Automorphic Forms on GL(2)

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Introduction

Two of the best known of Hecke's achievements are his theory of L-functions with grössencharakter, which are Dirichlet series which can be represented by Euler products, and his theory of the Euler products, associated to automorphic forms on $GL(2)$. Since a grössencharakter is an automorphic form on $GL(1)$ one is tempted to ask if the Euler products associated to automorphic forms on $GL(2)$ play a role in the theory of numbers similar to that played by the L-functions with grössencharakter. In particular do they bear the same relation to the Artin L-functions associated to two-dimensional representations of a Galois group as the Hecke L-functions bear to the Artin L-functions associated to one-dimensional representations? Although we cannot answer the question definitively one of the principal purposes of these notes is to provide some evidence that the answer is affirmative.

The evidence is presented in §12. It come from reexamining, along lines suggested by a recent paper of Weil, the original work of Hecke. Anything novel in our reexamination comes from our point of view which is the theory of group representations. Unfortunately the facts which we need from the representation theory of $GL(2)$ do not seem to be in the literature so we have to review, in Chapter I, the representation theory of $GL(2, F)$ when F is a local field. §7 is an exceptional paragraph. It is not used in the Hecke theory but in the chapter on automorphic forms and quaternion algebras.

Chapter I is long and tedious but there is nothing hard in it. Nonetheless it is necessary and anyone who really wants to understand L-functions should take at least the results seriously for they are very suggestive.

§9 and §10 are preparatory to the Hecke theory which is finally taken up in §11. We would like to stress, since it may not be apparent, that our method is that of Hecke. In particular the principal tool is the Mellin transform. The success of this method for $GL(2)$ is related to the equality of the dimensions of a Cartan subgroup and the unipotent radical of a Borel subgroup of $PGL(2)$. The implication is that our methods do not generalize. The results, with the exception of the converse theorem in the Hecke theory, may.

The right way to establish the functional equation for the Dirichlet series associated to the automorphic forms is probably that of Tate. In §13 we verify, essentially, that this method leads to the same local factors as that of Hecke and in §14 we use the method of Tate to prove the functional equation for the L-functions associated to automorphic forms on the multiplicative group of a quaternion algebra. The results of §13 suggest a relation between the characters of representations of $GL(2)$ and the characters of representations of the multiplicative group of a quaternion algebra which is verified, using the results of §13, in §15. This relation was well-known for archimedean fields but its significance had not been stressed. Although our proof leaves something to be desired the result itself seems to us to be one of the more striking facts brought out in these notes.

Both §15 and §16 are after thoughts; we did not discover the results in them until the rest of the notes were almost complete. The arguments of §16 are only sketched and we ourselves have not verified all the details. However the theorem of §16 is important and its proof is such a beautiful illustration of the power and ultimate simplicity of the Selberg trace formula and the theory of harmonic analysis on semi-simple groups that we could not resist adding it. Although we are very dissatisfied with the methods of the first fifteen paragraphs we see no way to improve on those of §16. They are perhaps the methods with which to attack the question left unsettled in §12.

We hope to publish a sequel to these notes which will include, among other things, a detailed proof of the theorem of §16 as well as a discussion of its implications for number theory. The theorem has, as these things go, a fairly long history. As far as we know the first forms of it were assertions about the representability of automorphic forms by theta series associated to quaternary quadratic forms.

As we said before nothing in these notes is really new. We have, in the list of references at the end of each chapter, tried to indicate our indebtedness to other authors. We could not however acknowledge completely our indebtedness to R. Godement since many of his ideas were communicated orally to one of us as a student. We hope that he does not object to the company they are forced to keep.

The notes* were typed by the secretaries of Leet Oliver Hall. The bulk of the work was done by Miss Mary Ellen Peters and to her we would like to extend our special thanks. Only time can tell if the mathematics justifies her great efforts.

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Chapter I: Local Theory

§1 **Weil representations.** Before beginning the study of automorphic forms we must review the representation theory of the general linear group in two variables over a local field. In particular we have to prove the existence of various series of representations. One of the quickest methods of doing this is to make use of the representations constructed by Weil in [1]. We begin by reviewing his construction adding, at appropriate places, some remarks which will be needed later.

In this paragraph F will be a local field and K will be an algebra over F of one of the following types:

(i) The direct sum $F \oplus F$.

(ii) A separable quadratic extension of F.

(iii) The unique quaternion algebra over F. K is then a division algebra with centre F.

(iv) The algebra $M(2, F)$ of 2×2 matrices over F.

In all cases we identify F with the subfield of K consisting of scalar multiples of the identity. In particular if $K = F \oplus F$ we identify F with the set of elements of the form (x, x). We can introduce an involution ι of K, which will send x to x^ι, with the following properties:

(i) It satisfies the identities $(x + y)^\iota = x^\iota + y^\iota$ and $(xy)^\iota = y^\iota x^\iota$.

(ii) If x belongs to F then $x = x^\iota$.

(iii) For any x in K both $\tau(x) = x + x^\iota$ and $\nu(x) = xx^\iota = x^\iota x$ belong to F.

If $K = F \oplus F$ and $x = (a, b)$ we set $x^\iota = (b, a)$. If K is a separable quadratic extension of F the involution ι is the unique non-trivial automorphism of K over F. In this case $\tau(x)$ is the trace of x and $\nu(x)$ is the norm of x. If K is a quaternion algebra a unique ι with the required properties is known to exist. τ and ν are the reduced trace and reduced norm respectively. If K is $M(2, F)$ we take ι to be the involution sending

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to

$$x = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then $\tau(x)$ and $\nu(x)$ are the trace and determinant of x.

If $\psi = \psi_F$ is a given non-trivial additive character of F then $\psi_K = \psi_F \circ \tau$ is a non-trivial additive character of K. By means of the pairing

$$\langle x, y \rangle = \psi_K(xy)$$

we can identify K with its Pontrjagin dual. The function ν is of course a quadratic form on K which is a vector space over F and $f = \psi_F \circ \nu$ is a character of second order in the sense of [1]. Since

$$\nu(x + y) - \nu(x) - \nu(y) = \tau(xy^\iota)$$

and

$$f(x + y)f^{-1}(x)f^{-1}(y) = \langle x, y^\iota \rangle$$

the isomorphism of K with itself associated to f is just ι. In particular ν and f are nondegenerate.

Let $\mathcal{S}(K)$ be the space of Schwartz-Bruhat functions on K. There is a unique Haar measure dx on K such that if Φ belongs to $\mathcal{S}(K)$ and

$$\Phi'(x) = \int_K \Phi(y)\,\psi_K(xy)\,dy$$

then

$$\Phi(0) = \int_K \Phi'(x)\,dx.$$

The measure dx, which is the measure on K that we shall use, is said to be self-dual with respect to ψ_K.

Since the involution ι is measure preserving the corollary to Weil's Theorem 2 can in the present case be formulated as follows.

Lemma 1.1. *There is a constant γ which depends on the ψ_F and K, such that for every function Φ in $\mathcal{S}(K)$*

$$\int_K (\Phi * f)(y)\,\psi_K(yx)\,dy = \gamma f^{-1}(x^\iota)\,\Phi'(x)$$

$\Phi * f$ is the convolution of Φ and f. The values of γ are listed in the next lemma.

Lemma 1.2 (i) *If $K = F \oplus F$ or $M(2,F)$ then $\gamma = 1$.*

(ii) *If K is the quaternion algebra over F then $\gamma = -1$.*

(iii) *If $F = \mathbb{R}$, $K = \mathbb{C}$, and*

$$\psi_F(x) = e^{2\pi i a x},$$

then

$$\gamma = \frac{a}{|a|}\,i$$

(iv) *If F is non-archimedean and K is a separable quadratic extension of F let ω be the quadratic character of F^* associated to K by local class-field theory. If U_F is the group of units of F^* let $m = m(\omega)$ be the smallest non-negative integer such that ω is trivial on*

$$U_F^m = \{a \in U_F \mid \alpha \equiv 1 \,(\operatorname{mod} \mathfrak{p}_F^m)\}$$

and let $n = n(\psi_F)$ be the largest integer such that ψ_F is trivial on the ideal $\mathfrak{p}_F^{-n}$. If a is any generator on the ideal $\mathfrak{p}_F^{m+n}$ then

$$\gamma = \omega(a)\frac{\int_{U_F} \omega^{-1}(\alpha)\,\psi_F(\alpha a^{-1})\,d\alpha}{\left|\int_{U_F} \omega^{-1}(\alpha)\,\psi_F(\alpha a^{-1})\,d\alpha\right|}.$$

The first two assertions are proved by Weil. To obtain the third apply the previous lemma to the function

$$\Phi(z) = e^{-2\pi z z^\iota}.$$

We prove the last. It is shown by Weil that $|\gamma| = 1$ and that if ℓ is sufficiently large γ differs from

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(x x^\iota)\,dx$$

by a positive factor. This equals

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\iota)\,|x|_K\,d^\times x = \int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\iota)|xx^\iota|_F\,d^\times x$$

if $d^\times x$ is a suitable multiplicative Haar measure. Since the kernel of the homomorphism ν is compact the integral on the right is a positive multiple of

$$\int_{\nu(\mathfrak{p}_K^{-\ell})} \psi_F(x)\,|x|_F\,d^\times x.$$

Set $k = 2\ell$ if K/F is unramified and set $k = \ell$ if K/F is ramified. Then $\nu(\mathfrak{p}_K^{-\ell}) = \mathfrak{p}_F^{-k} \cap \nu(K)$. Since $1 + \omega$ is twice the characteristic function of $\nu(K^\times)$ the factor γ is the positive multiple of

$$\int_{\mathfrak{p}_F^{-k}} \psi_F(x)\,dx + \int_{\mathfrak{p}_F^{-k}} \psi_F(x)\,\omega(x)\,dx.$$

For ℓ and therefore k sufficiently large the first integral is 0. If K/F is ramified well-known properties of Gaussian sums allow us to infer that the second integral is equal to

$$\int_{U_F} \psi_F\left(\frac{\alpha}{a}\right)\,\omega\left(\frac{\alpha}{a}\right)\,d\alpha.$$

Since $\omega = \omega^{-1}$ we obtain the desired expression for γ by dividing this integral by its absolute value. If K/F is unramified we write the second integral as

$$\sum_{j=0}^{\infty}(-1)^{j-k}\left\{\int_{\mathfrak{p}_F^{-k+j}} \psi_F(x)\,dx - \int_{\mathfrak{p}_F^{-k+j+1}} \psi_F(x)\,dx\right\}$$

In this case $m = 0$ and

$$\int_{\mathfrak{p}_F^{-k+j}} \psi_F(x)\,dx$$

is 0 if $k - j > n$ but equals q^{k-j} if $k - j \leq n$, where q is the number of elements in the residue class field. Since $\omega(a) = (-1)^n$ the sum equals

$$\omega(a)\left\{q^m + \sum_{j=0}^{\infty}(-1)^j q^{m-j}\left(1 - \frac{1}{q}\right)\right\}$$

A little algebra shows that this equals $\frac{2\omega(a)q^{m+1}}{q+1}$ so that $\gamma = \omega(a)$, which upon careful inspection is seen to equal the expression given in the lemma.

In the notation of [19] the third and fourth assertions could be formulated as an equality

$$\gamma = \lambda(K/F, \psi_F).$$

It is probably best at the moment to take this as the definition of $\lambda(K/F, \psi_F)$.

If K is not a separable quadratic extension of F we take ω to be the trivial character.

Proposition 1.3 *There is a unique representation r of $SL(2,F)$ on $\mathcal{S}(K)$ such that*

(i) $r\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right)\Phi(x)=\omega(\alpha)\,|\alpha|_K^{1/2}\Phi(\alpha x)$

(ii) $r\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)\Phi(x)=\psi_F(z\nu(x))\Phi(x)$

(iii) $r\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\Phi(x)=\gamma\Phi'(x^\iota)$.

If $\mathcal{S}(K)$ is given its usual topology, r is continuous. It can be extended to a unitary representation of $SL(2,F)$ on $L^2(K)$, the space of square integrable functions on K. If F is archimedean and Φ belongs to $\mathcal{S}(K)$ then the function $r(g)\Phi$ is an indefinitely differentiable function on $SL(2,F)$ with values in $\mathcal{S}_(K)$.

This may be deduced from the results of Weil. We sketch a proof. $SL(2,F)$ is the group generated by the elements $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$, $\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$, and $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ with α in $F^\times$ and z in F subject to the relations

$$w\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}=\begin{pmatrix}\alpha^{-1} & 0\\ 0 & \alpha\end{pmatrix}w \tag{a}$$

$$w^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \tag{b}$$

$$w\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}w=\begin{pmatrix}-a^{-1} & 0\\ 0 & -a\end{pmatrix}\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & -a^{-1}\\ 0 & 1\end{pmatrix} \tag{c}$$

together with the obvious relations among the elements of the form $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$ and $\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$. Thus the uniqueness of r is clear. To prove the existence one has to verify that the mapping specified by (i), (ii), (iii) preserves all relations between the generators. For all relations except (a), (b), and (c) this can be seen by inspection. (a) translates into an easily verifiable property of the Fourier transform. (b) translates into the equality $\gamma^2=\omega(-1)$ which follows readily from Lemma 1.2.

If $a=1$ the relation (c) becomes

$$\int_K \Phi'(y^\iota)\,\psi_F(\nu(y))\langle y,x^\iota\rangle\,dy=\gamma\psi_F(-\nu(x))\int_K\Phi(y)\psi_F(-\nu(y))\langle y,-x^\iota\rangle\,dy \tag{1.3.1}$$

which can be obtained from the formula of Lemma 1.1 by replacing $\Phi(y)$ by $\Phi'(-y^\iota)$ and taking the inverse Fourier transform of the right side. If a is not 1 the relation (c) can again be reduced to (1.3.1) provided ψ_F is replaced by the character $x\to\psi_F(ax)$ and γ and dx are modifed accordingly. We refer to Weil's paper for the proof that r is continuous and may be extended to a unitary representation of $SL(2,F)$ in $L^2(K)$.

Now take F archimedean. It is enough to show that all of the functions $r(g)\Phi$ are indefinitely differentiable in some neighborhood of the identity. Let

$$N_F=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\,\middle|\,x\in F\right\}$$

and let

$$A_F = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \,\middle|\, \alpha \in F^\times \right\}$$

Then $N_F w A_F N_F$ is a neighborhood of the identity which is diffeomorphic to $N_F \times A_F \times N_F$. It is enough to show that

$$\phi(n, a, n_1) = r(nwan)\Phi$$

is infinitely differentiable as a function of n, as a function of a, and as a function of n_1 and that the derivations are continuous on the product space. For this it is enough to show that for all Φ all derivatives of $r(n)\Phi$ and $r(a)\Phi$ are continuous as functions of n and Φ or a and Φ. This is easily done.

The representation r depends on the choice of ψ_F. If a belongs to $F^\times$ and $\psi'_F(x) = \psi_F(ax)$ let r' be the corresponding representation. The constant $\gamma' = \omega(a)\gamma$.

Lemma 1.4 (i) *The representation r' is given by*

$$r'(g) = r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

(ii) *If b belongs to K^* let $\lambda(b)\Phi(x) = \Phi(b^{-1}x)$ and let $\rho(b)\Phi(x) = \Phi(xb)$. If $a = \nu(b)$ then*

$$r'(g)\lambda(b^{-1}) = \lambda(b^{-1})r(g)$$

and

$$r'(g)\rho(b) = \rho(b)r(g).$$

In particular if $\nu(b) = 1$ both $\lambda(b)$ and $\rho(b)$ commute with r.

We leave the verification of this lemma to the reader. Take K to be a separable quadratic extension of F or a quaternion algebra of centre F. In the first case $\nu(K^\times)$ is of index 2 in $F^\times$. In the second case $\nu(K^\times)$ is $F^\times$ if F is non-archimedean and $\nu(K^\times)$ has index 2 in $F^\times$ if F is $\mathbb{R}$.

Let K' be the compact subgroup of $K^\times$ consisting of all x with $\nu(x) = xx^\iota = 1$ and let G_+ be the subgroup of $GL(2, F)$ consisting of all g with determinant in $\nu(K^\times)$. G_+ has index 2 or 1 in $GL(2, F)$. Using the lemma we shall decompose r with respect to K' and extend r to a representation of G_+.

Let Ω be a finite-dimensional irreducible representation of $K^\times$ in a vector space U over $\mathbb{C}$. Taking the tensor product of r with the trivial representation of $SL(2, F)$ on U we obtain a representation on

$$\mathcal{S}(K) \otimes_{\mathbb{C}} U = \mathcal{S}(K, U)$$

which we still call r and which will now be the centre of attention.

Proposition 1.5 (i) *If $\mathcal{S}(K, \Omega)$ is the space of functions Φ in $\mathcal{S}(K, U)$ satisfying*

$$\Phi(xh) = \Omega^{-1}(h)\Phi(x)$$

for all h in K' then $\mathcal{S}(K, \Omega)$ is invariant under $r(g)$ for all g in $SL(2, F)$.

(ii) *The representation r of $SL(2, F)$ on $\mathcal{S}(K, \Omega)$ can be extended to a representation r_Ω of G_+ satisfying*

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = |h|_K^{1/2}\Omega(h)\Phi(xh)$$

if $a = \nu(h)$ belongs to $\nu(K^\times)$.

(iii) *If η is the quasi-character of $F^\times$ such that*

$$\Omega(a) = \eta(a)I$$

for a in $F^\times$ then

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a)I$$

(iv) *The representation r_Ω is continuous and if F is archimedean all factors in $\mathcal{S}(K,\Omega)$ are infinitely differentiable.*

(v) *If U is a Hilbert space and Ω is unitary let $L^2(K,U)$ be the space of square integrable functions from K to U with the norm*

$$\|\Phi\|^2 = \int \|\Phi(x)\|^2\,dx$$

If $L^2(K,\Omega)$ is the closure of $\mathcal{S}(K,\Omega)$ in $L^2(K,U)$ then r_Ω can be extended to a unitary representation of G_+ in $L^2(K,\Omega)$.

The first part of the proposition is a consequence of the previous lemma. Let H be the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with a in $\nu(K^\times)$. It is clear that the formula of part (ii) defines a continuous representation of H on $\mathcal{S}(K,\Omega)$. Moreover G_+ is the semi-direct of H and $SL(2,F)$ so that to prove (ii) we have only to show that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) r_\Omega(g)\, r_\Omega\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Let $a = \nu(h)$ and let r' be the representation associated $\psi'_F(x) = \psi_F(ax)$. By the first part of the previous lemma this relation reduces to

$$r'_\Omega(g) = \rho(h)\, r_\Omega(g)\, \rho^{-1}(h),$$

which is a consequence of the last part of the previous lemma.

To prove (iii) observe that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and that $a^2 = \nu(a)$ belongs to $\nu(K^\times)$. The last two assertions are easily proved.

We now insert some remarks whose significance will not be clear until we begin to discuss the local functional equations. We associate to every Φ in $\mathcal{S}(K,\Omega)$ a function

$$W_\Phi(g) = r_\Omega(g)\,\Phi(1) \tag{1.5.1}$$

on G_+ and a function

$$\varphi_\Phi(a) = W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{1.5.2}$$

on $\nu(K^\times)$. The both take values in U.

It is easily verified that

$$W_\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi_F(x) W_\Phi(g)$$

If $g \in G_+$ and F is a function on G_+ let $\rho(g)F$ be the function $h \to F(hg)$. Then

$$\rho(g)\, W_\Phi = W_{r_\Omega}(g)\Phi$$

Let B_+ be the group of matrices of the form

$$\begin{pmatrix}a & x\\ 0 & 1\end{pmatrix}$$

with a in $\nu(K^\times)$. Let ξ be the representation of B_+ on the space of functions on $\nu(K^\times)$ with values in U defined by

$$\xi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\varphi(b) = \varphi(ba)$$

and

$$\xi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi(b) = \psi_F(bx)\,\varphi(b).$$

Then for all b in B_+

$$\xi(b)\varphi_\Phi = \varphi_{r_\Omega}(b)\Phi. \tag{1.5.3}$$

The application $\Phi \to \varphi_\Phi$, and therefore the application $\Phi \to W_\Phi$, is injective because

$$\varphi_\Phi(\nu(h)) = |h|_K^{1/2}\,\Omega(h)\,\Phi(h). \tag{1.5.4}$$

Thus we may regard r_Ω as acting on the space V of functions φ_Φ, $\Phi \in \mathcal{S}(K,\Omega)$. The effect of a matrix in B_+ is given by (1.5.3). The matrix $\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}$ corresponds to the operator $\omega(a)\,\eta(a)I$. Since G_+ is generated by B_+, the set of scalar matrices, and $w = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ the representation r_Ω on V is determined by the action of w. To specify this we introduce, formally at first, the Mellin transform of $\varphi = \varphi_\Phi$.

If μ is a quasi-character of $F^\times$ let

$$\widehat{\varphi}(\mu) = \int_{\nu(K^\times)} \varphi(\alpha)\,\mu(\alpha)\,d^\times\alpha. \tag{1.5.5}$$

Appealing to (1.5.4) we may write this as

$$\widehat{\varphi}_\Phi(\mu) = \widehat{\varphi}(\mu) = \int_{K^\times} |h|_K^{1/2}\mu(\nu(h))\,\Omega(h)\,\Phi(h)\,d^\times h. \tag{1.5.6}$$

If λ is a quasi-character of $F^\times$ we sometimes write λ for the associated quasi-character $\lambda\circ\nu$ of $K^\times$. The tensor product $\lambda\otimes\Omega$ of λ and Ω is defined by

$$(\lambda\otimes\Omega)(h) = \lambda(h)\,\Omega(h).$$

If $\alpha_K : h \to |h|_K$ is the module of K then

$$\alpha_K^{1/2}\mu \otimes \Omega(h) = |h|_K^{1/2}\,\mu(\nu(h))\,\Omega(h).$$

We also introduce, again in a purely formal manner, the integrals

$$Z(\Omega, \Phi) = \int_{K^\times} \Omega(h)\,\Phi(h)\,d^\times h$$

and

$$Z(\Omega^{-1}, \Phi) = \int_{K^\times} \Omega^{-1}(h)\,\Phi(h)\,d^\times h$$

so that

$$\widehat{\varphi}(\mu) = Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi). \tag{1.5.7}$$

Now let $\varphi' = \varphi_{r_\Omega(w)\Phi}$ and let Φ' be the Fourier transform of Φ so that $r_\Omega(w)\,\Phi(x) = \gamma\Phi'(x^\iota)$. If $\mu_0 = \omega\eta$

$$\widehat{\varphi}'\left(\mu^{-1}\mu_0^{-1}\right) = Z\left(\mu^{-1}\mu_0^{-1}\alpha_K^{1/2} \otimes \Omega, r_\Omega(w)\Phi\right)$$

which equals

$$\gamma \int_K \mu^{-1}\mu_0^{-1}(\nu(h))\Omega(h)\,\Phi'(h^\iota)\,d^\times h.$$

Since $\mu_0(\nu(h)) = \eta(\nu(h)) = \Omega(h^\iota h) = \Omega(h^\iota)\Omega(h)$ this expression equals

$$\gamma \int_K \mu^{-1}(\nu(h))\Omega^{-1}(h^\iota)\,\Phi'(h^\iota)\,d^\times h = \gamma \int_K \mu^{-1}(\nu(h))\Omega^{-1}(h)\,\Phi'(h)\,d^\times h$$

so that

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = \gamma Z\left(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi'\right). \tag{1.5.8}$$

Take $\mu = \mu_1\alpha_F^s$ where μ_1 is a fixed quasi-character and s is complex number. If K is a separable quadratic extension of F the representation Ω is one-dimensional and therefore a quasi-character. The integral defining the function

$$Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi)$$

is known to converge for $\operatorname{Re} s$ sufficiently large and the function itself is essentially a local zeta-function in the sense of Tate. The integral defining

$$Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi')$$

converges for $\operatorname{Re} s$ sufficiently small, that is, large and negative. Both functions can be analytically continued to the whole s-plane as meromorphic functions. There is a scalar $C(\mu)$ which depends analytically on s such that

$$Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi) = C(\mu)Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi').$$

All these assertions are also known to be valid for quaternion algebras. We shall return to the verification later. The relation

$$\widehat{\varphi}(\mu) = \gamma^{-1}C(\mu)\widehat{\varphi}'(\mu^{-1}\mu_0^{-1})$$

determines φ' in terms of φ.

If λ is a quasi-character of $F^\times$ and $\Omega_1 = \lambda \otimes \Omega$ then $\mathcal{S}(K, \Omega_1) = \mathcal{S}(K, \Omega)$ and

$$r_{\Omega_1}(g) = \lambda(\det g) r_\Omega(g)$$

so that we may write

$$r_{\Omega_1} = \lambda \otimes r_\Omega$$

However the space V_1 of functions on $\nu(K^\times)$ associated to r_{Ω_1} is not necessarily V. In fact

$$V_1 = \{\lambda_\varphi \mid \varphi \in V\}$$

and $r_{\Omega_1}(g)$ applied to λ_φ is the product of $\lambda(\det g)$ with the function $\lambda \cdot r_\Omega(g)_\varphi$. Given Ω one can always find a λ such that $\lambda \otimes \Omega$ is equivalent to a unitary representation.

If Ω is unitary the map $\Phi \to \varphi_\Phi$ is an isometry because

$$\int_{\nu(K^\times)} \|\varphi_\Phi(a)\|^2 \, d^\times a = \int_{K^\times} \|\Omega(h)\, \Phi(h)\|^2 |h|_K \, d^\times h = \int_K \|\Phi(h)\|^2 \, dh$$

if the measures are suitably normalized.

We want to extend some of these results to the case $K = F \oplus F$. We regard the element of K as defining a row vector so that K becomes a right module for $M(2, F)$. If Φ belongs to $\mathcal{S}(K)$ and g belongs to $GL(2, F)$, we set

$$\rho(g)\, \Phi(x) = \Phi(xg).$$

Proposition 1.6 (i) *If $K = F \oplus F$ then r can be extended to a representation r of $GL(2, F)$ such that*

$$r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \Phi = \rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \Phi$$

for a in $F^\times$.

(ii) *If $\widetilde{\Phi}$ is the partial Fourier transform*

$$\widetilde{\Phi}(a, b) = \int_F \Phi(a, y)\, \psi_F(by)\, dy$$

and the Haar measure dy is self-dual with repsect to ψ_F then

$$[r(g)\widetilde{\Phi}] = \rho(g)\widetilde{\Phi}$$

for all Φ in $\mathcal{S}(K)$ and all g in G_F.

It is easy to prove part (ii) for g in $SL(2, F)$. In fact one has just to check it for the standard generators and for these it is a consequence of the definitions of Proposition 1.3. The formula of part (ii) therefore defines an extension of r to $GL(2, F)$ which is easily seen to satisfy the condition of part (i).

Let Ω be a quasi-character of $K^\times$. Since $K^\times = F^\times \times F^\times$ we may identify Ω with a pair (ω_1, ω_2) of quasi-characters of $F^\times$. Then r_Ω will be the representation defined by

$$r_\Omega(g) = |\det g|_F^{1/2} \omega_1(\det g) r(g).$$

If x belongs to $K^\times$ and $\nu(x) = 1$ then x is of the form (t, t^{-1}) with t in $F^\times$. If Φ belongs to $\mathcal{S}(K)$ set

$$\theta(\Omega, \Phi) = \int_{F^\times} \Omega((t, t^{-1}))\, \Phi((t, t^{-1}))\, d^\times t.$$

Since the integrand has compact support on $F^\times$ the integral converges. We now associate to Φ the function

$$W_\Phi(g) = \theta(\Omega, r_\Omega(g)\Phi) \tag{1.6.1}$$

on $GL(2, F)$ and the function

$$\varphi_\Phi(a) = W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{1.6.2}$$

on $F^\times$. We still have

$$\rho(g)W_\Phi = W_{r_\Omega}(g)\Phi.$$

If

$$B_F = \left\{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in F^\times,\, x \in F\right\}$$

and if the representations ξ of B_F on the space of functions on $F^\times$ is defined in the same manner as the representation ξ of B_+ then

$$\xi(b)\varphi_\Phi = \varphi_{r_\Omega(b)\Phi}$$

for b in B_F. The applications $\Phi \to W_\Phi$ and $\Phi \to \varphi_\Phi$ are no longer injective.

If μ_0 is the quasi-character defined by

$$\mu_0(a) = \Omega((a, a)) = \omega_1(a)\,\omega_2(a)$$

then

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \mu_0(a)\, W_\Phi(g).$$

It is enough to verify this for $g = e$.

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \theta\left(\Omega, r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\Phi\right)$$

and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

so that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\Phi(x, y) = |a^2|_F^{1/2}\omega_1(a^2)|a|_K^{-1/2}\,\Phi(ax, a^{-1}y).$$

Consequently

$$\begin{aligned} W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) &= \int_{F^\times} \omega_1(a^2)\omega_1(x)\omega_2^{-1}(x)\Phi(ax, a^{-1}x^{-1})\, d^\times x \\ &= \omega_1(a)\omega_2(a)\int_{F^\times} \omega_1(x)\omega_2^{-1}(x)\Phi(x, x^{-1})\, d^\times x \end{aligned}$$

which is the required result.

Again we introduce in a purely formal manner the distribution

$$Z(\Omega, \Phi) = Z(\omega_1, \omega_2 \Phi) = \int \Phi(x_1, x_2)\, \omega_1(x_2)\, \omega_2(x_2)\, d^\times x_2\, d^\times x_2.$$

If μ is a quasi-character of $F^\times$ and $\varphi = \varphi_\Phi$ we set

$$\widehat{\varphi}(\mu) = \int_{F^\times} \varphi(\alpha)\, \mu(\alpha)\, d^\times \alpha.$$

The integral is

$$\begin{aligned}&\int_{F^\times} \mu(\alpha)\theta\left(\Omega, r_\Omega\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\Phi\right) d^\times \alpha\\ &\qquad = \int_{F^\times} \mu(\alpha)\left\{\int_{F^\times} r_\Omega\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\Phi(x, x^{-1})\omega_1(x)\omega_2^{-1}(x)\, d^\times x\right\} d^\times \alpha\end{aligned}$$

which in turn equals

$$\int_{F^\times} \mu(\alpha)\omega_1(\alpha)|\alpha|_F^{1/2}\left\{\int_{F^\times} \Phi(\alpha x, x^{-1})\omega_1(x)\omega_2^{-1}(x)\, d^\times x\right\} d^\times \alpha.$$

Writing this as a double integral and then changing variables we obtain

$$\int_{F^\times}\int_{F^\times} \Phi(\alpha, x)\mu\omega_1(\alpha)\mu\omega_2(x)|\alpha x|_F^{1/2}\, d^\times \alpha\, d^\times x$$

so that

$$\widehat{\varphi}(\mu) = Z\left(\mu\omega_1\alpha_F^{1/2}, \mu\omega_2\alpha_F^{1/2}, \Phi\right). \tag{1.6.3}$$

Let $\varphi' = \varphi_{r_\Omega(w)\Phi}$. Then

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = Z\left(\mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, r_\Omega(w)\Phi\right)$$

which equals

$$\iint \Phi'(y, x)\mu^{-1}\omega_2^{-1}(x)\mu^{-1}\omega_1^{-1}(y)|xy|_F^{1/2}\, d^\times x\, d^\times y$$

so that

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = Z(\mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \Phi'). \tag{1.6.4}$$

Suppose $\mu = \mu_1\alpha_F^s$ where μ_1 is a fixed quasi-character and s is a complex number. We shall see that the integral defining the right side of (1.6.3) converges for $\operatorname{Re} s$ sufficiently large and that the integral defining the right side of (1.6.4) converges for $\operatorname{Re} s$ sufficiently small. Both can be analytically continued to the whole complex plane as meromorphic functions and there is a meromorphic function $C(\mu)$ which is independent of Φ such that

$$Z(\mu\omega_1\alpha_F^{1/2}, \mu\omega_2\alpha_F^{1/2}) = C(\mu)Z(\mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \Phi').$$

Thus

$$\widehat{\varphi}(\mu) = C(\mu)\widehat{\varphi}'(\mu^{-1}\mu_0^{-1})$$

The analogy with the earlier results is quite clear.

§**2 Representations of $GL(2,F)$ in the non-archimedean case.** In this and the next two paragraphs the ground field F is a non- archimedean local field. We shall be interested in representations π of $G_F = GL(2,F)$ on a vector space V over $\mathbb{C}$ which satisfy the following condition.

(2.1) *For every vector v in V the stabilizer of v in G_F is an open subgroup of G_F.*

Those who are familiar with such things can verify that this is tantamount to demanding that the map $(g,v) \to \pi(g)v$ of $G_F \times V$ into V is continuous if V is given the trivial locally convex topology in which every semi-norm is continuous. A representation of G_F satisfying (2.1) will be called admissible if it also satisfies the following condition

(2.2) *For every open subgroup G' of $GL(2,O_F)$ the space of vectors v in V stablizied by G' is finite-dimensional.*

O_F is the ring of integers of F.

Let $\mathcal{H}_F$ be the space of functions on G_F which are locally constant and compactly supported. Let dg be that Haar measure on G_F which assigns the measure 1 to $GL(2,O_F)$. Every f in $\mathcal{H}_F$ may be identified with the measure $f(g)\,dg$. The convolution product

$$f_1 * f_2(h) = \int_{G_F} f_1(g)\, f_2(g^{-1}h)\, dg$$

turns $\mathcal{H}_F$ into an algebra which we refer to as the Hecke algebra. Any locally constant function on $GL(2,O_F)$ may be extended to G_F by being set equal to 0 outside of $GL(2,O_F)$ and therefore may be regarded as an element of $\mathcal{H}_F$. In particular if π_i, $1 \le i \le r$, is a family of inequivalent finite-dimensional irreducible representations of $GL(2,O_F)$ and

$$\xi_i(g) = \dim(\pi_i)\,\mathrm{tr}\,\pi_i(g^{-1})$$

for g in $GL(2,O_F)$ we regard ξ_i as an element of $\mathcal{H}_F$. The function

$$\xi = \sum_{i=1}^{r} \xi_i$$

is an idempotent of $\mathcal{H}_F$. Such an idempotent will be called elementary.

Let π be a representation satisfying (2.1). If f belongs to $\mathcal{H}_F$ and v belongs to V then $f(g)\,\pi(g)v$ takes on only finitely many values and the integral

$$\int_{G_F} f(g)\,\pi(g)v\,dg = \pi(f)v$$

may be defined as a finite sum. Alternatively we may give V the trivial locally convex topology and use some abstract definition of the integral. The result will be the same and $f \to \pi(f)$ is the representation of $\mathcal{H}_F$ on V. If g belongs to G_F then $\lambda(g)f$ is the function whose value at h is $f(g^{-1}h)$. It is clear that

$$\pi(\lambda(g)f) = \pi(g)\,\pi(f).$$

Moreover

(2.3) *For every v in V there is an f in $\mathcal{H}_F$ such that $\pi f(v) = v$.*

In fact f can be taken to be a multiple of the characteristic function of some open and closed neighborhood of the identity. If π is admissible the associated representation of $\mathcal{H}_F$ satisfies

(2.4) *For every elementary idempotent ξ of $\mathcal{H}_F$ the operator $\pi(\xi)$ has a finite-dimensional range.*

We now verify that from a representation π of $\mathcal{H}_F$ satisfying (2.3) we can construct a representation π of G_F satisfying (2.1) such that

$$\pi(f) = \int_{G_F} f(g)\,\pi(g)\,dg.$$

By (2.3) every vector v in V is of the form

$$v = \sum_{i=1}^{r} \pi(f_i)\,v_i$$

with v_i in V and f_i in $\mathcal{H}_F$. If we can show that

$$\sum_{i=1}^{r} \pi(f_i)\,v_i = 0 \tag{2.3.1}$$

implies that

$$w = \sum_{i=1}^{r} \pi\big(\lambda(g) f_i\big) v_i$$

is 0 we can define $\pi(g)v$ to be

$$\sum_{i=1}^{r} \pi\big(\lambda(g) f_i\big) v_i$$

π will clearly be a representation of G_F satisfying (2.1).

Suppose that (2.3.1) is satisifed and choose f in $\mathcal{H}_F$ so that $\pi(f)w = w$. Then

$$w = \sum_{i=1}^{r} \pi\big(f * \lambda(g) f_i\big) v_i.$$

If $\rho(g)f(h) = f(hg)$ then

$$f * \lambda(g) f_i = \rho(g^{-1}) f * f_i$$

so that

$$w = \sum_{i=1}^{r} \pi\big(\rho(g^{-1}) f * f_i\big) v_i = \pi\big(\rho(g^{-1}) f\big) \left\{ \sum_{i=1}^{r} \pi(f_i) v_i \right\} = 0.$$

It is easy to see that the representation of G_F satisfies (2.2) if the representation of $\mathcal{H}_F$ satisfies (2.4). A representation of $\mathcal{H}_F$ satisfying (2.3) and (2.4) will be called admissible. There is a complete correspondence between admissible representations of G_F and of $\mathcal{H}_F$. For example a subspace is invariant under G_F if and only if it is invariant under H_F and an operator commutes with the action of G_F if and only if it commutes with the action of $\mathcal{H}_F$.

>From now on, unless the contrary is explicitly stated, an irreducible representation of G_F or $\mathcal{H}_F$ is to be assumed admissible. If π is irreducible and acts on the space V then any linear transformation of V commuting with $\mathcal{H}_F$ is a scalar. In fact if V is assumed, as it always will be, to be different from 0 there is an elementary idempotent ξ such that $\pi(\xi) \neq 0$. Its range is a finite-dimensional space invariant under A. Thus A has at least one eigenvector and is consequently a scalar. In particular there is a homomorphism ω of $F^\times$ into $\mathbb{C}$ such that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for all a in $F^\times$. By (2.1) the function ω is 1 near the identity and is therefore continuous. We shall refer to a continuous homomorphism of a topological group into the multiplicative group of complex numbers as a quasi-character.

If χ is a quasi-character of $F^\times$ then $g \to \chi(\det g)$ is a quasi-character of G_F. It determines a one-dimensional representation of G_F which is admissible. It will be convenient to use the letter χ to denote this associated representation. If π is an admissible reprentation of G_F on V then $\chi \otimes \pi$ will be the reprenentation of G_F on V defined by

$$(\chi \otimes \pi)(g) = \chi(\det g)\pi(g).$$

It is admissible and irreducible if π is.

