Finite Models for Percolation†

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Dedicated to the memory of Lawrence Corwin

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1. Introduction. The notion of percolation as a subject of concern to mathematicians and theoretical physicists will not be familiar to many readers of this collection; the notion of finite models of percolation is familiar to almost no-one. Therefore, although the basic questions leading to the introduction of such models are discussed at length in [Con], we shall review them, but very briefly. We shall also repeat, for convenience, the definitions of the finite models appearing in [L]. The principal purpose of the paper is, however, to describe some numerical results that support the hope that further mathematical investigation of the finite models will be of some value in the study of the three basic questions of §2. Almost all of the computations were performed, with independent programming, by both of the authors. The first author, less adept with the computer, is grateful to Dennis Hejhal for some very useful advice; both are grateful for conversations with Yvan Saint-Aubin.

2. Questions. Consider in the plane the graph whose sites are the points of $\mathbb{Z}^2$ and whose bonds are given by joining nearest neighbors in it. Thus there is a bond joining $(0,0)$ to each of the four points, $(0,\pm1)$ and $(\pm1,0)$, but no bond joining it to other points. Percolation by sites on this graph is the study of a particular class of events with respect to a simple probability measure on the set of functions on the sites with values in the set $\{0,1\}$, thus on the set

$$X = \prod_{\mathbb{Z}^2}\{0,1\}.$$

Such a function is referred to as a configuration, and is usually represented graphically, as in [Con], by depicting the open sites as small disks, the closed sites as small circles with blank interiors, and by drawing all bonds between nearest neighbors.

Fix a p with $0 \le p \le 1$. The measure on X is the product over $\mathbb{Z}^2$ of the measures on $\{0,1\}$ that assign the measure p to $\{1\}$ and $1-p$ to $\{0\}$. If $x \in X$ is a given configuration then the site $s \in \mathbb{Z}^2$ is said to be open with respect to x if $x(s) = 1$. An event of particular interest is the crossing of a square of side n, where n is a positive integer.

The configuration x is said to admit a horizontal crossing of the square given as

$$S_n = \{s = (a,b) | 1 \le a, b \le n\}$$

if it is possible to start at a site on the side $a = 1$ open for x and to pass from one site to a nearest neighbor, all sites touched being open for x, and all lying within the square S_n, finally reaching the side $a = n$. The probability $\pi_h^{(n)}$ of the event that a crossing exists is defined and depends on p:

$$\pi_h^{(n)} = \pi_h^{(n)}(p).$$

In general the collection of all points in $\mathbb{Z}^2$ that can be reached in the above manner from the same starting point while remaining inside a given set S is referred to as a cluster in S.

A basic fact of percolation is that there exists a critical probability $p = p_c$ such that

$$\lim_{n\to\infty} \pi_h^{(n)}(p) = 1$$

for $p > p_c$, and

$$\lim_{n\to\infty} \pi_h^{(n)}(p) = 0$$

for $p < p_c$.

The first question is whether

$$\lim_{n\to\infty} \pi_h^{(n)}(p)$$

exists for $p = p_c$. If it exists, it is neither 0 nor 1.

For n very large, the graph of $\pi_h^{(n)}(p)$ as a function of p on the interval $[0,1]$ will rise sharply from 0 to 1 around the point p_c. Thus A_n, the derivative of this function with respect to p at $p = p_c$, can be expected to be large. The second question is whether there exists a constant ν such that the limit of

$$\frac{A_n}{n^\nu}$$

exists as $n \to \infty$ and is neither 0 nor ∞.

In order to simplify the discussion, we have considered here a specific model of percolation in two dimensions. There are many such models, and the third question is whether the exponent ν is the same for all of them.

The evidence suggests that the response to all questions is affirmative, but none has been answered rigorously. There are, however, heuristic arguments of *renormalization* that provide at least a tentative theoretical approach to the questions. The use of *crossing probabilities* allows renormalization to be discussed in very elementary terms.

3. Crossing probabilities. The probability of a horizontal crossing is the first example of a crossing probability. They are defined more generally by a simple closed curve C and arcs

$$\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2, \ldots, \gamma_n, \delta_1, \delta_2, \ldots, \delta_n$$

on C. The numbers m and n are integers, positive or zero. If A is a large real number let C' be the dilation of C by the factor A with respect to some fixed but irrelevant point. Define α_i', β_i', γ_j', and δ_j' in a similar fashion.

The notion that a configuration x for the model of percolation by sites on $\mathbb{Z}^2$ admits a crossing within C' from α_i' to β_i' for each $1 \le i \le m$ but none from γ_j' to δ_j' for $1 \le j \le n$ can easily be made precise ([Con]), and the numerical evidence suggests that the limit as A approaches ∞ of the probability of this event exists for $p = p_c$. (It will exist for other values of p as well, but $p = p_c$ is the sole value of p with which we are presently concerned.) We use the notation

$$\pi(E) = \lim_{A\to\infty} \pi(C', \alpha_1', \ldots, \delta_n'), \tag{3.1}$$

thinking of E as the event defined by the data C, α_i, β_i, γ_j and δ_j. Assuming that the limits (3.1) exist, they are referred to as crossing probabilities.

The hypothesis of conformal invariance examined in [Con] implies that all crossing probabilities can be obtained from those attached to a square of side 1. The arcs α_i, β_i, γ_j, and δ_j then appear as arcs on this square. If l is an integer, divide each side of the square into l intervals of length $1/l$. Call these the basic intervals, and denote the set of basic intervals by $\mathfrak{A}_l$. It is clear that simple approximation arguments will permit all the probabilities $\pi(E)$ for the square to be calculated as limits, in which l is allowed to grow, of those associated to events defined by arcs α_i, β_i, γ_j and δ_j that are unions of basic intervals.

We fix l and consider those events defined by unions of intervals in $\mathfrak{A}_l$. Let y be a function on $\mathfrak{A}_l \times \mathfrak{A}_l$ with values in $\{0,1\}$. Each such function y defines an event E_y in which there is a crossing between two basic intervals α and β, or more informally they are joined, if and only if $y(\alpha,\beta) = 1$. Moreover all probabilities $\pi(E)$ for events defined by unions of intervals in $\mathfrak{A}_l$ can be calculated in terms of the probabilites $\pi(E_y)$; so we consider these alone.

