathbf{A}_f^p)h$, and the group H_γ will be a twisted form of $G_{\gamma'}$ over $\mathbf{Q}$. Comparing (7) with (5) and (6), we see that, if the properties of Tamagawa numbers are taken into account, the problem is reduced to a comparison of the functions $\varphi^{(n)}(\gamma)$ with the orbital integrals.

The orbital integrals can without much difficulty be computed explicitly for the groups under consideration; so the burden of the problem is to compute the functions $\varphi^{(n)}(\gamma)$. The sets Y_K are indexed by equivalence classes of pairs of Frobenius type [5]. To define these one fixes an imbedding of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_p$ which defines the prime $\mathfrak{p}$. The orbits of Φ_p in I correspond to primes $\mathfrak{q}$ of F dividing p. Let $b_\mathfrak{q}$ be the number of points in the orbit $\mathfrak{q}$ which lie in J. Working out the definition of [5] one finds that there is one Y_K corresponding to each triple consisting of: a totally imaginary quadratic extension F' of F; a non-empty subset S of the primes dividing p at which F' splits, which is such that $b_\mathfrak{q}$ is even if F' splits at $\mathfrak{q}$ and $\mathfrak{q}$ is not in S; and, for each $\mathfrak{q} \in S$, a pair of non-negative integers $k_\mathfrak{q}, k'_\mathfrak{q}$ with $k_\mathfrak{q} \neq k'_\mathfrak{q}, k_\mathfrak{q} + k'_\mathfrak{q} = b_\mathfrak{q}$. There is also one additional Y_K. To be more precise, $k_\mathfrak{q}, k'_\mathfrak{q}$ are associated to the two primes of F' dividing $\mathfrak{q}$, and two triples which can be obtained from each other by an automorphism over F are not to be distinguished. For the Y_K defined by a triple, H is the group associated to the multiplicative group of F'; for the remaining Y_K it is the group associated to the multiplicative group of a quaternion algebra over F, ramified everywhere at infinity, at the primes at which D ramifies, and at the primes for which $b_\mathfrak{q}$ is odd, but nowhere else.

The space X is a product

$$X = \prod X_\mathfrak{q}$$

and so is the auxiliary group $\overline{G}(\mathbf{Q}_p)$

$$\overline{G}(\mathbf{Q}_p) = \prod \overline{G}_\mathfrak{q}(\mathbf{Q}_p).$$

Fix a point ι in the orbit $\mathfrak{q}$; then $X_\mathfrak{q}$ consists of sequences $\{L_i \mid i \in \mathbf{Z}\}$ of lattices in the two-dimensional space over the maximal unramified extension $\mathbf{Q}_p^{un}$ of $\mathbf{Q}_p$ with the following properties:

(i) $L_{i-1} = L_i$ if $\Phi^{1-i}\iota \notin J$;

(ii) $L_i \underset{\neq}{\supset} L_{i-1} \underset{\neq}{\supset} pL_i$ if $\Phi^{1-i}\iota \in J$;

(iii) if $m = m_\mathfrak{q}$ is the number of elements in the orbit and σ is the Frobenius on $\mathbf{Q}_p^{un}$ then $dL_{j+m} = \sigma^{-m}L_j$.

Here d is a two-by-two matrix. For a Y_K parametrized by a triple,

$$d = \begin{bmatrix} p^{k_\mathfrak{q}} & 0 \\ 0 & p^{k'_\mathfrak{q}} \end{bmatrix}$$

if $\mathfrak{q} \in S$ and

$$d = \begin{bmatrix} p^{b_\mathfrak{q}/2} & 0 \\ 0 & p^{b_\mathfrak{q}/2} \end{bmatrix}$$

if $\mathfrak{q} \notin S$ but F' splits at $\mathfrak{q}$. In the first case $\overline{G}_{\mathfrak{q}}(\mathbf{Q}_p)$ is the group of diagonal matrices over $F_{\mathfrak{q}}$. In the second it is $GL(2, F_{\mathfrak{q}})$. If F' does not split at $\mathfrak{q}$ or if we are dealing with the extra Y_K then d is some fixed element of $GL(2, F_{\mathfrak{q}})$ whose order is $b_{\mathfrak{q}}$. The group $\overline{G}_{\mathfrak{q}}(\mathbf{Q}_p)$ is $GL(2, F_{\mathfrak{q}})$ or the multiplicative group of a quaternion algebra over $F_{\mathfrak{q}}$ according as $b_{\mathfrak{q}}$ is even or odd. There is one condition omitted from the description of X in [5]. If χ in X corresponds to $g \in G(\mathfrak{k})$ then it must also be demanded that

$$|\lambda(b^{\sigma} g)| = |p^{\langle \lambda, \mu^{\wedge} \rangle}||\lambda(g)|$$

if λ is a rational character of G over k. I also observe that all references to projective limits over K should be expunged from [5].

Each point of $X_{\mathfrak{q}}$ defines an infinite path in the tree associated by Bruhat-Tits to the group $GL(2, \mathbf{Q}_p^{un})$ and the computation of the functions $\varphi^{(n)}(\gamma)$ becomes thereby a combinatorial exercise, not however a completely trivial one.

†My first attempts to master the techniques described above were presented in a report in the Antwerp Summer School on Modular Functions (Springer Lecture Notes 349). That report is complicated, partly because I was dealing with unfamiliar material, and partly because a great deal of extra discussion is required in order to deal with the cusps or with the ramified primes. A case, perhaps the only one, which is transparent in its representation-theoretic or combinatorial aspects is that of the present paper when $F = \mathbf{Q}$. It is to be hoped that a suitable occasion will be found to discuss it. My ambition now is to publish in this Journal, the Editors willing, a detailed treatment of the results for an arbitrary totally real field. Several papers will be required. The central one, *On the zeta-functions of some simple Shimura varieties*, will treat the combinatorics and representation theory carefully. However, since I want to show at the same time how it happens that L-functions associated to groups other than G itself must be used to obtain the zeta-function of S_K, this paper has to be preceded by another, written jointly with J.-P. Labesse and entitled *L-indistinguishability for $SL(2)$*. If we are ever to be able to treat zeta-functions for general Shimura varieties, the notion of L-indistinguishability will have to be defined for all reductive groups. Some simple definitions and lemmas to this purpose, employed even in the study of $SL(2)$, will be collected in *Stable Conjugacy: Definitions and Lemmas*. Finally one paper will have to be devoted to a detailed statement of the conjectures of [5], and perhaps another to establishing them in the cases accessible to present techniques.

† Added at the request of the referee

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