Let π be an admissible representation of G_F on V and let V^* be the space of all linear forms on V. We define a representation π^* of $\mathcal{H}_F$ on V^* by the relation

$$\langle v, \pi^*(f)v^* \rangle = \langle \pi(\check{f})v, v^* \rangle$$

where $\check{f}\nu(g) = f(g^{-1})$. Since π^* will not usually be admissible, we replace V^* by $\widetilde{V} = \pi^*(\mathcal{H}_F)V^*$. The space $\widetilde{V}$ is invariant under $\mathcal{H}_F$. For each f in $\mathcal{H}_F$ there is an elementary idempotent ξ such that $\xi * f = f$ and therefore the restriction $\widetilde{\pi}$ of π^* to $\widetilde{V}$ satisfies (2.3). It is easily seen that if ξ is an elementary idempotent so is $\check{\xi}$. To show that $\widetilde{\pi}$ is admissible we have to verify that

$$\widetilde{V}(\xi) = \widetilde{\pi}(\xi)\widetilde{V} = \pi^*(\xi)V^*$$

is finite-dimensional. Let $V(\check{\xi}) = \pi(\check{\xi})V$ and let $V_c = \big(1 - \pi(\check{\xi})\big)V$. V is clearly the direct sum of $V(\check{\xi})$, which is finite-dimensional, and V_c. Moreover $\widetilde{V}(\xi)$ is orthogonal to V_c because

$$\langle v - \pi(\check{\xi})v, \widetilde{\pi}(\xi)\widetilde{v} \rangle = \langle \pi(\check{\xi})v - \pi(\check{\xi})v, \widetilde{v} \rangle = 0.$$

It follows immediately that $\widetilde{V}(\xi)$ is isomorphic to a subspace of the dual of $V(\check{\xi})$ and is therefore finite-dimensional. It is in fact isomorphic to the dual of $V(\check{\xi})$ because if v^* annihilates V_c then, for all v in V,

$$\langle v, \pi^*(\xi)v^* \rangle - \langle v, v^* \rangle = -\langle v - \pi(\check{\xi})v, v^* \rangle = 0$$

so that $\pi^*(\xi)v^* = v^*$.

$\widetilde{\pi}$ will be called the representation contragradient to π. It is easily seen that the natural map of V into $\widetilde{V}^*$ is an isomorphism and that the image of this map is $\widetilde{\pi}^*(\mathcal{H}_F)\widetilde{V}^*$ so that π may be identified with the contragredient of $\widetilde{\pi}$.

If V_1 is an invariant subspace of V and $V_2 = V_1 \setminus V$ we may associate to π representations π_1 and π_2 on V_1 and V_2. They are easily seen to be admissible. It is also clear that there is a natural embedding of $\widetilde{V}_2$ in $\widetilde{V}$. Moreover any element $\widetilde{v}_1$ of $\widetilde{V}_1$ lies in $\widetilde{V}_1(\xi)$ for some ξ and therefore is determined by its effect on $V_1(\check{\xi})$. It annihilates $\big(I - \pi(\check{\xi})\big)V_1$. There is certainly a linear function $\widetilde{v}$ on V which annihilates $\big(I - \pi(\check{\xi})\big)V$ and agrees with $\widetilde{V}_1$ on $V_1(\check{\xi})$. $\widetilde{v}$ is necessarily in $\widetilde{V}$ so that $\widetilde{V}_1$ may be identified with $\widetilde{V}_2 \setminus \widetilde{V}$. Since every representation is the contragredient of its contragredient we easily deduce the following lemma.

Lemma 2.5 *(a). Suppose V_1 is an invariant subspace of V. If $\widetilde{V}_2$ is the annihilator of V_1 in $\widetilde{V}$ then V_1 is the annihilator of $\widetilde{V}_2$ in V.*
(b) π is irreducible if and only if $\widetilde{\pi}$ is.

Observe that for all g in G_F

$$\langle \pi(g)v, \widetilde{v} \rangle = \langle v, \widetilde{\pi}(g^{-1})\widetilde{v} \rangle.$$

If π is the one-dimensional representation associated to the quasi-character χ then $\widetilde{\pi} = \chi^{-1}$. Moreover if χ is a quasi-character and π any admissible representation then the contragredient of $\chi \otimes \pi$ is $\chi^{-1} \otimes \widetilde{\pi}$.

Let V be a separable complete locally convex space and π a continuous representation of G_F on V. The space $V_0 = \pi(\mathcal{H}_F)V$ is invariant under G_F and the restriction π_0 of π to V_0 satisfies (2.1). Suppose that it also satisfies (2.2). Then if π is irreducible in the topological sense π_0 is algebraically irreducible. To see this take any two vectors v and w in V_0 and choose an elementary idempotent ξ so that $\pi(\xi)v = v$. v is in the closure of $\pi(\mathcal{H}_F)w$ and therefore in the closure of $\pi(\mathcal{H}_F)w \cap \pi(\xi)V$. Since, by assumption, $\pi(\xi)V$ is finite dimensional, v must actually lie in $\pi(\mathcal{H}_F)w$.

The equivalence class of π is not in general determined by that of π_0. It is, however, when π is unitary. To see this one has only to show that, up to a scalar factor, an irreducible admissible representation admits at most one invariant hermitian form.

Lemma 2.6 *Suppose π_1 and π_2 are irreducible admissible representations of G_F on V_1 and V_2 respectively. Suppose $A(v_1, v_2)$ and $B(v_1, v_2)$ are non-degenerate forms on $V_1 \times V_2$ which are linear in the first variable and either both linear or both conjugate linear in the second variable. Suppose moreover that, for all g in G_F*

$$A\big(\pi_1(g)v_1, \pi_2(g)v_2\big) = A(v_1, v_2)$$

and

$$B\big(\pi_1(g)v_1, \pi_2(g)v_2\big) = B(v_1, v_2)$$

Then there is a complex scalar λ such that

$$B(v_1, v_2) = \lambda A(v_1, v_2)$$

Define two mappings S and T of V_2 into $\widetilde{V}_1$ by the relations

$$A(v_1, v_2) = \langle v_1, Sv_2 \rangle$$

and

$$B(v_1, v_2) = \langle v_1, Tv_2 \rangle,$$

Since S and T are both linear or conjugate linear with kernel 0 they are both embeddings. Both take V_2 onto an invariant subspace of $\widetilde{V}_1$. Since $\widetilde{V}_1$ has no non-trivial invariant subspaces they are both isomorphisms. Thus $S^{-1}T$ is a linear map of V_2 which commutes with G_F and is therefore a scalar λI. The lemma follows.

An admissible representation will be called unitary if it admits an invariant positive definite hermitian form.

We now begin in earnest the study of irreducible admissible representations of G_F. The basic ideas are due to Kirillov.

Proposition 2.7. *Let π be an irreducible admissible representation of G_F on the vector space V.*

(a) If V is finite-dimensional then V is one-dimensional and there is a quasi-character χ of $F^\times$ such that

$$\pi(g) = \chi(\det g)$$

(b) If V is infinite dimensional there is no nonzero vector invariant by all the matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $x \in F$.

If π is finite-dimensional its kernel H is an open subgroup. In particular there is a positive number ϵ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to H if $|x| < \epsilon$. If x is any element of F there is an a in $F^\times$ such that $|ax| < \epsilon$. Since

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to H for all x in F. For similar reasons the matrices

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

do also. Since the matrices generate $SL(2,F)$ the group H contains $SL(2,F)$. Thus $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$ for all g_1 and g_2 in G_F. Consequently each $\pi(g)$ is a scalar matrix and $\pi(g)$ is one-dimensional. In fact

$$\pi(g) = \chi(\det g)I$$

where χ is a homorphism of $F^\times$ into $\mathbb{C}^\times$. To see that χ is continuous we need only observe that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi(a)I.$$

Suppose V contains a nonzero vector v fixed by all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Let H be the stabilizer of the space $\mathbb{C}v$. To prove the second part of the proposition we need only verify that H is of finite index in G_F. Since it contains the scalar matrices and an open subgroup of G_F it will be enough to show that it contains $SL(2,F)$. In fact we shall show that H_0, the stabilizer of v, contains $SL(2,F)$. H_0 is open and therefore contains a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$. It also contains

$$\begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b_0 \\ c & 0 \end{pmatrix} = w_0.$$

If $x = \frac{b_0}{c}\, y$ then

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0^{-1}$$

also belongs to H_0. As before we see that H_0 contains $SL(2,F)$.

Because of this lemma we can confine our attention to infinite-dimensional representations. Let $\psi = \psi_F$ be a nontrivial additive character of F. Let B_F be the group of matrices of the form

$$b = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with a in $F^\times$ and x in F. If X is a complex vector space we define a representation ξ_ψ of B_F on the space of all functions of $F^\times$ with values in X by setting

$$\big(\xi_\psi(b)\varphi\big)(\alpha) = \psi(\alpha x)\varphi(\alpha a).$$

ξ_ψ leaves the invariant space $\mathcal{S}(F^\times, X)$ of locally constant compactly supported functions. ξ_ψ is continuous with respect to the trivial topology on $\mathcal{S}(F^\times, X)$.

Proposition 2.8. *Let π be an infinite dimensional irreducible representation of G_F on the space V. Let $\mathfrak{p} = \mathfrak{p}_F$ be the maximal ideal in the ring of integers of F, and let V' be the set of all vectors v in V such that*

$$\int_{\mathfrak{p}^{-n}} \psi_F(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx = 0$$

for some integer n. Then

(i) The set V' is a subspace of V.

(ii) Let $X = V' \setminus V$ and let A be the natural map of V onto X. If v belongs to V let φ_v be the function defined by

$$\varphi_v(a) = A\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) v\right).$$

The map $v \to \varphi_v$ is an injection of V into the space of locally constant functions on $F^\times$ with value in X.

(iii) If b belongs to B_F and v belongs to V then

$$\varphi_{\pi(b)v} = \xi_\psi(b)\varphi_v.$$

If $m \geq n$ so that $\mathfrak{p}^{-m}$ contains $\mathfrak{p}^{-n}$ then

$$\int_{\mathfrak{p}^{-m}} \psi(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx$$

is equal to

$$\sum_{y \in \mathfrak{p}^{-m}/\mathfrak{p}^{-n}} \psi(-y)\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx.$$

Thus if the integral of the lemma vanishes for some integer n it vanishes for all larger integers. The first assertion of the proposition follows immediately.

To prove the second we shall use the following lemma.

Lemma 2.8.1 *Let $\mathfrak{p}^{-m}$ be the largest ideal on which ψ is trivial and let f be a locally constant function on $\mathfrak{p}^{-\ell}$ with values in some finite dimensional complex vector space. For any integer $n \le \ell$ the following two conditions are equivalent*
(i) f is constant on the cosets of $\mathfrak{p}^{-n}$ in $\mathfrak{p}^{-\ell}$
(ii) The integral

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx$$

is zero for all a outside of $\mathfrak{p}^{-m+n}$.

Assume (i) and let a be an element of $F^\times$ which is not in $\mathfrak{p}^{-m+n}$. Then $x \to \psi(-ax)$ is a non-trivial character of $\mathfrak{p}^{-n}$ and

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = \sum_{y \in \mathfrak{p}^{-\ell}/\mathfrak{p}^{-n}} \psi(-ay) \left\{ \int_{\mathfrak{p}^{-n}} \psi(-ax)\, dx \right\} f(y) = 0.$$

f may be regarded as a locally constant function on F with support in $\mathfrak{p}^{-\ell}$. Assuming (ii) is tantamount to assuming that the Fourier transform F' of f has its support in $\mathfrak{p}^{-m+n}$. By the Fourier inversion formula

$$f(x) = \int_{\mathfrak{p}^{-m+n}} \psi(-xy)\, f'(y)\, dy.$$

If y belongs to $\mathfrak{p}^{-m+n}$ the function $x \to \psi(-xy)$ is constant on cosets of $\mathfrak{p}^{-n}$. It follows immediately that the second condition of the lemma implies the first.

To prove the second assertion of the proposition we show that if φ_v vanishes identically then v is fixed by the operator $\pi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ for all x in F and then appeal to Proposition 2.7.

Take

$$f(x) = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v.$$

The restriction of f to an ideal in F takes values in a finite-dimensional subspace of V. To show that f is constant on the cosets of some ideal $\mathfrak{p}^{-n}$ it is enough to show that its restriction to some ideal $\mathfrak{p}^{-\ell}$ containing $\mathfrak{p}^{-n}$ has this property.

By assumption there exists an n_0 such that f is constant on the cosets of $\mathfrak{p}^{-n_0}$. We shall now show that if f is constant on the cosets of $\mathfrak{p}^{-n+1}$ it is also constant on the cosets of $\mathfrak{p}^{-n}$. Take any ideal $\mathfrak{p}^{-\ell}$ containing $\mathfrak{p}^{-n}$. By the previous lemma

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = 0$$

if a is not in $\mathfrak{p}^{-m+n-1}$. We have to show that the integral on the left vanishes if a is a generator of $\mathfrak{p}^{-m+n-1}$.

If U_F is the group of units of O_F the ring of integers of F there is an open subgroup U_1 of U_F such that

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) v = v$$

for b in U_1. For such b

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right)_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = \int_{\mathfrak{p}^{-\ell}} \psi(-ax) \pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx$$

is equal to

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\pi\left(\begin{pmatrix}1 & bx\\ 0 & 1\end{pmatrix}\right)\pi\left(\begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}\right)v\,dx = \int_{\mathfrak{p}^{-\ell}} \psi\left(-\frac{a}{b}x\right)f(x)\,dx.$$

Thus it will be enough to show that for some sufficiently large ℓ the integral vanishes when a is taken to be one of a fixed set of representatives of the cosets of U_1 in the set of generators of $\mathfrak{p}^{-m+n-1}$. Since there are only finitely many such cosets it is enough to show that for each a there is at least one ℓ for which the integral vanishes.

By assumption there is an ideal $\mathfrak{a}(a)$ such that

$$\int_{\mathfrak{a}(a)} \psi(-x)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)v\,dx = 0$$

But this integral equals

$$|a|\pi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\int_{a^{-1}\mathfrak{a}(a)} \psi(-ax)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx$$

so that $\ell = \ell(a)$ could be chosen to make

$$\mathfrak{p}^{-\ell} = a^{-1}\mathfrak{a}(a).$$

To prove the third assertion we verify that

$$A\left(\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v\right) = \psi(y)\,A(v) \tag{2.8.2}$$

for all v in V and all y in F. The third assertion follows from this by inspection. We have to show that

$$\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v - \psi(y)v$$

is in V' or that, for some n,

$$\int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v\,dx - \int_{\mathfrak{p}^{-n}} \psi(-x)\,\psi(y)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx$$

is zero. The expression equals

$$\int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}\right)v\,dx - \int_{\mathfrak{p}^{-n}} \psi(-x+y)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx.$$

If $\mathfrak{p}^{-n}$ contains y we may change the variables in the first integral to see that it equals the second.

It will be convenient now to identify v with φ_v so that V becomes a space of functions on $F^\times$ with values in X. The map A is replaced by the map $\varphi \to \varphi(1)$. The representation π now satisfies

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. There is a quasi-character ω_0 of $F^\times$ such that

$$\pi\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right) = \omega_0(a)\,I.$$

If

$$w = \left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)$$

the representation is determined by ω_0 and $\pi(w)$.

Proposition 2.9 (i) *The space V contains*

$$V_0 = \mathcal{S}(F^\times, X)$$

(ii) *The space V is spanned by V_0 and $\pi(w)V_0$.*

For every φ in V there is a positive integer n such that

$$\pi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)\varphi = \varphi$$

if x and $a-1$ belong to $\mathfrak{p}^n$. In particular $\varphi(\alpha a) = \varphi(a)$ if α belongs to $F^\times$ and $a-1$ belongs to $\mathfrak{p}^n$. The relation

$$\psi(\alpha x)\varphi(\alpha) = \varphi(\alpha)$$

for all x in $\mathfrak{p}^n$ implies that $\varphi(\alpha) = 0$ if the restriction of ψ to $\alpha\mathfrak{p}^n$ is not trivial. Let $\mathfrak{p}^{-m}$ be the largest ideal on which ψ is trivial. Then $\varphi(\alpha) = 0$ unless $|\alpha| \le |\varpi|^{-m-n}$ if ϖ is a generator of $\mathfrak{p}$.

Let V_0 be the space of all ϖ in V such that, for some integer ℓ depending on φ, $\varphi(\alpha) = 0$ unless $|\alpha| > |\varpi|^\ell$. To prove (i) we have to show that $V_0 = \mathcal{S}(F^\times, X)$. It is at least clear that $\mathcal{S}(F^\times, X)$ contains V_0. Moreover for every φ in V and every x in F the difference

$$\varphi' = \varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in V_0. To see this observe that

$$\varphi'(\alpha) = \big(1 - \psi(\alpha x)\big)\varphi(\alpha)$$

is identically zero for $x = 0$ and otherwise vanishes at least on $x^{-1}\mathfrak{p}^{-m}$. Since there is no function in V invariant under all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

the space V_0 is not 0.

Before continuing with the proof of the proposition we verify a lemma we shall need.

Lemma 2.9.1 *The representation ξ_ψ of B_F in the space $\mathcal{S}(F^\times)$ of locally constant, compactly supported, complex-valued functions on $F^\times$ is irreducible.*

For every character μ of U_F let φ_μ be the function on $F^\times$ which equals μ on U_F and vanishes off U_F. Since these functions and their translates span $\mathcal{S}(F^\times)$ it will be enough to show that any non-trivial invariant subspace contains all of them. Such a space must certainly contain some non-zero function φ which satisfies, for some character ν of U_F, the relation

$$\varphi(a\epsilon) = \nu(\epsilon)\,\varphi(a)$$

for all a in $F^\times$ and all ϵ in U_F. Replacing φ by a translate if necessary we may assume that $\varphi(1) \ne 0$. We are going to show that the space contains φ_μ if μ is different from ν. Since U_F has at least two characters we can then replace φ by some φ_μ with μ different from ν, and replace ν by μ and μ by ν to see it also contains φ_ν.

Set

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\right)\xi_\psi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi\, d\epsilon$$

where x is still to be determined. μ is to be different form ν. φ' belongs to the invariant subspace and

$$\varphi'(a\epsilon) = \mu(\epsilon)\varphi'(a)$$

for all a in $F^\times$ and all ϵ in U_F. We have

$$\varphi'(a) = \varphi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi(ax\epsilon)\, d\epsilon$$

The character $\mu^{-1}\nu$ has a conductor $\mathfrak{p}^n$ with n positive. Take x to be of order $-n-m$. The integral, which can be rewritten as a Gaussian sum, is then, as is well-known, zero if a is not in U_F but different from zero if a is in U_F. Since $\varphi(1)$ is not zero φ' must be a nonzero multiple of φ_μ.

To prove the first assertion of the proposition we need only verify that if u belongs to X then V_0 contains all functions of the form $\alpha \to \eta(\alpha)u$ with η in $\mathcal{S}(F^\times)$. There is a φ in V such that $\varphi(1) = u$. Take x such that $\psi(x) \neq 1$. Then

$$\varphi' = \varphi - \pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi$$

is in V_0 and $\varphi'(1) = \big(1-\psi(x)\big)u$. Consequently every u is of the form $\varphi(1)$ for some φ in V_0.

If μ is a character of U_F let $V_0(\mu)$ be the space of functions φ in V_0 satisfying

$$\varphi(a\epsilon) = \mu(\epsilon)\varphi(a)$$

for all a in $F^\times$ and all ϵ in U_F. V_0 is clearly the direct sum of the space $V_0(\mu)$. In particular every vector u in X can be written as a finite sum

$$u = \sum \varphi_i(1)$$

where φ_i belongs to some $V_0(\mu_i)$.

If we make use of the lemma we need only show that if u can be written as $u = \varphi(1)$ where φ is in $V_0(\nu)$ for some ν then there is at least one function in V_0 of the form $\alpha \to \eta(\alpha)u$ where η is a nonzero function in $\mathcal{S}(F^\times)$. Choose μ different from ν and let $\mathfrak{p}^n$ be the conductor of $\mu^{-1}\nu$. We again consider

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi\, d\epsilon$$

where x is of order $-n-m$. Then

$$\varphi'(a) = \varphi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi_F(ax\epsilon)\, d\epsilon$$

The properties of Gaussian sums used before show that φ' is a function of the required kind.

The second part of the proposition is easier to verify. Let P_F be the group of upper-triangular matrices in G_F. Since V_0 is invariant under P_F and V is irreducible under G_F the space V is spanned by V_0 and the vectors

$$\varphi' = \pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\pi(w)\varphi$$

with φ in V_0. But

$$\varphi' = \{\varphi' - \pi(w)\varphi\} + \pi(w)\varphi$$

and as we saw, $\varphi' - \pi(w)\varphi$ is in V_0. The proposition is proved.

To study the effect of w we introduce a formal Mellin transform. Let ϖ be a generator of $\mathfrak{p}$. If φ is a locally constant function on $F^\times$ with values in X then for every integer n the function $\epsilon \to \varphi(\epsilon\varpi^n)$ on U_F takes its values in a finite-dimensional subspace of X so that the integral

$$\int_{U_F} \varphi(\epsilon\varpi^n)\nu(\epsilon) = \widehat{\varphi}_n(\nu)$$

is defined. In this integral we take the total measure of U_F to be 1. It is a vector in X. $\widehat{\varphi}(\nu, t)$ will be the Formal Laurent series

$$\sum_t t^n \widehat{\varphi}_n(\nu)$$

If φ is in V the series has only a finite number of terms with negative exponent. Moreover the series $\widehat{\varphi}(\nu, t)$ is different from zero for only finitely many ν. If φ belongs to V_0 these series have only finitely many terms. It is clear that if φ is locally constant and all the formal series $\widehat{\varphi}(\nu, t)$ vanish then $\varphi = 0$.

Suppose φ takes values in a finite-dimensional subspace of X, ω is a quasi-character of $F^\times$, and the integral

$$\int_{F^\times} \omega(a)\varphi(a)\, d^\times a \tag{2.10.1}$$

is absolutely convergent. If ω' is the restriction of ω to U_F this integral equals

$$\sum_n z^n \int_{U_F} \varphi(\varpi^n\epsilon)\, \omega'(\epsilon)\, d\epsilon = \sum_n z^n\, \widehat{\varphi}_n(\omega')$$

if $z = \omega(\varpi)$. Consequently the formal series $\widehat{\varphi}(\omega', t)$ converges absolutely for $t = z$ and the sum is equal to (2.10.1). We shall see that X is one dimensional and that there is a constant $c_0 = c_0(\varphi)$ such that if $|\omega(\varpi)| = |\varpi|^c$ with $c > c_0$ then the integral (2.10.1) is absolutely convergent. Consequently all the series $\widehat{\varphi}(\nu, t)$ have positive radii of convergence.

If $\psi = \psi_F$ is a given non-trivial additive character of F, μ any character of U_F, and x any element of F we set

$$\eta(\mu, x) = \int_{U_F} \mu(\epsilon)\, \psi(\epsilon x)\, d\epsilon$$

The integral is taken with respect to the normalized Haar measure on U_F. If g belongs to G_F, φ belongs to V, and $\varphi' = \pi(g)\varphi$ we shall set

$$\pi(g)\, \widehat{\varphi}(\nu, t) = \widehat{\varphi}'(\nu, t).$$

Proposition 2.10 (i) *If δ belongs to U_F and ℓ belongs to $\mathbb{Z}$ then*

$$\pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = t^{-\ell}\nu^{-1}(\delta)\, \widehat{\varphi}(\nu, t)$$

(ii) *If x belongs to F then*

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = \sum_n t^n \left\{ \sum_\mu \eta(\mu^{-1}\nu, \varpi^n x)\widehat{\varphi}_n(\mu) \right\}$$

where the inner sum is taken over all characters of U_F

(iii) *Let ω_0 be the quasi-character defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)\, I$$

for a in $F^\times$. Let ν_0 be its restriction to U_F and let $z_0 = \omega_0(\varpi)$. For each character ν of U_F there is a formal series $C(\nu, t)$ with coefficients in the space of linear operators on X such that for every φ in V_0

$$\pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = C(\nu, t)\, \widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1} z_0^{-1}).$$

Set

$$\varphi' = \pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \nu(\epsilon)\, \varphi(\varpi^{n+\ell}\, \delta\epsilon)\, d\epsilon.$$

Changing variables in the integration and in the summation we obtain the first formula of the proposition.

Now set

$$\varphi' = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \psi(\varpi^n \epsilon x)\, \nu(\epsilon)\, \varphi(\varpi^n \epsilon)\, d\epsilon.$$

By Fourier inversion

$$\varphi(\varpi^n \epsilon) = \sum_\mu \widehat{\varphi}_n(\mu)\, \mu^{-1}(\epsilon).$$

The sum on the right is in reality finite. Substituting we obtain

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \left\{ \sum_\mu \int_{U_F} \mu^{-1}\nu(\epsilon)\, \psi(\epsilon\varpi^n x)\, d\epsilon\, \widehat{\varphi}_n(\mu) \right\}$$

as asserted.

Suppose ν is a character of U_F and φ in V_0 is such that $\widehat{\varphi}(\mu, t) = 0$ unless $\mu = \nu^{-1}\nu_0^{-1}$. This means that

$$\varphi(a\epsilon) \equiv \nu\nu_0(\epsilon)\, \varphi(a)$$

or that

$$\pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi = \nu\nu_0(\epsilon)\varphi$$

for all ϵ in U_F. If $\varphi' = \pi(w)\varphi$ then

$$\pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi' = \pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi = \pi(w)\pi\left(\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\right)\varphi.$$

Since

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\right) = \left(\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\right)\left(\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

the expression on the right is equal to

$$\nu^{-1}(\epsilon)\pi(w)\varphi = \nu^{-1}(\epsilon)\varphi',$$

so that $\widehat{\varphi}'(\mu, t) = 0$ unless $\mu = \nu$.

Now take a vector u in X and a character ν of U_F and let φ be the function in V_0 which is zero outside of U_F and on U_F is given by

$$\varphi(\epsilon) = \nu(\epsilon)\,\nu_0(\epsilon)u. \tag{2.10.2}$$

If $\varphi' = \pi(w)\varphi$ then $\widehat{\varphi}'_n$ is a function of n, ν, and u which depends linearly on u and we may write

$$\widehat{\varphi}'_n(\nu) = C_n(\nu)u$$

where $C_n(\nu)$ is a linear operator on X.

We introduce the formal series

$$C(\nu, t) = \sum t^n C_n(\nu).$$

We have now to verify the third formula of the proposition. Since φ is in V_0 the product on the right is defined. Since both sides are linear in φ we need only verify it for a set of generators of V_0. This set can be taken to be the functions defined by (2.10.2) together with their translates of power ϖ. For functions of the form (2.10.2) the formula is valid because of the way the various series $C(\nu, t)$ were defined. Thus all we have to do is show that if the formula is valid for a given function φ it remains valid when φ is replaced by

$$\pi\left(\begin{pmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

By part (i) the right side is replaced by

$$z_0^\ell t^\ell C(\nu, t)\,\widetilde{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$

Since

$$\pi(w)\pi\left(\begin{pmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi = \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{pmatrix}\right)\pi(w)\varphi$$

and $\pi(w)\widehat{\varphi}(\nu, t)$ is known we can use part (i) and the relation

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{pmatrix} = \begin{pmatrix} \varpi^\ell & 0 \\ 0 & \varpi^\ell \end{pmatrix}\begin{pmatrix} \varpi^{-\ell} & 0 \\ 0 & 1 \end{pmatrix}$$

to see that the left side is replaced by

$$z_0^\ell t^\ell \pi(w)\widehat{\varphi}(\nu, t) = z_0^\ell t^\ell C(\nu, t)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$

For a given u in X and a given character ν of U_F there must exist a φ in V such that

$$\widehat{\varphi}(\nu, t) = \sum t^n C_n(\nu)u$$

Consequently there is an n_0 such that $C_n(\nu)u = 0$ for $n < n_0$. Of course n_0 may depend on u and ν. This observation together with standard properties of Gaussian sums shows that the infinite sums occurring in the following proposition are meaningful, for when each term is multiplied on the right by a fixed vector in X all but finitely many disappear.

Proposition 2.11 *Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial.*

(i) *Let ν and ρ be two characters of U_F such that $\nu\rho\nu_0$ is not 1. Let $\mathfrak{p}^m$ be its conductor. Then*

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma)$$

is equal to

$$\eta(\nu^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell})z_0^{m+\ell}\nu\rho\nu_0(-1)C_{n-m-\ell}(\nu)C_{p-m-\ell}(\rho)$$

for all integers n and p.

(ii) *Let ν be any character of U_F and let $\widetilde{\nu} = \nu^{-1}\nu_0^{-1}$. Then*

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\widetilde{\nu}, \varpi p)C_{p+n}(\sigma)$$

is equal to

$$z_0^p\nu_0(-1)\delta_{n,p} + (|\varpi| - 1)^{-1}z_0^{\ell+1}C_{n-1-\ell}(\nu)C_{p-1-\ell}(\widetilde{\nu}) - \sum_{-2-\ell}^{-\infty} z^{-r}C_{n+r}(\nu)C_{p+r}(\widetilde{\nu})$$

for all integers n and p.

The left hand sums are taken over all characters σ of U_F and $\delta_{n,p}$ is Kronecker's delta. The relation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\pi(w)\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi$$

for all φ in V_0. Since $\pi(w)\varphi$ is not necessarily in V_0 we write this relation as

$$\pi(w)\left\{\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi - \pi(w)\varphi\right\} + \pi^2(w)\varphi = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

The term $\pi^2(w)\varphi$ is equal to $\nu_0(-1)\varphi$.

We compute the Mellin transforms of both sides

$$\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n\left\{\sum_\rho \eta(\rho^{-1}\nu, -\varpi^n)\widehat{\varphi}_n(\rho)\right\}$$

and

$$\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n \sum_{p,\rho} \eta(\rho^{-1}\nu^{-1}\nu_0^{-1}, -\varpi^p)z_0^{-P}C_{p+n}(\nu)\widehat{\varphi}_p(\rho)$$

so that the Mellin transform of the right side is

$$\nu_0(-1)\sum_n t^n \sum_{p,\rho,\sigma} \eta(\sigma^{-1}\nu, -\varpi^n)\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p)z_0^{-p}C_{p+n}(\sigma)\widehat{\varphi}_p(\rho). \tag{2.11.1}$$

On the other hand

$$\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_p z_0^{-p}C_{p+n}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})$$

and

$$\pi\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_{p,\rho} z_0^{-p}\eta(\rho^{-1}\nu,\varpi^n)C_{p+n}(\rho)\widehat{\varphi}_p(\rho^{-1}\nu_0^{-1})$$

so that

$$\pi\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu,t)-\pi(w)\widehat{\varphi}(\nu,t)$$

is equal to

$$\sum_n t^n\sum_{p,\rho} z_0^{-p}[\eta(\rho\nu\nu_0,\varpi^n)-\delta(\rho\nu\nu_0)]C_{p+n}(\rho^{-1}\nu_0^{-1})\widehat{\varphi}_p(\rho).$$

Here $\delta(\rho\nu\nu_0)$ is 1 if $\rho\nu\nu_0$ is the trivial character and 0 otherwise. The Mellin transform of the left hand side is therefore

$$\sum t^n\sum_{p,r,\rho} z_0^{-p-r}[\eta(\rho\nu^{-1},\varpi^r)-\delta(\rho\nu^{-1})]C_{n+r}(\nu)C_{p+r}(\rho^{-1}\nu_0^{-1})\widehat{\varphi}_p(\rho)+\nu_0(-1)\sum t^n\widehat{\varphi}_n(\nu). \quad (2.11.2)$$

The coefficient of $t^n\widehat{\varphi}_p(\rho)$ in (2.11.1) is

$$\nu_0(-1)\sum_\sigma \eta(\sigma^{-1}\nu,-\varpi^n)\,\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1},-\varpi^p)z_0^{-1}C_{p+n}(\sigma) \quad (2.11.3)$$

and in (2.11.2) it is

$$\sum_r[\eta(\rho\nu^{-1},\varpi^r)-\delta(\rho\nu^{-1})]z_0^{-p-r}C_{n+r}(\nu)C_{p+r}(\rho^{-1}\nu_0^{-1})+\nu_0(-1)\delta_{n,\rho}\delta(\rho\nu^{-1})I \quad (2.11.4)$$

These two expressions are equal for all choice of n, p, ρ, and ν.

If $\rho\neq\nu$ and the conductor of $\nu\rho^{-1}$ is $\mathfrak{p}^m$ the gaussian sum $\eta(\rho\nu^{-1},\varpi^r)$ is zero unless $r=-m-\ell$. Thus (2.11.4) reduces to

$$\eta(\rho\nu^{-1},\varpi^{-m-\ell})z_0^{-p-m-\ell}C_{n-m-\ell}(\nu)C_{p-m-\ell}(\rho^{-1}\nu_0^{-1}).$$

Since

$$\eta(\mu,-x)=\mu(-1)\,\eta(\mu,x)$$

the expression (2.11.3) is equal to

$$\rho^{-1}\nu(-1)\sum_\sigma\eta(\sigma^{-1}\nu,\varpi^n)\,\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}\varpi^p)z_0^{-p}C_{p+n}(\sigma).$$

Replacing ρ by $\rho^{-1}\nu_0^{-1}$ we obtain the first part of the proposition.

If $\rho=\nu$ then $\delta(\rho\nu^{-1})=1$. Moreover, as is well-known and easily verified, $\eta(\rho\nu^{-1},\varpi^r)=1$ if $r\geq-\ell$,

$$\eta(\rho\nu^{-1},\varpi^{-\ell-1})=|\varpi|(|\varpi|-1)^{-1}$$

and $\eta(\rho\nu^{-1},\varpi^r)=0$ if $r\leq-\ell-2$. Thus (2.11.4) is equal to

$$\nu_0(-1)\delta_{n,p}I+(|\varpi|-1)^{-1}z_0^{-p+\ell+1}C_{n-\ell-1}(\nu)C_{n-\ell-1}(\nu^{-1}\nu_0^{-1})-\sum_{r=-\ell-2}^{-\infty}z_0^{-p-r}C_{n+r}(\nu)C_{n+r}(\nu^{-1}\nu_0^{-1}).$$

The second part of the proposition follows.

Proposition(2.12) (i) *For every n, p, ν and ρ*

$$C_n(\nu)C_p(\rho) = C_p(\rho)C_n(\nu)$$

(ii) *There is no non-trivial subspace of X invariant under all the operators $C_n(\nu)$.*
(iii) *The space X is one-dimensional.*

Suppose $\rho\nu\nu_0 \neq 1$. The left side of the first identity in the previous proposition is symmetric in the two pairs (n,ν) and (p,ρ). Since $(\eta^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell})$ is not zero we conclude that

$$C_{n-m-\ell}(\nu)\, C_{p-m-\ell}(\rho) = C_{p-m-\ell}(\rho)\, C_{n-m-\ell}(\nu)$$

for all choices of n and p. The first part of the proposition is therefore valid in $\rho \neq \widetilde{\nu}$.

Now suppose $\rho = \widetilde{\nu}$. We are going to that if (p,n) is a given pair of integers and u belongs to X then

$$C_{n+r}(\nu)C_{p+r}(\widetilde{\nu})u = C_{p+r}(\widetilde{\nu})C_{n+r}(\nu)u$$

for all r in $\mathbb{Z}$. If $r \ll 0$ both sides are 0 and the relation is valid so the proof can proceed by induction on r. For the induction one uses the second relation of Proposition 2.11 in the same way as the first was used above.

Suppose X_1 is a non-trivial subspace of X invariant under all the operators $C_n(\nu)$. Let V_1 be the space of all functions in V_0 which take values in X_1 and let V_1' be the invariant subspace generated by V_1. We shall show that all functions in V_1' take values in X_1 so that V_1' is a non-trivial invariant subspace of V. This will be a contradiction. If φ in V takes value in X_1 and g belongs to P_F then $\pi(g)\varphi$ also takes values in X_1. Therefore all we need to do is show that if φ is in V_1 then $\pi(w)\varphi$ takes values in X_1. This follows immediately from the assumption and Proposition 2.10.

To prove (iii) we show that the operators $C_n(\nu)$ are all scalar multiples of the identity. Because of (i) we need only show that every linear transformation of X which commutes with all the operators $C_n(\nu)$ is a scalar. Suppose T is such an operator. If φ belongs to V let T_φ be the function from $F^\times$ to X defined by

$$T\varphi(a) = T\bigl(\varphi(a)\bigr).$$

Observe that $T\varphi$ is still in V. This is clear if φ belongs to V_0 and if $\varphi = \pi(w)\varphi_0$ we see on examining the Mellin transforms of both sides that

$$T\varphi = \pi(w)T\varphi_0.$$

Since $V = V_0 + \pi(w)V_0$ the observation follows. T therefore defines a linear transformation of V which clearly commutes with the action of any g in P_F. If we can show that it commutes with the action of w it will follow that it and, therefore, the original operator on X are scalars. We have to verify that

$$\pi(w)T\varphi = T\pi(w)\varphi$$

at least for φ on V_0 and for $\varphi = \pi(w)\varphi_0$ with φ_0 in V_0. We have already seen that the identity holds for φ in V_0. Thus if $\varphi = \pi(w)\varphi_0$ the left side is

$$\pi(w)T\pi(w)\varphi_0 = \pi^2(w)T\varphi_0 = \nu_0(-1)T\varphi_0$$

and the right side is

$$T\pi^2(w)\varphi_0 = \nu_0(-1)T\varphi_0.$$

Because of this proposition we can identify X with $\mathbb{C}$ and regard the operators $C_n(\nu)$ as complex numbers. For each r the formal Laurent series $C(\nu,t)$ has only finitely many negative terms. We now want to show that the realization of π on a space of functions on $F^\times$ is, when certain simple conditions are imposed, unique so that the series $C(\nu,t)$ are determined by the class of π and that conversely the series $C(\nu,t)$ determine the class of π.

Theorem 2.13 *Suppose an equivalence class of infinite-dimensional irreducible admissible representations of G_F is given. Then there exists exactly one space V of complex-valued functions on $F^\times$ and exactly one representation π of G_F on V which is in this class and which is such that*

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F and φ is in V.

We have proved the existence of one such V and π. Suppose V' is another such space of functions and π' a representation of G_F on V' which is equivalent to π. We suppose of course that

$$\pi'(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F and φ is in V'. Let A be an isomorphism of V with V' such that $A\pi(g) = \pi'(g)A$ for all g. Let L be the linear functional

$$L(\varphi) = A\varphi(1)$$

on V. Then

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = A\varphi(a)$$

so that A is determined by L. If we could prove the existence of a scalar λ such that $L(\varphi) = \lambda\varphi(1)$ it would follow that

$$A\varphi(a) = \lambda\varphi(a)$$

for all a such that $A\varphi = \lambda\varphi$. This equality of course implies the theorem.