4. Basic events. The events E_y are basic events, and it will be useful to know which have positive probability and which probability 0. The first observation is that if α and β are neighboring intervals in $\mathfrak{A}_l$ then α and β are joined with probability 1, so that the basic event E_y has positive probability only if $y(\alpha,\beta) = 1$ whenever α and β are neighbors. Although we have been somewhat imprecise about our conventions, we can suppose that the probability that α and β are joined is taken to be the limit for large n of the probability that two linear collections of points in $\mathbb{Z}^2$ are joined: either the two collections

$$\{(a,b),(a+1,b),\ldots,(a+n,b)\} \quad \text{and} \quad \{(a+n+1,b),(a+n+2,b),\ldots,(a+2n,b)\};$$

or

$$\{(a,b),(a+1,b),\dots,(a+n,b)\} \quad \text{and} \quad \{(a+n,b+1),(a+n,b+2)\dots,(a+n,b+n)\}.$$

The probability is independent of the choice of a and b. The first two collections are obtained by dilating two contiguous intervals on the same horizontal side of the square of side 1 by the factor n, and the other two collections are obtained by dilating two intervals on this square that meet at a corner. Other positions of the initial intervals are possible, but it clearly suffices to treat these two.

In [K] (pp.174-178) it is proved that at $p = p_c$ the probability that within the square of side n about $(a+n,b)$ punctured at $(a+n,b)$ there is a cluster that completely surrounds this point approaches 1 as $n \to \infty$. Such a cluster clearly joins the two collections.

Kesten's proof proceeds by showing that the probability of a cluster within a rectangular annulus of fixed shape but arbitrary size is bounded below, and then representing the punctured square as the union of such annuli. One can therefore expect that the cluster joining two adjacent intervals joins them by a path that starts from a point on one interval (or rather on its dilation, but the principles are clearer if the square of side 1 is kept fixed and the lattice scaled down so that its mesh becomes $1/n$) that is very close to their meeting point and then passes to a nearby point on the other.

As a result, there is very little chance that such a cluster meets and fuses with a large cluster joining one of the two intervals to a third. Thus it is of no *dynamic* significance. For the purposes of the finite models to be treated in this paper, it is best to suppress these joins. If y' is such that $y'(\alpha,\beta)=1$ whenever α and β are adjacent, we associate to y' the function y that takes the same values as y' on pairs of intervals that are not adjacent, but the value 0 on adjacent intervals. Clearly y' can be recovered immediately from y. We introduce a probability on the set of these y by

$$\eta(y) = \eta_l(y) = \pi(E_{y'}).$$

There are other conditions on the functions y, two that are trivial and were implicit from the beginning:

(1) for all $\alpha \in \mathfrak{A}_l$, the value of $y(\alpha,\alpha)$ is 1;

(2) for all α and β the two values $y(\alpha,\beta)$ and $y(\beta,\alpha)$ are equal;

as well as the one just imposed:

(3) if α and β are adjacent then $y(\alpha,\beta)=0$.

There is a fourth, geometric condition. A sequence of four distinct intervals α, β, γ, and δ will be said to be cyclic if the intervals are met in this order upon traversing the square in or the other of the two possible senses. Without troubling ourselves about the proof, which would presumably require a good deal of attention to detail, we assert that if, for a given configuration x, there is a cluster within the square joining the intervals α and γ and the intervals β and δ of a cyclic sequence then it joins them all. Thus it is natural to impose the following, fourth condition on the function y:

(4) if α, β, γ, δ is a cyclic sequence in $\mathfrak{A}_l$ and if $y(\alpha,\gamma)=1$ and $y(\beta,\delta)=1$ but α and β are not adjacent then $y(\alpha,\beta)=1$.

The set of functions y satisfying these four conditions will be denoted $\mathfrak{Y}_l$, and we shall be concerned with probability measures on it. One such measure η_l has already been defined.

We suppressed the joins between adjacent intervals because they have a purely local significance with no global consequences. There are, however, some such joins that are more important than others, and it is useful to put them back in. If $y \in \mathfrak{Y}_l$ is given we let $\bar{y}$ be the function that is equal to y on all pairs that are not adjacent, but that is equal to 1 on an adjacent pair (α,β) if there is a cyclic sequence α, β, γ, δ for which $y(\alpha,\gamma)=1$ and $y(\beta,\delta)=1$. The use of $\bar{y}$ in the construction of §6 is essential.

There is, of course, considerable redundance in the use of all three functions y, y', and $\bar{y}$. The function y' is introduced only to define η; the function y because it is derived so simply from y' and, having the fewest joins, is the most easily represented graphically; but it is the function $\bar{y}$ that appears directly in the dynamical construction.

5. Coarsening. For each positive integer we have introduced the set $\mathfrak{Y}_l$. Suppose that k divides l. If y' is a function in $\mathfrak{Y}_l$, we define a coarsening $y = \Gamma^l_k(y')$ of it. Each interval α' in $\mathfrak{A}_l$ is contained in a unique interval α

of $\mathfrak{A}_k$. Set $y(\alpha,\beta)=1$ if and only if α and β are not adjacent and α contains an interval α' and β an interval β' such that $y'(\alpha',\beta')=1$. Otherwise $y(\alpha,\beta)=0$. It is easily verified ([L]) that $y\in\mathfrak{A}_k$ if $y'\in\mathfrak{A}_l$. The map Γ_k^l also acts on measures.

6. Heaping. Let n and l be two positive integers. We introduce as well a mapping $\Phi=\Phi_l^{(n)}$ from the n^2-fold product of $\mathfrak{Y}_l$ with itself to $\mathfrak{Y}_{nl}$. The composition $\Theta=\Theta_l=\Theta_l^{(n)}=\Gamma_l^{nl}\circ\Phi_l^{(n)}$ is then a map from the n^2-fold product of $\mathfrak{Y}_l$ with itself to $\mathfrak{Y}_l$, and therefore defines a map from the set of all probability measures on $\mathfrak{Y}_l$ to the same set. It is the map on measures that is of primary interest, and we also denote it by Θ. Our purpose in this paper is to discuss $\Theta_1^{(2)}$ and $\Theta_2^{(2)}$. Observe that Θ is intended to be a renormalization!

