Ankara

April 18, 1968

Dear Harish-Chandra,

I have been wanting to write you for some time to thank you for your letter and to assure you that everything is satisfactory here. However, I had begun to form some suspicions about representations of reductive algebraic groups over local fields and I wanted at least to check these carefully in the case of $GL_2$ before writing to you. I see now that I will have left here before I finish so I started to write you anyway and voice the suspicions in a premature form.

I remind you that if $k$ is a local field and $K$ a normal extension then $G_{K,k}$ (the Weil group of $K/k$) is a certain extension of $K^*$ by the Galois group of $K/k$. If $k \subseteq K \subseteq L$ with $K$ and $L$ normal there is a homomorphism $\varphi$ of $G_{L,k}$ into $G_{K,k}$. This homomorphism is not uniquely determined but if $\rho$ is a representation of $G_{K,k}$ the equivalence class of $\rho \circ \varphi$ is uniquely determined. Thus if $\rho_1$ and $\rho_2$ are representations of $G_{K_1,k}$ and $G_{K_2,k}$ respectively I can call them equivalent if they become equivalent when lifted to $G_{K_1K_2,k}$. Also I can speak of the representations of the local group of $k$ without being explicit about the choice of $K$. Now in my old letter to Weil I made a rough attempt to define the dual group of a reductive group. The attempt was not satisfactory but provides a basis for thought. I have come to believe that associated to almost every equivalence class of continuous representations of the local group in this dual group, a complex group, which is such that $\rho(\sigma)$ is semi-simple for all $\sigma$ there should correspond an equivalence class of representations of the algebraic group. In particular to *every* unitary representation of the Weil group in this local group should correspond a unitary representation of the algebraic group over $k$. To give some basis to this belief and to complete the things I was doing with the Jacquet I wanted to check this out completely for $GL_2$. In this case the dual group is $GL(2,\mathbb{C})$.

If the local field is non-archimedean and the characteristic of the residue class field is different from 2 then there are basically only two ways of getting a representation of the Weil group in $GL(2,\mathbb{C})$. Either take $K = k$ so that $G_{K,k} = k^*$ and send $\alpha \to \begin{pmatrix} \chi_1(\alpha) & 0 \\ 0 & \chi_2(\alpha) \end{pmatrix}$ where $\chi_1$ and $\chi_2$ are two characters of $k^*$ or take $K$ to be a quadratic extension of $k$, take a character $\chi$ of $K^*$, and let $\chi$ induce a representation of $G_{K,k}$. Now give $\chi_1, \chi_2$ or $\chi$ we know how to construct a representation of $GL(2,k)$. The only thing to check is that if the representations give equivalent representations of the Weil group in the sense mentioned above then the representations of $GL(2,k)$ are equivalent. This I have done, although I should probably look at the proof again. It can, for example, happen that $K$ and $K'$ are distinct quadratic extensions while the representations induced from $\chi$ and $\chi'$ give equivalent representations of the Weil group. Archimedean fields are even simpler.

However, it is very likely that if the characteristic of the residue class field is 2 there are two dimensional representations of the Weil group which are not abelian and are not associated to a character of a quadratic

extension of $k$. This is why I told Jacquet that I considered it unlikely that we had exhausted the representations in this case. Although my work in this is proceeding at a reasonably steady pace it will be a while before it is finished. I hope to be able to tell you something definite when I return in August.

All the best,

Bob Langlands