Observe that

$$L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)A\varphi(1) = \psi(x)L(\varphi). \tag{2.13.1}$$

Thus we need the following lemma.

Lemma 2.13.2 *If L is a linear functional on V satisfying (2.13.1) there is a scalar λ such that*

$$L(\varphi) = \lambda\varphi(1).$$

This is a consequence of a slightly different lemma.

Lemma 2.13.3 *Suppose L is a linear functional on the space $\mathcal{S}(F^\times)$ of locally constant compactly supported functions on $F^\times$ such that*

$$L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \psi(x)\,L(\varphi)$$

for all φ in $\mathcal{S}(F^\times)$ and all x in F. Then there is a scalar λ such that $L(\varphi) = \lambda\varphi(1)$.

Suppose for a moment that the second lemma is true. Then given a linear functional L on V satisfying (2.13.1) there is a λ such that $L(\varphi) = \lambda\varphi(1)$ for all φ in $V_0 = \mathcal{S}(F^\times)$. Take x in F such that $\psi(x) \neq 1$ and φ in V. Then

$$L(\varphi) = L\left(\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) + L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right).$$

Since

$$\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in V_0 the right side is equal to

$$\lambda\varphi(1) - \lambda\psi(x)\varphi(1) + \psi(x)L(\varphi)$$

so that

$$\big(1-\psi(x)\big)\,L(\varphi) = \lambda\big(1-\psi(x)\big)\,\varphi(1)$$

which implies that $L(\varphi) = \lambda\varphi(1)$.

To prove the second lemma we have only to show that $\varphi(1) = 0$ implies $L(\varphi) = 0$. If we set $\varphi(0) = 0$ then φ becomes a locally constant function with compact support in F. Let φ' be its Fourier transform so that

$$\varphi(a) = \int_F \psi(ba)\,\varphi'(-b)\,db.$$

Let Ω be an open compact subset of $F^\times$ containing 1 and the support of φ. There is an ideal $\mathfrak{a}$ in F so that for all a in Ω the function $\varphi'(-b)\psi(ba)$ is constant on the cosets of $\mathfrak{a}$ in F. Choose an ideal $\mathfrak{b}$ containing $\mathfrak{a}$ and the support of φ'. If S is a set of representatives of $\mathfrak{b}/\mathfrak{a}$ and if c is the measure of $\mathfrak{a}$ then

$$\varphi(a) = c\sum_{b\in S}\psi(ba)\varphi'(-b).$$

If φ_0 is the characteristic function of Ω this relation may be written

$$\varphi = \sum_{b\in S}\lambda_b\xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

with $\lambda_b = c\varphi'(-b)$. If $\varphi(1) = 0$ then

$$\sum_{b\in S}\lambda_b\psi(b) = 0$$

so that

$$\varphi = \sum\lambda_b\left\{\xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0 - \psi(b)\varphi_0\right\}$$

It is clear that $L(\varphi) = 0$.

The representation of the theorem will be called the Kirillov model. There is another model which will be used extensively. It is called the Whittaker model. Its properties are described in the next theorem.

Theorem 2.14 (i) *For any φ in V set*

$$W_\varphi(g) = \big(\pi(g)\varphi\big)(1)$$

so that W_φ is a function in G_F. Let $W(\pi,\psi)$ be the space of such functions. The map $\varphi \to W_\varphi$ is an isomorphism of V with $W(\pi,\psi)$. Moreover

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

(ii) *Let $W(\psi)$ be the space of all functions W on G_F such that*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) g = \psi(x)W(g)$$

for all x in F and g in G. Then $W(\pi,\psi)$ is contained in $W(\psi)$ and is the only invariant subspace which transforms according to π under right translations.

Since

$$W_\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)(1) = \varphi(a)$$

the function W_φ is 0 only if φ is. Since

$$\rho(g)W(h) = W(hg)$$

the relation

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

is clear. Then $W(\pi,\psi)$ is invariant under right translation and transforms according to π.

Since

$$W_\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\pi(g)\varphi\right)(1) = \psi(x)\{\pi(g)\varphi(1)\}$$

the space $W(\pi,\psi)$ is contained in $W(\psi)$. Suppose W is an invariant subspace of $W(\psi)$ which transforms according to π. There is an isomorphism A of V with W such that

$$A\big(\pi(g)\varphi\big) = \rho(g)(A\varphi).$$

Let

$$L(\varphi) = A\varphi(1).$$

Since

$$L\big(\pi(g)\varphi\big) = A\pi(g)\varphi(1) = \rho(g)A\varphi(1) = A\varphi(g)$$

the map A is determined by L. Also

$$L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = A\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(x)A\varphi(1) = \psi(x)L(\varphi)$$

so that by Lemma 2.13.2 there is a scalar λ such that

$$L(\varphi) = \lambda\varphi(1).$$

Consequently $A\varphi = \lambda W_\varphi$ and $W = W(\pi, \psi)$.

The realization of π on $W(\pi,\psi)$ will be called the Whittaker model. Observe that the representation of G_F on $W(\psi)$ contains no irreducible finite-dimensional representations. In fact any such representation is of the form

$$\pi(g) = \chi(\det g).$$

If π were contained in the representation on $W(\psi)$ there would be a nonzero function W on G_F such that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\chi(\det g)W(e)$$

In particular taking $g = e$ we find that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(x)W(e)$$

However it is also clear that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi\left(\det\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)W(e) = W(e)$$

so that $\psi(x) = 1$ for all x. This is a contradiction. We shall see however that π is a constituent of the representation on $W(\psi)$. That is, there are two invariant subspaces W_1 and W_2 of $W(\psi)$ such that W_1 contains W_2 and the representation of the quotient space W_1/W_2 is equivalent to π.

Proposition 2.15 *Let π and π' be two infinite-dimensional irreducible representations of G_F realized in the Kirillov form on spaces V and V'. Assume that the two quasi-characters defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

are the same. Let $\{C(\nu,t)\}$ and $\{C'(\nu,t)\}$ be the families of formal series associated to the two representations. If

$$C(\nu,t) = C'(\nu,t)$$

for all ν then $\pi = \pi'$.

If φ belongs to $\mathcal{S}(F^\times)$ then, by hypothesis,

$$\pi(w)\widehat{\varphi}(\nu,t) = \pi'(w)\widehat{\varphi}(\nu,t)$$

so that $\pi(w)\varphi = \pi'(w)\varphi$. Since V is spanned by $\mathcal{S}(F^\times)$ and $\pi(w)\mathcal{S}(F^\times)$ and V' is spanned by $\mathcal{S}(F^\times)$ and $\pi'(w)\mathcal{S}(F^\times)$ the spaces V and V' are the same. We have to show that

$$\pi(g)\varphi = \pi'(g)\varphi$$

for all φ in V and all g in G_F. This is clear if g is in P_F so it is enough to verify it for $g = w$. We have already observed that $\pi(w)\varphi_0 = \pi'(w)\varphi_0$ if φ_0 is in $\mathcal{S}(F^\times)$ so we need only show that $\pi(w)\varphi = \pi'(w)\varphi$ if φ is of the form $\pi(w)\varphi_0$ with φ_0 in $\mathcal{S}(F^\times)$. But $\pi(w)\varphi = \pi^2(w)\varphi_0 = \omega(-1)\varphi_0$ and, since $\pi(w)\varphi_0 = \pi'(w)\varphi_0$, $\pi'(w)\varphi = \omega'(-1)\varphi_0$.

Let N_F be the group of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with x in F and let $\mathcal{B}$ be the space of functions on G_F invariant under left translations by elements of N_F. $\mathcal{B}$ is invariant under right translations and the question of whether or not a given irreducible representation π is contained in $\mathcal{B}$ arises. The answer is obviously positive when $\pi = \chi$ is one-dimensional for then the function $g \to \chi(\det g)$ is itself contained in $\mathcal{B}$.

Assume that the representation π which is given in the Kirillov form acts on $\mathcal{B}$. Then there is a map A of V into $\mathcal{B}$ such that

$$A\pi(g)\varphi = \rho(g)A\varphi$$

If $L(\varphi) = A\varphi(1)$ then

$$L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = L(\varphi) \tag{2.15.1}$$

for all φ in V and all x in F. Conversely given such a linear form the map $\varphi \to A\varphi$ defined by

$$A\varphi(g) = L\big(\pi(g)\varphi\big)$$

satisfies the relation $A\pi(g) = \rho(g)A$ and takes V into $\mathcal{B}$. Thus π is contained in $\mathcal{B}$ if an only if there is a non-trivial linear form L on V which satisfies (2.15.1).

Lemma 2.15.2 *If L is a linear form on $\mathcal{S}(F^\times)$ which satisfies (2.15.1) for all x in F and for all φ in $\mathcal{S}(F^\times)$ then L is zero.*

We are assuming that L annihilates all functions of the form

$$\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi - \varphi$$

so it will be enough to show that they span $\mathcal{S}(F^\times)$. If φ belongs to $\mathcal{S}(F^\times)$ we may set $\varphi(0) = 0$ and regard φ as an element of $\mathcal{S}(F)$. Let φ' be its Fourier transform so that

$$\varphi(x) = \int_F \varphi'(-b)\psi(bx)\,db.$$

Let Ω be an open compact subset of $F^\times$ containing the support of φ and let $\mathfrak{p}^{-n}$ be an ideal containing Ω. There is an ideal $\mathfrak{a}$ of F such that $\varphi'(-b)\psi(bx)$ is, as a function of b, constant on cosets of $\mathfrak{a}$ for all x in $\mathfrak{p}^{-n}$. Let $\mathfrak{b}$ be an ideal containing both $\mathfrak{a}$ and the support of φ'. If S is a set of representatives for the cosets of $\mathfrak{a}$ in $\mathfrak{b}$, if c is the measure of $\mathfrak{a}$, and if φ_0 is the characteristic function of Ω then

$$\varphi(x) = \sum_{b\in S}\lambda_b\psi(bx)\varphi_0(x)$$

if $\lambda_b = c\varphi'(-b)$. Thus

$$\varphi = \sum_b \lambda_b\xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0.$$

Since $\varphi(0) = 0$ we have

$$\sum_b \lambda_b = 0$$

so that

$$\varphi = \sum_b \lambda_b \left\{ \xi_\psi \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0 - \varphi_0 \right\}$$

as required.

Thus any linear form on V verifying (2.15.1) annihilates $\mathcal{S}(F^\times)$. Conversely any linear form on V annihilating $\mathcal{S}(F^\times)$ satisfies (2.15.1) because

$$\xi_\psi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi - \varphi$$

is in $\mathcal{S}(F^\times)$ if φ is in V. We have therefore proved

Proposition 2.16 *For any infinite-dimensional irreducible representation π the following two properties are equivalent:*

(i) *π is not contained in $\mathcal{B}$.*

(ii) *The Kirillov model of π is realized in the space $\mathcal{S}(F^\times)$.*

A representation satisfying these two conditions will be called absolutely cuspidal.

Lemma 2.16.1 *Let π be an infinite-dimensional irreducible representation realized in the Kirillov form on the space V. Then $V_0 = \mathcal{S}(F^\times)$ is of finite codimension in V.*

We recall that $V = V_0 + \pi(w)V_0$. Let V_1 be the space of all φ in V_0 with support in U_F. An element of $\pi(w)V_0$ may always be written as a linear combination of functions of the form

$$\pi \left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix} \right) \pi(w)\varphi$$

with φ in V_1 and p in $\mathbb{Z}$. For each character μ of U_F let φ_μ be the function in V_1 such that $\varphi_\mu(\epsilon) = \mu(\epsilon)\nu_0(\epsilon)$ for ϵ in U_F. Then

$$\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu\nu_0)$$

and

$$\pi(w)\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu^{-1})C(\nu, t).$$

Let $\eta_\mu = \pi(w)\varphi_\mu$. The space V is spanned by V_0 and the functions

$$\pi \left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix} \right) \eta_\mu$$

For the moment we take the following two lemmas for granted.

Lemma 2.16.2 *For any character μ of $\widehat{U}_F$ there is an integer n_0 and a family of constants λ_i, $1 \le i \le p$, such that*

$$C_n(\mu) = \sum_{i=1}^{p} \lambda_i C_{n-i}(\mu)$$

for $n \ge n_0$.

Lemma 2.16.3 *There is a finite set S of characters of U_F such that for ν not in S the numbers $C_n(\nu)$ are 0 for all but finitely many n.*

If μ is not in S the function η_μ is in V_0. Choose μ in S and let V_μ be the space spanned by the functions

$$\pi\left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix}\right)\eta_\mu$$

and the functions φ in V_0 satisfying $\varphi(a\epsilon) = \varphi(a)\mu^{-1}(\epsilon)$ for all a in $F^\times$ and all ϵ in U_F. It will be enough to show that $V_\mu/V_\mu \cap V_0$ is finite-dimensional.

If φ is in V_μ then $\widehat{\varphi}(\nu, t) = 0$ unless $\nu = \mu$ and we may identify φ with the sequence $\{\widehat{\varphi}_n(\mu)\}$. The elements of $V_\mu \cap V_0$ are the elements corresponding to sequences with only finitely many nonzero terms. Referring to Proposition 2.10 we see that all of the sequences satisfying the recursion relation

$$\widehat{\varphi}_n(\mu) = \sum_{i=1}^{p} \lambda_a \widehat{\varphi}_{n-i}(\mu)$$

for $n \ge n_1$. The integer n_1 depends on φ.

Lemma 2.16.1 is therefore a consequence of the following elementary lemma whose proof we postpone to Paragraph 8.

Lemma 2.16.4 *Let λ_i, $1 \le i \le p$, be p complex numbers. Let A be the space of all sequences $\{a_n\}$, $n \in \mathbb{Z}$ for which there exist two integers n_1 and n_2 such that*

$$a_n = \sum_{1 \le i \le p} \lambda_i a_{n-i}$$

for $n \ge n_1$ and such that $a_n = 0$ for $n \le n_2$. Let A_0 be the space of all sequences with only a finite number of nonzero terms. Then A/A_0 is finite-dimensional.

We now prove Lemma 2.16.2. According to Proposition 2.11

$$\sum_{\sigma} \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\widetilde{\nu}, \varpi^p) C_{p+n}(\sigma)$$

is equal to

$$z_0^p \nu_0(-1)\delta_{n,p} + (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\widetilde{\nu}) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\widetilde{\nu}).$$

Remember that $\mathfrak{p}^{-\ell}$ is the largest ideal on which ψ is trivial. Suppose first that $\widetilde{\nu} = \nu$.

Take $p = -\ell$ and $n > -\ell$. Then $\delta(n-p) = 0$ and

$$\eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\nu, \varpi^p) = 0$$

unless $\sigma = \nu$. Hence

$$C_{n-\ell}(\nu) = (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{-2\ell-1}(\nu) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{-\ell+r}(\nu)$$

which, since almost all of the coefficients $C_{-\ell+r}(\nu)$ in the sum are zero, is the relation required.

If $\nu \neq \widetilde{\nu}$ take $p \geq -\ell$ and $n > p$. Then $\eta(\sigma^{-1}\nu, \varpi^n) = 0$ unless $\sigma = \nu$ and $\eta(\sigma^{-1}\nu, \varpi^p) = 0$ unless $\sigma = \widetilde{\nu}$. Thus

$$(|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\widetilde{\nu}) - \sum_{2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\widetilde{\nu}) = 0. \tag{2.16.5}$$

There is certainly at least one i for which $C_i(\widetilde{\nu}) \neq 0$. Take $p - 1 - \ell \geq i$. Then from (2.16.5) we deduce a relation of the form

$$C_{n+r}(\nu) = \sum_{i=1}^{q} \lambda_i C_{n+r-i}(\nu)$$

where r is a fixed integer and n is any integer greater than p.

Lemma 2.16.3 is a consequence of the following more precise lemma. If $\mathfrak{p}^m$ is the conductor of a character ρ we refer to m as the order of ρ.

Lemma 2.16.6 *Let m_0 be of the order ν_0 and let m_1 be an integer greater than m_0. Write ν_0 in any manner in the form $\nu_0 = \nu_1^{-1}\nu_2^{-1}$ where the orders of ν_1 and ν_2 are strictly less than m_1. If the order m of ρ is large enough*

$$C_{-2m-2\ell}(\rho) = \nu_2^{-1}\rho(-1) z_0^{-m-\ell} \frac{\eta(\nu_1^{-1}\rho, \varpi^{-m-\ell})}{\eta(\nu_2\rho^{-1}, \varpi^{-m-\ell})}$$

and $C_p(\rho) = 0$ if $p \neq -2m - 2\ell$.

Suppose the order of ρ is at least m_1. Then $\rho\nu_1\nu_0 = \rho\nu_2^{-1}$ is still of order m. Applying Proposition 2.11 we see that

$$\sum_{\sigma} \eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell}) \eta(\sigma^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\sigma)$$

is equal to

$$\eta(\nu_1^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell}) z_0^{m+\ell} \nu_1\rho\nu_0(-1) C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho)$$

for all integers n and p. Choose n such that $C_n(\nu_1) \neq 0$. Assume also that $m + n + \ell \geq -\ell$ or that $m \geq -2\ell - n$. Then $\eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell}) = 0$ unless $\sigma = \nu_1$ so that

$$\eta(\nu_1^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\nu_1) = \eta(\nu_2\rho^{-1}, \varpi^{-m-\ell} z_0^{m+\ell} \nu_1\rho\nu_0(-1) C_n(\nu_1) C_p(\rho).$$

Since $\nu_1^{-1}\rho$ is still of order m the left side is zero unless $p = -2m - 2\ell$. The only term on the right side that can vanish is $C_p(\rho)$. On the other hand if $p = -2m - 2\ell$ we can cancel the terms $C_n(\nu_1)$ from both side to obtain the relation of the lemma.

Apart from Lemma 2.16.4 the proof of Lemma 2.16.1 is complete. We have now to discuss its consequences. If ω_1 and ω_2 are two quasi-characters of $F^\times$ let $\mathcal{B}(\omega_1, \omega_2)$ be the space of all functions φ on G_F which satisfy

(i) For all g in G_F, a_1, a_2 in $F^\times$, and x in F

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a \end{pmatrix} g\right) = \omega_1(a_1)\omega_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}\varphi(g).$$

(ii) There is an open subgroup U of $GL(2, O_F)$ such that $\varphi(gu) \equiv \varphi(g)$ for all u in U.
Since

$$G_F = N_F\, A_F\, GL(2, O_F)$$

where A_F is the group of diagonal matrices the elements of $\mathcal{B}(\omega_1, \omega_2)$ are determined by their restrictions to $GL(2, O_F)$ and the second condition is tantamount to the condition that φ be locally constant. $\mathcal{B}(\omega_1, \omega_2)$ is invariant under right translations by elements of G_F so that we have a representation $\rho(\omega_1, \omega_2)$ of G_F on $\mathcal{B}(\omega_1, \omega_2)$. It is admissible.

Proposition 2.17 *If π is an infinite-dimensional irreducible representation of G_F which is not absolutely cuspidal then for some choice of μ_1 and μ_2 it is contained in $\rho(\mu_1, \mu_2)$.*

We take π in the Kirillov form. Since V_0 is invariant under the group P_F the representation π defines a representation σ of P_F on the finite-dimensional space V/V_0. It is clear that σ is trivial on N_F and that the kernel of σ is open. The contragredient representation has the same properties. Since P_F/N_F is abelian there is a nonzero linear form L on V/V_0 such that

$$\widetilde{\sigma}\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) L = \mu_1^{-1}(a_1)\mu_1^{-1}(a_2)L$$

for all a_1, a_2, and x. μ_1 and μ_2 are homomorphisms of $F^\times$ into $\mathbb{C}^\times$ which are necessarily continuous. L may be regarded as a linear form on V. Then

$$L\left(\pi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \mu_1(a_1)\mu_2(a_2)L(\varphi).$$

If φ is in V let $A\varphi$ be the function

$$A\varphi(g) = L\big(\pi(g)\varphi\big)$$

on G_F. A is clearly an injection of V into $\mathcal{B}(\mu_1, \mu_2)$ which commutes with the action of G_F.

Before passing to the next theorem we make a few simple remarks. Suppose π is an infinite-dimensional irreducible representation of G_F and ω is a quasi-character of $F^\times$. It is clear that $W(\omega \otimes \pi, \psi)$ consists of the functions

$$g \to W(g)\omega(\det g)$$

with W on $W(\pi, \psi)$. If V is the space of the Kirillov model of π the space of the Kirillov model of $\omega \otimes \pi$ consists of the functions $a \to \varphi(a)\omega(a)$ with φ in V. To see this take π in the Kirillov form and observe first of all that the map $A : \varphi \to \varphi\omega$ is an isomorphism of V with another space V' on which G_F acts by means of the representation $\pi' = A(\omega \otimes \pi)A^{-1}$. If

$$b\begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}$$

belongs to B_F and $\varphi' = \varphi\omega$ then

$$\pi'(b)\varphi'(a) = \omega(a)\{\omega(\alpha)\psi(ax)\varphi(\alpha a)\} = \psi(ax)\varphi'(\alpha a)$$

so that $\pi'(b)\varphi' = \xi_\psi(b)\varphi'$. By definition then π' is the Kirillov model of $\omega \otimes \pi$. Let ω_1 be the restriction of ω to U_F and let $z_1 = \omega(\varpi)$. If $\varphi' = \varphi\omega$ then

$$\widehat{\varphi}'(\nu, t) = \widehat{\varphi}(\nu\omega_1, z_1 t).$$

Thus

$$\pi'(w)\varphi'(\nu, t) = \pi(w)\widehat{\varphi}(\nu\omega_1, z_1 t) = C(\nu\omega_1, z_1 t)\widehat{\varphi}(v^{-1}\omega_1^{-1}\nu_0^{-1}, z_0^{-1}z_1^{-1}t^{-1}).$$

The right side is equal to

$$C(\nu\omega_1, z_1 t)\widehat{\varphi}'(\nu^{-1}\nu_0^{-1}\omega_1^{-1}, z_0^{-1}z_1^{-2}t^{-1})$$

so that when we replace π by $\omega \otimes \pi$ we replace $C(\nu, t)$ by $C(\nu\omega_1, z_1 t)$.

Suppose $\psi'(x) = \psi(bx)$ with b in $F^\times$ is another non-trivial additive character. Then $W(\pi, \psi')$ consists of the functions

$$W'(g) = W\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

with W in $W(\pi, \psi)$.

The last identity of the following theorem is referred to as the local functional equation. It is the starting point of our approach to the Hecke theory.

Theorem 2.18 *Let π be an irreducible infinite-dimensional admissible representation of G_F.*

(i) *If ω is the quasi-character of G_F defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

then the contragredient representation $\widetilde{\pi}$ is equivalent to $\omega^{-1} \otimes \pi$.

(ii) *There is a real number s_0 such that for all g in G_F and all W in $W(\pi, \psi)$ the integrals*

$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\, d^\times a = \Psi(g, s, W)$$

$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\omega^{-1}(a)\, d^\times a = \widetilde{\Psi}(g, s, W)$$

converge absolutely for $\operatorname{Re} s > s_0$.

(iii) *There is a unique Euler factor $L(s, \pi)$ with the following property: if*

$$\Psi(g, s, W) = L(s, \pi)\Phi(g, s, W)$$

then $\Phi(g,s,W)$ is a holomorphic function of s for all g and all W and there is at least one W in $W(\pi,\psi)$ such that

$$\Phi(e,s,W)=a^s$$

where a is a positive constant.

(iv) *If*

$$\widetilde{\Psi}(g,s,W)=L(s,\widetilde{\pi})\widetilde{\Phi}(g,s,W)$$

there is a unique factor $\epsilon(s,\pi,\psi)$ which, as a function of s, is an exponential such that

$$\widetilde{\Phi}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,1-s,W\right)=\epsilon(s,\pi,\psi)\Phi(g,s,W)$$

for all g in G_F and all W in $W(\pi,\psi)$.

To say that $L(s,\pi)$ is an Euler product is to say that $L(s,\pi)=P^{-1}(q^{-s})$ where P is a polynomial with constant term 1 and $q=|\varpi|^{-1}$ is the number of elements in the residue field. If $L(s,\pi)$ and $L'(s,\pi)$ were two Euler factors satisfying the conditions of the lemma their quotient would be an entire function with no zero. This clearly implies uniqueness.

If ψ is replaced by ψ' where $\psi'(x)=\psi(bx)$ the functions W are replaced by the functions W' with

$$W'(g)=W\left(\begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}g\right)$$

and

$$\Psi(g,s,W')=|b|^{1/2-s}\Psi(g,s,W)$$

while

$$\Psi(g,s,W')=|b|^{1/2-s}\omega(b)\widetilde{\Psi}(g,s,W).$$

Thus $L(s,\pi)$ will not depend on ψ. However

$$\epsilon(s,\pi,\psi')=\omega(b)\,|b|^{2s-1}\epsilon(s,\pi,\psi).$$

According to the first part of the theorem if W belongs to $W(\pi,\psi)$ the function

$$\widetilde{W}(g)=W(g)\omega^{-1}(\det g)$$

is in $W(\widetilde{\pi},\psi)$. It is clear that

$$\widetilde{\Psi}(g,s,W)=\omega(\det g)\Psi(g,s,\widetilde{W})$$

so that if the third part of the theorem is valid when π is replaced by $\widetilde{\pi}$ the function $\widetilde{\Phi}(g,s,W)$ is a holomorphic function of s. Combining the functional equation for π and for $\widetilde{\pi}$ one sees that

$$\epsilon(s,\pi,\psi)\epsilon(1-s,\widetilde{\pi},\psi)=\omega(-1).$$

Let V be the space on which the Kirillov model of π acts. For every W in $W(\pi,\psi)$ there is a unique φ in V such that

$$W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)=\varphi(a).$$

If π is itself the canonical model

$$\pi(w)\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w\right)$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If χ is any quasi-character of $F^\times$ we set

$$\widehat{\varphi}(\chi) = \int_{F^\times} \varphi(a)\chi(a)\, d^\times a$$

if the integral converges. If χ_0 is the restriction of χ to U_F then

$$\widehat{\varphi}(\chi) = \widehat{\varphi}\big(\chi_0, \chi(\varpi)\big).$$

Thus if α_F is the quasi-character $\alpha_F(x) = |x|$ and the appropriate integrals converge

$$\begin{aligned} \Psi(e, s, W) &= \widehat{\varphi}(\alpha_F^{s-1/2}) = \widehat{\varphi}(1, q^{1/2-s}) \\ \Psi(e, s, W) &= \widehat{\varphi}(\alpha_F^{s-1/2}\omega^{-1}) = \widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{1/2-s}) \end{aligned}$$

if ν_0 is the restriction of ω to U_F and $z_0 = \omega(\varphi)$. Thus if the theorem is valid the series $\widehat{\varphi}(1, t)$ and $\widehat{\varphi}(\nu_0^{-1}, t)$ have positive radii of convergence and define functions which are meromorphic in the whole t-plane.

It is also clear that

$$\Psi(w, 1-s, W) = \pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{s-1/2}).$$

If φ belongs to V_0 then

$$\pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{-1/2}t) = C(\nu_0^{-1}, z_0^{-1/2}q^{-1/2}t)\widehat{\varphi}(1, q^{1/2}t^{-1}).$$

Choosing φ in V_0 such that $\widehat{\varphi}(1, t) \equiv 1$ we see that $C(\nu_0^{-1}, t)$ is convergent in some disc and has an analytic continuation to a function meromorphic in the whole plane.

Comparing the relation

$$\pi(w)\widehat{\varphi}(\nu_0^{-1}z_0^{-1}q^{-1/2}q^s) = C(\nu_0^{-1}, z_0^{-1/2}q^{-1/2}q^s)\widehat{\varphi}(1, q^{1/2}q^{-s})$$

with the functional equation we see that

$$C(\nu_0^{-1}, z_0^{-1}q^{-1/2}q^s) = \frac{L(1-s, \tilde{\pi})\epsilon(s, \pi, \psi)}{L(s, \pi)}. \tag{2.18.1}$$

Replacing π by $\chi \otimes \pi$ we obtain the formula

$$C(\nu_0^{-1}\chi_0^{-1}, z_0^{-1}z_1^{-1}q^{-1/2}q^s) = \frac{L(1-s, \chi^{-1} \otimes \widetilde{\pi})\epsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)}.$$

Appealing to Proposition 2.15 we obtain the following corollary.

Corollary 2.19 *Let π and π' be two irreducible infinite-dimensional representations of G_F. Assume that the quasi-characters ω and ω' defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

are equal. Then π and π' are equivalent if and only if

$$\frac{L(1-s,\chi^{-1}\otimes\widetilde{\pi})\epsilon(s,\chi\otimes\pi,\psi)}{L(s,\chi\otimes\pi)} = \frac{L(1-s,\chi^{-1}\otimes\widetilde{\pi}')\epsilon(s,\chi\otimes\pi',\psi)}{L(s,\chi\otimes\pi')}$$

for all quasi-characters.

We begin the proof of the first part of the theorem. If φ_1 and φ_2 are numerical functions on $F^\times$ we set

$$\langle\varphi_1,\varphi_2\rangle = \int \varphi_1(a)\varphi_2(-a)\,d^\times a.$$

The Haar measure is the one which assigns the measure 1 to U_F. If one of the functions is in $\mathcal{S}(F^\times)$ and the other is locally constant the integral is certainly defined. By the Plancherel theorem for U_F

$$\langle\varphi,\varphi'\rangle = \sum_n\sum_\nu \nu(-1)\widehat{\varphi}_n(\nu)\widehat{\varphi}'_n(\nu^{-1}).$$

The sum is in reality finite. It is easy to se that if b belongs to B

$$\langle\xi_\psi(b)\varphi,\xi_\psi(b)\varphi')\rangle = \langle\varphi,\varphi'\rangle.$$

Suppose π is given in the Kirillov form and acts on V. Let π', the Kirillov model of $\omega^{-1}\otimes\pi$, act on V'. To prove part (i) we have only to construct an invariant non-degenerate bilinear form β on $V\times V'$. If φ belongs to V_0 and φ' belongs to V' or if φ belongs to V and φ' belongs to V_0' we set

$$\beta(\varphi,\varphi') = \langle\varphi,\varphi'\rangle.$$

If φ and φ' are arbitrary vectors in V and V' we may write $\varphi = \varphi_1 + \pi(w)\varphi_2$ and $\varphi' = \varphi_1 + \pi'(w)\varphi_2'$ with φ, φ_2 in V_0 and φ_1', φ_2' in V_0'. We want to set

$$\beta(\varphi,\varphi') = \langle\varphi_1,\varphi_1'\rangle + \langle\varphi_1,\pi'(w)\varphi_2'\rangle + \langle\pi(w)\varphi_2,\varphi_1'\rangle + \langle\varphi_2,\varphi_2'\rangle.$$

The second part of the next lemma shows that β is well defined.

Lemma 2.19.1 *Let φ and φ' belong to V_0 and V_0' respectively. Then*

(i)

$$\langle\pi(w)\varphi,\varphi'\rangle = \nu_0(-1)\langle\varphi,\pi'(w)\varphi'\rangle$$

(ii) *If either $\pi(w)\varphi$ belongs to V_0 or $\pi'(w)\varphi'$ belongs to V_0' then*

$$\langle\pi(w)\varphi,\pi'(w)\varphi'\rangle = \langle\varphi,\varphi'\rangle.$$

The relation
$$\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_p C_{n+p}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})z_0^{-p}$$
implies that
$$\langle\pi(w)\varphi,\varphi'\rangle=\sum_{n,p,\nu}\nu(-1)C_{n+p}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})z_0^{-p}\widehat{\varphi}'_n(\nu^{-1}). \tag{2.19.2}$$
Replacing π by π' replaces ω by ω^{-1}, ν_0 by ν_0^{-1}, z_0 by z_0^{-1}, and $C(\nu,t)$ by $C(\nu\nu_0^{-1},z_0^{-1}t)$. Thus
$$\langle\varphi,\pi(w)\varphi'\rangle=\sum_{n,p,\nu}\nu(-1)C_{n+p}(\nu\nu_0^{-1})z_0^{-n}\widehat{\varphi}'_p(\nu^{-1}\nu_0)\widehat{\varphi}_n(\nu^{-1}). \tag{2.19.3}$$
Replacing ν by $\nu\nu_0$ in (2.19.3) and comparing with (2.19.2) we obtain the first part of the lemma.

Because of the symmetry it will be enough to consider the second part when $\pi(w)\varphi$ belongs to V_0. By the first part
$$\langle\pi(w)\varphi,\pi'(w)\varphi'\rangle=\nu_0(-1)\langle\pi^2(w)\varphi,\varphi'\rangle=\langle\varphi,\varphi'\rangle.$$

It follows immediately from the lemma that
$$\beta\big(\pi(w)\varphi,\pi'(w)\varphi'\big)=\beta(\varphi,\varphi')$$
so that to establish the invariance of β we need only show that
$$\beta\big(\pi(p)\varphi,\pi'(p)\varphi'\big)=\beta(\varphi,\varphi')$$
for all triangular matrices p. If φ is in V_0 or φ' is in V_0' this is clear. We need only verify it for φ in $\pi(w)V_0$ and φ' in $\pi'(w)V_0'$.

If φ is in V_0, φ' is in V_0' and p is diagonal then
$$\beta\big(\pi(p)\pi(w)\varphi,\pi'(p)\pi'(w)\varphi'\big)=\beta\big(\pi(w)\pi(p_1)\varphi,\pi'(w)\pi'(p_1)\varphi'\big)$$
where $p_1=w^{-1}pw$ is also diagonal. The right side is equal to
$$\beta\big(\pi(p_1)\varphi,\pi'(p_1)\varphi'\big)=\beta(\varphi,\varphi')=\beta\big(\pi(w)\varphi,\pi'(w)p'\big).$$

Finally we have to show that*
$$\beta\left(\pi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi,\pi'\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi'\right)=\beta(\varphi,\varphi') \tag{2.19.2}$$
for all x in F and all φ and φ'. Let φ_i, $1<i<r$, generate V modulo V_0 and let φ_j', $1\le j\le r'$, generate V' modulo V_0'. There certainly is an ideal $\mathfrak{a}$ of F such that
$$\pi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi_i=\varphi_i$$

* The tags on Equations 2.19.2 and 2.19.3 have inadvertently been repeated.

and

$$\pi'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi'_j = \varphi'_j$$

for all i and j if x belongs to $\mathfrak{a}$. Then

$$\beta\left(\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi_i, \pi'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi'_j\right) = \beta(\varphi_i, \varphi_j).$$

Since 2.19.2 is valid if x is in $\mathfrak{a}$ and φ is in V_0 or φ' is in V'_0 it is valid for all φ and φ' provided that x is in $\mathfrak{a}$. Any y in F may be written as ax with a in $F^\times$ and x in $\mathfrak{a}$. Then

$$\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}$$

and it follows readily that

$$\beta\left(\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\varpi, \pi'\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\varphi'\right) = \beta(\varphi, \varphi').$$

Since β is invariant and not identically zero it is non-degenerate. The rest of the theorem will now be proved for absolutely cuspidal representations. The remaining representations will be considered in the next chapter. We observe that since $W(\pi,\psi)$ is invariant under right translations the assertions need only be established when g is the identity matrix e.

If π is absolutely cuspidal then $V = V_0 = \mathcal{S}(F^\times)$ and $W\left(\left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right)\right) = \varphi(a)$ is locally constant with compact support. Therefore the integrals defining $\Psi(e,s,W)$ and $\widetilde{\Psi}(e,s,W)$ are absolutely convergent for all values of s and the two functions are entire. We may take $L(s,\pi) = 1$. If φ is taken to be the characteristic function of U_F then $\Phi(e,s,W) = 1$.

Referring to the discussion preceding Corollary 2.19 we see that if we take

$$\epsilon(s,\pi,\psi) = C(\nu_0^{-1}, z_0^{-1}q^{-1/2}q^s)$$

the local functional equation of part (iv) will be satisfied. It remains to show that $\epsilon(s,\pi,\psi)$ is an exponential function or, what is at least ast strong, to show that, for all ν, $C(\nu,t)$ is a multiple of a power of t. It is a finite linear combination of powers of t and if it is not of the form indicated it has a zero at some point different form 0. $C(\nu\nu_0^{-1}, z_0^{-1}t^{-1})$ is also a linear combination of powers of t and so cannot have a pole except at zero. To show that $C(\nu,t)$ has the required form we have only to show that

$$C(\nu,t)C(\nu^{-1}\nu_0^{-1}, z_0^{-1}t^{-1}) = \nu_0(-1). \tag{2.19.3}$$

Choose φ in V_0 and set $\varphi' = \pi(w)\varphi$. We may suppose that $\varphi'(\nu,t) \neq 0$. The identity is obtained by combining the two relations

$$\widehat{\varphi}'(\nu,t) = C(\nu,t)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, z_0^{-1}t^{-1})$$

and

$$\nu_0(-1)\widehat{\varphi}(\nu^{-1}\nu_0^{-1},t) = C(\nu^{-1}\nu_0^{-1},t)\widehat{\varphi}'(\nu, z_0^{-1}t^{-1}).$$

We close this paragraph with a number of facts about absolutely cuspidal representations which will be useful later.

Proposition 2.20 *Let π be an absolutely cuspidal representation of G_F. If the quasi-character ω defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a) I$$

is actually a character then π is unitary.

As usual we take π and $\widetilde{\pi}$ in the Kirillov form. We have to establish the existence of a positive-definite invariant hermitian form on V. We show first that if φ belongs to V and $\widetilde{\varphi}$ belongs to $\widetilde{V}$ then there is a compact set Ω in G_F such that if

$$Z_F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a \in F \right\}$$

the support of $\langle \pi(g)\varphi, \widetilde{\varphi}\rangle$, a function of g, is contained in $Z_F\Omega$. If A_F is the group of diagonal matrices $G_F = GL(2, O_F)\, A_F\, GL(2, O_F)$. Since φ and $\widetilde{\varphi}$ are both invariant under subgroups of finite index in $GL(2, O_F)$ it is enough to show that the function $\langle \pi(b)\varphi, \widetilde{\varphi}\rangle$ on A_F has support in a set $Z_F\Omega$ with Ω compact. Since

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} b\right)\varphi, \widetilde{\varphi}\rangle = \omega(a)\langle \pi(b)\varphi, \widetilde{\varphi}\rangle$$

it is enough to show that the function

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi, \widetilde{\varphi}\rangle$$

has compact support in $F^\times$. This matrix element is equal to

$$\int_{F^\times} \varphi(ax)\widetilde{\varphi}(-x)\, d^\times x.$$

Since φ and $\widetilde{\varphi}$ are functions with compact support the result is clear.