To heap we first take the square of side 1 and divide it into n^2 smaller squares of side $1/n$. The perimeter of each of these smaller squares is then divided into $4l$ intervals of length $1/nl$. Each of the small squares is labeled by a pair of indices (i,j), $1\le i,j\le n$ and the set $\mathfrak{A}_{i,j}$ of intervals obtained from one small square is to be identified with $\mathfrak{A}_l$. An element of the n^2-fold product of $\mathfrak{Y}_l$ with itself is a collection of functions $y_{i,j}$, one for each square. Thus we must attach to the collection $\{y_{i,j}\}$ a function y in $\mathfrak{Y}_{nl}$. We recall from [L] the construction of y, referring to that paper for the elementary proof that the result does in fact lie in $\mathfrak{Y}_{nl}$.

We distinguish between those intervals that lie in the interior of the large square and those that lie on its boundary, referring to them as interior and exterior intervals respectively. The collection of exterior intervals is to be identified with $\mathfrak{A}_{nl}$. If α and β are two exterior intervals then an admissible path from α to β is a sequence

$$(i_0,j_0),(i_2,j_2),\ldots,(i_{2r},j_{2r}),$$

and a sequence

$$\alpha_{-1},\alpha_1,\ldots,\alpha_{2r+1}$$

that satisfy the following conditions.

(1) For $0\le k<r$, let $i=i_{2k+1}$, $i'=i_{2k}$, and $i''=i_{2k+2}$. Define j, j', and j'' in a similar manner. The interval α_i lies in $\mathfrak{A}_{i',j'}$ and in $\mathfrak{A}_{i'',j''}$. Moreover $\alpha_{-1}=\alpha\in\mathfrak{A}_{i_0.j_0}$ and $\alpha_{2r+1}=\beta\in A_{i_{2r},j_{2r}}$.

(2) For $0\le k<r$ the interval $\alpha_{2k+1,2k+1}$ is inner.

(3) For $0\le k\le r$ the value of $\bar{y}_{i,j}(\alpha_{2k-1},\alpha_{2k+1})$ is 1.

The function y is defined on pairs α, β that are not adjacent by the condition that $y(\alpha,\beta)=1$ if and only if there is an admissible path from α to β.

7. A goal. As observed the mapping Θ_l acts on probability measures on $\mathfrak{Y}_l$. It appears useful to consider not the full set of such measures but the subset Π_l introduced in [L] and defined by the FKG inequality.

The set $\mathfrak{Y}_l$ is clearly ordered. The element y_1 is greater than or equal to y_2 if it takes the value 1 on every pair at which y_2 takes the value 1. Thus there is a notion of monotonicity on the set of functions on $\mathfrak{Y}_l$. The function f is monotone increasing if $f(y_1)\ge f(y_2)$ whenever $y_1\ge y_2$. The measure π satisfies the FKG inequality if for any two monotone increasing functions f and g,

$$\int fg\,d\pi\ge\int f\,d\pi\int g\,d\pi.$$

As verified in [L], the map Θ_l takes the set Π_l of measures satisfying this inequality to itself, and it seems prudent to consider only measures in Π_l. The measure η_l lies in this set ([K,§4.1]).

We refer to the sets Π_l provided with the maps $\Theta_l=\Theta^{(n)}=\Theta_l^{(n)}$ as finite models for percolation. For $n=1$ the map Θ_l is of little interest. We observe only that the distinction between interior and exterior intervals implies that all points of Π_l are fixed by $\Theta^{(1)}$ and that for general n a fixed point ν of $\Theta^{(n)}$ is not necessarily concentrated on the set of functions y for which $y(\alpha,\beta)=1$ and $y(\beta,\gamma)=1$ implies that $y(\alpha,\gamma)=1$. This would defeat the goal about to be described.

Fix $n > 1$. An ultimate purpose is to show that there is a "natural" sequence of measures $\nu_l = \nu_l^{(n)} \in \Pi_l$, one for each integer l, each fixed by $\Theta_l^{(n)}$, and such that

$$\lim_{l\to\infty} \Gamma_k^l(\nu_l) = \eta_k \tag{7.1}$$

for each positive integer k. The conclusion would then be that η_k was associated to the finite models and the renormalization of them, but not to a particular lattice model of percolation, and therefore universal.

The goal of this paper is far more modest. It is to establish numerically the existence of $\nu_l^{(n)}$ for $n = 2$ and $l = 1, 2$, and to discover whether the numerical solutions offer any positive evidence for (7.1). The constraint $n = 2$ will be in force from now on, and will not be mentioned explicitly again.

8. Level one. If $l = 1$ the intervals of $\mathfrak{A}_l$ are the four sides of the square, and as observed in [L] there are four elements in $\mathfrak{Y}_1$: the element y_0 with only the trivial joins (of an interval with itself); the element y_h that joins the two opposing vertical sides; the element y_v that joins the top to the bottom; and the element y_{hv} that joins both pairs of opposing sides. Thus y_h and y_{hv} are the only two elements that admit a horizontal crossing.

A measure is defined by its values π_0, π_h, π_v, and π_{hv} on these four elements. Since

$$\pi_0 + \pi_h + \pi_v + \pi_{hv} = 1,$$

the set of such measures forms a simplex of dimension three. For the moment we consider the action of Θ on the full set of probability measures, examining the effect of the FKG inequality later.

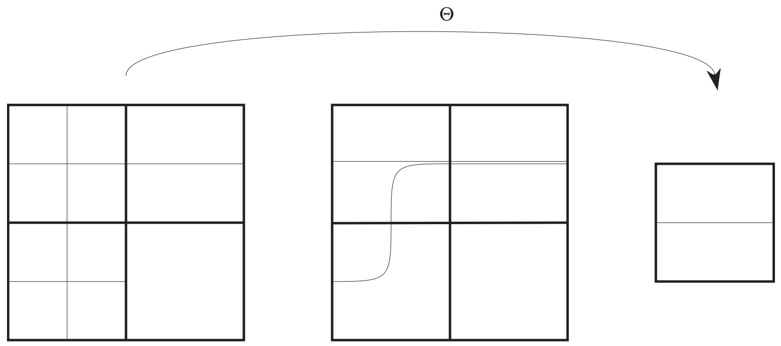


Diagram A

Suppose $\Theta(\pi) = \pi'$, and consider $\sigma'_h = \pi'_h + \pi'_{hv}$. It is the sum over all collections $(y_{11}, y_{12}, y_{21}, y_{22})$ whose image y under Θ admits a horizontal crossing of

$$\pi(y_{11})\pi(y_{12})\pi(y_{21})\pi(y_{22}).$$

Consider, for example, the configuration of Diagram A in which $y_{11} = y_{21} = y_{hv}$, $y_{12} = y_h$, and $y_{22} = y_0$. The mapping Θ sends this configuration to $y = y_h$. There are several admissible paths that yield the horizontal join between the two vertical sides. Two are shown schematically in Diagram A. One of them occurs entirely within the top row of two squares. A little reflection makes it clear that whenever (for $l = 1$ and $n = 2$) there is a join between vertical sides then it can be effected by an admissible path that remains within one of the two rows. As a consequence,

$$\sigma'_h = 2\sigma_h^2 - \sigma_h^4, \qquad \sigma_h = \pi_h + \pi_{hv}. \tag{8.1}$$