Choose $\widetilde{\varphi}_0 \neq 0$ in $\widetilde{V}$ and set

$$(\varphi_1, \varphi_2) = \int_{Z_F\backslash G_F} \langle \pi(g)\varphi_1, \widetilde{\varphi}_0\rangle \overline{\langle \pi(g)\varphi_2, \widetilde{\varphi}_0\rangle}\, dg.$$

This is a positive invariant hermitian form on V.

We have incidentally shown that π is square-integrable. Observe that even if the absolutely cuspidal representation π is not unitary one can choose a quasi-character χ such that $\chi \otimes \pi$ is unitary.

If π is unitary there is a conjugate linear map $A : V \to \widetilde{V}$ defined by

$$(\varphi_1, \varphi_2) = \langle \varphi_1, A\varphi_2\rangle.$$

Clearly $A\xi_\psi(b) = \xi_\psi(b)A$ for all b in B_F. The map A_0 defined by

$$A_0\varphi(a) = \overline{\varphi}(-a)$$

has the same properties. We claim that

$$A = \lambda A_0$$

with λ in $\mathbb{C}^\times$. To see this we have only to apply the following lemma to $A_0^{-1}A$.

Lemma 2.21.1. *Let T be a linear operator on $\mathcal{S}(F^\times)$ which commutes with $\xi_\psi(b)$ for all b in B_F. Then T is a scalar.*

Since ξ_ψ is irreducible it will be enough to show that T has an eigenvector. Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial. Let μ be a non-trivial charcter of U_F and let $\mathfrak{p}^n$ be its conductor. T commutes with the operator

$$S = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \varpi^{-\ell-n}\\ 0 & 1\end{pmatrix}\right) d\epsilon$$

and it leaves the range of the restriction of S to the functions invariant under U_F invariant. If φ is such a function

$$S\varphi(a) = \varphi(a)\int_{U_F} \mu^{-1}(\epsilon)\psi(a\epsilon\varpi^{-\ell-n})\, d\epsilon.$$

The Gaussian sum is 0 unless a lies in U_F. Therefore $S\varphi$ is equal to $\varphi(1)$ times the function which is zero outside of U_F and equals μ on U_F. Since T leaves a one-dimensional space invariant it has an eigenvector.

Since $A = \lambda A_0$ the hermitian form (φ_1, φ_2) is equal to

$$\lambda \int_{F^\times} \varphi_1(a)\overline{\varphi}_2(a)\, d^\times a.$$

Proposition 2.21.2. *Let π be an absolutely cuspidal representation of G_F for which the quasi-character ω defined by*

$$\pi\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right) = \omega(a)I$$

is a character.

(i) *If π is in the Kirillov form the hermitian form*

$$\int_{F^\times} \varphi_1(a)\overline{\varphi}_2(a)\, d^\times a$$

is invariant.

(ii) *If $|z| = 1$ then $|C(\nu, z)| = 1$ and if* $\mathrm{Re}\, s = 1/2$

$$|\epsilon(s, \pi, \psi)| = 1.$$

Since $|z_0| = 1$ the second relation of part (ii) follows from the first and the relation

$$\epsilon(s, \pi, \psi) = C(\nu_0^{-1}, q^{s-1/2}z_0^{-1}).$$

If n is in $\mathbb{Z}$ and ν is a character of U_F let

$$\varphi(\epsilon\varpi^m) = \delta_{n,m}\nu(\epsilon)\nu_0(\epsilon)$$

for m in $\mathbb{Z}$ and ϵ in U_F. Then

$$\int_{F^\times} |\varphi(a)|^2\, da = 1.$$

If $\varphi' = \pi(w)\varphi$ and $C(\nu, t) = C_\ell(\nu)t^\ell$ then

$$\varphi'(\epsilon\varpi^m) = \delta_{\ell-n,m}C_\ell(\nu)z_0^{-n}\nu^{-1}(\epsilon).$$

Since $|z_0| = 1$

$$\int_{F^\times} |\varphi'(a)|^2\, da = |C_\ell(\nu)|^2.$$

Applying the first part of the lemma we see that, if $|z| = 1$, both $|C_\ell(\nu)|^2$ and $|C(\nu, z)|^2 = |C_\ell(\nu)|^2\, |z|^{2\ell}$ are 1.

Proposition 2.22. *Let π be an irreducible representation of G_F. It is absolutely cuspidal if and only if for every vector v there is an ideal $\mathfrak{a}$ in F such that*

$$\int_{\mathfrak{a}} \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\,dx = 0.$$

It is clear that the condition cannot be satisfied by a finite dimensional representation. Suppose that π is infinite dimensional and in the Kirillov form. If φ is in V then

$$\int_{\mathfrak{a}} \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \varphi\,dx = 0$$

if and only if

$$\varphi(a) \int_{\mathfrak{a}} \psi(ax)\,dx = 0$$

for all a. If this is so the character $x \to \psi(ax)$ must be non-trivial on $\mathfrak{a}$ for all a in the support of φ. This happens if and only if φ is in $\mathcal{S}(F^\times)$. The proposition follows.

Proposition 2.23. *Let π be an absolutely cuspidal representation and assume the largest ideal on which ψ is trivial is O_F. Then, for all characters ν, $C_n(\nu) = 0$ if $n \geq -1$.*

Take a character ν and choose n_1 such that $C_{n_1}(\nu) \neq 0$. Then $C_n(\nu) = 0$ for $n \neq n_1$. If $\widetilde{\nu} = \nu^{-1}\nu_0^{-1}$ then, as we have seen,

$$C(\nu, t)C(\widetilde{\nu}, t^{-1}z_0^{-1}) = \nu_0(-1)$$

so that

$$C_n(\widetilde{\nu}) = 0$$

for $n \neq n_1$ and

$$C_{n_1}(\nu)C_{n_1}(\widetilde{\nu}) = \nu_0(-1)z_0^{n_1}.$$

In the second part of Proposition 2.11 take $n = p = n_1 + 1$ to obtain

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^{n_1+1})\eta(\sigma^{-1}\widetilde{\nu}, \varpi^{n_1+1})C_{2n_1+2}(\sigma) = z_0^{n_1+1}\nu_0(-1) + (|\varpi| - 1)^{-1}z_0 C_{n_1}(\nu)C_{n_1}(\widetilde{\nu}).$$

The right side is equal to

$$z_0^{n_1+1}\nu_0(-1) \cdot \frac{|\varpi|}{|\varpi| - 1}.$$

Assume $n_1 \geq -1$. Then $\eta(\sigma^{-1}\nu, \varpi^{n_1+1})$ is 0 unless $\sigma = \nu$ and $\eta(\sigma^{-1}\widetilde{\nu}, \varpi^{n_1+1})$ is 0 unless $\sigma = \widetilde{\nu}$. Thus the left side is 0 unless $\nu = \widetilde{\nu}$. However if $\nu = \widetilde{\nu}$ the left side equals $C_{2n_1+2}(\nu)$. Since this cannot be zero $2n_1 + 2$ must be equal n_1 so that $n_1 = -2$. This is a contradiction.

§**3. The principal series for non-archimedean fields.** In order to complete the discussion of the previous pragraph we have to consider representations which are not absolutely cuspidal. This we shall now do. We recall that if μ_1, μ_2 is a pair of quasi-characters of $F^\times$ the space $\mathcal{B}(\mu_1, \mu_2)$ consists of all locally constant functions f on G_F which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) \tag{3.1}$$

for all g in G_F, a_1, a_2, in $F^\times$, and x in F. $\rho(\mu_1, \mu_2)$ is the representation of G_F on $\mathcal{B}(\mu_a, \mu_2)$.

Because of the Iwasawa decomposition $G_F = P_F GL(2, O_F)$ the functions in $\mathcal{B}(\mu_1, \mu_2)$ are determined by their restrictions to $GL(2, O_F)$. The restriction can be any locally constant function on $GL(2, O_F)$ satisfying

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2) f(g)$$

for all g in $GL(2, O_F)$, a_1, a_2 in U_F, and x in O_F. If U is an open subgroup of $GL(2, O_F)$ the restriction of any function invariant under U is a function on $GL(2, O_F)/U$ which is a finite set. Thus the space of all such functions is finite dimensional and as observed before $\rho(\mu_1, \mu_2)$ is admissible.

Let $\mathcal{F}$ be the space of continuous functions f on G_F which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| f(g)$$

for all g in G_F, a_1, a_2 in $F^\times$, and x in F. We observe that $\mathcal{F}$ contains $\mathcal{B}(\alpha_F^{1/2}, \alpha_F^{-1/2})$. G_F acts on $\mathcal{F}$. The Haar measure on G_F if suitably normalized satisfies

$$\int_{G_F} f(g)\, dg = \int_{N_F}\int_{A_F}\int_{GL(2,O_F)} \left|\frac{a_1}{a_2}\right|^{-1} f(nak)\, dn\, da\, dk$$

if

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

It follows easily from this that

$$\int_{GL(2,O_F)} f(k)\, dk$$

is a G_F-invariant linear form on $\mathcal{F}$. There is also a positive constant c such that

$$\int_{G_F} f(g)\, dg = c\int_{N_F}\int_{A_F}\int_{N_F} \left|\frac{a_1}{a_2}\right|^{-1} f\left(na\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n_1\right) dn\, da\, dn_1.$$

Consequently

$$\int_{GL(2,O_F)} f(k)\, dk = c\int_F f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx.$$

If φ_1 belongs to $\mathcal{B}(\mu_1, \mu_2)$ and φ_2 belongs to $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ then $\varphi_1\varphi_2$ belongs to $\mathcal{F}$ and we set

$$\langle \varphi_1, \varphi_2 \rangle = \int_{GL(2,O_F)} \varphi_1(k)\, \varphi_2(k)\, dk.$$

Clearly

$$\langle \rho(g)\varphi_1, \rho(g)\varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle$$

so that this bilinear form is invariant. Since both φ_1 and φ_2 are determined by their restrictions to $GL(2, O_F)$ it is also non-degenerate. Thus $\rho(\mu_1^{-1}, \mu_2^{-1})$ is equivalent to the contragredient of $\rho(\mu_1, \mu_2)$.

In Proposition 1.6 we introduced a representation r of G_F and then we introduced a representation $r_\Omega = r_{\mu_1,\mu_2}$. Both representations acted on $\mathcal{S}(F^2)$. If

$$\Phi^\sim(a, b) = \int_F \Phi(a, y)\psi(by)\, dy$$

is the partial Fourier transform

$$[r(g)\Phi]^\sim = \rho(g)\Phi^\sim \tag{3.1.1}$$

and

$$r_{\mu_1,\mu_2}(g) = \mu_1(\det g)\, |\det g|^{1/2} r(g). \tag{3.1.2}$$

We also introduced the integral

$$\theta(\mu_1, \mu_2; \Phi) = \int \mu_1(t)\mu_2^{-1}(t)\Phi(t, t^{-1})\, d^\times t$$

and we set

$$W_\Phi(g) = \theta(\mu_1, \mu_2; r_{\mu_1,\mu_2}(g)\Phi). \tag{3.1.3}$$

The space of functions W_Φ is denoted $W(\mu_1, \mu_2; \psi)$.

If ω is a quasi-character of $F^\times$ and if $|\omega(\varpi)| = |\varpi|^s$ with $s > 0$ the integral

$$z(\omega, \Phi) = \int_{F^\times} \Phi(0, t)\omega(t)\, d^\times t$$

is defined for all Φ in $\mathcal{S}(F^2)$. In particular if $|\mu_1(\varpi)\mu_1^{-1}(\varpi)| = |\varpi|^s$ with $s > -1$ we can consider the function

$$f_\Phi(g) = \mu_1(\det g)\, |\det g|^{1/2} z(\alpha_F \mu_1 \mu_2^{-1}, \rho(g)\Phi)$$

on G_F. Recall that $\alpha_F(a) = |a|$. Clearly

$$\rho(h) f_\Phi = f_\Psi \tag{3.1.4}$$

if

$$\Psi = \mu_1(\det h)\, |\det h|^{1/2} \rho(h)\Phi.$$

We claim that f_Φ belongs to $\mathcal{B}(\mu_1, \mu_2)$. Since the stabilizer of every Φ under the representation $g \to \mu_1(\det g)\, |\det g|^{1/2}\rho(g)$ is an open subgroup of G_F the functions f_Φ are locally constant. Since the space of functions f_Φ is invariant under right translations we need verify (3.1) only for $g = e$.

$$f_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) = z\left(\mu_1\mu_2^{-1}\alpha_F, \rho\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right)\Phi\right) \mu(a_1 a_2)\, |a_1 a_2|^{1/2}.$$

By definition the right side is equal to

$$\mu_1(a_1 a_2)|a_1 a_2|^{1/2} \int \mu_1(t)\mu_2^{-1}(t)\, |t|\, \Phi(0, a_2 t)\, d^\times t.$$

Changing variables we obtain

$$\mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} \int \mu_1(t)\mu_2^{-1}(t)\, |t|\, \Phi(0, t)\, d^\times t.$$

The integral is equal to $f_\Phi(e)$. Hence our assertion.

Proposition 3.2. *Assume* $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ *with* $s > -1$.

(i) *There is a linear transformation* A *of* $W(\mu_1, \mu_2; \psi)$ *into* $\mathcal{B}(\mu_1, \mu_2)$ *which for all* Φ *in* $\mathcal{S}(F^2)$, *sends* W_Φ *to* $f_{\Phi^\sim}$.

(ii) A *is bijective and commutes with right translations.*

To establish the first part of the proposition we have to show that $f_{\Phi^\sim}$ is 0 if W_Φ is. Since $N_F A_F \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_F$ is a dense subset of G_F this will be a consequence of the following lemma.

Lemma 3.2.1. *If the assumptions of the proposition are satisfied then, for all* Φ *in* $\mathcal{S}(F^2)$, *the function*

$$a \longrightarrow \mu_2^{-1}(a)\,|a|^{-1/2} W_\Phi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

is integrable with respect to the additive Haar measure on F *and*

$$\int W_\Phi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a)\,|a|^{-1/2} \psi(ax)\,da = f_{\Phi^\sim} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right).$$

By definition

$$f_{\Phi^\sim} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \int \Phi^\sim(t, tx) \mu_1(t) \mu_2^{-1}(t)\,|t|\,d^\times t$$

while

$$W_\Phi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a)\,|a|^{-1/2} = \mu_1(a)\mu_2^{-1}(a) \int \Phi(at, t^{-1})\,\mu_1(t)\,\mu_2^{-1}(t)\,d^\times t. \tag{3.2.2}$$

After a change of variable the right side becomes

$$\int \Phi(t, at^{-1}) \mu_1(t) \mu_2^{-1}(t)\,d^\times t.$$

Computing formally we see that

$$\int W_\Phi \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a)\,|a|^{-1/2} \psi(ax)\,da$$

is equal to

$$\int_F \psi(ax) \left\{ \int_{F^\times} \Phi(t, at^{-1}) \mu_1(t) \mu_2^{-1}(t)\,d^\times t \right\} da = \int_{F^\times} \mu_1(t)\mu_2^{-1}(t) \left\{ \int_F \Phi(t, at^{-1}) \psi(ax)\,da \right\} d^\times t$$

which in turn equals

$$\int_{F^\times} \mu_1(t) \mu_2^{-1}(t)\,|t| \left\{ \int_F \Phi(t, a) \psi(axt)\,da \right\} d^\times t = \int_{F^\times} \Phi^\sim(t, xt) \mu_1(t) \mu_2^{-1}(t)\,|t|\,d^\times t.$$

Our computation will be justified and the lemma proved if we show that the integral

$$\int_{F^\times}\int_F |\Phi(t,at^{-1})\mu_1(t)|\,d^\times t\,da$$

is convergent. It equals

$$\int_{F^\times}\int_F |\Phi(t,a)|\,|t|^{s+1}\,d^\times t\,da$$

which is finite because s is greater than -1.

To show that A is surjective we show that every function f in $\mathcal{B}(\mu_1,\mu_2)$ is of the form f_Φ for some Φ in $\mathcal{S}(F^2)$. Given f let $\Phi(x,y)$ be 0 if (x,y) is not of the form $(0,1)g$ for some g in $GL(2,O_F)$ but if (x,y) is of this form let $\Phi(x,y)=\mu_1^{-1}(\det g)f(g)$. It is easy to see that Φ is well-defined and belongs to $\mathcal{S}(F^2)$. To show that $f=f_\Phi$ we need only show that $f(g)=f_\Phi(g)$ for all g in $GL(2,O_F)$. If g belongs to $GL(2,O_F)$ then $\Phi((0,t)g)=0$ unless t belongs to U_F so that

$$f_\Phi(g)=\mu_1(\det g)\int_{U_F}\Phi((0,t)g)\mu_1(t)\mu_2^{-1}(t)\,dt.$$

Since

$$\Phi((0,t)g)=\mu_1^{-1}(t)\mu_2^{-1}(\det g)f\left(\begin{pmatrix}1&0\\0&t\end{pmatrix}g\right)=\mu_1^{-1}(t)\mu_2(t)\mu_2^{-1}(\det g)f(g)$$

the required equality follows.

Formulae (3.1.2) to (3.1.4) show that A commutes with right translations. Thus to show that A is injective we have to show that $W_\Phi(e)=0$ if $f_{\tilde\Phi}$ is 0. It follows from the previous lemma that

$$W_\Phi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)$$

is zero for almost all a if $f_{\tilde\Phi}$ is 0. Since $W_\Phi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)$ is a locally constant function on $F^\times$ it must vanish everywhere.

We have incidentally proved the following lemma.

Lemma 3.2.3 *Suppose $|\mu_1(\varpi)\mu_2^{-1}(\varpi)|=|\varpi|^s$ with $s>-1$ and W belongs to $W(\mu_1,\mu_2;\psi)$. If*

$$W\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)=0$$

for all a in $F^\times$ then W is 0.

Theorem 3.3 *Let μ_1 and μ_2 be two quasi-characters of $F^\times$.*

(i) *If neither $\mu_1\mu_2^{-1}$ nor $\mu_1^{-1}\mu_2$ is α_F the representations $\rho(\mu_1,\mu_2)$ and $\rho(\mu_2,\mu_2)$ are equivalent and irreducible.*

(ii) *If $\mu_1\mu_2^{-1}=\alpha_F$ write $\mu_1=\chi\alpha_F^{1/2}$, $\mu_2=\chi\alpha_F^{-1/2}$. Then $\mathcal{B}(\mu_1,\mu_2)$ contains a unique proper invariant subspace $\mathcal{B}_s(\mu_1,\mu_2)$ which is irreducible. $\mathcal{B}(\mu_2,\mu_1)$ also contains a unique proper invariant subspace $\mathcal{B}_f(\mu_2,\mu_1)$. It is one-dimensional and contains the function $\chi(\det g)$. Moreover the G_F-modules $\mathcal{B}_s(\mu_1,\mu_2)$ and $\mathcal{B}(\mu_2,\mu_1)/\mathcal{B}_f(\mu_2,\mu_1)$ are equivalent as are the modules $\mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_s(\mu_1,\mu_2)$ and $\mathcal{B}_f(\mu_2,\mu_1)$.*

We start with a simple lemma.

Lemma 3.3.1 *Suppose there is a non-zero function f in $\mathcal{B}(\mu_1,\mu_2)$ invariant under right translations by elements of N_F. Then there is a quasi-character χ such that $\mu_1=\chi\alpha_F^{-1/2}$ and $\mu_2=\chi\alpha_F^{1/2}$ and f is a multiple of χ.*

Since $N_F A_F \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) N_F$ is an open subset of G_F the function f is determined by its value at $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Thus if μ_1 and μ_2 have the indicated form it must be a multiple of χ.

If c belongs to $F^\times$ then

$$\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}=\begin{pmatrix}c^{-1} & 1\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & c^{-1}\\ 0 & 1\end{pmatrix}.$$

Thus if f exists and $\omega=\mu_2\mu_1^{-1}\alpha_F^{-1}$

$$f\left(\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right)=\omega(c)f\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right).$$

Since f is locally constant there is an ideal $\mathfrak{a}$ in F such that ω is constant on $\mathfrak{a}-\{0\}$. It follows immediately that ω is identically 1 and that μ_1 and μ_2 have the desired form.

The next lemma is the key to the theorem.

Lemma 3.3.2. *If $|\mu_1\mu_2(\varpi)|=|\varpi|^s$ with $s>-1$ there is a minimal non-zero invariant subspace X of $\mathcal{B}(\mu_1,\mu_2)$. For all f in $\mathcal{B}(\mu_1,\mu_2)$ and all n in N_F the difference $f-\rho(n)f$ belongs to X.*

By Proposition 3.2 it is enough to prove the lemma when $\mathcal{B}(\mu_1,\mu_2)$ is replaced by $W(\mu_1,\mu_2;\psi)$. Associate to each function W in $W(\mu_1,\mu_2;\psi)$ a function

$$\varphi(a)=W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)$$

on $F^\times$. If φ is 0 so is W. We may regard $\pi=\rho(\mu_1,\mu_2)$ as acting on the space V of such functions. If b is in B_F

$$\pi(b)\varphi=\xi_\psi(b)\varphi.$$

Appealing to (3.2.2) we see that every function φ in V has its support in a set of the form

$$\{a\in F^\times \mid |a|\le c\}$$

where $c=c(\varphi)$ is a constant. As in the second paragraph the difference $\varphi-\pi(n)\varphi=\varphi-\xi_\psi(n)\varphi$ is in $\mathcal{S}(F^\times)$ for all n in N_F. Thus $V\cap\mathcal{S}(F^\times)$ is not 0. Since the representation ξ_ψ of B_F on $\mathcal{S}(F^\times)$ is irreducible, V and every non-trivial invariant subspace of V contains $\mathcal{S}(F^\times)$. Taking the intersection of all such spaces we obtain the subspace of the lemma.

We first prove the theorem assuming that $|\mu_1(\varpi)\mu_2^{-1}(\varpi)|=|\varpi|^s$ with $s>-1$. We have defined a non-degenerate pairing between $\mathcal{B}(\mu_1,\mu_2)$ and $\mathcal{B}(\mu_1^{-1},\mu_2^{-1})$. All elements of the orthogonal complement of X are invariant under N_F. Thus if $\mu_1\mu_2^{-1}$ is not α_F the orthogonal complement is 0 and X is $\mathcal{B}(\mu_1,\mu_2)$ so that the representation is irreducible. The contragredient representation $\rho(\mu_1^{-1},\mu_2^{-1})$ is also irreducible.

If $\mu_1\mu_2^{-1}=\alpha_F$ we write $\mu_1=\chi\alpha_F^{1/2}$, $\mu_2=\chi\alpha_F^{-1/2}$. In this case X is the space of the functions orthogonal to the function χ^{-1} in $\mathcal{B}(\mu_1^{-1},\mu_2^{-1})$. We set $\mathcal{B}_s(\mu_1,\mu_2)=X$ and we let $\mathcal{B}_f(\mu_1^{-1},\mu_2^{-1})$ be the space of scalar multiples of χ^{-1}. The representation of G_F on $\mathcal{B}_s(\mu_1,\mu_2)$ is irreducible. Since

$\mathcal{B}_s(\mu_1, \mu_2)$ is of codimension one it is the only proper invariant subspace of $\mathcal{B}(\mu_1, \mu_2)$. Therefore $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ is the only proper invariant subspace of $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$.

If $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ then $|\mu_1^{-1}(\varpi)\mu_2(\varpi)| = |\varpi|^{-s}$ and either $s > -1$ or $-s > -1$. Thus if $\mu_1^{-1}\mu_2$ is neither α_F nor α_F^{-1} the representation $\pi = \rho(\mu_1, \mu_1)$ is irreducible. If $\omega = \mu_1\mu_2$ then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

so that π is equivalent to $\omega \otimes \tilde{\pi}$ or to $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$. It is easily seen that $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$ is equivalent to $\rho(\omega\mu_1^{-1}, \omega\mu_1^{-1}) = \rho(\mu_2, \mu_1)$.

If $\mu_1\mu_2^{-1} = \alpha_F$ and π is the restriction of ρ to $\mathcal{B}_s(\mu_1, \mu_2)$ then $\tilde{\pi}$ is the representation on $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})/\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ defined by $\rho(\mu_1^{-1}, \mu_2^{-1})$. Thus π is equivalent to the tensor product of $\omega = \mu_1\mu_2$ and this representation. The tensor product is of course equivalent to the representation on $\mathcal{B}(\mu_2, \mu_1)/\mathcal{B}_f(\mu_2, \mu_1)$. If $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$ the representations on $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$ and $\mathcal{B}_f(\mu_2, \mu_1)$ are both equivalent to the representations $g \to \chi(\det g)$.

The space $W(\mu_1, \mu_2; \psi)$ has been defined for all pairs μ_1, μ_2.

Proposition 3.4 (i) *For all pairs* μ_1, μ_2

$$W(\mu_1, \mu_2;\, \psi) = W(\mu_2, \mu_1;\, \psi)$$

(ii) *In particular if* $\mu_1\mu_2^{-1} \neq \alpha_F^{-1}$ *the representation of* G_F *on* $W(\mu_1, \mu_2;\, \psi)$ *is equivalent to* $\rho(\mu_1, \mu_2)$.

If Φ is a function on F^2 define Φ^ι by

$$\Phi^\iota(x, y) = \Phi(y, x).$$

To prove the proposition we show that, if Φ is in $\mathcal{S}(F^2)$,

$$\mu_1(\det g)\, |\det g|^{1/2}\theta\big(\mu_1, \mu_2;\, r(g)\Phi^\iota\big) = \mu_2(\det g)\, |\det g|^{1/2}\theta\big(\mu_2, \mu_1;\, r(g)\Phi\big).$$

If g is the identity this relation follows upon inspection of the definition of $\theta(\mu_1, \mu_2;\, \Phi^\iota)$. It is also easily seen that

$$r(g)\Phi^\iota = [r(g)\Phi]^\iota$$

if g is in $SL(2, F)$ so that it is enough to prove the identity for

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

It reduces to

$$\mu_1(a)\int \Phi^\iota(at, t^{-1})\mu_1(t)\mu_2^{-1}(t)\, d^\times t = \mu_2(a)\int \Phi(at, t^{-1})\mu_2(t)\mu_2^{-1}(t)\, d^\times t.$$

The left side equals

$$\mu_1(a)\int \Phi(t^{-1}, at)\mu_1(t)\mu_2^{-1}(t)\, d^\times t$$

which, after changing the variable of integration, one sees is equal to the right side.

If $\mu_1\mu_2^{-1}$ is not α_F or α_F^{-1} so that $\rho(\mu_1, \mu_2)$ is irreducible we let $\pi(\mu_1, \mu_2)$ be any representation in the class of $\rho(\mu_1, \mu_2)$. If $\rho(\mu_1, \mu_2)$ is reducible it has two constituents one finite dimensional and one infinite dimensional. A representation in the class of the first will be called $\pi(\mu_1, \mu_2)$. A representation in the class of the second will be called $\sigma(\mu_1, \mu_2)$. Any irreducible representation which is not absolutely cuspidal is either a $\pi(\mu_1, \mu_2)$ or a $\sigma(\mu_1, \mu_2)$. The representations $\sigma(\mu_1, \mu_2)$ which are defined only for certain values of μ_1 and μ_2 are called special representations.

Before proceeding to the proof of Theorem 2.18 for representations which are not absolutely cuspidal we introduce some notation. If ω is an unramified quasi-character of $F^\times$ the associated L-function is

$$L(s, \omega) = \frac{1}{1 - \omega(\varpi)\,|\varpi|^s}.$$

It is independent of the choice of the generator ϖ of $\mathfrak{p}$. If ω is ramified $L(s, \omega) = 1$. If φ belongs to $\mathcal{S}(F)$ the integral

$$Z(\omega\alpha_F^s, \varphi) = \int_{F^\times} \varphi(\alpha)\omega(\alpha)\,|\alpha|^s\, d^\times\alpha$$

is absolutely convergent in some half-plane $\mathrm{Re}\, s > s_0$ and the quotient

$$\frac{Z(\omega\alpha_F^s, \varphi)}{L(s, \omega)}$$

can be analytically continued to a function holomorphic in the whole complex plane. Moreover for a suitable choice of φ the quotient is 1. If ω is unramified and

$$\int_{U_F} d^\times\alpha = 0$$

one could take the characteristic function of O_F. There is a factor $\varepsilon(s, \omega, \psi)$ which, for a given ω and ψ, is of the form ab^s so that if $\widehat{\varphi}$ is the Fourier transform of φ

$$\frac{Z(\omega^{-1}\alpha_F^{1-s}, \widehat{\varphi})}{L(1-s, \omega^{-1})} = \varepsilon(s, \omega, \psi)\frac{Z(\omega\alpha_F^s, \varphi)}{L(s, \omega)}.$$

If ω is unramified and O_F is the largest ideal on which ψ is trivial $\varepsilon(s, \omega, \psi) = 1$.

Proposition 3.5 *Suppose μ_1 and μ_2 are two quasi-characters of $F^\times$ such that neither $\mu_1^{-1}\mu_2$ nor $\mu_1\mu_2^{-1}$ is α_F and π is $\pi(\mu_1, \mu_2)$. Then*

$$W(\pi, \psi) = W(\mu_1, \mu_2;\, \psi)$$

and if

$$\begin{aligned} L(s, \pi) &= L(s, \mu_1)\, L(s, \mu_2) \\ L(s, \tilde{\pi}) &= L(s, \mu_1^{-1})\, L(s, \mu_2^{-1}) \\ \varepsilon(s, \pi, \psi) &= \epsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi) \end{aligned}$$

all assertions of Theorem 2.18 are valid. In particular if $|\mu_1(\varpi)| = |\varpi|^{-s_1}$ and $|\mu_2(\varpi)| = |\varpi|^{-s_2}$ the integrals defining $\Psi(g, s, W)$ are absolutely convergent if $\mathrm{Re}\, s > \max\{s_1, s_2\}$. If μ_1 and μ_2 are *unramifed and O_F is the largest ideal of F on which ψ is trivial there is a unique function W_0 in $W(\pi,\psi)$ which is invariant under $GL(2,O_F)$ and assumes the value 1 at the identity. If*

$$\int_{U_F} d^\times\alpha = 1$$

then $\Phi(e,s,W_0)=1$.

That $W(\pi,\psi)=W(\mu_1,\mu_2;\psi)$ is of course a consequence of the previous proposition. As we observed the various assertions need be established only for $g=e$. Take Φ in $\mathcal{S}(F^2)$ and let $W=W_\Phi$ be the corresponding element of $W(\pi,\psi)$. Then

$$\varphi(a)=W\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right)$$

belongs to the space of the Kirillov model of π. As we saw in the closing pages of the first paragraph

$$\Psi(e,s,W)=\int_{F^\times} W\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right)|a|^{s-1/2}\,d^\times a=\widehat{\varphi}(\alpha_F^{s-1/2})$$

is equal to

$$Z(\mu_1\alpha_F^s,\mu_2\alpha_F^s,\Phi)$$

if the last and therefore all of the integrals are defined.

Also

$$\widetilde{\Psi}(e,s,W)=Z(\mu_2^{-1}\alpha_F^s,\mu_1^{-1}\alpha_F^s,\Phi).$$

Any function in $\mathcal{S}(F^2)$ is a linear combination of functions of the form

$$\Phi(x,y)=\varphi_1(x)\varphi_2(y).$$

Since the assertions to be proved are all linear we need only consider the functions Φ which are given as products. Then

$$Z(\mu_1\alpha_F^s,\mu_2\alpha_F^s,\Phi)=Z(\mu_1\alpha_F^s,\varphi_1)Z(\mu_2\alpha_F^s,\varphi_2)$$

so that the integral does converge in the indicated region. Moreover

$$Z(\mu_2^{-1}\alpha_F^s,\mu_1^{-1}\alpha_F^s,\Phi)=Z(\mu_2^{-1}\alpha_F^s,\varphi_1)Z(\mu_1^{-1}\alpha_F^s,\varphi_2)$$

also converges for $\operatorname{Re} s$ sufficiently large. $\Phi(e,s,W)$ is equal to

$$\frac{Z(\mu_1\alpha_F^s,\varphi_1)}{L(s,\mu_1)}\frac{Z(\mu_2\alpha_F^s,\varphi_2)}{L(s,\mu_2)}$$

and is holomorphic in the whole complex plane. We can choose φ_1 and φ_2 so that both factors are 1.

It follows from the Iwasawa decomposition $G_F=P_F\,GL(2,O_F)$ that if both μ_1 and μ_2 are unramified there is a non-zero function on $\mathcal{B}(\mu_1,\mu_2)$ which is invariant under $GL(2,O_F)$ and that it is unique up to a scalar factor. If the largest ideal on which ψ is trivial is O_F, if Φ_0 is the characteristic function of O_F^2, and if $\Phi_0^\sim$ is the partial Fourier transform introduced in Proposition 1.6 then $\Phi_0^\sim=\Phi_0$. Consequently

$$r_{\mu_1,\mu_2}(g)\Phi_0=\Phi_0$$

for all g in $GL(2, O_F)$. If $W_0 = W_{\Phi_0}$ then, since Φ_0 is the product of the characteristic function of O_F with itself, $\Phi(e, s, W_0) = 1$ if

$$\int_{U_F} d^\times \alpha = 1.$$

The only thing left to prove is the local functional equation. Observe that

$$\widetilde{\Phi}(w, s, W) = \widetilde{\Phi}\big(e, s, \rho(w)W\big),$$

that if $W = W_\Phi$ then $\rho(w)W = W_{r(w)\Phi}$, and that $r(w)\Phi(x, y) = \Phi'(y, x)$ if Φ' is the Fourier transform of Φ. Thus if $\Phi(x, y)$ is a product $\varphi_1(x)\varphi_2(y)$

$$\widetilde{\Phi}(w, s, W) = \frac{Z(\mu_1^{-1}\alpha_F^s, \widehat{\varphi}_1)}{L(s, \mu_1^{-1})}\frac{Z(\mu_2^{-1}\alpha_F^s, \widehat{\varphi}_2)}{L(s, \mu_2^{-1})}.$$

The functional equation follows immediately.

If $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1} and $\pi = \pi(\mu_1, \mu_2)$ we still set

$$L(s, \pi) = L(s, \mu_1)\, L(s, \mu_2)$$

and

$$\varepsilon(s, \pi, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi).$$

Since $\tilde{\pi}$ is equivalent to $\pi(\mu_1^{-1}, \mu_2^{-1})$

$$L(s, \tilde{\pi}) = L(s, \mu_1^{-1})L(s, \mu_2^{-1}).$$

Theorem 2.18 is not applicable in this case. It has however yet to be proved for the special representations. Any special representation σ is of the form $\sigma(\mu_1, \mu_2)$ with $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$. The contragredient representation of $\widetilde{\sigma}$ is $\sigma(\mu_2^{-1}, \mu_1^{-1})$. This choice of μ_1 and μ_2 is implicit in the following proposition.

Proposition 3.6 *$W(\sigma, \psi)$ is the space of functions $W = W_\Phi$ in $W(\mu_1, \mu_2;\, \psi)$ for which*

$$\int_F \Phi(x, 0)\, dx = 0.$$

Theorem 2.18 will be valid if we set $L(s, \sigma) = L(s, \widetilde{\sigma}) = 1$ and $\varepsilon(s, \sigma, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi)$ when χ is ramified and we set $L(s, \sigma) = L(s, \mu_1)$, $L(s, \widetilde{\sigma}) = L(s, \mu_2^{-1})$, and

$$\varepsilon(s, \sigma, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi)\, \frac{L(1 - s, \mu_1^{-1})}{L(s, \mu_2)}$$

when χ is unramified.

$W(\sigma, \psi)$ is of course the subspace of $W(\mu_1, \mu_2;\, \psi)$ corresponding to the space $\mathcal{B}_s(\mu_1, \mu_2)$ under the transformation A of Proposition 3.2. If $W = W_\Phi$ then A takes W to the function $f = f_{\Phi^\sim}$ defined by

$$f(g) = z\big(\mu_1\mu_2^{-1}\alpha_F, \rho(g)\Phi^\sim\big)\mu_1(\det g)\, |\det g|^{1/2}.$$

f belongs to $\mathcal{B}_s(\mu_1,\mu_2)$ if and only if

$$\int_{GL(2,O_F)} \chi^{-1}(g) f(g)\, dg = 0.$$

As we observed this integral is equal to a constant times

$$\int_F \chi^{-1}\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) f\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx = \int_F f\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx$$

which equals

$$\int z\left(\alpha_F^2, \rho(w)\,\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\Phi^\sim\right) dx = \int\left\{\int \Phi^\sim(-t,-tx)\,|t|^2\, d^\times t\right\} dx.$$

The double integral does converge and equals, apart from a constant factor,

$$\iint \Phi^\sim(t,tx)\,|t|\, dt\, dx = \iint \Phi^\sim(t,x)\, dt\, dx$$

which in turn equals

$$\int \Phi(t,0)\, dt.$$

We now verify not only the remainder of the theorem but also the following corollary.

Corollary 3.7 (i) *If $\pi=\pi(\mu_1,\mu_2)$ then*

$$\varepsilon(s,\sigma,\psi)\,\frac{L(1-s,\widetilde{\sigma})}{L(s,\sigma)} = \varepsilon(s,\pi,\psi)\,\frac{L(1-s,\tilde{\pi})}{L(s,\pi)}$$

(ii) *The quotient*

$$\frac{L(s,\pi)}{L(s,\sigma)}$$

is holomorphic

(iii) *For all Φ such that*

$$\int \Phi(x,0)\, dx = 0$$

the quotient

$$\frac{Z(\mu_1\alpha_F^s,\mu_2\alpha_F^s,\Phi)}{L(s,\sigma)}$$

is holomorphic and there exists such a Φ for which the quotient is one.

The first and second assertions of the corollary are little more than matters of definition. Although $W(\mu_1,\mu_2\psi)$ is not irreducible we may still, for all W in this space, define the integrals

$$\Psi(g,s,W) = \int W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-1/2}\, d^\times a$$
$$\widetilde{\Psi}(g,s,W) = \int W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-1/2}\omega^{-1}(a)\, d^\times a.$$

They may be treated in the same way as the integrals appearing in the proof of Proposition 3.5. In particular they converge to the right of some vertical line and if $W = W_\Phi$

$$\Psi(e,s,W) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)$$
$$\widetilde{\Psi}(e,s,W) = Z(\mu_2^{-1}\alpha_F^s, \mu_1^{-1}\alpha_F^s, \Phi).$$

Moreover

$$\frac{\Psi(g,s,W)}{L(s,\pi)}$$

is a holomorphic function of s and

$$\frac{\widetilde{\Psi}(g,1-s,W)}{L(1-s,\tilde{\pi})} = \varepsilon(s,\pi,\psi)\frac{\Psi(g,s,W)}{L(s,\pi)}.$$

Therefore

$$\Phi(g,s,W) = \frac{\Psi(g,s,W)}{L(s,\sigma)}$$

and

$$\widetilde{\Phi}(g,s,W) = \frac{\widetilde{\Psi}(g,s,W)}{L(s,\widetilde{\sigma})}$$

are meromorphic functions of s and satisfy the local functional equation

$$\widetilde{\Phi}(wg,1-s,W) = \varepsilon(s,\sigma,\psi)\,\Phi(g,s,W).$$

To compete the proof of the theorem we have to show that $\varepsilon(s,\sigma,\psi)$ is an exponential function of s and we have to verify the third part of the corollary. The first point is taken care of by the observation that $\mu_1^{-1}(\varpi)\,|\varpi| = \mu_2^{-1}(\varpi)$ so that

$$\frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)} = \frac{1-\mu_2(\varpi)\,|\varpi|^s}{1-\mu_1^{-1}(\varpi)\,|\varpi|^{1-s}} = -\mu_1(\varpi)\,|\varpi|^{s-1}.$$

If χ is ramified so that $L(s,\sigma) = L(s,\pi)$ the quotient part (iii) of the corollary is holomorphic. Moreover a Φ in $\mathcal{S}(F^2)$ for which

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi) = L(s,\sigma) = 1$$

can be so chosen that

$$\Phi(\varepsilon x, \eta y) = \chi(\varepsilon\eta)\Phi(x,y)$$

for ε and η in U_F. Then

$$\int_F \Phi(x,0)\,dx = 0.$$

Now take χ unramified so that $\chi(a) = |a|^r$ for some complex number r. We have to show that if

$$\int_F \Phi(x,0)\,dx = 0$$

then

$$\frac{Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)}{L(s, \mu_1)}$$

is a holomorphic function of s. Replacing s by $s - r + 1/2$ we see that it is enough to show that

$$(1 - |\varpi|^{s+1}) \iint \Phi(x, y)\, |x|^{s+1}\, |y|^s\, d^\times x\, d^\times y$$

is a holomorphic function of s. Without any hypothesis on Φ the integral converges for $\operatorname{Re} s > 0$ and the product has an analytic continuation whose only poles are at the roots of $|\varpi|^s = 1$. To see that these poles do not occur we have only to check that there is no pole at $s = 0$. For a given Φ in $\mathcal{S}(F^2)$ there is an ideal $\mathfrak{a}$ such that

$$\Phi(x, y) = \Phi(x, 0)$$

for y in $\mathfrak{a}$. If $\mathfrak{a}'$ is the complement of $\mathfrak{a}$

$$\iint \Phi(x, y)\, |x|^{s+1}\, |y|^s\, d^\times x\, d^\times y$$

is equal to the sum of

$$\int_F \int_{\mathfrak{a}'} \Phi(x, y)\, |x|^{s+1}\, |y|^s\, d^\times x\, d^\times y$$

which has no pole at $s = 0$ and a constant times

$$\left\{\int_F \Phi(x, 0)\, |x|^s\, dx\right\} \left\{\int_{\mathfrak{a}} |y|^s\, d^\times y\right\}$$

If $\mathfrak{a} = \mathfrak{p}^n$ the second integral is equal to

$$|\varpi|^{ns}(1 - |\varpi|^s)^{-1}$$

If

$$\int_F \Phi(x, 0)\, dx = 0$$

the first term, which defines a holomorphic function of s, vanishes at $s = 0$ and the product has no pole there.

If φ_0 is the characteristic function of O_F set

$$\Phi(x, y) = \left\{\varphi_0(x) - |\varpi|^{-1}\varphi_0(\varpi^{-1}x)\right\} \varphi_0(y).$$

Then

$$\int_F \Phi(x, 0)\, dx = 0$$

and

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)$$

is equal to

$$\left\{\int \left(\varphi_0(x) - |\varpi|^{-1}\varphi_0(\varpi^{-1}x)\right) \mu_1(x)\, |x|^s\, d^\times x\right\} \left\{\int \varphi_0(y)\, \mu_2(y)\, |y|^s\, d^\times y\right\}$$

The second integral equals $L(s, \mu_2)$ and the first equals

$$(1 - \mu_1(\varpi)\, |\varpi|^{s-1})\, L(s, \mu_1)$$

so their product is $L(s, \mu_1) = L(s, \sigma)$.

Theorem 2.18 is now completely proved. The properties of the local L-functions $L(s, \pi)$ and the factors $\epsilon(s, \pi, \psi)$ described in the next proposition will not be used until the paragraph on extraordinary representations.

Proposition 3.8 (i) *If π is an irreducible representation there is an integer m such that if the order of χ is greater than m both $L(s, \chi \otimes \pi)$ and $L(s, \chi \otimes \tilde{\pi})$ are 1.*

(ii) *Suppose π_1 and π_2 are two irreducible representations of G_F and that there is a quasi-character ω such that*

$$\pi_1 \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) I \qquad \pi_2 \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) I$$

Then there is an integer m such that if the order of χ is greater than m

$$\varepsilon(s, \chi \otimes \pi_1, \psi) = \varepsilon(s, \chi \otimes \pi_2, \psi)$$

(iii) *Let π be an irreducible representation and let ω be the quasi-character defined by*

$$\pi \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) I$$

Write ω in any manner as $\omega = \mu_1 \mu_2$. Then if the order of χ is sufficiently large in comparison to the orders of μ_1 and μ_2

$$\varepsilon(s, \chi \otimes \pi, \psi) = \varepsilon(s, \chi\mu_1, \psi)\, \varepsilon(s, \chi\mu_2, \psi).$$

It is enough to treat infinite-dimensional representations because if $\sigma = \sigma(\mu_1, \mu_2)$ and $\pi = \pi(\mu_1, \mu_2)$ are both defined $L(s, \chi \otimes \sigma) = L(s, \chi \otimes \pi)$, $L(s, \chi \otimes \widetilde{\sigma}) = L(s, \chi \otimes \tilde{\pi})$, and $\varepsilon(s, \chi \otimes \sigma, \psi) = \varepsilon(s, \chi \otimes \pi, \psi)$ if the order of χ is sufficiently large.

If π is not absolutely cuspidal the first part of the proposition is a matter of definition. If π is absolutely cuspidal we have shown that $L(s, \chi \otimes \pi) = L(s, \chi \otimes \tilde{\pi}) = 1$ for all π.

According to the relation (2.18.1)

$$\varepsilon(s, \chi \otimes \pi, \psi) = C(\nu_0^{-1} \nu_1^{-1}, z_0^{-1} z_1^{-1} q^{-1/2} z^{-1})$$

if the order of χ is so large that $L(s, \chi \otimes \pi) = L(s, \chi^{-1} \otimes \tilde{\pi}) = 1$. Thus to prove the second part we have only to show that if $\{C_1(\nu, t)\}$ and $\{C_2(\nu, t)\}$ are the series associated to π_1 and π_2 then

$$C_1(\nu, t) = C_2(\nu, t)$$

if the order of ν is sufficiently large. This was proved in Lemma 2.16.6. The third part is also a consequence of that lemma but we can obtain it by applying the second part to $\pi_1 = \pi$ and to $\pi_2 = \pi(\mu_1, \mu_2)$.

We finish up this paragraph with some results which will be used in the Hecke theory to be developed in the second chapter.

Lemma 3.9 *The restriction of the irreducible representation π to $GL(2,O_F)$ contains the trivial representation if and only if there are two unramified characters μ_1 and μ_2 such that $\pi = \pi(\mu_1,\mu_2)$.*

This is clear if π is one-dimensional so we may as well suppose that π is infinite dimensional. If $\pi = \pi(\mu_1,\mu_2)$ we may let $\pi = \rho(\mu_1,\mu_2)$. It is clear that there is a non-zero vector in $\mathcal{B}(\mu_1,\mu_2)$ invariant under $GL(2,O_F)$ if and only if μ_1 and μ_2 are unramified and that if there is such a vector it is determined up to a scalar factor. If $\pi = \sigma(\mu_1,\mu_2)$ and $\mu_1\mu_2^{-1} = \alpha_F$ we can suppose that π is the restriction of $\rho(\mu_1,\mu_2)$ to $\mathcal{B}_S(\mu_1,\mu_2)$. The vectors in $\mathcal{B}(\mu_1,\mu_2)$ invariant under $GL(2,O_F)$ clearly do not lie in $\mathcal{B}_s(\mu_1,\mu_2)$ so that the restriction of π to $GL(2,O_F)$ does not contain the trivial representation. All that we have left to do is to show that the restiction of an absolutely cuspidal representation to $GL(2,O_F)$ does not contain the trivial representation.

Suppose the infinite-dimensional irreducible representation π is given in the Kirillov form with respect to an additive character ψ such that O_F is the largest ideal on which ψ is trivial. Suppose the non-zero vector φ is invariant under $GL(2,O_F)$. It is clear that if

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

then ω is unramified, that $\varphi(\nu,t) = 0$ unless $\nu = 1$ is the trivial character, and that $\varphi(\nu,t)$ has no pole at $t = 0$. Suppose π is absolutely cuspidal so that φ belongs to $\mathcal{S}(F^\times)$. Since $\pi(w)\varphi = \varphi$ and the restriction of ω to U_F is trivial

$$\widehat{\varphi}(1,t) = C(1,t)\,\widehat{\varphi}(1,z_0^{-1}t^{-1})$$

if $z_0 = \omega(\varpi)$. Since $C(1,t)$ is a constant times a negative power of t the series on the left involves no negative powers of t and that on the right involves only negative powers. This is a contradiction.

Let $\mathcal{H}_0$ be the subalgebra of the Hecke algebra formed by the functions which are invariant under left and right translations by elements of $GL(2,O_F)$. Suppose the irreducible representation π acts on the space X and there is a non-zero vector x in X invariant under $GL(2,O_F)$. If f is in $\mathcal{H}_0$ the vector $\pi(f)x$ has the same property and is therefore a multiple $\lambda(f)x$ of x. The map $f \to \lambda(f)$ is a non-trivial homomorphism of $\mathcal{H}_0$ into the complex numbers.

Lemma 3.10 *Suppose $\pi = \pi(\mu_1,\mu_2)$ where μ_1 and μ_2 are unramified and λ is the associated homomorphism of $\mathcal{H}_0$ into $\mathbb{C}$. There is a constant c such that*

$$|\lambda(f)| \le c\int_{G_F} |f(g)|\,dg \tag{3.10.1}$$

for all f in $\mathcal{H}$ if and only if $\mu_1\mu_2$ is a character and $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ with $-1 \le s \le 1$.

Let $\tilde{\pi}$ act on $\widetilde{X}$ and let $\widetilde{x}$ in $\widetilde{X}$ be such that $\langle x,\widetilde{x}\rangle \ne 0$. Replacing $\widetilde{x}$ by

$$\int_{GL(2,O_F)} \tilde{\pi}(g)\widetilde{x}\,dg$$

if necessary we may suppose that $\widetilde{x}$ is invariant under $GL(2,O_F)$. We may also assume that $\langle x,\widetilde{x}\rangle = 1$. If $\eta(g) = \langle \pi(g)x,\widetilde{x}\rangle$ then

$$\lambda(f)\,\eta(g) = \int_{G_F} \eta(gh)\,f(h)\,dh$$

for all f in $\mathcal{H}_0$. In particular

$$\lambda(f) = \int_{G_F} \eta(h)\, f(h)\, dh.$$

If $|\eta(h)| \le c$ for all h the inequality (3.10.1) is certainly valid. Conversely, since η is invariant under left and right translations by $GL(2, O_F)$ we can, if the inequality holds, apply it to the characteristic functions of double cosets of this group to see that $|\eta(h)| \le c$ for all h. Since

$$\eta\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} h\right) = \mu_1(a)\,\mu_2(a)\,\eta(h)$$

the function η is bounded only if $\mu_1\mu_2$ is a character as we now assume it to be. The finite dimensional representations take care of themselves so we now assume π is infinite-dimensional.

Since π and $\tilde{\pi}$ are irreducible the function $\langle \pi(g)x, \widetilde{x}\rangle$ is bounded for a given pair of non-zero vectors if and only if it is bounded for all pairs. Since $G_F = GL(2, O_F)\, A_F\, GL(2, O_F)$ and $\mu_1\mu_2$ is a character these functions are bounded if and only if the functions

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) x, \widetilde{x}\rangle$$

are bounded on $F^\times$. Take π and $\tilde{\pi}$ in the Kirillov form. If φ is in V and $\widetilde{\varphi}$ is in V then

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \tilde{\pi}(w)\widetilde{\varphi}\rangle$$

is equal to

$$\langle \pi^{-1}(w)\, \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \widetilde{\varphi}\rangle = \mu_1(a)\,\mu_2(a) < \pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \pi^{-1}(w)\varphi, \widetilde{\varphi} >$$

Thus $\eta(g)$ is bounded if and only if the functions

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \widetilde{\varphi}\rangle$$

are bounded for all φ in V and all $\widetilde{\varphi}$ in $\mathcal{S}(F^\times)$.

It is not necessary to consider all $\widetilde{\varphi}$ in $\mathcal{S}(F^\times)$ but only a set which together with its translates by the diagonal matrices spans $\mathcal{S}(F^\times)$. If μ is a character of U_F let φ_μ be the function on $F^\times$ which is 0 outside of U_F and equals μ on U_F. It will be sufficient to consider the functions $\widetilde{\varphi} = \varphi_\mu$ and all we need show is that

$$\langle \pi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \varphi_\mu\rangle \tag{3.10.2}$$

is a bounded function of n for all μ and all φ. The expression (3.10.2) is equal to $\widehat{\varphi}_n(\mu)$. If φ belongs to $\mathcal{S}(F^\times)$ the sequence $\{\widehat{\varphi}_n(\mu)\}$ has only finitely many non-zero terms and there is no problem. If $\varphi = \pi(w)\varphi_0$ then

$$\sum_n \widehat{\varphi}_n(\mu)t^n = C(\mu, t)\,\eta(t)$$

where $\eta(t)$ depends on φ_0 and is an arbitrary finite Laurent series. We conclude that (3.10.1) is valid if and only if $\mu_1\mu_2$ is a character and the coefficients of the Laurent series $C(\mu, t)$ are bounded for every choice of μ.

It follows from Proposition 3.5 and formula (2.18.1) that, in the present case, the series has only one term if μ is ramified but that if μ is trivial

$$C\big(\mu, |\varpi|^{1/2}\,\mu_1^{-1}(\varpi)\,\mu_2^{-1}(\varpi)t\big) = \frac{\big(1-\mu_1(\varpi)t^{-1}\big)\big(1-\mu_2(\varpi)t^{-1}\big)}{\big(1-\mu_1^{-1}(\varpi)\,|\varpi|\,t\big)\big(1-\mu_2^{-1}(\varpi)\,|\varpi|\,t\big)}.$$

The function on the right has zeros at $t = \mu_1(\varpi)$ and $t = \mu_2(\varpi)$ and poles at $t = 0$, $t = |\varpi|^{-1}\mu_1(\varpi)$, and $t = |\varpi|^{-1}\mu_2(\varpi)$. A zero can cancel a pole only if $\mu_2(\varpi) = |\varpi|^{-1}\mu_1(\varpi)$ or $\mu_1(\varpi) = |\varpi|^{-1}\mu_2(\varpi)$. Since μ_1 and μ_2 are unramified this would mean that $\mu_1^{-1}\mu_2$ equals α_F or α_F^{-1} which is impossible when $\pi = \pi(\mu_1, \mu_2)$ is infinite dimensional.

If $C(\mu, t)$ has bounded coefficients and $\mu_1\mu_2$ is a character the function on the right has no poles for $|t| < |\varpi|^{-1/2}$ and therfore $|\mu_1(\varpi)| \geq |\varpi|^{1/2}$ and $|\mu_2(\varpi)| \geq |\varpi|^{1/2}$. Since

$$|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\mu_1(\varpi)|^2 = |\mu_2^{-1}(\varpi)|^2$$

where $\mu_1\mu_2$ is a character these two inequalities are equivalent to that of the lemma. Conversely if these two inequalities are satisifed the rational function on the right has no pole except that at 0 inside the circle $|t| = |\varpi|^{-1/2}$ and at most simple poles on the circle itself. Applying, for example, partial fractions to find its Laurent series expansion about 0 one finds that the coefficients of $C(\mu, t)$ are bounded.

Lemma 3.11 *Suppose μ_1 and μ_2 are unramified, $\mu_1\mu_2$ is a character, and $\pi = \pi(\mu_1, \mu_2)$ is infinite dimensional. Let $|\mu_1(\varpi)| = |\varpi|^r$ where r is real so that $|\mu_2(\varpi)| = |\varpi|^{-r}$. Assume O_F is the largest ideal on which ψ is trivial and let W_0 be that element of $W(\pi, \psi)$ which is invariant under $GL(2, O_F)$ and takes the value 1 at the identity. If $s > |r|$ then*

$$\int_{F^\times} \left| W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| |a|^{s-1/2}\, d^\times a \leq \frac{1}{(1-|\varpi|^{s+r})(1-|\varpi|^{s-r})}$$

if the Haar measure is so normalized that the measure of U_F is one.

If Φ is the characteristic function of O_F^2 then

$$W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_1(a)\,|a|^{1/2} \int_{F^\times} \Phi(at, t^{-1})\,\mu_1(t)\,\mu_2^{-1}\, d^\times t$$

and

$$\int_{F^\times} \left| W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| |a|^{s-1/2}\, d^\times a \leq \iint \Phi(at, t^{-1})\,|a|^{s+r}|t|^{2r}\, d^\times a\, d^\times t.$$

Changing variables in the left-hand side we obtain

$$\int_{O_F}\int_{O_F} |a|^{s+r}|b|^{s-r}\, d^\times a\, d^\times b = \frac{1}{(1-|\varpi|^{s+r})(1-|\varpi|^{s-r})}.$$

§**4. Examples of absolutely cuspidal representations** In this paragraph we will use the results of the first paragraph to construct some examples of absolutely cuspidal representations.

First of all let K be a quaternion algebra over F. K is of course unique up to isomorphism. As in the first paragraph Ω will denote a continuous finite-dimensional representation of $K^\times$ the multiplicative group of K. If χ is a quasi-character of $F^\times$ and ν is the reduced norm on K we denote the one-dimensional representation $g \to \chi(\nu(g))$ of $K^\times$ by χ also. If Ω is any representation $\chi \otimes \Omega$ is the representation $g \to \chi(g)\,\Omega(g)$. If Ω is irreducible all operators commuting with the action of $K^\times$ are scalars. In particular there is a quasi-character Ω of $F^\times$ such that

$$\Omega(a) = \omega(a)I$$

for all a in $F^\times$ which is of course a subgroup of $K^\times$. If Ω is replaced by $\chi \otimes \Omega$ then ω is replaced by $\chi^2\omega$. $\widetilde{\Omega}$ will denote the representation contragredient to Ω.

Suppose Ω is irreducible, acts on V, and the quasi-character ω is a character. Since $K^\times/F^\times$ is compact there is a positive definite hermitian form on V invariant under $K^\times$. When this is so we call Ω unitary.

It is a consequence of the following lemma that any one-dimensional representation of $K^\times$ is the representation associated to a quasi-character of $F^\times$.

Lemma 4.1 *Let K_1 be the subgroup of $K^\times$ consisting of those x for which $\nu(x) = 1$. Then K_1 is the commutator subgroup, in the sense of group theory, of $K^\times$.*

K_1 certainly contains the commutator subgroup. Suppose x belongs to K_1. If $x = x^\iota$ then $x^2 = xx^\iota = 1$ so that $x = \pm 1$. Otherwise x determines a separable quadratic extension of F. Thus, in all cases, if $xx^\iota = 1$ there is a subfield L of K which contains x and is quadratic and separable over L. By Hilbert's Theorem 90 there is a y in L such that $x = yy^{-\iota}$. Moreover there is an element σ in K such that $\sigma z \sigma^{-1} = z^\iota$ for all z in L. Thus $x = y\sigma y^{-1}\sigma^{-1}$ is in the commutator subgroup.

In the first paragraph we associated to Ω a representation r_Ω of a group G_+ on the space $\mathcal{S}(K,\Omega)$. Since F is now non-archimedean the group G_+ is now $G_F = GL(2,F)$.

Theorem 4.2 (i) *The representation r_Ω is admissible.*

(ii) *Let $d = \mathrm{degree}\,\Omega$. Then r_Ω is equivalent to the direct sum of d copies of an irreducible representation $\pi(\Omega)$.*

(iii) *If Ω is the representation associated to a quasi-character χ of $F^\times$ then $\pi(\Omega) = \sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.*

(iv) *If $d > 1$ the representation $\pi(\Omega)$ is absolutely cuspidal.*

If n is a natural number we set

$$G_n = \{g \in GL(2, O_F) \,|\, g = I (\mathrm{mod}\, \mathfrak{p}^n)\}$$

We have first to show that if Φ is in $\mathcal{S}(K,\Omega)$ there is an n such that $r_\Omega(g)\Phi = \Phi$ if g is in G_n and that for a given n the space of Φ in $\mathcal{S}(K,\Omega)$ for which $r_\Omega(g)\Phi = \Phi$ for all g in G_n is finite dimensional.

Any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in G_n may be written as

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b' \\ 0 & d' \end{pmatrix}$$

and both the matrices on the right are in G_n. Thus G_n is generated by the matrices of the forms

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad w\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}w^{-1} \qquad w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}w^{-1}$$

with $a \equiv 1 \pmod{\mathfrak{p}^n}$ and $x \equiv 0 \pmod{\mathfrak{p}^n}$. It will therefore be enough to verify the following three assertions.

(4.2.1) Given Φ there is an $n > 0$ such that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

if $a \equiv 1 \pmod{\mathfrak{p}^n}$

(4.2.2) Given Φ there is an $n > 0$ such that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

if $x \equiv 0 \pmod{\mathfrak{p}^n}$.

(4.2.3) Given $n > 0$ the space of Φ in $\mathcal{S}(K, \Omega)$ such that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

and

$$r_\Omega(w^{-1})\, r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) r_\Omega(w)\Phi = \Phi$$

for all x in $\mathfrak{p}^n$ is finite-dimensional.

If $a = \nu(h)$ then

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = |h|_K^{1/2}\,\Omega(h)\,\Phi(xh).$$

Since Φ has compact suport in K and is locally constant there is a neighborhood U of 1 in $K^\times$ such that

$$\Omega(h)\,\Phi(xh)\,|h|_K^{1/2} = \Phi(x)$$

for all h in U and all x in K. The assertion (4.2.1) now follows from the observation that ν is an open mapping of $K^\times$ onto $F^\times$.

We recall that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi(z) = \psi\big(x\nu(z)\big)\,\Phi(z)$$

Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial and let $\mathfrak{p}_K$ be the prime ideal of K. Since $\nu(\mathfrak{p}_K^m) = \mathfrak{p}_F^m$

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

for all x in $\mathfrak{p}^n$ if and only if the support of Φ is contained in $\mathfrak{p}_K^{-n-\ell}$. With this (4.2.2) is established.

Φ satisfies the two conditions of (4.2.3) if and only if both Φ and $r(w)\Phi$ have support in $\mathfrak{p}_K^{-n-\ell}$ or, since $r(w)\Phi = -\Phi'$, if and only if Φ and Φ', its Fourier transform, have support in this set. There is certainly a natural number k such that $\psi\big(\tau(y)\big) = 1$ for all y in $\mathfrak{p}_K^k$. Assertion (4.2.3) is therefore a consequence of the following simple lemma.

Lemma 4.2.4 *If the support of Φ is contained in $\mathfrak{p}_K^{-n}$ and $\psi\big(\tau(y)\big) = 1$ for all y in $\mathfrak{p}_K^k$ the Fourier transform of Φ is constant on cosets of $\mathfrak{p}_K^{k+n}$.*

Since

$$\Phi'(x) = \int_{\mathfrak{p}_K^{-n}} \Phi(y)\, \psi\big(\tau(x,y)\big)\, dy$$

the lemma is clear.

We prove the second part of the theorem for one-dimensional Ω first. Let Ω be the representation associated to χ. $\mathcal{S}(K,\Omega)$ is the space of Φ in $\mathcal{S}(K)$ such that $\Phi(xh) = \Phi(x)$ for all h in K_1. Thus to every Φ in $\mathcal{S}(K,\Omega)$ we may associate the function φ_Φ on $F^\times$ defined by

$$\varphi_\Phi(a) = |h|_K^{1/2}\, \Omega(h)\, \Phi(h)$$

if $a = \nu(h)$. The map $\Phi \to \varphi_\Phi$ is clearly injective. If φ belongs to $\mathcal{S}(F^\times)$ the function Φ defined by

$$\Phi(h) = |h|_K^{-1/2}\, \Omega^{-1}(h)\, \varphi\big(\nu(h)\big)$$

if $h \neq 0$ and by

$$\Phi(0) = 0$$

belongs to $\mathcal{S}(K,\Omega)$ and $\varphi = \varphi_\Phi$. Let $\mathcal{S}_0(K,\Omega)$ be the space of functions obtained in this way. It is the space of functions in $\mathcal{S}(K,\Omega)$ which vanish at 0 and therefore is of codimension one. If Φ belongs to $\mathcal{S}_0(K,\Omega)$, is non-negative, does not vanish identically and Φ' is its Fourier transform then

$$\Phi'(0) = \int \Phi(x)\, dx \neq 0.$$

Thus $r_\Omega(w)\Phi$ does not belong to $\mathcal{S}_0(K,\Omega)$ and $\mathcal{S}_0(K,\Omega)$ is not invariant. Since it is of codimension one there is no proper invariant subspace containing it.

Let V be the image of $\mathcal{S}(K,\omega)$ under the map $\Phi \to \varphi_\Phi$. We may regard r_Ω as acting in V. >From the original definitions we see that

$$r_\Omega(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. If V_1 is a non-trivial invariant subspace of V the difference

$$\varphi - r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in $V_0 \cap V_1$ for all φ in V_1 and all x in F. If φ is not zero we can certainly find an x for which the difference is not zero. Consequently $V_0 \cap V_1$ is not 0 so that V_1 contains V_0 and hence all of V.

r_Ω is therefore irreducible and when considered as acting on V it is in the Kirillov form. Since V_0 is not V it is not absolutely cuspidal. It is thus a $\pi(\mu_1,\mu_2)$ or a $\sigma(\mu_1,\mu_2)$. To see which we have to find a linear form on V which is trivial on V_0. The obvious choice is

$$L(\varphi) = \Phi(0)$$

if $\varphi = \varphi_\Phi$. Then

$$L\left(r_\Omega\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\varphi\right)\right) = \chi(a_1 a_2)\left|\frac{a_1}{a_2}\right| L(\varphi).$$

To see this we have only to recall that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \Omega(a)I = \chi^2(a)I$$

and that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\,\Phi(0) = |h|_K^{1/2}\,\Omega(h)\,\Phi(0)$$

where $a = \nu(h)$ so that $|h|_K^{1/2} = |a|_F^2$ and $\Omega(h) = \chi(a)I$. Thus if

$$A\varphi(g) = L(r_\Omega(g)\varphi)$$

A is an injection of V into an irreducible invariant subspace of $\mathcal{B}(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$. The only such subspace is $\mathcal{B}_s(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ and r_Ω is therefore $\sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.

Suppose now that Ω is not one-dimensional. Let Ω act on U. Since K_1 is normal and K/K_1 is abelian there is no non-zero vector in U fixed by every element of K_1. If Φ is in $\mathcal{S}(K,\Omega)$ then

$$\Phi(xh) = \Omega^{-1}(h)\,\Phi(x)$$

for all h in K_1. In particular $\Phi(0)$ is fixed by every element in K_1 and is therefore 0. Thus all functions in $\mathcal{S}(K,\Omega)$ have compact supports in $K^\times$ and if we associate to every Φ in $\mathcal{S}(K,\Omega)$ the function

$$\varphi_\Phi(a) = |h|_K^{1/2}\,\Omega(h)\,\Phi(h)$$

where $a = \nu(h)$ we obtain a bijection from $\mathcal{S}(K,\Omega)$ to $\mathcal{S}(F^\times, U)$. It is again clear that

$$\varphi_{\Phi_1} = \xi_\psi(b)\varphi_\Phi$$

if b is in B_F and $\Phi_1 = r_\Omega(b)\Phi$.

Lemma 4.2.5 *Let Ω be an irreducible representation of $K^\times$ in the complex vector space U. Assume that U has dimension greater than one.*

(i) *For any Φ in $\mathcal{S}(K,U)$ the integrals*

$$Z(\alpha_F^s\otimes\Omega,\Phi) = \int_{K^\times} |a|_K^{s/2}\,\Omega(a)\,\Phi(a)\,d^\times a$$
$$Z(\alpha_F^s\otimes\Omega^{-1},\Phi) = \int_{K^\times} |a|_K^{s/2}\,\Omega^{-1}(a)\,\Phi(a)\,d^\times a$$

are absolutely convergent in some half-plane $\operatorname{Re} s > s_0$.

(ii) *The functions $Z(\alpha_F^s\otimes\Omega,\Phi)$ and $Z(\alpha_F^s\otimes\Omega^{-1},\Phi)$ can be analytically continued to functions meromorphic in the whole complex plane.*

(iii) *Given u in U there is a Φ in $\mathcal{S}(K,U)$ such that*

$$Z(\alpha_F^s\otimes\Omega,\Phi) \equiv u.$$

(iv) *There is a scalar function $\varepsilon(s,\Omega,\psi)$ such that for all Φ in $\mathcal{S}(K,U)$*

$$Z(\alpha_F^{3/2-s}\otimes\Omega^{-1},\Phi') = -\varepsilon(s,\Omega,\psi)\,Z(\alpha_F^{s+1/2}\otimes\Omega,\Phi)$$

if Φ' is the Fourier transform of Φ. Moreover, as a function of s, $\varepsilon(s,\Omega,\psi)$ is a constant times an exponential.

There is no need to verify the first part of the lemma. Observe that $\alpha_F\big(\nu(x)\big) = |\nu(x)|_F = |x|_K^{1/2}$ so that

$$(\alpha_F^s \otimes \Omega)(x) = |x|_K^{s/2}\,\Omega(x).$$

If Φ belongs to $\mathcal{S}(K,U)$ set

$$\Phi_1(x) = \int_{K_1} \Omega(h)\,\Phi(xh).$$

The integration is taken with respect to the normalized Haar measure on the compact group K_1. Φ_1 clearly belongs to $\mathcal{S}(K,U)$ and

$$Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1) \tag{4.2.6}$$

and the Fourier transform Φ_1' of Φ_1 is given by

$$\Phi_1'(x) = \int_{K_1} \Omega(h^{-1})\,\Phi'(hx)$$

The function $\Phi_1'(x^\iota)$ belongs to $\mathcal{S}(K,\Omega)$ and

$$Z(\alpha_F^s \otimes \Omega^{-1}, \Phi') = Z(\alpha_F^s \otimes \Omega^{-1}, \Phi_1'). \tag{4.2.7}$$

Since Φ_1 and Φ_1' both have compact support in $K^\times$ the second assertion is clear.

If u is in U and we let Φ_u be the function which is O outside of U_K, the group of units of O_K, and on U_K is given by $\Phi_u(x) = \Omega^{-1}(x)u$ then

$$Z(\alpha_F^s \otimes \Omega, \Phi_u) = cu$$

if

$$c = \int_{U_K} d^\times a.$$

If φ belongs to $\mathcal{S}(K^\times)$ let $A(\varphi)$ and $B(\varphi)$ be the linear transformations of U defined by

$$A(\varphi)U = Z(\alpha_F^{s+1/2} \otimes \Omega, \varphi^u)$$
$$B(\varphi)u = Z(\alpha_F^{-s+3/2} \otimes \Omega^{-1}, \varphi' u)$$

where φ' is the Fourier transform of φ. If $\lambda(h)\,\varphi(h) = \varphi(h^{-1}x)$ and $\rho(h)\,\varphi(x) = \varphi(xh)$ then

$$A(\lambda(h)\varphi) = |h|_K^{s/2+1/4}\Omega(h)\,A(\varphi)$$

and

$$A(\rho(h)\varphi) = |h|_K^{-s/2-1/4}A(\varphi)\Omega^{-1}(h).$$

Since the Fourier transform of $\lambda(h)\varphi$ is $|h|_K\rho(h)\varphi'$ and the Fourier transform of $\rho(h)\varphi$ is $|h|_K^{-1}\lambda(h)\varphi'$, the map $\varphi \to B(\varphi)$ has the same two properties. Since the kernel of Ω is open it is easily seen that $A(\varphi)$ and $B(\varphi)$ are obtained by integrating φ against locally constant functions α and β. They will of course take values in the space of linear transformations of U. We will have

$$\alpha(ha) = |h|_K^{s/2+1/4}\Omega(h)\,\alpha(a)$$

and

$$\alpha(ah^{-1}) = |h|_K^{-s/2-1/4}\alpha(a)\,\Omega^{-1}(h)$$

β will satisfy similar identities. Thus

$$\alpha(h) = |h|_K^{s/2+1/4}\,\Omega(h)\,\alpha(1)$$
$$\beta(h) = |h|_K^{s/2+1/4}\,\Omega(h)\,\beta(1)$$

$\alpha(1)$ is of course the identity. However $\beta(1)$ must commute with $\Omega(h)$ for all h in $K^\times$ and therefore it is a scalar multiple of the identity. Take this scalar to be $-\varepsilon(s,\Omega,\psi)$.

The identity of part (iv) is therefore valid for Φ in $\mathcal{S}(K^\times,U)$ and in particular for Φ in $\mathcal{S}(K,\Omega)$. The general case follows from (4.2.6) and (4.2.7). Since

$$\varepsilon(s,\Omega,\psi) = -\frac{1}{c}\,Z(\alpha_F^{3/2-s}\otimes\Omega^{-1},\Phi'_u)$$

the function $\varepsilon(s,\Omega,\psi)$ is a finite linear combination of powers $|\varpi|^s$ if ϖ is a generator of $\mathfrak{p}_F$. Exchanging the roles of Φ_u and Φ'_u we see that $\varepsilon^{-1}(s,\Omega,\psi)$ has the same property. $\varepsilon(s,\Omega,\psi)$ is therefore a multiple of some power of $|\varpi|^s$.

We have yet to complete the proof of the theorem. Suppose $\varphi=\varphi_\Phi$ belongs to $\mathcal{S}(F^\times,U)$ and $\varphi'=\varphi_{r_\Omega(w)\Phi}$. We saw in the first paragraph that if χ is a quasi-character of $F^\times$ then

$$\widehat{\varphi}(\chi) = Z(\alpha_F\chi\otimes\Omega,\Phi) \tag{4.2.8}$$

and, if $\Omega(a)=\omega(a)I$ for a in $F^\times$,

$$\widehat{\varphi}'(\chi^{-1}\omega^{-1}) = -Z(\alpha_F\chi^{-1}\otimes\Omega^{-1},\Phi'). \tag{4.2.9}$$

Suppose U_0 is a subspace of U and φ takes its values in U_0. Then, by the previous lemma, $\widehat{\varphi}(\chi)$ and $\widehat{\varphi}'(\chi^{-1}\omega^{-1})$ also lie in U_0 for all choices of χ. Since φ' lies in $\mathcal{S}(F^\times,U)$ we may apply Fourier inversion to the multiplicative group to see that φ' takes values in U_0.

We may regard r_Ω as acting on $\mathcal{S}(F^\times,U)$. Then $\mathcal{S}(F^\times,U_0)$ is invariant under $r_\Omega(w)$. Since $r_\Omega(b)\varphi=\xi_\psi(b)\varphi$ for b in B_F it is also invariant under the action of B_F. Finally $r_\Omega\left(\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right)\right)\varphi=\omega(a)\varphi$ so that $\mathcal{S}(F^\times,U_0)$ is invariant under the action of G_F itself. If we take U_0 to have dimension one then $\mathcal{S}(F^\times,U_0)$ may be identified with $\mathcal{S}(F^\times)$ and the representation r_Ω restricted to $\mathcal{S}(F^\times,U_0)$ is irreducible. From (4.2.8) and (4.2.9) we obtain

$$\widehat{\varphi}(\alpha_F^{s-1/2}\chi) = Z(\alpha_F^{s+1/2}\chi\otimes\Omega,\Phi)$$
$$\widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1}) = -Z(\alpha_F^{-s+3/2}\chi^{-1}\otimes\Omega^{-1},\Phi')$$

so that

$$\widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1} = \varepsilon(s,\chi\otimes\Omega,\psi)\,\widehat{\varphi}(\alpha_F^{s-1/2}\chi).$$

Thus if π_0 is the restriction of r_Ω to $\mathcal{S}(F^\times, U_0)$

$$\varepsilon(s, \chi \otimes \pi_0, \psi) = \varepsilon(s, \chi \otimes \Omega, \psi)$$

so that $\pi_0 = \pi(\Omega)$ is, apart from equivalence, independent of U_0. The theorem follows.

Let Ω be any irreducible finite-dimensional representation of $K^\times$ and let Ω act on U. The contragredient representation $\widetilde{\Omega}$ acts on the dual space $\widetilde{U}$ of U. If u belongs to U and $\widetilde{u}$ belongs to $\widetilde{U}$

$$\langle u, \widetilde{\Omega}(h)\widetilde{u}\rangle = \langle \Omega^{-1}(h)u, \widetilde{u}\rangle.$$

If Φ belongs to $\mathcal{S}(K)$ set

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi; u, \widetilde{u}) = \int_{K^\times} |\nu(h)|^s \, \Phi(h) \, \langle \Omega(h)u, \widetilde{u}\rangle \, d^\times h$$

and set

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi; u, \widetilde{u}) = \int_{K^\times} |\nu(h)|^s \, \Phi(h) \, \langle u, \widetilde{\Omega}(h)\widetilde{u}\rangle \, d^\times h.$$

Theorem 4.3 *Let Ω be an irreducible representation of $K^\times$ in the space U.*

(i) *For any quasi-character χ of $F^\times$*

$$\pi(\chi \otimes \Omega) = \chi \otimes \pi(\Omega).$$

(ii) *There is a real number s_0 such that for all u, $\widetilde{u}$ and Φ and all s with $\operatorname{Re} s > s_0$ the integral defining $Z(\alpha_F^s \otimes \Omega, \Phi; u, \widetilde{u})$ is absolutely convergent.*

(iii) *There is a unique Euler factor $L(s, \Omega)$ such that the quotient*

$$\frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \widetilde{u})}{L(s, \Omega)}$$

is holomorphic for all u, $\widetilde{u}$, Φ and for some choice of these variables is a non-zero constant.