Therefore at a fixed point π of Θ, one has

$$\sigma_h = 2\sigma_h^2 - \sigma_h^4, \tag{8.2}$$

as well as the analogous equation

$$\sigma_v = 2\sigma_v^2 - \sigma_v^4, \qquad \sigma_v = \pi_v + \pi_{hv}. \tag{8.3}$$

These equations have several solutions. First of all, the equation (8.2) alone has the four solutions: $\sigma_h = 1$, $\sigma_h = 0$, $\sigma_h = (-1 + \sqrt{5})/2 = \delta \sim .618$, and $\sigma_h = (-1 - \sqrt{5})/2$. The last root is negative and thus can be discarded.

If $\sigma_h = \sigma_v = 1$ then $\pi_0 = \pi_h = \pi_v = 0$ and $\pi_{hv} = 1$. This is a fixed point of Θ in which every join occurs with probability 1 except those between adjacent intervals, and these we had deliberately excluded. In some sense it corresponds to percolation for $p > p_c$. In the same way, $\sigma_h = \sigma_v = 0$ yields a fixed point with $\pi_h = \pi_v = \pi_{hv} = 0$ and $\pi_0 = 1$, and corresponds to $p < p_c$. If $\sigma_h = 1$ and $\sigma_v = 0$ then $\pi_0 = \pi_v = \pi_{hv} = 0$ and $\pi_h = 1$. This is again a fixed point, but is extremely asymmetric. There is a horizontal crossing with probability 1 but a vertical crossing with probability 0. Such extreme asymmetry does not normally occur in percolation.

If $\sigma_h = \delta$ and $\sigma_v = 1$ then $\pi_0 = \pi_h = 0$ and $\pi_{hv} = \delta$, $\pi_v = 1 - \delta$ is also a fixed point. So is that given by $\sigma_h = \delta$ and $\sigma_v = 0$, and defined by $\pi_0 = 1 - \delta$, $\pi_v = \pi_{hv} = 0$, and $\pi_h = \delta$. Neither these points nor those obtained from them by interchanging horizontal and vertical directions appear to be of particular interest.

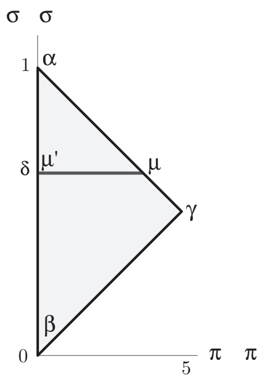


Diagram B

This leaves the possibility that $\sigma_h = \sigma_v = \delta$ to be examined. The point π then necessarily lies in the set of symmetric measures defined by $\pi_h = \pi_v$ that form the two-dimensional simplex of Diagram B. The three vertices are α with $\alpha_{hv} = 1$, β with $\beta_0 = 1$, and γ with $\gamma_h = \gamma_v = .5$. The interval I is that defined by

$$\pi_h + \pi_{hv} = \pi_v + \pi_{hv} = \delta.$$

It is mapped to itself by Θ, and we verify that it contains a single fixed point.

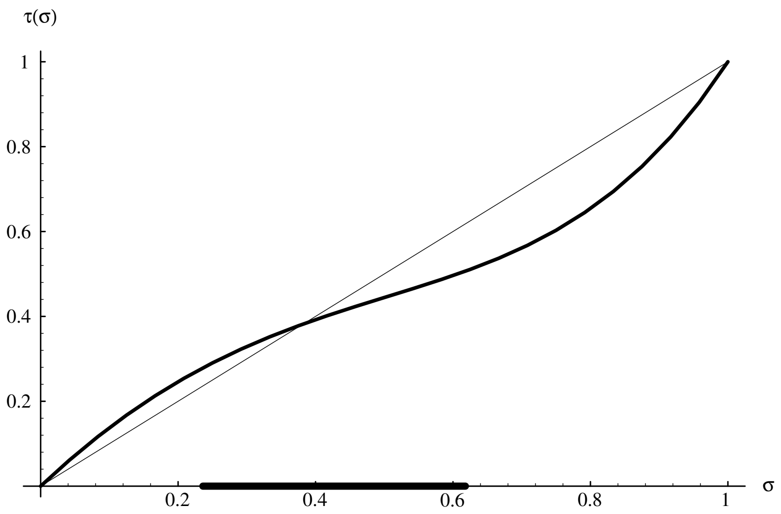


Diagram C

In Diagram C the restriction θ of Θ to the interval is drawn, as well as the identity map. It is easy to verify that θ does indeed have the form shown. On the interval the measure π is determined by π_{hv} and for brevity we have denoted this parameter by σ. Thus $2\delta - 1 \le \sigma \le \delta$. At μ the value of σ is $2\delta - 1$ and at μ' it is δ.

We shall verify that

$$\theta(\sigma) = \sigma(4\delta^2(1-\sigma) + \sigma^3). \tag{8.4}$$

As a consequence

$$\theta'(\sigma) = 4(\sigma^3 + \delta^2(1-2\sigma))$$
$$\theta''(\sigma) = 12\sigma^3 - 8\delta^2$$
$$\theta'''(\sigma) = 36\sigma$$

We may also write

$$\theta'(\sigma) = 4\delta^2(\sigma - 2\delta + 1) + 4(\sigma + 2\delta)(\sigma - \delta)^2,$$

so that θ' is positive on I. We infer also that θ' is convex. Its value at δ is $4\delta^2(1-\delta) \sim .584 < 1$ and at $2\delta - 1$ is $4(1-\delta)^2(4\delta - 1) \sim .859$. Thus its value throughout the interval I lies between 0 and 1.

At μ' the value of $\theta(\sigma)/\sigma$ is

$$4\delta^2(1-\delta) + \delta^3 = \delta^2(4 - 3\delta) = (1-\delta)(4-3\delta) \sim .820 < 1,$$

and at μ it is

$$8\delta^2(1-\delta) + (1-2\delta)^3 \sim 1.180 > 1.$$

Thus Diagram C is qualitatively correct.