(iv) *There is a functional equation*

$$\frac{Z(\alpha_F^{3/2-s} \otimes \widetilde{\Omega}, \Phi', u, \widetilde{u})}{L(1-s, \widetilde{\Omega})} = -\varepsilon(s, \Omega, \psi) \, \frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \widetilde{u})}{L(s, \Omega)}$$

where $\varepsilon(s, \Omega, \psi)$ is, as a function of s, an exponential.

(v) *If $\Omega(a) = \omega(a)I$ for a in $F^\times$ and if $\pi = \pi(\Omega)$ then*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I.$$

Moreover $L(s, \pi) = L(s, \Omega)$, $L(s, \widetilde{\pi}) = L(s, \widetilde{\Omega})$ and $\varepsilon(s, \pi, \psi) = \varepsilon(s, \Omega, \psi)$.

The first assertion is a consequence of the definitions. We have just proved all the others when Ω has a degree greater than one. Suppose then that $\Omega(h) = \chi\big(\nu(h)\big)$ where χ is a quasi-character of $F^\times$. Then $\pi(\Omega) = \pi(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ and if the last part of the theorem is to hold $L(s, \Omega)$, which is of course uniquely determined by the conditions of part (iii), must equal $L(s,\pi) = L(s, \chi\alpha_F^{1/2})$. Also $L(s, \widetilde{\Omega})$ must equal $L(s, \tilde{\pi}) = L(s, \chi^{-1}\alpha_F^{1/2})$.

In the case under consideration $U = \mathbb{C}$ and we need only consider

$$Z(\alpha_F^s \otimes \Omega, \Phi; 1, 1) = Z(\alpha_F^s \otimes \Omega, \Phi).$$

As before the second part is trivial and

$$Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1)$$

if

$$\Phi_1(x) = \int_{K_1} \Phi(xh).$$

The Fourier transform of Φ_1 is

$$\Phi_1'(x) = \int_{K_1} \Phi'(hx) = \int_{K_1} \Phi'(xh)$$

and

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi') = Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi_1').$$

It is therefore enough to consider the functions in $\mathcal{S}(K, \Omega)$.

If $\varphi = \varphi_\Phi$ is defined as before then φ lies in the space on which the Kirillov model of π acts and

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = A(\alpha_F^{s+1/2} \otimes \Omega, \Phi).$$

The third assertion follows from the properties of $L(s,\pi)$. The fourth follows from the relation

$$\widehat{\varphi}'(\alpha_F^{1/2-s}\omega^{-1}) = -Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi'),$$

which was proved in the first paragraph, and the relation

$$\frac{\widehat{\varphi}'(\alpha_F^{1/2-s}\omega^{-1})}{L(1-s, \tilde{\pi})} = \varepsilon(s, \pi, \psi)\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\pi)},$$

which was proved in the second, if we observe that $\widetilde{\Omega}(h) = \Omega^{-1}(h)$. φ' is of course $\pi(w)\varphi$.

Corollary 4.4 *If $\pi = \pi(\Omega)$ then $\tilde{\pi} = \pi(\widetilde{\Omega})$.*

This is clear if Ω if of degree one so suppose it is of degree greater than one. Combining the identity of part (iv) with that obtained upon interchanging the roles of Ω and $\widetilde{\Omega}$ and of Φ and Φ' we find that

$$\varepsilon(s, \Omega, \psi)\, \varepsilon(1-s, \widetilde{\Omega}, \psi) = \omega(-1).$$

The same considerations show that

$$\varepsilon(s, \pi, \psi)\, \varepsilon(1-s, \tilde{\pi}, \psi) = \omega(-1).$$

Consequently

$$\varepsilon(s, \tilde{\pi}, \psi) = \varepsilon(s, \widetilde{\Omega}, \psi).$$

Replacing Ω by $\chi \otimes \Omega$ we see that

$$\varepsilon(s, \chi^{-1} \otimes \tilde{\pi}, \psi) = \varepsilon(s, \chi^{-1} \otimes \widetilde{\Omega}, \psi) = \varepsilon\big(s, \chi^{-1}\pi(\widetilde{\Omega}), \psi\big)$$

for all quasi-characters χ. Since $\tilde{\pi}$ and $\pi(\widetilde{\Omega})$ are both absolutely cuspidal they are equivalent.

There is a consequence of the theorem whose significance we do not completely understand.

Proposition 4.5 *Let Ω be an irreducible representation of $K^\times$ on the space U and suppose that the dimension of U is greater than one. Let $\widetilde{U}$ be the dual space of U. Let π be the Kirillov model of $\pi(\Omega)$, let φ lie in $\mathcal{S}(F^\times)$, and let $\varphi' = \pi(w)\varphi$. If u belongs to U and $\widetilde{u}$ belong to $\widetilde{U}$ the function Φ on K which vanishes at 0 and on $K^\times$ is defined by*

$$\Phi(x) = \varphi\big(\nu(x)\big)\,|\nu(x)|^{-1}\langle u, \widetilde{\Omega}(x)\widetilde{u}\rangle$$

is in $\mathcal{S}(K)$ and its Fourier transform Φ' vanishes at 0 and on $K^\times$ is given by

$$\Phi'(x) = -\varphi'\big(\nu(x)\big)\,|\nu(x)|^{-1}\omega^{-1}\big(\nu(x)\big)\,\langle \Omega(x)u, \widetilde{u}\rangle$$

if $\Omega(a) = \omega(a)I$ for a in $F^\times$.

It is clear that Φ belongs not merely to $\mathcal{S}(K)$ but in fact to $\mathcal{S}(K^\times)$. So does the function Φ_1 which we are claiming is equal to Φ'. The Schur orthogonality relations for the group K_1 show that $\Phi'(0) = 0$ so that Φ' also belongs to $\mathcal{S}(K^\times)$.

We are going to show that for every irreducible representation of Ω' of $K^\times$

$$\int \frac{\Phi_1(x), \langle u', \widetilde{\Omega}'(x)\widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x}{L(1-s, \widetilde{\Omega}')} = -\int \frac{\varepsilon(s, \Omega', \psi)\,\Phi(x)\,\langle \Omega'(x)u', \widetilde{u}'\rangle\,|\nu(x)|^{s+1/2}\,d^\times x}{L(s, \Omega')}$$

for all choices of u' and $\widetilde{u}'$. Applying the theorem we see that

$$\int \{\Phi_1(x) - \Phi'(x)\}\,\langle u', \widetilde{\Omega}'(x)\widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x = 0$$

for all choices of Ω', u', $\widetilde{u}'$, and all s. An obvious and easy generalization of the Peter-Weyl theorem, which we do not even bother to state, shows that $\Phi_1 = \Phi'$.

If

$$\Psi(x) = \int_{K_1} \langle u, \widetilde{\Omega}(hx)\widetilde{u}\rangle\,\langle \Omega'(hx)u', \widetilde{u}'\rangle\,dh$$

then

$$\int_{K^\times} \Phi(x)\langle \Omega'(x)u', \widetilde{u}'\rangle\,|\nu(x)|^{s+1/2}\,d^\times x = \int_{K^\times/K_1} \varphi\big(\nu(x)\big)\,|\nu(x)|^{s-1/2}\,\Psi(x)\,d^\times x$$

while

$$\int_{K^\times} \Phi_1(x)\,\langle u'\widetilde{\Omega}'(x), \widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x = -\int_{K^\times/K_1} \varphi'\big(\nu(x)\big)\,\omega^{-1}\big(\nu(x)\big)\,|\nu(x)|^{1/2-s}\,\Psi(x^{-1})\,d^\times x$$

If Ψ is 0 for all choice of u' and $\widetilde{u}'$ the required identity is certainly true. Suppose then Ψ is different from 0 for some choice u' and $\widetilde{u}'$.

Let U be the intersection of the kernels of Ω' and Ω. It is an open normal subgroup of $K^\times$ and $H = U\,K_1\,F^\times$ is open, normal, and of finite index in $K^\times$. Suppose that $\Omega'(a) = \omega'(a)I$ for a in $F^\times$. If h belongs to H

$$\Psi(xh) = \chi_0(h)\,\Psi(x)$$

where χ_0 is a quasi-character of H trivial on U and K_1 and equal to $\omega'\omega^{-1}$ on $F^\times$. Moreover χ_0 extends to a quasi-character χ of $K^\times$ so that

$$\int_{K^\times/H} \Psi(x)\,\chi^{-1}(x) = \int_{K^\times/F^\times} \psi(x)\,\chi^{-1}(x) \neq 0$$

χ may of course be identified with a quasi-character of $F^\times$.

Lemma 4.5.1 *If*

$$\int_{K^\times/F^\times} \Psi(x)\,\chi^{-1}(x) \neq 0$$

then Ω' is equivalent to $\chi\otimes\Omega$.

Ω' and $\chi\otimes\Omega$ agree on $F^\times$ and

$$\int_{K^\times/F^\times} \langle u, \widetilde{\chi\otimes\Omega}(x)\widetilde{u}\rangle\,\langle\Omega'(x)u', \widetilde{u}'\rangle \neq 0.$$

The lemma follows from the Schur orthogonality relations.

We have therefore only to prove the identity for $\Omega' = \chi\otimes\Omega$. Set

$$F(x) = \int_{K_1} \langle u, \widetilde{\Omega}(hx)\widetilde{u}\rangle\,\langle\Omega(hx)u', \widetilde{u}'\rangle\,dh.$$

u' and $\widetilde{u}'$ now belong to the spaces U and $\widetilde{U}$. There is a function f on $F^\times$ such that

$$F(x) = f\big(\nu(x)\big)$$

The identity we are trying to prove may be written as

$$\frac{\int \varphi'(a)\,\chi^{-1}(a)\,\omega^{-1}(a)\,f(a^{-1})\,|a|^{1/2-s}\,d^\times a}{L(1-s,\chi^{-1}\otimes\tilde{\pi})} = \varepsilon(s,\chi\otimes\pi,\psi)\frac{\int \varphi(a)\,\chi(a)\,f(a)\,|a|^{s-1/2}\,d^\times a}{L(s,\chi\otimes\pi)}. \tag{4.5.2}$$

Let H be the group constructed as before with U taken as the kernel of Ω. The image F' of H under ν is a subgroup of finite index in $F^\times$ and f, which is a function on $F^\times/F'$, may be written as a sum

$$f(a) = \sum_{i=1}^{p} \lambda_k\,\chi_i(a)$$

where $\{\chi_1,\cdots,\chi_p\}$ are the characters of $F^\times/F'$ which are not orthogonal to f. By the lemma Ω is equivalent to $\chi_i\otimes\Omega$ for $1\leq i\leq p$ and therefore π is equivalent to $\chi_i\otimes\pi$. Consequently

$$\varepsilon(s,\chi\otimes\pi,\psi) = \varepsilon(s,\chi\chi_i\otimes\pi,\psi)$$

and

$$\frac{\int_{F^\times} \varphi'(a)\,\chi^{-1}(a)\,\chi_1^{-1}(a)\,\omega^{-1}(a)\,|a|^{1/2-s}\,d^\times a}{L(1-s,\chi^{-1}\otimes\tilde{\pi})} = \varepsilon(s,\chi\otimes\pi,\psi)\frac{\int_{F^\times} \varphi(a)\,\chi(a)\,\chi_i(a)\,|a|^{s-1/2}\,d^\times a}{L(s,\chi\otimes\pi)}.$$

The identity (4.5.2) follows.

Now let K be a separable quadratic extension of F. We are going to associate to each quasi-character ω of $K^\times$ an irreducible representation $\pi(\omega)$ of G_F. If G_+ is the set of all g in G_F whose determinants belong to $\nu(K^\times)$ we have already, in the first paragraph, associated to ω a representation r_ω of G_+. To emphasize the possible dependence of r_ω on ψ we now denote it by $\pi(\omega,\psi)$. G_+ is of index 2 in G_F. Let $\pi(\omega)$ be the representation of G_F induced from $\pi(\omega,\psi)$.

Theorem 4.6 (i) *The representation $\pi(\omega, \psi)$ is irreducible.*

(ii) *The representation $\pi(\omega)$ is admissible and irreducible and its class does not depend on the choice of ψ.*

(iii) *If there is no quasi-character χ of $F^\times$ such that $\omega = \chi_0\nu$ the representation $\pi(\omega)$ is absolutely cuspidal.*

(iv) *If $\omega = \chi_0\nu$ and η is the character of $F^\times$ associated to K by local class field theory then $\pi(\omega)$ is $\pi(\chi, \chi_\eta)$.*

It is clear what the notion of admissibility for a representation of G_+ should be. The proof that $\pi(\omega, \psi)$ is admissible proceeds like the proof of the first part of Theorem 4.2 and there is little point in presenting it.

To every Φ in $\mathcal{S}(K, \omega)$ we associate the function φ_Φ on $F_+ = \nu(K^\times)$ defined by

$$\varphi_\Phi(a) = \omega(h)\, |h|_K^{1/2}\, \Phi(h)$$

if $a = \nu(h)$. Clearly $\varphi_\Phi = 0$ if and only if $\Phi = 0$. Let V_+ be the space of functions on F_+ obtained in this manner. V_+ clearly contains the space $\mathcal{S}(F_+)$ of locally constant compactly supported functions on F_+. In fact if φ belongs to $\mathcal{S}(F_+)$ and

$$\Phi(h) = \omega^{-1}(h)\, |h|_K^{-1/2}\, \varphi\big(\nu(h)\big)$$

then $\varphi = \varphi_\Phi$. If the restriction of ω to the group K_1 of elements of norm 1 in $K^\times$ is not trivial so that every element of $\mathcal{S}(K, \omega)$ vanishes at 0 then $V_+ = \mathcal{S}(F_+)$. Otherwise $\mathcal{S}(F_+)$ is of codimension one in V_+.

Let B_+ be the group of matrices of the form

$$\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with a in F_+ and x in F. In the first paragraph we introduced a representation $\xi = \xi_\psi$ of B_+ on the space of functions on F_+. It was defined by

$$\xi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi(b) = \varphi(ba)$$

and

$$\xi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \varphi(b) = \psi(bx)\, \varphi(b).$$

We may regard $\pi(\omega, \psi)$ as acting on V_+ and if we do the restriction of $\pi(\omega, \psi)$ to B_+ is ξ_ψ.

Lemma 4.6.1 *The representation of B_F induced from the representation ξ_ψ of B_+ on $\mathcal{S}(F_+)$ is the representation ξ_ψ of B_F. In particular the representation ξ_ψ of B_+ is irreducible.*

The induced representation is of course obtained by letting B_F act by right translations on the space of all functions $\widetilde{\varphi}$ on B_F with values in $\mathcal{S}(F_+)$ which satisfy

$$\widetilde{\varphi}(b_1 b) = \xi_\psi(b_1)\, \widetilde{\varphi}(b)$$

for all b_1 in B_+. Let L be the linear functional in $\mathcal{S}(F_+)$ which associates to a function its value at 1. Associate to $\widetilde{\varphi}$ the function

$$\varphi(a) = L\left(\widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = L\left(\rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\widetilde{\varphi}(e)\right)$$

The value of $\widetilde{\varphi}\left(\left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ at α in F_+ is

$$\begin{aligned} L\left(\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & \alpha x \\ 0 & 1 \end{pmatrix}\right)\right) &= L\left(\xi_\psi\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\right)\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \psi(ax) L\left(\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = \psi(\alpha x)\varphi(\alpha a). \end{aligned}$$

Since $F^\times/F_+$ is finite it follows immediately that φ is in $\mathcal{S}(F^\times)$ and that $\widetilde{\varphi}$ is 0 if φ is. It also shows that φ can be any function in $\mathcal{S}(F^\times)$ and that if $\widetilde{\varphi}' = \rho(b)\,\widetilde{\varphi}$ then $\varphi' = \xi(b)\,\varphi$ for all b in B_F. Since a representation obtained by induction cannot be irreducible unless the original representation is, the second assertion follows from Lemma 2.9.1.

If the restriction of ω to K_1 is not trivial the first assertion of the theorem follows immediately. If it is then, by an argument used a number of times previously, any non-zero invariant subspace of V_+ contains $\mathcal{S}(F_+)$ so that to prove the assertion we have only to show that $\mathcal{S}(F_+)$ is not invariant.

As before we observe that if Φ in $\mathcal{S}(K,\omega) = \mathcal{S}(K)$ is taken to vanish at 0 but to be non-negative and not identically 0 then

$$r_\omega(w)\,\Phi(0) = \gamma \int_K \Phi(x)\,dx \neq 0$$

so that φ_Φ is in $\mathcal{S}(F_+)$ but $\varphi_{r_\omega(w)\Phi}$ is not.

The representation $\pi(\omega)$ is the representation obtained by letting G_+ act to the right on the space of functions $\widetilde{\varphi}$ on G_+ with values in V_+ which satisfy

$$\widetilde{\varphi}(hg) = \pi(\omega,\psi)(h)\,\widetilde{\varphi}(g)$$

for h in G_+. Replacing the functions $\widetilde{\varphi}$ by the functions

$$\widetilde{\varphi}'(g) = \widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

we obtain an equivalent representation, that induced from the representation

$$g \to \pi(\omega,\psi)\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

of G_+. It follows from Lemma 1.4 that this representation is equivalent to $\pi(\omega,\psi')$ if $\psi'(x) = \psi(ax)$. Thus $\pi(\omega)$ is, apart from equivalence, independent of ψ.

Since

$$G_F = \left\{ g\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid g \in G_+,\, a \in F^\times \right\}$$

$\widetilde{\varphi}$ is determined by its restrictions to B_F. This restriction, which we again call $\widetilde{\varphi}$, is any one of the functions considered in Lemma 4.6.1. Thus, by the construction used in the proof of that lemma, we can associate to any $\widetilde{\varphi}$ a function φ on $F^\times$. Let V be the space of functions so obtained. We can regard $\pi = \pi(\omega)$ as acting on V. It is clear that, for all φ in V,

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. Every function on F_+ can, by setting it equal to 0 outside of F_+, be regarded as a function $F^\times$. Since

$$\widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(\alpha) = \varphi(\alpha a)$$

V is the space generated by the translates of the functions in V_+. Thus if $V_+ = \mathcal{S}(F_+)$ then $V = \mathcal{S}(F^\times)$ and if $\mathcal{S}(F_+)$ is of codimension one in V_+ then $\mathcal{S}(F^\times)$ is of codimension two in V.

It follows immediately that $\pi(\omega)$ is irreducible and absolutely cuspidal if the restriction of ω to K_1 is not trivial.

The function φ in V_+ corresponds to the function $\widetilde{\varphi}$ which is 0 outside of G_+ and on G_+ is givne by

$$\widetilde{\varphi}(g) = \pi(\omega, \psi)(g)\varphi.$$

It is clear that

$$\pi(\omega)(g)\varphi = \pi(\omega, \psi)(g)\varphi$$

if g is in G_+. Any non-trivial invariant subspace of V will have to contain $\mathcal{S}(F^\times)$ and therefore $\mathcal{S}(F_+)$. Since $\pi(\omega, \psi)$ is irreducible it will have to contain V_+ and therefore will be V itself. Thus $\pi(\omega)$ is irreducible for all ω.

If the restriction of ω to K_1 is trivial there is a quasi-character χ of $F^\times$ such that $\omega = \chi \circ \nu$. To establish the last assertion of the lemma all we have to do is construct a non-zero linear form L on V which annihilates $\mathcal{S}(F^\times)$ and satisfies

$$L\left(\pi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \chi(a_1 a_2)\,\eta(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} L(\varphi)$$

if $\pi = \pi(\omega)$. We saw in Proposition 1.5 that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi = \chi^2(a)\,\eta(a)\varphi$$

so will only have to verify that

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi\right) = \chi(a)|a|^{1/2}L(\varphi)$$

If $\varphi = \varphi_\Phi$ is in V_+ we set

$$L(\varphi) = \Phi(0)$$

so that if a is in F_+

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(0) = \chi(a)\,|a|^{1/2}\,L(\varphi).$$

If ε is in $F^\times$ but not in F_+ any function φ in V can be written uniquely as

$$\varphi = \varphi_1 + \pi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_2$$

with φ_1 and φ_2 in V_+. We set

$$L(\varphi) = L(\varphi_1) + \chi(\varepsilon)\, L(\varphi_2).$$

Theorem 4.7 (i) *If $\pi = \pi(\omega)$ then $\pi = \pi(\omega^\iota)$ if $\omega^\iota(a) = \omega(a^\iota)$, $\tilde\pi = \pi(\omega^{-1})$ and $\chi \otimes \pi = \pi(\omega\chi')$ if χ is a quasi-character of $F^\times$ and $\chi' = \chi \circ \nu$.*

(ii) *If a is in $F^\times$ then*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a) I.$$

(iii) *$L(s,\pi) = L(s,\omega)$ and $L(s,\tilde\pi) = L(s,\omega^{-1})$. Moreover if $\psi_K(x) = \psi_F\big(\xi(x)\big)$ for x in K and if $\lambda(K/F,\psi_F)$ is the factor introduced in the first paragraph then*

$$\varepsilon(s,\pi,\psi_F) = \varepsilon(s,\omega,\psi_K)\,\lambda(K/F,\psi_F)$$

It is clear that $\chi\otimes\pi(\omega,\psi)$ of G_+. However by its very construction $\chi\otimes\pi(\omega,\psi) = \pi(\omega\chi',\psi)$. The relation

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a) I$$

is a consequence of part (iii) of Proposition 1.5 and has been used before. Since $\eta' = \eta\circ\nu$ is trivial and $\omega\big(\nu(a)\big) = \omega(a)\,\omega^\iota(a)$

$$\tilde\pi = \omega^{-1}\eta^{-1}\otimes\pi = \pi(\omega^{-\iota})$$

To complete the proof of the first part of the theorem we have to show that $\pi(\omega) = \pi(\omega^\iota)$. It is enough to verify that $\pi(\omega,\psi) = \pi(\omega^\iota,\psi)$. If Φ belongs to $\mathcal{S}(K)$ let $\Phi^\iota(x) = \Phi(x^\iota)$. $\Phi \to \Phi^\iota$ is a bijection of $\mathcal{S}(K,\omega)$ with $\mathcal{S}(K,\omega^\iota)$ which changes $\pi(\omega,\psi)$ into $\pi(\omega^\iota,\psi)$. Observe that here as elsewhere we have written an equality when we really mean an equivalence.

We saw in the first paragraph that if $\varphi = \varphi_\Phi$ is in V_+ then

$$\widehat\varphi(\alpha_F^{s-1/2}) = Z(\alpha_K^s\omega,\Phi)$$

and that if $\varphi\prime = \pi(w)\varphi$ and Φ' is the Fourier transform of Φ then, if $\omega_0(a) = \omega(a)\,\eta(a)$ for a in $F^\times$,

$$\widehat\varphi'(\omega_0^{-1}\alpha_F^{s-1/2}) = \gamma Z(\alpha_K^{1-s}\omega^{-1},\Phi')$$

if $\gamma = \lambda(K/F,\psi_F)$. Thus for all φ in V_+ the quotient

$$\frac{\widehat\varphi(\alpha_F^{s-1/2})}{L(s,\omega)}$$

has an analytic continuation as a holomorphic function of s and for some φ it is a non-zero constant. Also

$$\frac{\widehat\varphi'(w_0^{-1}\alpha_F^{1/2-s})}{L(1-s,\omega^{-1})} = \lambda(K/F,\psi_F)\,\varepsilon(s,\omega,\psi_K)\,\frac{\widehat\varphi(\alpha_F^{s-1/2})}{L(s,\omega)}.$$

To prove the theorem we have merely to check that these assertions remain valid when φ is allowed to vary in V. In fact we need only consider functions of the form

$$\varphi = \pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

where φ_0 is in V_+ and ε is not in F_+. Since

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = |\varepsilon|^{1/2-s}\,\widehat{\varphi}_0(\alpha_F^{s-1/2})$$

the quotient

$$\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\omega)}$$

is certainly holomorphic in the whole plane. Since

$$\widehat{\varphi}'(\omega_0^{-1}\alpha_F^{1/2-s}) = \omega_0(\varepsilon)\,\omega_0^{-1}(\varepsilon)\,|\varepsilon|^{1/2-s}\widehat{\varphi}_0'(\omega_0^{-1}\alpha_F^{1/2-s}) = |\varepsilon|^{\frac{1}{2}-s}\widehat{\varphi}_0'(\omega_0^{-1}\alpha_F^{1/2-s})$$

the functional equation is also satisfied.

Observe that if $\omega = \chi \circ \nu$ then $\pi(\omega) = \pi(\chi, \chi_\eta)$ so that

$$L(s,\omega) = L(s,\chi)\,L(s,\chi_\eta)$$

and

$$\varepsilon(s,\omega,\psi_K)\,\lambda(K/F,\psi_F) = \varepsilon(s,\chi,\psi_F)\,\varepsilon(s,\chi_\eta,\psi_F)$$

These are special cases of the identities of [19].

§**5. Representations of** $GL(2,\mathbb{R})$**.** We must also prove a local functional equation for the real and complex fields. In this paragraph we consider the field $\mathbb{R}$ of real numbers. The standard maximal compact subgroup of $GL(2,\mathbb{R})$ is the orthogonal group $O(2,\mathbb{R})$. Neither $GL(2,\mathbb{R})$ nor $O(2,\mathbb{R})$ is connected.

Let $\mathcal{H}_1$ be the space of infinitely differentiable compactly supported functions on $GL(2,\mathbb{R})$ which are $O(2,\mathbb{R})$ finite on both sides. Once a Haar measure on $G_\mathbb{R} = GL(2,\mathbb{R})$ has been chosen we may regard the elements of $\mathcal{H}_1$ as measures and it is then an algebra under convolution.

$$f_1 \times f_2(g) = \int_{G_\mathbb{R}} f_1(gh^{-1})\, f_2(h)\, dh.$$

On $O(2,\mathbb{R})$ we choose the normalized Haar measure. Then every function ξ on $O(2,\mathbb{R})$ which is a finite sum of matrix elements of irreducible representations of $O(2,\mathbb{R})$ may be identified with a measure on $O(2,\mathbb{R})$ and therefore on $GL(2,\mathbb{R})$. Under convolution these measures form an algebra $\mathcal{H}_2$. $\mathcal{H}_\mathbb{R}$ will be the sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is also an algebra under convolution of measures. In particular if ξ belongs to $\mathcal{H}_2$ and f belongs to $\mathcal{H}_1$

$$\xi * f(g) = \int_{O(2,\mathbb{R})} \xi(u)\, f(u^{-1}g)\, du$$

and

$$f * \xi(g) = \int_{O(2,\mathbb{R})} f(gu^{-1})\, \xi(u)\, du.$$

If σ_i, $1 \le i \le p$, is a family of inequivalent irreducible representations of $O(2,\mathbb{R})$ and

$$\xi_i(u) = \dim \sigma_i \ \operatorname{trace} \sigma_i(u^{-1})$$

then

$$\xi = \sum_{i=1}^{p} \xi_i$$

is an idempotent of $\mathcal{H}_\mathbb{R}$. Such an idempotent is called elementary.

It is a consequence of the definitions that for any f in $\mathcal{H}_1$ there is an elementary idempotent ξ such that

$$\xi * f = f * \xi = f.$$

Moreover for any elementary idempotent ξ

$$\xi * \mathcal{H}_1 * \xi = \xi * C_c^\infty(G_\mathbb{R}) * \xi$$

is a closed subspace of $C_c^\infty(G_\mathbb{R})$, in the Schwartz topology. We give it the induced topology.

A representation π of the algebra $\mathcal{H}_\mathbb{R}$ on the complex vector space V is said to be admissible if the following conditions are satisfied.

(5.1) Every vector v in V is of the form

$$v = \sum_{i=1}^{r} \pi(f_i) v_i$$

with f_i in $\mathcal{H}_1$ and v_i in V.

(5.2) For every elementary idempotent ξ the range of $\pi(\xi)$ is finite dimensional.

(5.3) For every elementary idempotent ξ and every vector v in $\pi(\xi)V$ the map $f \to \pi(f)v$ of $\xi\mathcal{H}_1\xi$ into the finite dimensional space $\pi(\xi)V$ is continuous.

If $v = \sum_{i=1}^{r} \pi(f_i)v_i$ we can choose an elementary idempotent ξ so that $\xi f_i = f_i\xi = f_i$ for $1 \leq i \leq r$. Then $\pi(\xi)v = v$. Let $\{\varphi\}$ be a sequence in $C_c^\infty(G_\mathbb{R})$ which converges, in the space of distributions, towards the Dirac distribution at the origin. Set $\varphi_n' = \xi * \varphi_n * \xi$. For each i the sequence $\{\varphi_n' * f_i\}$ converges to f_i in the space $\xi\mathcal{H}_1\xi$. Thus by (5.3) the sequence $\{\pi(\varphi_n')v\}$ converges to v in the finite dimensional space $\pi(\xi)v$. Thus v is in the closure of the subspace $\pi(\xi\mathcal{H}_1\xi)v$ and therefore belongs to it.

As in the second paragraph the conditions (5.1) and (5.2) enable us to define the representation $\tilde{\pi}$ contragredient to π. Up to equivalence it is characterized by demanding that it satisfy (5.1) and (5.2) and that there be a non-degenerate bilinear form on $V \times \widetilde{V}$ satisfying

$$\langle \pi(f)v, \widetilde{v}\rangle = \langle v, \pi(\check{f})\widetilde{v}\rangle$$

for all f in $\mathcal{H}_\mathbb{R}$. $\widetilde{V}$ is the space on which $\tilde{\pi}$ acts and $\check{f}$ is the image of the measure f under the map $g \to g^{-1}$. Notice that we allow ourselves to use the symbol f for all elements of $\mathcal{H}_\mathbb{R}$. The condition (5.3) means that for every v in V and every $\widetilde{v}$ in $\widetilde{V}$ the linear form

$$f \to \langle \pi(f)v, \widetilde{v}\rangle$$

is continuous on each of the spaces $\xi\mathcal{H}_1\xi$. Therefore $\tilde{\pi}$ is also admissible.

Choose ξ so that $\pi(\xi)v = v$ and $\tilde{\pi}(\check{\xi})\widetilde{v} = \widetilde{v}$. Then for any f in $\mathcal{H}_1$

$$\langle \pi(f)v, \widetilde{v}\rangle = \langle \pi(\xi f\xi)v, \widetilde{v}\rangle.$$

There is therefore a unique distribution μ on $G_\mathbb{R}$ such that

$$\mu(f) = \langle \pi(f)v, \widetilde{v}\rangle$$

for f in $\mathcal{H}_1$. Choose φ in $\xi\mathcal{H}_1\xi$ so that $\pi(\varphi)v = v$. Then

$$\mu(f\varphi) = \mu(\xi f\varphi\xi) = \mu(\xi f\xi\varphi) = \langle \pi(\xi f\xi\varphi)v, \widetilde{v}\rangle = \langle \pi(\xi f\xi)v, \widetilde{v}\rangle$$

so that $\mu(f\varphi) = \mu(f)$. Consequently the distribution μ is actually a function and it is not unreasonable to write it as $g \to \langle \pi(g)v, \widetilde{v}\rangle$ even though π is not a representation of $G_\mathbb{R}$. For a fixed g, $\langle \pi(g)v, \widetilde{v}\rangle$ depends linearly on v and $\widetilde{v}$. If the roles of π and $\tilde{\pi}$ are reversed we obtain a function $\langle v, \tilde{\pi}(g)\widetilde{v}\rangle$. It is clear from the definition that

$$\langle \pi(g)v, \widetilde{v}\rangle = \langle v, \tilde{\pi}(g^{-1})\widetilde{v}\rangle.$$

Let $\mathfrak{g}$ be the Lie algebra of $G_\mathbb{R}$ and let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \bigotimes_\mathbb{R} \mathbb{C}$. Let $\mathfrak{A}$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. If we regard the elements of $\mathfrak{A}$ as distributions on $G_\mathbb{R}$ with support at the identity we can take their convolution product with the elements of $C_c^\infty(G_\mathbb{R})$. More precisely if X belongs to $\mathfrak{g}$

$$X * f(g) = \frac{d}{dt} f\big(\exp(-tX)\big)\big|_{t=0}$$

and

$$f * X(g) = \frac{d}{dt} f\big(g\exp(-tX)\big)\big|_{t=0}$$

If f belongs to $\mathcal{H}_1$ so do $f * X$ and $X * f$.

We want to associate to the representation π of $\mathcal{H}_\mathbb{R}$ on V a representation π of $\mathfrak{A}$ on V such that

$$\pi(X)\,\pi(f) = \pi(X * f)$$

and

$$\pi(f)\,\pi(X) = \pi(f * X)$$

for all X in $\mathfrak{A}$ and all f in $\mathcal{H}_1$. If $v = \sum \pi(f_i)\,v_i$ we will set

$$\pi(X)v = \sum_i \pi(X * f_i)v_i$$

and the first condition will be satisfied. However we must first verify that if

$$\sum_i \pi(f_i)\,v_i = 0$$

then

$$w = \sum_i \pi(X * f_i)v_i$$

is also 0. Choose f so that $w = \pi(f)w$. Then

$$w = \sum_i \pi(f)\,\pi(X * f_i)v_i = \sum_i \pi(f * X * f_i)v_i = \pi(f * X)\{\sum \pi(f_i)v_i\} = 0.$$

>From the same calculation we extract the relation

$$\pi(f)\left\{\sum_i \pi(X * f_i)v_i\right\} = \pi(f * X)\left\{\sum \pi(f_i)v_i\right\}$$

for all f so that $\pi(f)\pi(X) = \pi(f * X)$.

If g is in $G_\mathbb{R}$ then $\lambda(g)\,f = \delta_g * f$ if δ_g is the Dirac function at g. If g is in $O(2,\mathbb{R})$ or in $Z_\mathbb{R}$, the groups of scalar matrices, $\delta_g * f$ is in $\mathcal{H}_1$ if f is, so that the same considerations allow us to associate to π a representation π of $O(2,\mathbb{R})$ and a representation π of $Z_\mathbb{R}$. It is easy to see that if h is in either of these groups then

$$\pi(\mathrm{Ad}hX) = \pi(h)\,\pi(X)\,\pi(h^{-1}).$$

To dispel any doubts about possible ambiguities of notation there is a remark we should make. For any f in $\mathcal{H}_1$

$$\langle \pi(f)v, \widetilde{v}\rangle = \int_{G_\mathbb{R}} f(g)\langle \pi(g)v, \widetilde{v}\rangle\, dg.$$

Thus if h is in $O(2,\mathbb{R})$ or $Z_\mathbb{R}$

$$\langle \pi(f * \delta_h)v, \widetilde{v}\rangle = \int_{G_\mathbb{R}} f(g)\,\langle \pi(gh)v, \widetilde{v}\rangle\, dg$$

and
$$\langle \pi(f)\pi(h)v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\langle \pi(g)\pi(h)v, \widetilde{v}\rangle\, dg$$
so that
$$\langle \pi(gh)v, \widetilde{v}\rangle = \langle \pi(g)\,\pi(h)v, \widetilde{v}\rangle.$$
A similar argument shows that
$$\langle \pi(hg)v, \widetilde{v}\rangle = \langle \pi(g)v, \widetilde{\pi}(h^{-1}\widetilde{v}\rangle.$$
It is easily seen that the function $\langle \pi(g)v, \widetilde{v}\rangle$ takes the value of $\langle v, \widetilde{v}\rangle$ at $g = e$. Thus if h belongs to $O(2,\mathbb{R})$ or $Z_{\mathbb{R}}$ the two possible interpretations of $\langle \pi(h)v, \widetilde{v}\rangle$ give the same result.

It is not possible to construct a representation of $G_{\mathbb{R}}$ on V and the representation of $\mathfrak{A}$ is supposed to be a substitute. Since $G_{\mathbb{R}}$ is not connected, it is not adequate and we introduce instead the notion of a representation π_1 of the system $\{\mathfrak{A}, \varepsilon\}$ where
$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
It is a representation π_1 of $\mathfrak{A}$ and an operator $\pi_1(\varepsilon)$ which satisfy the relations
$$\pi_1^2(\varepsilon) = I$$
and
$$\pi_1(\mathrm{Ad}\varepsilon X) = \pi_1(\varepsilon)\pi_1(X)\,\pi_1(\varepsilon^{-1}).$$
Combining the representation π with $\mathfrak{A}$ with the operator $\pi(\varepsilon)$ we obtain a representation of the system $\{\mathfrak{A}, \varepsilon\}$.

There is also a representation $\tilde{\pi}$ of $\mathfrak{A}$ associated to $\tilde{\pi}$ and it is not difficult to see that
$$\langle \pi(X)v, \widetilde{v}\rangle = \langle v, \tilde{\pi}(\check{X})\widetilde{v}\rangle$$
if $X \to \check{X}$ is the automorphism of $\mathfrak{A}$ which sends X in $\mathfrak{g}$ to $-X$.