To verify (8.4), we observe that if the heaping of four states yields y_{hv}, then there is a horizontal crossing in the two upper squares or the two lower squares, and a vertical crossing in the two left squares or the two right squares. The probability of, say an upper horizontal crossing and a left vertical crossing is clearly $\sigma\delta^2$, since the element in the upper left corner must be y_{hv}. The probability that there is both an upper and a lower horizontal crossing together with a left vertical crossing is, for similar reasons, $\delta^2\sigma^2$. The probability that there are two horizontal and two vertical crossings is σ^4. Thus the probability $\theta(\sigma)$ of y_{hv} upon heaping is

$$4\sigma\delta^2 - 4\delta^2\sigma^2 + \sigma^4 = \sigma(4\delta^2(1-\sigma) + \sigma^3).$$

A function f on $\mathfrak{Y}_1$ is monotone if

$$f(y_{hv}) \geq \max\{f(y_h), f(y_v)\} \geq \min\{f(y_h), f(y_v)\} \geq f(y_0).$$

It is therefore easily verified that a measure π satisfies the FKG inequality if and only if

$$\pi_{hv} \geq (\pi_h + \pi_{hv})(\pi_v + \pi_{hv}).$$

A measure π on the interval I associated to the parameter σ satisfies the inequality if and only if $\delta^2 \leq \sigma \leq \delta$. Since, as observed in [L], Θ takes Π_1 to itself, this interval is taken into itself by θ. As θ contracts the entire interval to the fixed point ν_1, this point lies in Π_1. It is in fact easily verified that the fixed point is $\sigma = \delta^2$, so that it lies on the boundary of Π_1.

Thus the fixed point $\nu = \nu_1$ of Θ is given by

$$\nu_0 = 2\delta^2 - \delta, \qquad \nu_h = \delta - \delta^2, \qquad \nu_v = \delta - \delta^2, \qquad \nu_{hv} = \delta^2,$$

or numerically

$$\nu_0 \sim .146, \qquad \nu_h \sim .236, \qquad \nu_v \sim .236, \qquad \nu_{hv} \sim .382. \tag{8.5}$$

As a first test of the equation (7.1) we can compare $\nu_1 = \Gamma_1^1(\nu_1)$ with the measure $\eta = \eta_1$ that is given by the first row of Table 3.2 of [Con].

$$\eta_0 \sim .322, \qquad \eta_h \sim .178, \qquad \eta_v \sim .178, \qquad \eta_{hv} \sim .322. \tag{8.6}$$

There is at best a qualitative similarity between the two measures, and ν lacks the symmetry of η.

The eigenvalues of the jacobian of Θ at the fixed point $\nu = \nu_1$ can also be calculated. It is best to calculate the matrix in terms of the coordinates σ_h, σ_v, and $\sigma = \pi_{hv}$, because of equation (8.1) and the corresponding equation for σ_v. Thus two eigenvalues are equal to

$$4\delta - 4\delta^3 \sim 1.528. \tag{8.7}$$

The third is obtained by substituting $\sigma = .382$ in equations (8.5) and (8.6) and is about $.584$.

Thus there is one stable, or in the language of renormalization, *irrelevant* direction. One of the other two eigenvalues corresponds to a direction symmetric upon interchange of the two axes. If it is λ then

$$\ln(2)/\ln(\lambda) \sim 1.635.$$

In so far as Θ_1 is an approximation to the "true" renormalization this should be an approximation to the correct value $4/3$ of the critical index ν of [Con]. It is not so good. We shall do better at the level $l = 2$.

Universality as formulated in [Con] suggests that in the limit there should be at least two eigenvalues equal to 1 in directions of asymmetry, because the pertinent fixed point of the "true" Θ is twofold degenerate. The value 1.635 obtained here for the asymmetric eigenvalue is perhaps to be considered as an approximation to 1. Once again we shall do better at level two. The second eigenvalue approximating 1 does not manifest itself at level one.

9. Level two. The set $\mathfrak{Y} = \mathfrak{Y}_2$ has 2274 elements. Since the calculation of $\Theta_2 = \Theta_2^{(2)}$ requires an elaborate construction for each of the 2274^4 elements in the fourfold product of $\mathfrak{Y}_l$ with itself, the mapping $\Theta = \Theta_2$ cannot be studied numerically without some preliminary simplification.

Set $\Pi = \Pi_2$. The reflections in the two axes, as well as the interchange of the two axes, define symmetries of Π. Let Π^R be the set of measures invariant under the two reflections, and $\Pi^S \subset \Pi^R$ the set of measures invariant in addition under the interchange of axes. The measure ν_2 that it is the purpose of this section to find numerically can be expected to lie in Π^S. We shall introduce a set $\mathfrak{X}$ with only 187 elements and a map $\varphi : \mathfrak{X} \to \mathfrak{Y}$ that induces, in the usual way, a map, again denoted φ from the set of probability measures on $\mathfrak{Y}$ to the set Σ of probability measures on $\mathfrak{X}$ and therefore from Π to Σ.

The restriction of Θ to Π^R will be defined by a commutative diagram

$$\begin{tikzcd}
\Pi^R \arrow[r] \arrow[d, "\varphi"'] & \Pi^R \times \Pi^R \times \Pi^R \times \Pi^R \arrow[r, "\Theta"] \arrow[d, "\varphi^{(4)}"'] & \Pi^R \arrow[d, "\varphi"] \\
\Sigma \arrow[r] & \Sigma \times \Sigma \times \Sigma \times \Sigma \arrow[r, "\Theta_{\mathrm{mod}}"'] \arrow[ur, "\Psi"] & \Sigma
\end{tikzcd}$$

Diagram D

in which Θ_{mod} is defined by a map

$$\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y} \xrightarrow{\varphi} \mathfrak{X}.$$

The map $\varphi^{(4)}$ is the fourfold product of φ with itself. If ν_{mod} is a fixed point of Θ_{mod} then $\nu = \nu_2 = \Psi(\nu_{mod})$ will be a fixed point of Θ.