Let
$$\varphi(g) = \langle \pi(g)v, \widetilde{v}\rangle.$$
φ is certainly infinitely differentiable. Integrating by parts we see that
$$\int_{G_{\mathbb{R}}} f(g)\varphi * X(g)\, dg = \int_{G_{\mathbb{R}}} f * \check{X}(g)\,\varphi(g)\, dg$$
The right side is
$$\langle \pi(f)\,\pi(\check{X})v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\langle \pi(g)\,\pi(\check{X})v, \widetilde{v}\rangle$$
so that
$$\varphi * \check{X}(g) = \langle \pi(g)\pi(\check{X})v, \widetilde{v}\rangle.$$

Assume now that the operators $\pi(X)$ are scalar if X is in the centre $\mathfrak{Z}$ of $\mathfrak{A}$. Then the standard proof, which uses the theory of elliptic operators, shows that the functions φ are analytic on $G_{\mathbb{R}}$. Since

$$\varphi * \check{X}(e) = \langle \pi(\check{X})v, \widetilde{v} \rangle$$
$$\varphi * \check{X}(\varepsilon) = \langle \pi(\varepsilon)\, \pi(\check{X})v, \widetilde{v} \rangle$$

and $G_{\mathbb{R}}$ has only two components, one containing e and the other containing ε. The function φ vanishes identically if $\langle \pi(\check{X})v, \widetilde{v} \rangle$ and $\langle \pi(\varepsilon)\, \pi(\check{X})v, \widetilde{v} \rangle$ are 0 for all X in $\mathfrak{A}$. Any subspace V_1 of V invariant under $\mathfrak{A}$ and ε is certainly invariant under $O(2, \mathbb{R})$ and therefore is determined by its annihilator in $\widetilde{V}$. If v is in V_1 and $\widetilde{v}$ annihilates V_1 the function $\langle \pi(g)v, \widetilde{v} \rangle$ is 0 so that

$$\langle \pi(f)v, \widetilde{v} \rangle = 0$$

for all f in $\mathcal{H}_1$. Thus $\pi(f)v$ is also in V_1. Since $\mathcal{H}_2$ clearly leaves V_1 invariant this space is left invariant by all of $\mathcal{H}_{\mathbb{R}}$.

By the very construction any subspace of V invariant under $\mathcal{H}_{\mathbb{R}}$ is invariant under $\mathfrak{A}$ and ε so that we have almost proved the following lemma.

Lemma 5.4 *The representation π of $\mathcal{H}_{\mathbb{R}}$ is irreducible if and only if the associated representation π of $\{\mathfrak{A}, \varepsilon\}$ is.*

To prove it completely we have to show that if the representation of $\{\mathfrak{A}, \varepsilon\}$ is irreducible the operator $\pi(X)$ is a scalar for all X in $\mathfrak{Z}$. As $\pi(X)$ has to have a non-zero eigenfunction we have only to check that $\pi(X)$ commutes with $\pi(Y)$ for Y in $\mathfrak{A}$ with $\pi(\varepsilon)$. It certainly commutes with $\pi(Y)$. X is invariant under the adjoint action not only of the connected component of $G_{\mathbb{R}}$ but also of the connected component of $GL(2, \mathbb{C})$. Since $GL(2, \mathbb{C})$ is conected and contains ε

$$\pi(\varepsilon)\, \pi(X)\, \pi^{-1}(\varepsilon) = \pi\big(\mathrm{Ad}\varepsilon(X)\big) = \pi(X).$$

Slight modifications, which we do not describe, of the proof of Lemma 5.4 lead to the following lemma.

Lemma 5.5 *Suppose π and π' are two irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$. π and π' are equivalent if and only if the associated representations of $\{\mathfrak{A}, \varepsilon\}$ are.*

We comment briefly on the relation between representations of $G_{\mathbb{R}}$ and representations of $\mathcal{H}_{\mathbb{R}}$. Let V be a complete separable locally convex topological space and π a continuous representation of $G_{\mathbb{R}}$ on V. Thus the map $(g, v) \to \pi(g)v$ of $G_{\mathbb{R}} \times V$ to V is continuous and for f in $C_c^{\infty}(G_{\mathbb{R}})$ the operator

$$\pi(f) = \int_{G_{\mathbb{R}}} f(x)\, \pi(x)\, dx$$

is defined. So is $\pi(f)$ for f in $\mathcal{H}_2$. Thus we have a representation of $\mathcal{H}_{\mathbb{R}}$ on V. Let V_0 be the space of $O(2, \mathbb{R})$-finite vectors in V. It is the union of the space $\pi(\xi)V$ as ξ ranges over the elementary idempotents and is invariant under $\mathcal{H}_{\mathbb{R}}$. Assume, as is often the case, that the representation π_0 of $\mathcal{H}_{\mathbb{R}}$ on V_0 is admissible. Then π_0 is irreducible if and only if π is irreducible in the topological sense.

Suppose π' is another continuous representation of $G_{\mathbb{R}}$ in a space V' and there is a continuous non-degenerate bilinear form on $V \times V'$ such that

$$\langle \pi(g)v, v' \rangle = \langle v, \pi'(g^{-1})v' \rangle.$$

Then the restriction of this form to $V_0 \times V_0'$ is non-degenerate and

$$\langle \pi(f)v, v'\rangle = \langle v, \pi'(\check{f})v'\rangle$$

for all f in $\mathcal{H}_{\mathbb{R}}$, v in V_0, and v' in V_0'. Thus π_0' is the contragredient of π_0. Since

$$\langle \pi_0(f)v, v'\rangle = \int_{G_{\mathbb{R}}} f(g)\,\langle \pi(g)v, v'\rangle$$

we have

$$\langle \pi_0(g)v, v'\rangle = \langle \pi(g)v, v'\rangle.$$

The special orthogonal group $SO(2,\mathbb{R})$ is abelian and so is its Lie algebra. The one-dimensional representation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \to e^{in\theta}$$

of $SO(2,\mathbb{R})$ and the associated representation of its Lie algebra will be both denoted by κ_n. A representation π of $\mathfrak{A}$ or of $\{\mathfrak{A},\varepsilon\}$ will be called admissible if its restrictions to the Lie algebra of $SO(2,\mathbb{R})$ decomposes into a direct sum of the representations κ_n each occurring with finite multiplicity. If π is an admissible representation of $\mathcal{H}_{\mathbb{R}}$ the corresponding representation of $\{\mathfrak{A},\varepsilon\}$ is also admissible. We begin the classification of the irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$ and of $\{\mathfrak{A},\varepsilon\}$ with the introduction of some particular representations.

Let μ_1 and μ_2 be two quasi-characters of $F^\times$. Let $\mathcal{B}(\mu_1,\mu_2)$ be the space of functions f on $G_{\mathbb{R}}$ which satisfy the following two conditions.

(i)

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\,\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g)$$

for all g in $G_{\mathbb{R}}$, a_1, a_2 in $\mathbb{R}^\times$, and x in $\mathbb{R}$.

(ii) f is $SO(2,\mathbb{R})$ finite on the right.

Because of the Iwasawa decomposition

$$G_{\mathbb{R}} = N_{\mathbb{R}}\,A_{\mathbb{R}}\,SO(2,\mathbb{R})$$

these functions are complete determined by their restrictions to $SO(2,\mathbb{R})$ and in particular are infinitely differentiable. Write

$$\mu_i(g) = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$$

where s_i is a complex number and m_i is 0 or 1. Set $s = s_1 - s_2$ and $m = |m_1 - m_2|$ so that $\mu_1\,\mu_2^{-1}(t) = |t|^s\,(\frac{t}{|t|})^m$. If n has the same parity as m let φ_n be the function in $\mathcal{B}(\mu_1,\mu_2)$ defined by

$$\varphi_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \mu_1(a_1)\,\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} e^{in\theta}.$$

The collection $\{\varphi_n\}$ is a basis of $\mathcal{B}(\mu_1,\mu_2)$.

For any infinitely differentiable function f on $G_{\mathbb{R}}$ and any compactly supported distribution μ we defined $\lambda(\mu)f$ by

$$\lambda(\mu)f(g) = \check{\mu}\big(\rho(g)f\big)$$

and $\rho(\mu)f$ by

$$\rho(\mu)f(g) = \mu\big(\lambda(g^{-1})f\big).$$

If, for example, μ is a measure

$$\lambda(\mu)f(g) = \int_{G_{\mathbb{R}}} f(h^{-1}g)\, d\mu(h)$$

and

$$\rho(\mu)f(g) = \int_{G_{\mathbb{R}}} f(gh)\, d\mu(h).$$

In all cases $\lambda(\mu)f$ and $\rho(\mu)f$ are again infinitely differentiable. For all f in $\mathcal{H}_{\mathbb{R}}$ the space $\mathcal{B}(\mu_1,\mu_2)$ is invariant under $\rho(f)$ so that we have a representation $\rho(\mu_1,\mu_2)$ of $\mathcal{H}_{\mathbb{R}}$ on $\mathcal{B}(\mu_1,\mu_2)$. It is clearly admissible and the associated representation $\rho(\mu_1,\mu_2)$ of $\{\mathfrak{A},\varepsilon\}$ is also defined by right convolution.

We introduce the following elements of $\mathfrak{g}$ which is identified with the Lie algebra of 2×2 matrices.

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as well as

$$D = X_+X_- + X_-X_+ + \frac{Z^2}{2},$$

which belongs to $\mathfrak{A}$.

Lemma 5.6 *The following relations are valid*

(i) $\rho(U)\varphi_n = in\varphi_n$ (ii) $\rho(\varepsilon)\varphi_n = (-1)^{m_1}\varphi_{-n}$

(iii) $\rho(V_+)\varphi_n = (s+1+n)\varphi_{n+2}$ (iv) $\rho(V_-)\varphi_n = (s+1-n)\varphi_{n-2}$

(v) $\rho(D)\varphi_n = \frac{s^2-1}{2}\varphi_n$ (vi) $\rho(J)\varphi_n = (s_1+s_2)\varphi_n$

The relations (i), (ii), and (vi) are easily proved. It is also clear that for all φ in $\mathcal{B}(\mu_1,\mu_2)$

$$\rho(Z)\,\varphi(e) = (s+1)\,\varphi(e)$$

and

$$\rho(X_+)\,\varphi(e) = 0.$$

The relations

$$\mathrm{Ad}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) V_+ = e^{2i\theta}V_+$$

and

$$\mathrm{Ad}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) V_- = e^{-2i\theta}V_-$$

show that $\rho(V_+)\varphi_n$ is a multiple of φ_{n+2} and that $\rho(V_-)\varphi_n$ is a multiple of φ_{n-2}. Since

$$V_+ = Z - iU + 2iX_+$$

and

$$V_- = Z + iU - 2iX_+$$

the value of $\rho(V_+)\varphi_n$ at the identity e is $s+1+n$ and that of $\rho(V_-)\varphi_n = s+1-n$. Relations (iii) and (iv) follow.

It is not difficult to see that D belongs to $\mathfrak{Z}$ the centre of $\mathfrak{A}$. Therefore $\rho(D)\varphi = \lambda(\check{D})\varphi = \lambda(D)\varphi$ since $D = \check{D}$. If we write D as

$$2X_-\,X_+ + Z + \frac{Z^2}{2}$$

and observe that $\lambda(X_+)\varphi = 0$ and $\lambda(Z)\varphi = -(s+1)\varphi$ if φ is in $\mathcal{B}(\mu_1,\mu_2)$ we see that

$$\rho(D)\varphi_n = \left\{-(s+1) + \frac{(s+1)^2}{2}\right\}\varphi = \frac{s^2-1}{2}\varphi_n.$$

Lemma 5.7 (i) *If $s-m$ is not an odd integer $\mathcal{B}(\mu_1,\mu_2)$ is irreducible under the action of $\mathfrak{g}$.*

(ii) *If $s-m$ is an odd integer and $s \geq 0$ the only proper subspaces of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ are*

$$\mathcal{B}_1(\mu_1,\mu_2) = \sum_{\substack{n \geq s+1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

$$\mathcal{B}_2(\mu_1,\mu_2) = \sum_{\substack{n \leq -s-1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

and, when it is different from $\mathcal{B}(\mu_1,\mu_2)$,

$$\mathcal{B}_s(\mu_1,\mu_2) = \mathcal{B}_1(\mu_1,\mu_2) + \mathcal{B}_2(\mu_1,\mu_2).$$

(iii) *If $s-m$ is an odd integer and $s < 0$ the only proper subspaces of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ are*

$$\mathcal{B}_1(\mu_1,\mu_2) = \sum_{\substack{n \geq s+1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

$$\mathcal{B}_2(\mu_1,\mu_2) = \sum_{\substack{n \leq -s-1 \\ n = (s+1) (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

and

$$\mathcal{B}_f(\mu_1,\mu_2) = \mathcal{B}_1(\mu_1,\mu_2) \cap \mathcal{B}_2(\mu_1,\mu_2).$$

Since a subspace of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ is spanned by those of the vectors φ_n that it contains this lemma is an easy consequence of the relations of Lemma 5.6.

Before stating the corresponding results for $\{\mathfrak{A}, \varepsilon\}$ we state some simple lemmas.

Lemma 5.8 *If π is an irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ there are two possibilities:*

(i) *The restriction of π to $\mathfrak{A}$ is irreducible and the representations $X \to \pi(X)$ and $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$ are equivalent.*

(ii) *The space V on which π acts decomposes into a direct sum $V_1 \oplus V_2$ where V_1 and V_2 are both invariant and irreducible under $\mathfrak{A}$. The representations π_1 and π_2 of $\mathfrak{A}$ on V_1 and V_2 are not equivalent but π_2 is equivalent to the representation $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$.*

If the restriction of π to $\mathfrak{A}$ is irreducible the representations $X \to \pi(X)$ and $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$ are certainly equivalent. If it is not irreducible let V_1 be a proper subspace invariant under $\mathfrak{A}$. If $V_2 = \pi(\varepsilon)V_1$ then $V_1 \cap V_2$ and $V_1 + V_2$ are all invariant under $\{\mathfrak{A}, \varepsilon\}$. Thus $V_1 \cap V_2 = \{0\}$ and $V = V_1 \oplus V_2$. If V_1 had a proper subspace V_1' invariant under $\mathfrak{A}$ the same considerations would show that $V = V_1' \oplus V_2'$ with $V_2' = \pi(\varepsilon)V_1'$. Since this is impossible V_1 and V_2 are irreducible under $\mathfrak{A}$.

If v_1 is in V_1

$$\pi_2(X)\,\pi(\varepsilon)v_1 = \pi(\varepsilon)\,\pi_1\big(\mathrm{ad}\,\varepsilon(X)\big)v_1$$

so that the representations $X \to \pi_2(X)$ and $X \to \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$ are equivalent. If π_1 and π_2 were equivalent there would be an invertible linear transformation A from V_1 to V_2 so that $A\pi_1(X) = \pi_2(X)A$. If v_1 is in V_1

$$A^{-1}\pi(\varepsilon)\,\pi_1(X)v_1 = A^{-1}\,\pi_2\big(\mathrm{ad}\,\varepsilon(X)\big)\,\pi(\varepsilon)\,v_1 = \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)\,A^{-1}\,\pi(\varepsilon)\,v_1$$

Consequently $\{A^{-1}\pi(\varepsilon)\}^2$ regarded as a linear transformation of V_1 commutes with $\mathfrak{A}$ and is therefore a scalar. There is no harm in supposing that it is the identity. The linear transformation

$$v_1 + v_2 \to A^{-1}v_2 + Av_1$$

then commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. This is a contradiction.

Let χ be a quasi-character of $\mathbb{R}^\times$ and let $\chi(t) = t^c$ for t positive. For any admissible representation π of $\mathfrak{A}$ and therefore of $\mathfrak{g}$ we define a representation $\chi \otimes \pi$ of $\mathfrak{g}$ and therefore $\mathfrak{A}$ by setting

$$\chi \otimes \pi(X) = \frac{c}{2}\,\mathrm{trace}\,X + \pi(X)$$

if X is in $\mathfrak{g}$. If π is a representation of $\{\mathfrak{A}, \varepsilon\}$ we extend $\chi \otimes \pi$ to $\{\mathfrak{A}, \varepsilon\}$ by setting

$$\chi \otimes \pi(\varepsilon) = \chi(-1)\,\pi(\varepsilon)$$

If π is associated to a representation π of $\mathcal{H}_{\mathbb{R}}$ then $\chi \otimes \pi$ is associated to the representation of $\mathcal{H}_{\mathbb{R}}$ defined by

$$\chi \otimes \pi(f) = \pi(\chi f)$$

if χf is the product of the functions χ and f.

Lemma 5.9 *Let π_0 be an irreducible admissible representation of $\mathfrak{A}$. Assume that π_0 is equivalent to the representation $X \to \pi_0\big(\mathrm{Ad}\varepsilon(X)\big)$. Then there is an irreducible representation π of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is π_0. If η is the non-trivial quadratic character of $\mathbb{R}^\times$ the representations π and $\eta \otimes \pi$ are not equivalent but any representatin of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is equivalent to π_0 is equivalent to one of them.*

Let π_0 act on V. There is an invertible linear transformation A of V such that $A\pi_0(X) = \pi_0\big(\mathrm{Ad}\varepsilon(X)\big)A$ for all X in $\mathfrak{A}$. Then A^2 commutes with all $\pi_0(X)$ and is therefore a scalar. We may suppose that $A^2 = I$. If we set $\pi(\varepsilon) = A$ and $\pi(X) = \pi_0(X)$ for X in $\mathfrak{A}$ we obtain the required representation. If we replace A by $-A$ we obtain the representation $\eta \otimes \pi$. π and $\eta \otimes \pi$ are not equivalent because any operator giving the equivalence would have to commute with all of the $\pi(X)$ and would therefore be a scalar. Any representation π' of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is equivalent to π_0 can be realized on V_0 in such a way that $\pi'(X) = \pi_0(X)$ for all X. Then $\pi'(\varepsilon) = \pm A$.

Lemma 5.10 *Let π_1 be an irreducible admissible representation of $\mathfrak{A}$. Assume that π_1 and π_2, with $\pi_2(X) = \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$, are not equivalent. Then there is an irreducible representation π of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is the direct sum of π_1 and π_2. Every irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ contains π_1 is equivalent to π. In particular $\eta \otimes \pi$ is equivalent to π.*

Let π_1 act on V_1. To construct π we set $V = V_1 \oplus V_2$ and we set

$$\pi(X)\,(v_1 \oplus v_2) = \pi_1(X)v_1 \oplus \pi_2(X)v_2$$

and

$$\pi(\varepsilon)\,(v_1 \oplus v_2) = v_2 \oplus v_1.$$

The last assertion of the lemma is little more than a restatement of the second half of Lemma 5.8.

Theorem 5.11 *Let μ_1 and μ_2 be two quasi-characters of $F^\times$.*

(i) *If $\mu_1\mu_2^{-1}$ is not of the form $t \to t^p \operatorname{sgn} t$ with p a non-zero integer the space $\mathcal{B}(\mu_1, \mu_2)$ is irreducible under the action of $\{\mathfrak{A}, \varepsilon\}$ or $\mathcal{H}_{\mathbb{R}}$. $\pi(\mu_1, \mu_2)$ is any representation equivalent to $\rho(\mu_1, \mu_2)$.*

(ii) *If $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$, where p is a positive integer, the space $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathcal{B}_s(\mu_1, \mu_2)$ invariant under $\{\mathfrak{A}, \varepsilon\}$. It is infinite dimensional and any representation of $\{\mathfrak{A}, \varepsilon\}$ equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathcal{B}_s(\mu_1, \mu_2)$ will be denoted by $\sigma(\mu_1, \mu_2)$. The quotient space*

$$\mathcal{B}_f(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$$

is finite-dimensional and $\pi(\mu_1, \mu_2)$ will be any representation equivalent to the representation of $\{\mathfrak{A}, \varepsilon\}$ on this quotient space.

(iii) *If $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$, where p is a negative integer, the space $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathcal{B}_f(\mu_1, \mu_2)$ invariant under $\{\mathfrak{A}, \varepsilon\}$. It is finite-dimensional and $\pi(\mu_1, \mu_2)$ will be any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathcal{B}_f(\mu_1, \mu_2)$. $\sigma(\mu_1, \mu_2)$ will be any representation equivalent to the representation on the quotient space*

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_f(\mu_1, \mu_2).$$

(iv) *A representation $\pi(\mu_1, \mu_2)$ is never equivalent to a representation $\sigma(\mu_1', \mu_2')$.*

(v) *The representations $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if either $(\mu_1, \mu_2) = (\mu_1', \mu_2')$ or $(\mu_1, \mu_2) = (\mu_2', \mu_1')$.*

(vi) *The representations $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if (μ_1, μ_2) is one of the four pairs (μ_1', μ_2'), (μ_2', μ_1'), $(\mu_1'\eta, \mu_2'\eta)$, or $(\mu_2'\eta, \mu_1'\eta)$.*

(vii) *Every irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ is either a $\pi(\mu_1, \mu_2)$ or a $\sigma(\mu_1, \mu_2)$.*

Let $\mu_1\mu_2^{-1}(t) = |t|^s(\frac{t}{|t|})^m$. $s - m$ is an odd integer if and only if s is an integer p and $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$. Thus the first three parts of the lemma are consequences of Lemma 5.6 and 5.7. The fourth follows from the observation that $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ cannot contain the same representations of the Lie algebra of $SO(2, \mathbb{R})$.

We suppose first that $s - m$ is not an odd integer and construct an invertible transformation T from $\mathcal{B}(\mu_1, \mu_2)$ to $\mathcal{B}(\mu_2, \mu_1)$ which commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. We have introduced a basis

$\{\varphi_n\}$ of $\mathcal{B}(\mu_1, \mu_2)$. Let $\{\varphi'_n\}$ be the analogous basis of $\mathcal{B}(\mu_2, \mu_1)$. T will have to take φ_n to a multiple $a_n\varphi'_n$ of φ'_n. Appealing to Lemma 5.6 we see that it commutes with the action of $\{\mathfrak{A}, \varepsilon\}$ if and only if

$$(s+1+n)a_{n+2} = (-s+1+n)a_n$$
$$(s+1-n)a_{n-2} = (-s+1-n)a_n$$

and

$$a_n = (-1)^m a_{-n}.$$

These relations will be satisfied if we set

$$a_n = a_n(s) = \frac{\Gamma\frac{(-s+1+n)}{2}}{\Gamma\frac{(s+1+n)}{2}}$$

Since $n = m(\operatorname{mod} 2)$ and $s - m - 1$ is not an even integer all these numbers are defined and different from 0.

If $s \leq 0$ and $s - m$ is an odd integer we set

$$a_n(s) = \lim_{z \to s} a_n(z)$$

The numbers $a_n(s)$ are still defined although some of them may be 0. The associated operator T maps $\mathcal{B}(\mu_1, \mu_2)$ into $\mathcal{B}(\mu_2, \mu_1)$ and commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. If $s = 0$ the operator T is non-singular. If $s < 0$ its kernel is $\mathcal{B}_f(\mu_1, \mu_2)$ and it defines an invertible linear transformation from $\mathcal{B}_s(\mu_1, \mu_2)$ to $\mathcal{B}_s(\mu_2, \mu_1)$. If $s > 0$ and $s - m$ is an odd integer the functions $a_n(z)$ have at most simple poles at s. Let

$$b_n(s) = \lim_{z \to s} (z - s)\, a_n(z)$$

The operator T associated to the family $\{b_n(s)\}$ maps $\mathcal{B}(\mu_1, \mu_2)$ into $\mathcal{B}(\mu_2, \mu_1)$ and commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. It kernel is $\mathcal{B}_s(\mu_1, \mu_2)$ so that it defines an invertible linear transformation from $\mathcal{B}_f(\mu_1, \mu_2)$ to $\mathcal{B}_f(\mu_2, \mu_1)$. These considerations together with Lemma 5.10 give us the equivalences of parts (v) and (vi).

Now we assume that $\pi = \pi(\mu_1, \mu_2)$ and $\pi' = \pi(\mu'_1, \mu'_2)$ or $\pi = \sigma(\mu_1, \mu_2)$ and $\pi' = \sigma(\mu'_1, \mu'_2)$ are equivalent. Let $\mu_i(T) = |t|^{s_i}(\frac{t}{|t|})^{m_i}$ and let $\mu'_i(t) = |t|^{s'_i}(\frac{t}{|t|})^{m'_i}$. Let $s = s_1 - s_2$, $m = |m_1 - m_2|$, $s' = s'_1 - s'_2$, $m' = |m'_1 - m'_2|$. Since the two representations must contain the same representations of the Lie algebra of $SO(2, \mathbb{R})$ the numbers m and m' are equal. Since $\pi(D)$ and $\pi'(D)$ must be the same scalar Lemma 5.6 shows that $s' = \pm s$. $\pi(J)$ and $\pi'(J)$ must also be the same scalar so $s'_1 + s'_2 = s_1 + s_2$. Thus if $\eta(t) = \operatorname{sgn} t$ the pair (μ_1, μ_2) must be one of the four pairs (μ'_1, μ'_2), (μ'_2, μ'_1), $(\eta\mu'_1, \eta\mu'_2)$, $(\eta\mu'_2, \eta\mu'_1)$. Lemma 5.9 shows that $\pi(\mu'_1\mu'_2)$ and $\pi(\eta\mu'_1, \eta\mu'_2)$ are not equivalent. Parts (v) and (vi) of the theorem follow immediately.

Lemmas 5.8, 5.9, and 5.10 show that to prove the last part of the theorem we need only show that any irreducible admissible representation π of $\mathfrak{A}$ is, for a suitable choice of μ_1 and μ_2, a constituent of $\rho(\mu_1, \mu_2)$. That is there should be two subspace $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}(\mu_1, \mu_2)$ invariant under $\mathfrak{A}$ so that $\mathcal{B}_1$ contains $\mathcal{B}_2$ and π is equivalent to the representation of $\mathfrak{A}$ on the quotient $\mathcal{B}_1/\mathcal{B}_2$. If χ is a quasi-character of $F^\times$ then π is a constituent of $\rho(\mu_1, \mu_2)$ if and only if $\chi \otimes \pi$ is a constituent of $\rho(\chi\mu_1, \chi\mu_2)$. Thus we may suppose that $\pi(J)$ is 0 so that π is actually a representation of $\mathfrak{A}_0$, the universal enveloping algebra of the Lie algebra of $Z_{\mathbb{R}} \setminus G_{\mathbb{R}}$. Since this group is semi-simple the desired result is a consequence of the general theorem of Harish–Chandra [6].

It is an immediate consequence of the last part of the theorem that every irreducible admissible representations of $\{\mathfrak{A}, \varepsilon\}$ is the representation associated to an irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. Thus we have classifed the irreducible admissible representations of $\{\mathfrak{A}, \varepsilon\}$ and of $\mathcal{H}_{\mathbb{R}}$. We can write such a representation of $\mathcal{H}_{\mathbb{R}}$ as $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$.

In the first paragraph we associated to every quasi-character ω of $\mathbb{C}^\times$ a representation of r_ω of G_+ the group of matrices with positive determinant. r_ω acts on the space of functions Φ in $\mathcal{S}(C)$ which satisfy

$$\Phi(xh) = \omega^{-1}(h)\,\Phi(x)$$

for all h such that $h\bar{h} = 1$. All elements of $\mathcal{S}(C, \omega)$ are infinitely differentiable vectors for r_ω so that r_ω also determines a representation, again called r_ω, of $\mathfrak{A}$. r_ω depended on the choice of a character of $\mathbb{R}$. If that character is

$$\psi(x) = e^{2\pi uxi}$$

then

$$r_\omega(X_+)\,\Phi(z) = (2\pi uz\bar{z}i)\Phi(z).$$

Lemma 5.12 *Let $\mathcal{S}_0(\mathbb{C}, \omega)$ be the space of functions Φ in $\mathcal{S}(C, \omega)$ of the form*

$$\Phi(z) = e^{-2\pi|u|z\bar{z}} P(z, \bar{z})$$

where $P(z, \bar{z})$ is a polynomial in z in $\bar{z}$. Then $\mathcal{S}_0(\mathbb{C}, \omega)$ is invariant under $\mathfrak{A}$ and the restriction of r_ω to $\mathcal{S}_0(\mathbb{C}, \omega)$ is admissible and irreducible.

It is well known and easily verified that the function $e^{-2\pi|u|z\bar{z}}$ is its own Fourier transform provided of course that the transform is taken with respect to the character

$$\psi_{\mathbb{C}}(z) = \psi(z + \bar{z})$$

and the self-dual measure for that character. From the elementary properties of the Fourier transform one deduces that the Fourier transform of a function

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}\, P(z, \bar{z})$$

where P is a polynomial in z and $\bar{z}$ is of the same form. Thus $r_\omega(w)$ leaves $\mathcal{S}_0(\mathbb{C}, \omega)$ invariant. Recall that

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\mathcal{S}_0(\mathbb{C}, \omega)$ is clearly invariant under $r_\omega(X_+)$. Since $X_- = \mathrm{Ad}w(X_+)$ it is also invariant under X. But $X_+\,X_- - X_-\,X_+ = Z$ so that it is also invariant under Z. We saw in the first paragraph that if ω_0 is the restriction of ω to $\mathbb{R}^\times$ then

$$r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = (\operatorname{sgn} a)\,\omega_0(a)I$$

thus $r_\omega(J) = cI$ if $\omega_0(a) = a^c$ for a positive a. In conclusion $\mathcal{S}_0(\mathbb{C}, \omega)$ is invariant under $\mathfrak{g}$ and therefore under $\mathfrak{A}$.

If

$$\omega(z) = (z\bar{z})^r \frac{z^m\,\bar{z}^n}{(z\bar{z})^{\frac{m+n}{2}}}$$

where r is a complex number and m and n are two integers, one 0 and the other non-negative, the functions

$$\Phi_p(z) = e^{-2\pi|u|z\bar{z}}\, z^{n+p}\, \bar{z}^{m+p},$$

with p a non-negative integer, form a basis of $\mathcal{S}_0(\mathbb{C},\omega)$. Suppose as usual that $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$ and that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$. Then the Fourier transform Φ'_p of Φ_p is given by

$$\Phi'_p(z) = \frac{1}{(2\pi i u)^{m+n+2p}}\,\frac{\partial^{n+p}}{\partial z^{n+p}}\,\frac{\partial^{m+p}}{\partial \bar{z}^{m+p}}\,e^{-2\pi|u|z\bar{z}}$$

which is a function of the form

$$(i\operatorname{sgn} u)^{m+n+2p}e^{-2\pi|u|z\bar{z}}\,\bar{z}^{n+p}\,z^{m+p} + \sum_{q=0}^{p-1} a_q e^{-2\pi|u|z\bar{z}}\,\bar{z}^{n+q}\,z^{m+q}.$$

Only the coefficient a_{p-1} interests us. It equals

$$\frac{(i\operatorname{sgn} u)^{m+n+2p-1}}{2\pi i u}\{p(n+m+1+p-1)\}.$$

Since

$$r_\omega(w)\,\Phi(z) = (i\operatorname{sgn} u)\,\Phi'(\bar{z})$$

and

$$r_\omega(X_-) = (-1)^{m+n}\,r_\omega(w)\,r_\omega(X_+)\,r(w)$$

while

$$r_\omega(X_+)\Phi_p = (2\pi u i)\Phi_{p+1}$$

we see that

$$r_\omega(X_-)\Phi_p = (2\pi u i)\Phi_{p+1} - (i\operatorname{sgn} u)(n+m+2p+1)\Phi_p + \sum_{q=0}^{p-1} b_q\Phi_q.$$

Since $U = X_+ - X_-$ we have

$$r_\omega(U)\Phi_p = (i\operatorname{sgn} u)(n+m+2p+1)\Phi_p - \sum_{1=0}^{p-1} b_q\Phi_q$$

and we can find the functions Ψ_p, $p = 0, 1, \cdots$, such that

$$\Psi_p = \Phi_p + \sum_{q=0}^{p-1} a_{pq}\,\Phi_q$$

while

$$r_\omega(U)\Psi_p = (i\operatorname{sgn} u)(n+m+2p+1)\Psi_p.$$

These functions form a basis of $\mathcal{S}_0(\mathbb{C},\omega)$. Consequently r_ω is admissible.

If it were not irreducible there would be a proper invariant subspace which may or not contain Φ_0. In any case if $\mathcal{S}_1$ is the intersection of all invariant subspaces containing Φ_0 and $\mathcal{S}_2$ is the sum of all invariant subspaces which do not contain Φ_0 both $\mathcal{S}_1$ and $\mathcal{S}_2$ are invariant and the representation π_1 of $\mathfrak{A}$ on $\mathcal{S}_1/\mathcal{S}_2 \cap \mathcal{S}_1$ is irreducible. If the restriction of π_1 to the Lie algebra of $SO(2,\mathbb{R})$ contains κ_p it does not contain κ_{-p}. Thus π_1 is not equivalent to the representation $X \to \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$. Consequently the irreducible representation π of $\{\mathfrak{A},\varepsilon\}$ whose restriction to $\mathfrak{A}$ is π_1 must be one of the special representations $\sigma(\mu_1,\mu_2)$ or a representation $\pi(\mu_1,\mu_2\eta)$. Examining these we see that since π contains κ_q with $q = \operatorname{sgn} u(n+m+1)$ it contains all the representations κ_q with $q = \operatorname{sgn} u(n+m+2p+1)$, $p = 0,1,2,\cdots$. Thus $\mathcal{S}_1$ contains all the functions Ψ_p and $\mathcal{S}_2$ contains none of them. Since this contradicts the assumption that $\mathcal{S}_0(\mathbb{C},\omega)$ contains a proper invariant subspace the representation r_ω is irreducible.

For the reasons just given the representation π of $\{\mathfrak{A},\varepsilon\}$ whose restriction to $\mathfrak{A}$ contains r_ω is either a $\sigma(\mu_1,\mu_2)$ or a $\pi(\mu_1,\mu_1\eta)$. It is a $\pi(\mu_1,\mu_1\eta)$ if and only if $n+m=0$. Since

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right) = \omega(a)\operatorname{sgn} aI = \omega(a)\,\eta(a)I,$$

we must have $\mu_1\mu_2 = \omega_0\eta$ in the first case and $\mu_1^2 = \omega_0$ in the second. ω_0 is the restriction of ω to $\mathbb{R}^\times$. Since the two solutions $\mu_1^2 = \omega_0$ differ by η they lead to the same representation. If $n+m=0$ then $\mu_1^2 = \omega_0$ if and only if $\omega(z) = \mu_1\big(\nu(z)\big)$ for all z in $\mathbb{C}^\times$. Of course $\nu(z) = z\bar{z}$.

Suppose $n+m>0$ so that π is a $\sigma(\mu_1,\mu_2)$. Let $\mu_i(t) = |t|^{s_i}(\frac{t}{|t|})^{m_i}$. Because of Theorem 5.11 we can suppose that $m_1 = 0$. Let $s = s_1 - s_2$. We can also suppose that s is non-negative. If $m = |m_1 - m_2|$ then $s-m$ is an odd integer so m and m_2 are determined by s. We know what representations of the Lie algebra of $SO(2,\mathbb{R})$ are contained in π. Appealing to Lemma 5.7 we see that $s = n+m$. Since $\mu_1\mu_2 = \eta\omega_0$ we have $s_1+s_2 = 2r$. Thus $s_1 = r + \frac{m+n}{2}$ and $s_2 = r - \frac{n+m}{2}$. In all cases the representation π is determined by ω alone and does not depend on ψ. We refer to it as $\pi(\omega)$. Every special representation $\sigma(\mu_1,\mu_2)$ is a $\pi(\omega)$ and $\pi(\omega)$ is equivalent to $\pi(\omega')$ if and only if $\omega = \omega'$ or $\omega'(z) = \omega(\bar{z})$.

We can now take the first step in the proof of the local functional equation.

Theorem 5.13 *Let π be an infinite-dimensional irreducible admissible representation of $\mathcal{H}_\mathbb{R}$. If ψ is a non-trivial additive character of $\mathbb{R}$ there exists exactly one space $W(\pi,\psi)$ of functions W on $G_\mathbb{R}$ with the following properties*

(i) *If W is in $W(\pi,\psi)$ then*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right) = \psi(x)\,W(g)$$

for all x in F.

(ii) *The functions W are continuous and $W(\pi,\psi)$ is invariant under $\rho(f)$ for all f in $\mathcal{H}_\mathbb{R}$. Moreover the representation of $\mathcal{H}_\mathbb{R}$ on $W(\pi,\psi)$ is equivalent to π.*

(iii) *If W is in $W(\pi,\psi)$ there is a positive number N such that*

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\right) = O(|t|^N)$$

as $|t| \to \infty$.

We prove first the existence of such a space. Suppose $\pi = \pi(\omega)$ is the representation associated to some quasi-character ω of $\mathbb{C}^\times$. An additive character ψ being given the restriction of π to $\mathfrak{A}$ contains the representation r_ω determined by ω and ψ. For any Φ in $\mathcal{S}(\mathbb{C},\omega)$ define a function W_Φ on G_+ by

$$W_\Phi(g) = r_\omega(g)\,\Phi(1)$$

Since $\rho(g)\,W_\Phi = W_{r_\omega(g)\Phi}$ the space of such functions is invariant under right translations. Moreover

$$W_\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)\,W_\Phi(g)$$

Every vector in $\mathcal{S}(\mathbb{C},\omega)$ is infinitely differentiable for the representation r_ω. Therefore the functions W_Φ are all infinitely differentiable and, if X is in $\mathfrak{A}$,

$$\rho(X)W_\Phi = W_{r_\omega(X)\Phi}.$$

In particular the space $W_1(\pi,\psi)$ of those W_Φ for which Φ is in $\mathcal{S}_0(\mathbb{C},\omega)$ is invariant under $\mathfrak{A}$. We set W_Φ equal to 0 outside of G_+ and regard it as a function on $G_\mathbb{R}$.

We want to take $W(\pi,\psi)$ to be the sum of $W_1(\pi,\psi)$ and its right translate by ε. If we do it will be invariant under $\{\mathfrak{A},\varepsilon\}$ and transform according to the representation π of $\{\mathfrak{A},\varepsilon\}$. To verify the second condition we have to show that it is invariant under $\mathcal{H}_\mathbb{R}$. For this it is enough to show that $\mathcal{S}_0(\mathbb{C},\omega)$ is invariant under the elements of $\mathcal{H}_\mathbb{R}$ with support in G_+. The elements certainly leave the space of functions in $\mathcal{S}(\mathbb{C},\omega)$ spanned by the functions transforming according to a one-dimensional representation of $SO(2,\mathbb{R})$ invariant. Any function in $\mathcal{S}(\mathbb{C},\omega)$ can be approximated uniformly on compact sets by a function in $\mathcal{S}_0(\mathbb{C},\omega)$. If in addition it transforms according to the representation κ_n of $SO(2,\mathbb{R})$ it can be approximated by functions in $\mathcal{S}_0(\mathbb{C},\omega)$ transforming according to the same representation. In other words it can be approximated by multiples of a single function in $\mathcal{S}_0(\mathbb{C},\omega)$ and therefore is already in $\mathcal{S}_0(\mathbb{C},\omega)$.

The growth condition need only be checked for the functions W_Φ in $W_1(\pi,\psi)$. If a is negative

$$W_\Phi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) = 0$$

but if a is positive and

$$\Phi(z) = e^{-2\pi|u|z\bar z}P(z,\bar z)$$

it is equal to

$$e^{-2\pi|u|a}P(a^{1/2},a^{1/2})\,\omega(a)\,|a|^{1/2},$$

and certainly satisfies the required condition.