In fact the core of Diagram D will be given by a diagram

$$\begin{tikzcd}
\Pi \times \Pi \times \Pi \times \Pi \arrow[r, "\Theta"] \arrow[d, "\varphi_{11} \times \varphi_{12} \times \varphi_{21} \varphi_{22}"'] & \Pi \\
\Sigma \times \Sigma \times \Sigma \times \Sigma \arrow[ur, "\Psi"] &
\end{tikzcd}$$

Diagram E

in which all four maps φ_{ij} have the same restriction to Π^R. All the maps φ_{ij} are obtained from φ_{11} by the action of symmetries, for example φ_{12} is obtained by reflecting φ_{11} in the axis of ordinates. Thus it suffices to describe φ_{11}, which is defined directly from a similar action on sets:

$$\varphi = \varphi_{11} : \mathfrak{Y} \to \mathfrak{X}.$$

To define φ we position an element y of $\mathfrak{Y} = \mathfrak{Y}_2$ in the upper left-hand square, and ask ourselves how we can simplify it before undertaking the heaping and coarsening. We first replace y by $\bar{y}$, because it is in terms of $\bar{y}$ that Θ is defined. The element $\bar{y}$ is a function on pairs of those intervals obtained by dividing each side of the square in two at its midpoint. First of all joins between exterior intervals are irrelevant. They have no effect on the heaping, except in so far as they force joins to be present in $\bar{y}$ that are not in y, and they are discarded in the coarsening. So they can be discarded immediately after passage to $\bar{y}$.

Moreover if α and α' are the two intervals into which an exterior side is divided then a join between α and an interior interval γ has the same effect on Θ as a join between α' and γ. Thus we may as well fuse α and α' into a single interval. The result is that we now consider functions on the product of a set of six elements with itself. The set contains two exterior intervals, thus the two exterior sides of the square, and four interior intervals. The two exterior intervals are not joined. Once y, or rather $\bar{y}$, is modified in this way to obtain a function z, we add joins to it in order to arrive at $x = \varphi(y)$.

The definition of z entailed passing to $\bar{y}$. Thus if α, β, γ, δ is a cyclic sequence, and $z(\alpha, \gamma) = z(\beta, \delta) = 1$ then $z(\alpha, \beta) = 1$ unless α and β are both exterior. If β is interior and $z(\alpha, \beta) = z(\gamma, \beta) = 1$ then we add the join from α to γ unless α and γ are both exterior. When all joins entailed by repeated application of this condition have been added, we have arrived at x.

Thus the set $\mathfrak{X}$ is defined as the set of functions x on the six intervals described satisfying the following conditions.

(1) For all α the value $x(\alpha, \alpha)$ is 1.

(2) If α and β are exterior and different then $x(\alpha, \beta) = 0$.

(3) If $\alpha, \beta, \gamma, \delta$ is cyclic, one of α and β is interior, and $x(\alpha, \gamma) = x(\beta, \delta) = 1$ then $x(\alpha, \beta) = 1$.

(4) If β is interior and α and γ are not both exterior then $x(\alpha,\beta) = x(\gamma,\beta) = 1$ implies that $x(\alpha,\gamma) = 1$.

The set $\mathfrak{X}$ contains 187 elements. We list them in Table I with the help of pictograms that give for each of the 187 elements all the nontrivial joins. Elements with fewer joins appear earlier. Thus the element of $\mathfrak{X}$ that contains all possible joins appears last, and the element that contains only the trivial joins is first. The symmetry of the square that exchanges the two exterior sides acts on $\mathfrak{X}$, and the relevant measures are all invariant under it. So an element and its reflection appear together.

There is an obvious method to search for the fixed point $\nu = \nu_2$. Since it is expected, or rather hoped, to be close to η_2, we start at an approximate value for η_2 or rather for $\varphi(\eta_2) = \eta_{mod}$ obtained by simulation, and then apply the method of Newton to Θ_{mod} and this approximate value. The method turns out to converge, or rather a very small number of iterations (five is enough) appear to yield a fixed point ν_{mod} of Θ_{mod}. The norm of the difference between the ν_{mod} of the table and its image is less than 2×10^{-11}. Applying Ψ to ν_{mod} we obtain an apparent fixed point of Θ that we take to be ν.

The values of η_{mod} and of ν_{mod} are given in Table I. Those of η_{mod} were obtained by sampling $1,000,000$ percolation states at critical probability on a 128×128 lattice. (For purposes of the table the probabilities of two states differing by the reflection have been averaged.) The distance between η_{mod} and ν_{mod} is about $.5$, and for measures on a set with 187 elements this is quite close. A more convincing comparison is to take the largest values of $\nu_{mod}(x)$ and of $\eta_{mod}(x)$, say all those over $\exp(-5) \sim .0067$ and to calculate $\ln(\eta_{mod}(x)/\nu_{mod}(x))$. This is done in Table II. The label in the first column is that attached to x in Table I. Thus the largest probability is that of the function in which all permissible joins appear, and the next largest that of the function with none but the trivial joins.

We can also compare the values of ν_{mod} with those of percolation by comparing $\nu' = \Gamma_1^2(\nu_2)$ with $\eta = \eta_1$. The values of ν' are given by

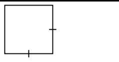
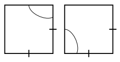
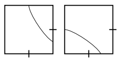
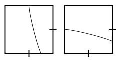
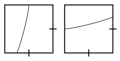
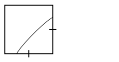
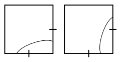
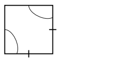
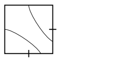
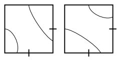
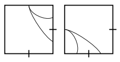
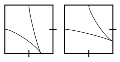
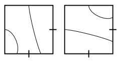
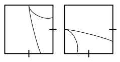
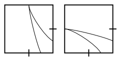
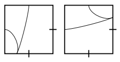
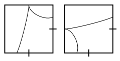
$$\nu'_0 \sim .226, \qquad \nu'_h \sim .164, \qquad \nu'_v \sim .164, \qquad \nu'_{hv} \sim .446. \tag{9.1}$$

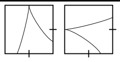
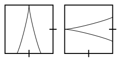
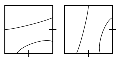
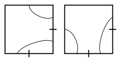
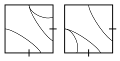
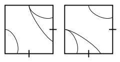
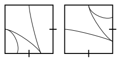
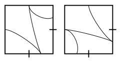
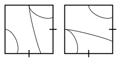
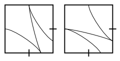
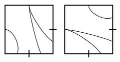
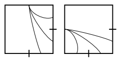
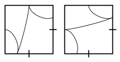
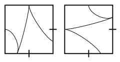
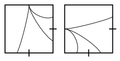
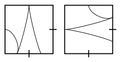
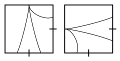
These values are all improvements on those of (8.5), in the sense that they better approximate those of η. The asymetry between ν_0 and ν_{hv} while smaller remains important. Observe that ν' satisfies the FKG inequality and lies not on the boundary but in the interior of Π_1.

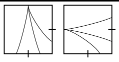
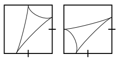
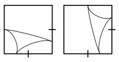
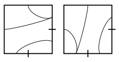
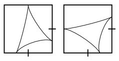
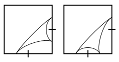
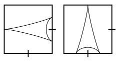
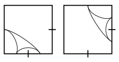
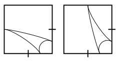
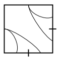
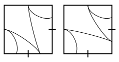
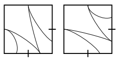
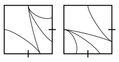
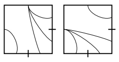
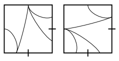
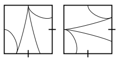
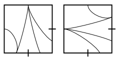
To find the measure ν_{mod} and thus ν we used the method of Newton, so that there is no assurance that the result satisfies the FKG inequality, and is thus in Π_2. This has to be verified numerically. As a result of Hilfsatz II.B.6 of [L], it suffices to verify that the measure ν_{mod} satisfies the FKG inequality,

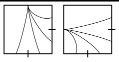
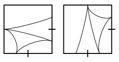
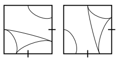
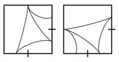
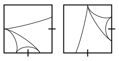
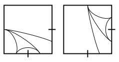
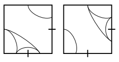
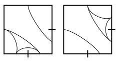
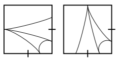
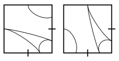
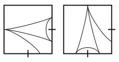
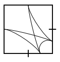
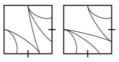
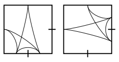
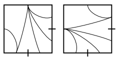
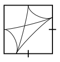
$$\int fg\, d\nu_{mod} \geq \int f\, d\nu_{mod} \int g\, d\nu_{mod}$$

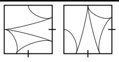
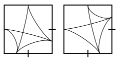
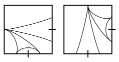
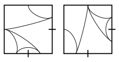
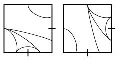
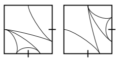
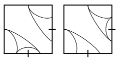
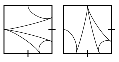
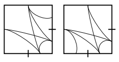
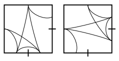
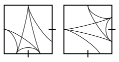
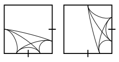
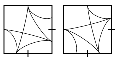
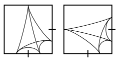
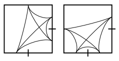
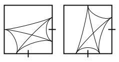
for all monotone increasing functions on $\mathfrak{X}$.

element	η_{mod}	ν_{mod}
	0.10824	0.15575471
	0.01633	0.00974049
	0.02170	0.01046274
	0.00109	0.00098032
	0.00258	0.00538282
	0.00036	0.00008398
	0.02228	0.01027501
	0.00298	0.00017456
	0.00434	0.00018587
	0.00335	0.00047248
	0.00098	0.00025440
	0.00402	0.00325969
	0.00014	0.00002285
	0.00001	0.00000119
	0.00188	0.00491799
	0.02438	0.04813036
	0.00009	0.00028812

element	η_{mod}	ν_{mod}
	0.00032	0.00019227
	0.00114	0.00037124
	0.00037	0.00018466
	0.00327	0.00036464
	0.00019	0.00000302
	0.00016	0.00000281
	0.00009	0.00000553
	0.00015	0.00001098
	0.00000	0.00000000
	0.00060	0.00005671
	0.00027	0.00016334
	0.00002	0.00000498
	0.00126	0.00037256
	0.00351	0.00185841
	0.00000	0.00000688
	0.00019	0.00012001
	0.00005	0.00000640

element	η_{mod}	ν_{mod}
	0.00012	0.00007972
	0.00153	0.00127150
	0.00457	0.00549868
	0.00451	0.00184827
	0.00336	0.00055956
	0.00138	0.00025717
	0.00127	0.00009518
	0.01114	0.00171819
	0.00183	0.00093122
	0.00001	0.00000001
	0.00000	0.00000000
	0.00002	0.00000063
	0.00000	0.00000007
	0.00001	0.00000002
	0.00004	0.00000981
	0.00000	0.00000013
	0.00014	0.00076725

element	η_{mod}	ν_{mod}
	0.00000	0.00000006
	0.00011	0.00008738
	0.00073	0.00017286
	0.00012	0.00000667
	0.00003	0.00003476
	0.00001	0.00000038
	0.00199	0.00002161
	0.00223	0.00002177
	0.00000	0.00001348
	0.00026	0.00000911
	0.00009	0.00000044
	0.01015	0.01449125
	0.00000	0.00000000
	0.02782	0.01397423
	0.00000	0.00000008
	0.01873	0.04793407

element	η_{mod}	ν_{mod}
	0.00037	0.00067187
	0.00791	0.01031797
	0.00002	0.00000025
	0.00076	0.00005930
	0.00000	0.00000000
	0.00006	0.00000034
	0.00010	0.00000007
	0.00008	0.00002708
	0.00030	0.00002900
	0.00124	0.00006625
	0.00371	0.00014819
	0.02568	0.04413746
	0.00031	0.00003418
	0.00415	0.00222361
	0.00422	0.00165513
	0.00465	0.00149362

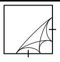
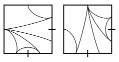
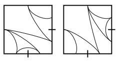
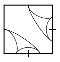
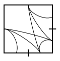
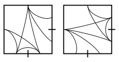
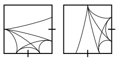
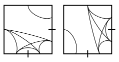
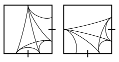
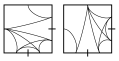
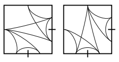
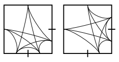
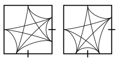
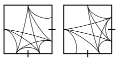
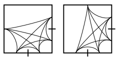
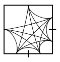
element	η_{mod}	ν_{mod}
	0.01500	0.00174528
	0.00000	0.00000001
	0.00000	0.00000000
	0.00126	0.00000066
	0.00001	0.00000000
	0.00003	0.00000049
	0.00021	0.00078817
	0.00397	0.00167168
	0.00004	0.00002214
	0.00263	0.00645557
	0.00065	0.00000270
	0.04194	0.06275602
	0.03360	0.02928257
	0.00133	0.00024089
	0.02669	0.01047879
	0.16214	0.18391669

Table I

The best we have been able to do is to establish that it is valid statistically. Once again, all monotone increasing functions are, after a constant function is subtracted from them, positive linear combinations of elementary monotone increasing functions, those that are equal to 1 on the set of all $x \in \mathfrak{X}$ greater than or equal to those in some given subset of $\mathfrak{X}$ and otherwise 0. Since the number of pairs of monotone increasing functions is greater than 10^{26}, they cannot all be examined as for $l = 1$. So we treated appropriately chosen samples.

We used two methods to determine the pairs $\{f, g\}$ of elementary functions. In the first method we construct a random sequence of functions and examine all pairs with elements taken from the sequence. To construct a term f of the sequence we begin by choosing randomly an element x_1 of $\mathfrak{X}$ and set $f(x_1)$ equal to 0 or 1 with equal probability $.5$. Since f is to be monotone increasing its value is then determined either on all $x \geq x_1$ if $f(x_1) = 1$ or on all $x \leq x$ if $f(x_1) = 0$. We then choose randomly a second x_2 on which the value of f is still free and set ot equal to 0 or 1 with equal probability $.5$. We constructed a sequence of over 4000 functions and found the inequality to be satisfied on the more than 8 million pairs built from them

There is a strong bias in this sample: the construction of the elementary functions favors those that are almost constant, for an initial choice of the value 0 on a point x with many joins or of 1 on a point with few strongly constrains the ensuing choices. If one of the functions in the FKG inequality is almost constant, it is more likely to be satisfied. An alternative is to construct simultaneously the functions f and g on which the inequality is to be tested.

First of all the elements of x are classified by the number of nontrivial joins they contain, varying between 0 and 14. The median number is 4, because there are 83 elements with less than four joins and 68 with more. We define $f(x)$ randomly and set $g(x) = 1 - f(x)$ for x in $\mathfrak{X}_4$, thus with four joins, in an attempt to make fg as small as possible so that the FKG inequality is more difficult to satisfy. On $\mathfrak{X}_3$ we define f and g so that $f(x) = 1 - g(x)$ when one or the other is not determined by its values on $\mathfrak{X}_4$. If $f(x)$ and $g(x)$ are both free, we choose $f(x)$ to be 0 or 1 with equal probability and $g(x) = 1 - f(x)$. Once the values on $\mathfrak{X}_3$ are fixed, we pass to $\mathfrak{X}_2$ and $\mathfrak{X}_1$, and finally take both f and g to be 0 on the sole element of $\mathfrak{X}_0$. Then we work our way up in a similar fashion from $\mathfrak{X}_5$ to $\mathfrak{X}_{10}$. Since $\mathfrak{X}_{11}$, $\mathfrak{X}_{12}$ and $\mathfrak{X}_{13}$ are empty sets, this leaves only $\mathfrak{X}_{14}$ containing a single element on which we take the value of both functions to be 1.

A sample of $1,000,000$ pairs yielded no violations of the inequality. Such statistical tests are, however, of

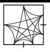
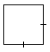
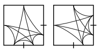
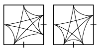
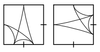
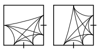
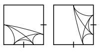
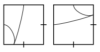
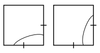
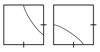
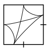
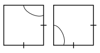
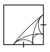
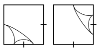
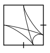
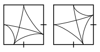
element	η_{mod}	ν_{mod}	$\ln(\eta/\nu)$
	0.16214	0.18391669	-0.1260
	0.10824	0.15575471	-0.3639
	0.04194	0.06275602	-0.4030
	0.03360	0.02928257	0.1375
	0.02782	0.01397423	0.6885
	0.02669	0.01047879	0.9349
	0.02568	0.04413746	-0.5416
	0.02438	0.04813036	-0.6801
	0.02228	0.01027501	0.7740
	0.02170	0.01046274	0.7295
	0.01873	0.04793407	-0.9397
	0.01633	0.00974049	0.5167
	0.01500	0.00174528	2.1513
	0.01114	0.00171819	1.8693
	0.01015	0.01449125	-0.3561
	0.00791	0.01031797	-0.2658

Table II

limited value because the calculations at level one suggest that even if the FKG inequality is satisfied there may be little room to spare, and sampling could well miss the critical cases.

The eigenvalues of Θ of largest absolute value are listed in Table III, as are the symmetries of the associated eigenvectors. The first two labels $+$ or $-$ give the sign with which the vector is multiplied upon reflection in the two axes. When the vector is also fixed up to sign by the interchange of the two axes this sign is the third entry in the sequence of signs.

The largest eigenvalue is $\lambda \sim 1.6346$, and

$$\ln(2)/\ln(\lambda) \sim 1.411.$$

This is a better approximation to $4/3$ than that obtained at the first level. The two succeeding eigenvalues are 1.1580 and $.4694$. The first of these may with some confidence be regarded as an approximation to 1. Whether this is also true of the second is open to doubt.

Otherwise the dynamics at level two appears to be a fair approximation to the dynamics expected in the infinite limit. We observe, in closing, that no effort was made to locate an approximation to the critical exponent η of [Con].

eigenvalue	symmetry
1.6345851	$+++$
1.1579551	$++-$
0.4693886	$--+$
0.4592117	$+-$
0.4592117	$-+$
0.4072630	$+++$
0.3642553	$+-$
0.3642553	$-+$
0.2640523	$+-$
0.2640523	$-+$
0.2583188	$++-$
0.2445117	$+++$
0.1983699	$--+$
0.1747358	$---$
0.1721207	$+++$
0.1286525	$--+$
0.1123677	$+++$

Table III

References

[K] H. Kesten, *Percolation theory for mathematicians*, Birkhäuser, 1982.

[Con] Robert Langlands, Philippe Pouliot, and Yvan Saint-Aubin, *Conformal invariance in two-dimensional percolation*, Bull. of the AMS, January, (1994), to appear.

[L] R. P. Langlands, *Dualität bei endlichen Modellen der Perkolation*, Math. Nach., Bd. 160 (1993), 7–58.