We have still to prove the existence of $W(\pi,\psi)$ when $\pi = \pi(\mu_1,\mu_2)$ and is infinite dimensional. As in the first paragraph we set

$$\theta(\mu_1,\mu_2,\Phi) = \int_{\mathbb{R}^\times} \mu_1(t)\,\mu_2^{-1}(t)\,\Phi(t,t^{-1})\,d^\times t$$

for Φ in $\mathcal{S}(\mathbb{R}^s)$ and we set

$$\begin{aligned} W_\Phi(g) &= \mu_1(\det g)\,|\det g|^{1/2}\,\theta\big(\mu_1,\mu_2,r(g)\Phi\big)\\ &= \theta\big(\mu_1,\mu_2,r_{\mu_1,\mu_2}(g)\Phi\big). \end{aligned}$$

r_{μ_1,μ_2} is the representation associated to the quasi-character $(a,b)\to\mu_1(a)\,\mu_2(b)$ of $\mathbb{R}^\times\times\mathbb{R}^\times$. If X is in $\mathfrak{A}$

$$\rho(X)\,W_\Phi(g) = W_{r_{\mu_1,\mu_2}(X)\Phi}(g)$$

Let $W(\mu_1,\mu_2;\psi)$ be the space of those W_Φ which are associated to $O(2,\mathbb{R})$-finite functions Φ. $W(\mu_1,\mu_2;\psi)$ is invariant under $\{\mathfrak{A},\varepsilon\}$ and under $\mathcal{H}_\mathbb{R}$.

Lemma 5.13.1 *Assume* $\mu_1(x)\,\mu_2^{-1}(x) = |x|^s\,(\frac{x}{|x|})^m$ *with* $\operatorname{Re} s > -1$ *and* m *equal to* 0 *or* 1. *Then there exists a bijection* A *of* $W(\mu_1, \mu_2; \psi)$ *with* $\mathcal{B}(\mu_1, \mu_2)$ *which commutes with the action of* $\{\mathfrak{A}, \varepsilon\}$.

We have already proved a lemma like this in the non-archimedean case. If Φ is in $\mathcal{S}(\mathbb{R}^2)$ and ω is a quasi-character of $\mathbb{R}^\times$ set

$$z(\omega, \Phi) = \int \Phi(0, t)\,\omega(t)\,d^\times(t)$$

The integral converges if $\omega(t) = |t|^r (\operatorname{sgn} t)^n$ with $r > 0$. In particular under the circumstances of the lemma

$$f_\Phi(g) = \mu_1(\det g)\,|\det g|^{1/2} z(\mu_1\mu_2^{-1}\alpha_{\mathbb{R}}, \rho(g)\Phi)$$

is defined. As usual $\alpha_{\mathbb{R}}(x) = |x|$. A simple calculation shows that

$$f_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\,\mu_2(a_2)\,\Big|\frac{a_1}{a_2}\Big|^{1/2} f_\Phi(g).$$

If $\Phi^\sim$ is the partial Fourier transform of Φ introduced in the first paragraph then

$$\rho(g)\,f_{\Phi^\sim} = f_{\Phi_1^\sim}$$

if $\Phi_1 = r_{\mu_1,\mu_2}(f)\Phi$. A similar relation will be valid for a function f in $\mathcal{H}_{\mathbb{R}}$, that is

$$\rho(f) f_{\Phi^\sim} = f_{\Phi_1^\sim}$$

if $\Phi_1 = r_{\mu_1,\mu_2}(f)\Phi$. In particular if $f_{\Phi^\sim}$ is $O(2,\mathbb{R})$-finite there is an elementary idempotent ξ such that $\rho(\xi) f_{\Phi^\sim} = f_{\Phi^\sim}$. Thus, if $\Phi_1 = r_{\mu_1,\mu_2}(\xi)\Phi$, $f_{\Phi^\sim} = f_{\Phi_1^\sim}$ and $\Phi_1^\sim$ is $O(2,\mathbb{R})$ finite. Of course $f_{\Phi^\sim}$ is $O(2,\mathbb{R})$-finite if and only if it belongs to $\mathcal{B}(\mu_1, \mu_2)$.

We next show that given any f in $\mathcal{B}(\mu_1, \mu_2)$ there is an $O(2,\mathbb{R})$-finite function Φ in $\mathcal{S}(\mathbb{R}^2)$ such that $f = f_{\Phi^\sim}$. According to the preceding observation together with the self-duality of $\mathcal{S}(\mathbb{R}^2)$ under Fourier transforms it will be enough to show that for some Φ in $\mathcal{S}(\mathbb{R}^2)$, $f = f_\Phi$. In fact, by linearity, it is sufficient to consider the functions φ_n in $\mathcal{B}(\mu_1, \mu_2)$ defined earlier by demanding that

$$\varphi_n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$$

n must be of the same parity as m. If $\delta = \operatorname{sgn} n$ set

$$\Phi(x, y) = e^{-\pi(x^2+y^2)}(x + i\delta y)^{|n|}$$

Then

$$\rho\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)\Phi = e^{in\theta}\Phi$$

Since $\rho(g) f_\Phi = f_{\rho(g)\Phi}$ when $\det g = 1$ the function f_Φ is a multiple of φ_n. Since

$$\begin{aligned} f_\Phi(e) &= (i)^{|n|}\int_{-\infty}^{\infty} e^{-\pi t^2} t^{|n|+s+1}\,d^\times t \\ &= (i)^n \frac{\pi^{\frac{-(|n|+s+1)}{2}}}{2}\,\Gamma\frac{(|n|+s+1)}{2} \end{aligned}$$

which is not 0, the function f_Φ is not 0.

The map A will transform the function W_Φ to $f_{\Phi^\sim}$. It will certainly commute with the action of $\{\mathfrak{A}, \varepsilon\}$. That A exists and is injective follows from a lemma which, together with its proof, is almost identical to the statement and proof of Lemma 3.2.1.

The same proof as that used in the non-archimedian case also shows that $W(\mu_1, \mu_2; \psi) = W(\mu_2, \mu_1; \psi)$ for all ψ. To prove the existence of $W(\pi, \psi)$ when $\pi = \pi(\mu_1, \mu_2)$ and is infinite-dimensional we need only show that when μ_1 and μ_2 satisfy the condition the previous lemma the functions W in $W(\mu_1, \mu_2; \psi)$ satisfy the growth condition of the theorem. We have seen that we can take $W = W_\Phi$ with

$$\Phi^\sim(x, y) = e^{-\pi(x^2+y^2)} P(x, y)$$

where $P(x, y)$ is a polynomial in x and y. Then

$$\Phi(x, y) = e^{-\pi(x^2+u^2y^2)} Q(x, y)$$

where $Q(x, y)$ is another polynomial. Recall that $\psi(x) = e^{2\pi i u x}$. Then

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_1(a)\,|a|^{1/2} \int_{-\infty}^{\infty} e^{-\pi(a^2t^2+u^2t^{-2})} Q(at, ut^{-1})\,|t|^s (\operatorname{sgn} t)^m\, d^\times t$$

The factor in front certainly causes no harm. If $\delta > 0$ the integrals from $-\infty$ to $-\delta$ and from δ to ∞ decrease rapidly as $|a| \to \infty$ and we need only consider integrals of the form

$$\int_0^\delta e^{-\pi(a^2t^2+u^2t^{-2})} t^r\, dt$$

where r is any real number and u is fixed and positive. If $v = \frac{u}{2}$ then $u^2 = v^2 + \frac{3u^2}{4}$ and $e^{-\frac{3}{4}\pi u^2 t^{-2}} t^r$ is bounded in the interval $[0, \delta]$ so we can replace u by v and suppose r is 0. We may also suppose that a and v are positive and write the integral as

$$e^{-2\pi a v} \int_0^\delta e^{-\pi(at+vt^{-1})^2}\, dt.$$

The integrand is bounded by 1 so that the integral is $O(1)$. In any case the growth condition is more than satisfied.

We have still to prove uniqueness. Suppose $W_1(\pi, \psi)$ is a space of functions satisfying the first two conditions of the lemma. Let κ_n be a representation of the Lie algebra of $SO(2, \mathbb{R})$ occurring in π and let W_1 be a function in $W_1(\pi, \psi)$ satisfying

$$W_1\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta} W_1(g).$$

If

$$\varphi_1(t) = W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right)$$

the function W_1 is completely determined by φ_1. It is easily seen that

$$\rho(U)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = i\,n\,\varphi_1(t)$$
$$\rho(Z)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = 2t\,\frac{d\varphi_1}{dt}$$
$$\rho(X_+)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = i\,u\,t\,\varphi_1(t).$$

Thus if φ_1^+ and φ_1^- correspond to $\rho(V_+)W_1$ and $\rho(V_-)W_1$

$$\varphi_1^+(t) = 2t\frac{d\varphi_1}{dt} - (2ut-n)\,\varphi_1$$

and

$$\varphi_1^-(t) = 2t\frac{d\varphi_1}{dt} + (2ut-n)\,\varphi_1(t).$$

Since

$$D = \frac{1}{2}V_-\,V_+ - iU - \frac{U^2}{2}$$

$\rho(D)W_1$ corresponds to

$$2t\,\frac{d}{dt}\left(t\,\frac{d\varphi_1}{dt} - 2t\,\frac{d\varphi_2}{dt}\right) + (2nut - 2u^2t^2)\varphi_1.$$

Finally $\rho(\varepsilon)W_1$ corresponds to $\varphi_1(-t)$.

Suppose that π is either $\pi(\mu_1,\mu_2)$ or $\sigma(\mu_1,\mu_2)$. Let $\mu_1\mu_2^{-1}(t) = |t|^s(\operatorname{sgn} t)^m$. If $s-m$ is an odd integer we can take $n = |s|+1$. From Lemma 5.6 we have $\rho(V_-)W_1 = 0$ so that φ_1 satisfies the equation

$$2t\,\frac{d\varphi_1}{dt} + (2ut-n)\varphi_1 = 0.$$

If the growth condition is to be satisfied φ_1 must be 0 for $ut < 0$ and a multiple of $|t|^{n/2}e^{-ut}$ for $ut > 0$. Thus W_1 is determined up to a scalar factor and the space $W(\pi,\psi)$ is unique.

Suppose $s-m$ is not an odd integer. Since $\rho(D)W_1 = \frac{s^2-1}{2}W_1$ the function φ_1 satisfies the equation

$$\frac{d^2\varphi_1}{dt^2} + \left\{-u^2 + \frac{nu}{t} + \frac{(1-s^2)}{4t^2}\right\}\varphi_1 = 0$$

We have already constructed a candidate for the space $W(\pi,\psi)$. Let's call this candidate $W_2(\pi,\psi)$. There will be a non-zero function φ_2 in it satisfying the same equation as φ_1. Now φ_1 and all of its derivatives go to infinity no faster than some power of $|t|$ as $t\to\infty$ while as we saw φ_2 and its derivations go to 0 at least exponentially as $|t|\to\infty$. Thus the Wronskian

$$\varphi_1\,\frac{d\varphi_2}{dt} - \varphi_2\,\frac{d\varphi_1}{dt}$$

goes to 0 as $|t| \to \infty$. By the form of the equation the Wronskian is constant. Therefore it is identically 0 and $\varphi_1(t) = \alpha\,\varphi_2(t)$ for $t > 0$ and $\varphi_1(t) = \beta\,\varphi_2(t)$ for $t < 0$ where α and β are two constants. The uniqueness will follow if we can show that for suitable choice of n we have $\alpha = \beta$. If $m = 0$ we can take $n = 0$. If $\mu_1(t) = |t|^{s_1}(\operatorname{sgn} t)^{m_1}$ then $\pi(\varepsilon)W_1 = (-1)^{m_2}W_1$ so that $\varphi_1(-t) = (-1)^{m_1}\varphi_1(t)$ and $\varphi_2(-t) = (-1)^{m_2}\varphi_2(t)$. Thus $\alpha = \beta$. If $m = 1$ we can take $n = 1$. From Lemma 5.6

$$\pi(V_{-1})W_1 = (-1)^{m_1} s\pi(\varepsilon)W_1$$

so that

$$2t\,\frac{d\varphi_1}{dt} + (2ut - 1)\varphi_1(t) = (-1)^{m_1} s\varphi_1(-t).$$

Since φ_2 satisfies the same equation $\alpha = \beta$.

If μ is a quasi-character of $\mathbb{R}^\times$ and ω is the character of $\mathbb{C}^\times$ defined by $\omega(z) = \mu(z\bar{z})$ then $\pi(\omega) = \pi(\mu, \mu\eta)$. We have defined $W(\pi(\omega), \psi)$ in terms of ω and also as $W(\mu_1, \mu_2; \psi)$. Because of the uniqueness the two resulting spaces must be equal.

Corollary 5.14 *Let m and n be two integers, one positive and the other 0. Let ω be a quasi-character of $\mathbb{C}^\times$ of the form*

$$\omega(z) = (z\bar{z})^{r - \frac{m+n}{2}}\, z^m\, \bar{z}^n$$

and let μ_1 and μ_2 be two quasi-characters of $\mathbb{R}^\times$ satisfying $\mu_1\mu_2(x) = |x|^{2r}(\operatorname{sgn} x)^{m+n+1}$ and $\mu_1\mu_2^{-1}(x) = x^{m+n}\operatorname{sgn} x$ so that $\pi(\omega) = \sigma(\mu_1, \mu_2)$. Then the subspace $\mathcal{B}_s(\mu_1, \mu_2)$ of $\mathcal{B}(\mu_1, \mu_2)$ is defined and there is an isomorphism of $\mathcal{B}(\mu_1, \mu_2)$ with $W(\mu_1, \mu_2; \psi)$ which commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. The image $W_s(\mu_1, \mu_2; \psi)$ of $\mathcal{B}_s(\mu_1, \mu_2)$ is $W(\pi(\omega), \psi)$. If Φ belongs to $\mathcal{S}(\mathbb{R}^2)$ and W_Φ belongs to $W(\mu_1, \mu_2; \psi)$ then W_Φ belongs to $W_s(\mu_1, \mu_2; \psi)$ if and only if

$$\int_{\infty}^{\infty} x^i\, \frac{\partial^j}{\partial y^j}\Phi(x, 0)\, dx = 0$$

for any two non-negative integers i and j with $i + j = m + n - 1$.

Only the last assertion is not a restatement of previously verified facts. To prove it we have to show that $f_{\Phi^\sim}$ belongs to $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if Φ satisfies the given relations. Let $f = f_{\Phi^\sim}$. It is in $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if it is orthogonal to the functions in $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$. Since $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ is finite-dimensional there is a non-zero vector f_0 in it such that $\rho(X_+)f_0 = 0$. Then

$$f_0\left(w\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) = f_0(w)$$

and f is orthogonal to f_0 if and only if

$$\int_{\mathbb{R}} f\left(w\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) dy = 0. \tag{5.14.1}$$

The dimension of $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-2})$ is $m + n$. It follows easily from Lemmas 5.6 and 5.7 that the vectors $\rho(X_+^p)\,\rho(w)\,f_0$, $0 \le p \le m + n - 1$ span it. Thus f is in $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if each of the functions $\rho(X_+^p)\,\rho(w)\,f$ satisfy (5.14.1). For f itself the left side of (5.14.1) is equal to

$$\int \left\{\int \Phi^\sim\left((0, t)\, w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \mu_1(t)\, \mu_2^{-1}(t)\, |t|\, d^\times t\right\} dx.$$

Apart from a positive constant which relates the additive and multiplicative Haar measure this equals

$$\iint \Phi^{\sim}(-t,-tx)t^{m+n}\operatorname{sgn} t\,dt\,dx$$

which is

$$(-1)^{m+n-1}\iint \Phi^{\sim}(t,x)t^{m+n-1}\,dt\,dx$$

or, in terms of Φ,

$$(-1)^{m+n-1}\int \Phi(t,0)\,t^{m+n-1}\,dt. \tag{5.14.2}$$

By definition

$$r_{\mu_1,\mu_2}(w)\,\Phi(x,y)=\Phi'(y,x)$$

and an easy calculation based on the definition shows that

$$r_{\mu_1,\mu_2}(X_+^p)\,\Phi(x,y)=(2i\pi uxy)^p\,\Phi(x,y).$$

Thus $r_{\mu_1,\mu_2}(X_+^p)\,r_{\mu_1,\mu_2}(w)\,\Phi$ is a non-zero scalar times

$$\frac{\partial^{2p}}{\partial x^p\,\partial y^p}\Phi'(y,x)$$

For this function (5.14.2) is the product of a non-zero scalar and

$$\iint \frac{\partial^{2p}}{\partial x^p\,\partial y^p}\Phi'(0,x)\,x^{m+n-1}\,dx.$$

Integrating by parts we obtain

$$\int \frac{\partial^{p}}{\partial y^p}\Phi'(0,x)\,x^{m+n-p-1}\,dx$$

except perhaps for sign. If we again ignore a non-zero scalar this can be expressed in terms of Φ as

$$\int \frac{\partial^{m+n-p-1}}{\partial y^{m+n-p-1}}\Phi(x,0)\,x^{p}\,dx.$$

The proof of the corollary is now complete.

Before stating the local functional equation we recall a few facts from the theory of local zeta-functions. If F is $\mathbb{R}$ or $\mathbb{C}$ and if Φ belongs to $\mathcal{S}(F)$ we set

$$Z(\omega\alpha_F^s,\Phi)=\int \Phi(a)\,\omega(a)\,|a|_F^s\,d^\times a.$$

ω is a quasi-character. The integral converges in a right half-plane. One defines functions $L(s,\omega)$ and $\varepsilon(s,\omega,\psi_F)$ with the following properties:

(a) For every Φ the quotient

$$\frac{Z(\omega\alpha_F^s,\Phi)}{L(s,\omega)}$$

has an analytic continuation to the whole complex plane as a holomorphic function. Moreover for a suitable choice of Φ it is an exponential function and in fact a constant.

(b) If Φ' is the Fourier transform of Φ with respect to the character ψ_F then

$$\frac{Z(\omega^{-1}\alpha_F^{1-s}, \Phi')}{L(1-s, \omega^{-1})} = \varepsilon(s, \omega, \psi_F)\frac{Z(\omega\alpha_F^s, \Phi)}{L(s, \omega)}.$$

If $F = \mathbb{R}$ and $\omega(x) = |x|_{\mathbb{R}}^r (\operatorname{sgn} x)^m$ with m equal to 0 or 1 then

$$L(s, \omega) = \pi^{-\frac{1}{2}(s+r+m)}\,\Gamma\Big(\frac{s+r+m}{2}\Big)$$

and if $\psi_F(x) = e^{2\pi i u x}$

$$\varepsilon(s, \omega, \psi_F) = (i \operatorname{sgn} u)^m\,|u|_{\mathbb{R}}^{s+r-\frac{1}{2}}.$$

If $F = \mathbb{C}$ and

$$\omega(x) = |x|_{\mathbb{C}}^r\, x^m \bar{x}^n$$

where m and n are non-negative integers, one of which is 0, then

$$L(s, \omega) = 2(2\pi)^{-(s+r+m+n)}\Gamma(s+r+m+n).$$

Recall that $|x|_{\mathbb{C}} = x\bar{x}$. If $\psi_F(x) = e^{4\pi i\,\mathrm{Re}(wz)}$

$$\varepsilon(s, \omega, \psi_F) = i^{m+n}\omega(w)\,|w|_{\mathbb{C}}^{s-1/2}.$$

These facts recalled, let π be an irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. If $\pi = \pi(\mu_1, \mu_2)$ we set

$$L(s, \pi) = L(s, \mu_1)\,L(s, \mu_2)$$

and

$$\varepsilon(s, \pi, \psi_{\mathbb{R}}) = \varepsilon(s, \mu_1, \psi_{\mathbb{R}})\,\varepsilon(s, \mu_2, \psi_{\mathbb{R}})$$

and if $\pi = \pi(\omega)$ where ω is a character of $\mathbb{C}^*$ we set

$$L(s, \pi) = L(s, \omega)$$

and

$$\varepsilon(s, \pi, \psi_{\mathbb{R}}) = \lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})\,\varepsilon(s, \omega, \psi_{\mathbb{C}/\mathbb{R}})$$

if $\psi_{\mathbb{C}/\mathbb{R}}(z) = \psi_{\mathbb{R}}(z + \bar{z})$. The factor $\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})$ was defined in the first paragraph. It is of course necessary to check that the two definitions coincide if $\pi(\omega) = \pi(\mu_1, \mu_2)$. This is an immediate consequence of the duplication formula.

Theorem 5.15 *Let π be an infinite-dimensional irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. Let ω be the quasi-character of $\mathbb{R}^{\times}$ defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

If W is in $W(\pi,\psi)$ set

$$\Psi(g,s,W) = \int_{\mathbb{R}^{\times}} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\, d^{\times}a$$
$$\widetilde{\Psi}(g,s,W) = \int_{\mathbb{R}^{\times}} W\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) \omega^{-1}(a)\, |a|^{s-1/2}\, d^{\times}a$$

and let

$$\Psi(g,s,W) = L(s,\pi)\,\Phi(g,s,W)$$
$$\widetilde{\Psi}(g,s,W) = L(s,\tilde{\pi})\,\widetilde{\Phi}(g,s,W).$$

(i) *The integrals defined $\Psi(g,s,W)$ and $\widetilde{\Psi}(g,s,W)$ are absolutely convergent in some right half-plane.*
(ii) *The functions $\Phi(g,s,W)$ and $\widetilde{\Phi}(g,s,W)$ can be analytically continued to the whole complex plane as meromorphic functions. Moreover there exists a W for which $\Phi(e,s,W)$ is an exponential function of s.*
(iii) *The functional equation*

$$\widetilde{\Phi}(wg,1-s,W) = \varepsilon(s,\pi,\psi)\,\Phi(g,s,W)$$

is satisifed.
(iv) *If W is fixed $\Psi(g,s,W)$ remains bounded as g varies in a compact set and s varies in the region obtained by removing discs centred at the poles of $L(s,\pi)$ from a vertical strip of finite width.*

We suppose first that $\pi = \pi(\mu_1,\mu_2)$. Then $W(\pi,\psi) = W(\mu_1,\mu_2;\psi)$. Each W in $W(\mu_1,\mu_2;\psi)$ is of the form $W = W_\Phi$ where

$$\Phi(x,y) = e^{-\pi(x^2+u^2y^2)}P(x,y)$$

with $P(x,y)$ a polynomial. However we shall verify the assertions of the theorem not merely for W in $W(\pi,\psi)$ but for any function $W = W_\Phi$ with Φ in $\mathcal{S}(\mathbb{R}^2)$. Since this class of functions is invariant under right translations most of the assertions need then be verified only for $g = e$.

A computation already performed in the non-archimedean case shows that

$$\Psi(e,s,W) = Z(\mu_1\alpha_{\mathbb{R}}^{s}, \mu_2\alpha_{\mathbb{R}}^{s}, \Phi)$$

the integrals defining these functions both being absolutely convergent in a right half-plane. Also for s in some left half-plane

$$\widetilde{\Psi}(w,1-s,W) = Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s}, \mu_2^{-1}\alpha_{\mathbb{R}}^{1-s}, \Phi')$$

if Φ' is the Fourier transform of Φ.

Since Φ can always be taken to be a function of the form $\Phi(x,y) = \Phi_1(x)\,\Phi_2(y)$ the last assertion of part (ii) is clear. All other assertions of the theorem except the last are consequence of the following lemma.

Lemma 5.15.1 *For every Φ in $\mathcal{S}(\mathbb{R}^2)$ the quotient*

$$\frac{Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)}{L(s,\mu_1)\,L(s,\mu_2)}$$

is a holomorphic function of (s_1,s_2) and

$$\frac{Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2},\Phi')}{L(1-s_1,\mu_1^{-1})\,L(1-s_2,\mu_2^{-1})}$$

is equal to

$$\varepsilon(s_1,\mu_1,\psi)\,\varepsilon(s_2,\mu_2,\psi)\,\frac{Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)}{L(s_1,\mu_1)\,L(s_2,\mu_2)}.$$

We may as well assume that μ_1 and μ_2 are characters so that the integrals converge for $\operatorname{Re} s_1 > 0$ and $\operatorname{Re} s_2 > 0$. We shall show that when $0 < \operatorname{Re} s_1 < 1$ and $0 < \operatorname{Re} s_2 < 1$

$$Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)\,Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2}\Psi')$$

is equal to

$$Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2},\Phi')\,Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Psi)$$

if Φ and Ψ belong to $\mathcal{S}(\mathbb{R}^2)$.

The first of these expressions is equal to

$$\int \Phi(x,y)\Psi'(u,v)\mu_1\left(\frac{x}{u}\right)\mu_2\left(\frac{y}{v}\right)\left|\frac{x}{u}\right|^{s_1}\left|\frac{y}{v}\right|^{s_2}\,d^\times x\,d^\times y\,du\,dv$$

if we assume, as we may, that $d^\times x = |x|^{-1}\,dx$. Changing variables we obtain

$$\int \mu_1(x)\,\mu_2(y)\,|x|^{s_1}\,|y|^{s_2}\left\{\int \Phi(xu,yv)\,\Psi'(u,v)\,du\,dv\right\}\,d^\times x\,d^\times y$$

The second expression is equal to

$$\int \mu_1^{-1}(x)\,\mu_2^{-1}(y)\,|x|^{1-s_1}\,|y|^{1-s_1}\left\{\int \Phi'(xu,yv)\,\Psi(u,v)\,du\,dv\right\}\,d^\times x\,d^\times y$$

which equals

$$\int \mu_1(x)\,\mu_2(y)\,|x|^{s_1}\,|y|^{s_2}\left\{\int |xy|^{-1}\,\Phi'(x^{-1}u,y^{-1}v)\,\Psi(u,v)\,du\,dv\right\}\,d^\times x\,d^\times y.$$

Since the Fourier transform of the function $(u,v)\to\Phi(xu,yv)$ is the function $|xy|^{-1}\Phi'(x^{-1}u,y^{-1}v)$ the Plancherel theorem implies that

$$\int \Phi(xu,yv)\,\Psi'(u,v)\,du\,dv = \int |xy|^{-1}\Phi'(x^{-1}u,y^{-1}v)\,\Psi(u,v)\,du\,dv.$$

The desired equality follows.

Choose Φ_1 and Φ_2 in $\mathcal{S}(\mathbb{R})$ such that

$$L(s,\mu_i) = Z(\mu_i\alpha_{\mathbb{R}}^{s},\Phi_i)$$

and take $\Psi(x,y) = \Phi_1(x)\,\Phi_2(y)$. The functional equation of the lemma follows immediately if $0 < s_1 < 1$ and $0 < s_2 < 1$. The expression on one side of the equation is holomorphic for $0 < \operatorname{Re} s_1$ and $0 < \operatorname{Re} s_2$. The expression on the other side is holomorphic for $\operatorname{Re} s_1 < 1$ and $\operatorname{Re} s_2 < 1$. Standard and easily proved theorems in the theory of functions of several complex variables show that the function they define is actually an entire function of s_1 and s_2. The lemma is completely proved.

For $\pi = \pi(\mu_1,\mu_2)$ the final assertion of the theorem is a consequence of the following lemma.

Lemma 5.15.2 *Let Ω be a compact subset of $\mathcal{S}(\mathbb{R}^2)$ and C a domain in $\mathbb{C}^2$ obtained by removing balls about the poles of $L(s_1,\mu_1)\,L(s_2,\mu_2)$ from a tube $a_1 \le \operatorname{Re} s_1 \le b_1$, $a_2 \le \operatorname{Re} s_2 \le b_2$. Then*

$$Z(\mu_1\alpha_{\mathbb{R}}^{s_1}, \mu_2\alpha_{\mathbb{R}}^{s_2}, \Phi)$$

remains bounded as Φ varies in Ω and (s_1, s_2) varies in C.

The theorems in the theory of functions alluded to earlier show that it is enough to prove this when either both a_1 and a_2 are greater than 0 or both b_1 and b_2 are less than 1. On a region of the first type the functions $Z(\mu_1\alpha_{\mathbb{R}}^{s}, \mu_2\alpha_{\mathbb{R}}^{s}, \Phi)$ is defined by a definite integral. Integrating by parts as in the theory of Fourier transforms one finds that

$$Z(\mu_1\alpha_{\mathbb{R}}^{\sigma_1+i\tau_1}, \mu_2\alpha_{\mathbb{R}}^{\sigma_2+i\tau_2}, \Phi) = O(\tau_1^2+\tau_2^2)^{-n}$$

as $\tau_1^2+\tau_2^2 \to \infty$ uniformly for Φ in Ω and $a_1 \le \sigma_1 \le b_1$, $a_2 \le \sigma_2 \le b_2$ which is a much stronger estimate than required. For a region of the second type one combines the estimates just obtained with the functional equation and the known asymptotic behavior of the Γ-function.

Now let ω be a quasi-character of $\mathbb{C}^\times$ which is not of the form $\omega(z) = \chi(z\bar{z})$ with χ a quasi-character of $\mathbb{R}^\times$ and let $\pi = \pi(\omega)$. $W(\pi,\psi)$ is the sum of $W_1(\pi,\psi)$ and its right translate by ε. It is easily seen that

$$\Phi(g, s, \rho(\varepsilon)W) = \omega(-1)\,\Phi(\varepsilon^{-1}g\varepsilon, s, W)$$

and that

$$\widetilde{\Phi}(wg, s, \rho(\varepsilon)W) = \omega(-1)\,\widetilde{\Phi}(w\varepsilon^{-1}g\varepsilon, s, W)$$

Thus it will be enough to prove the theorem for W in $W_1(\pi,\psi)$. Since

$$\Phi(\varepsilon g, s, W) = \Phi(g, s, W)$$

and

$$\widetilde{\Phi}(w\varepsilon g, s, W) = \widetilde{\Phi}(wg, s, W)$$

we can also take g in G_+. $W_1(\pi,\psi)$ consists of the functions W_Φ with Φ in $\mathcal{S}_0(\mathbb{C},\omega)$. We prove the assertions for functions W_Φ with Φ in $\mathcal{S}(\mathbb{C},\omega)$. Since this class of functions is invariant under right translations by elements of G_+ we may take $g = e$.

As we observed in the first paragraph we will have

$$\Psi(e, s, W) = Z(\omega\alpha_{\mathbb{C}}^{s}, \Phi)$$
$$\widetilde{\Psi}(w, 1-s, W) = \lambda(\mathbb{C}/\mathbb{R},\psi)\,Z(\omega^{-1}\alpha_{\mathbb{C}}^{1-s}, \Phi')$$

in some right half plane and the proof proceeds as before. If $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$ and $p - q = n - m$ the function

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}\, z^p\,\bar{z}^q$$

belongs to $\mathcal{S}_0(\mathbb{C},\omega)$ and

$$\begin{aligned} Z(\omega\alpha_{\mathbb{C}}^{s}, \Phi) &= 2\pi\int_0^\infty e^{-2\pi|u|t^2}\, t^2(s+r+p+m)\,dt \\ &= \pi(2\pi|u|)^{-(s+r+p+m)}\,\Gamma(s+r+p+m) \end{aligned}$$

Taking $p = n$ we obtain an exponential times $L(s,\omega)$. The last part of the theorem follows from an analogue of Lemma 5.15.2.

The local functional equation which we have just proved is central to the Hecke theory. We complete the paragraph with some results which will be used in the paragraph on extraordinary representations and the chapter on quaternion algebras.

Lemma 5.16 *Suppose μ_1 and μ_2 are two quasi-characters for which both $\pi = \pi(\mu_1, \mu_2)$ and $\sigma = \sigma(\mu_1, \mu_2)$ are defined. Then*

$$\frac{L(1-s, \widetilde{\sigma})\, \varepsilon(s, \sigma, \psi)}{L(s, \sigma)} = \frac{L(1-s, \tilde{\pi})\, \varepsilon(s, \pi, \psi)}{L(s, \pi)}$$

and the quotient

$$\frac{L(s, \sigma)}{L(s, \pi)}$$

is an exponential times a polynomial.

Interchanging μ_1 and μ_2 if necessary we may suppose that $\mu_1 \mu_2^{-1}(x) = |x|^s (\operatorname{sgn} x)^m$ with $s > 0$. According to Corollary 5.14, $W(\sigma, \psi)$ is a subspace of $W(\mu_1, \mu_2, \psi)$. Although $W(\mu_1, \mu_2, \psi)$ is not irreducible it is still possible to define $\Psi(g, s, W)$ and $\widetilde{\Psi}(g, s, W)$ when W lies in $W(\mu_1, \mu_2, \psi)$ and to use the method used to prove Theorem 5.15 to show that

$$\frac{\widetilde{\Psi}(wg, 1-s, W)}{L(1-s, \tilde{\pi})}$$

is equal to

$$\varepsilon(s, \pi\psi) \frac{\Psi(g, s, W)}{L(s, \pi)}$$

Applying the equality to an element of $W(\sigma, \psi)$ we obtain the first assertion of the lemma. The second is most easily obtained by calculation. Replacing μ_1 and μ_2 by $\mu_1 \alpha_{\mathbb{R}}^t$ and $\mu_2 \alpha_{\mathbb{R}}^t$ is equivalent to a translation in s so we may assume μ_2 is of the form $\mu_s(x) = (\operatorname{sgn} x)^{m_2}$. There is a quasi-character ω of $\mathbb{C}^\times$ such that $\sigma = \pi(\omega)$. If $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$ then $\mu_1(x) = |x|^{2r+m+n} (\operatorname{sgn} x)^{m+n+m_2+1}$, $\mu_1(x) = x^{m+n} (\operatorname{sgn} x)^{m_2+1}$ so that $r = 0$. Apart from an exponential factor $L(s, \sigma)$ is equal to $\Gamma(s+m+n)$ while $L(s, \pi)$ is, again apart from an exponential factor,

$$\Gamma\left(\frac{s+m+n+m_1}{2}\right) \Gamma\left(\frac{s+m_2}{2}\right) \tag{5.16.1}$$

where $m_1 = m + n + m_2 + 1 \pmod 2$. Since $m + n > 0$ the number

$$k = \frac{1}{2}(m + n + 1 + m_1 - m_2) - 1$$

is a non-negative integer and $m_2 + 2k = m + n + m_1 - 1$. Thus

$$\Gamma\left(\frac{s+m_2}{2}\right) = \left\{ \frac{1}{2^{k+1}} \prod_{j=0}^{k} (s + m_2 + 2j) \right\}^{-1} \Gamma\left(\frac{s+m+n+m_1+1}{2}\right).$$

By the duplication formula the product (5.16.1) is a constant times an exponential times

$$\frac{\Gamma(s+m+n+m_1)}{\prod_{j=0}^{k} (s + m_2 + 2j)}.$$

If $m_1 = 0$ the lemma follows immediately. If $m_1 = 1$

$$\Gamma(s+m+n+m_1) = (s+m+n)\, \Gamma(s+m+n)$$

and $m_2 + 2k = m + n$. The lemma again follows.

Lemma 5.17 *Suppose $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$ is a quasi-charaacter of $\mathbb{C}^\times$ with $mn = 0$ and $m + n > 0$. Suppose μ_1 and μ_2 are two quasi-characters of $F^\times$ with $\mu_1\mu_2(x) = |x|^{2r} x^{m+n} \operatorname{sgn} x$ and $\mu_1\mu_2^{-1}(x) = x^{m+n} \operatorname{sgn} x$. Then for every Φ in $\mathcal{S}(\mathbb{R}^2)$ such that*

$$\int x^i \frac{\partial^j \Phi}{\partial y j}(x, 0)\, dx = 0$$

for $i > 0$, $j \geq 0$, and $i + j + 1 = m + n$ the quotient

$$\frac{Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)}{L\big(s, \pi(w)\big)}$$

is a holomorphic function of s and for some Φ it is an exponential.

If W_Φ belongs to $W(\mu_1, \mu_2, \psi)$ this is a consequence of Corollary 5.14 and Theorem 5.15. Unfortunately we need the result for all Φ. The observations made during the proof of Lemma 5.16 show that if $\pi = \pi(\mu_1, \mu_2)$ the quotient

$$\frac{Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)}{L(s, \pi)}$$

is holomorphic. Since $L(s, \pi)$ and $L(s, \sigma)$ have no zeros we have only to show that the extra poles of $L(s, \pi)$ are not really needed to cancel poles of $Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$. As in the proof of Lemma 5.16 we may take $r = 0$. We have to show that $Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$ is holomorphic at $s = -m_2 - 2j$, $0 \leq j \leq k$ if $m_1 = 0$ and at $s = -m_2 - 2j$, $0 \leq j \leq k$ if $m_1 = 1$. We remark first that if μ_1 and μ_2 are two quasi-characters of $\mathbb{R}^\times$, Φ belongs to $\mathcal{S}(\mathbb{R}^2)$, and $\operatorname{Re} s$ is sufficiently large then, by a partial integration,

$$\int \mu_1(x)\, \mu_2(y)\, |x|^s\, |y|^s\, \Phi(x, y)\, d^\times x\, d^\times y = -\frac{1}{s} \int \mu_1(x)\, \mu_2(y)\, \eta(y)\, |x|^s\, |y|^{s+1}\, \frac{\partial \Phi}{\partial y}(x, y)\, d^\times x\, d^\times y$$

if $\eta(y) = \operatorname{sgn} y$. Integrating by parts again we obtain

$$\int \mu_1(x)\mu_2(y)\, |x|^s\, |y|^s\, \Phi(x, y)\, d^\times x\, d^\times y = \frac{1}{s(s+1)} \int \mu_1(x)\, \mu_2(y)\, |x|^s\, |y|^{s+2}\, \frac{\partial^2 \Phi}{\partial y^2}(x, y)\, d^\times x\, d^\times y.$$

If Φ belongs to $\mathcal{S}(\mathbb{R}^2)$ the function defined by

$$\int \Phi(x, y)\, |x|^{s+1}\, |y|^s\, d^\times x\, d^y y \tag{5.17.1}$$

is certainly holomorphic for $\operatorname{Re} s > 0$. We have to show that if

$$\int \Phi(x, 0)\, dx = 0$$

it is holomorphic for $\operatorname{Re} s > -1$. Suppose first that $\Phi(x, 0) \equiv 0$. Since

$$\Phi(x, y) = y \frac{\partial \Phi}{\partial y}(x, 0) + \int_0^y (y - u) \frac{\partial^2 \Phi}{\partial y^2}(x, u)\, du$$

the function

$$\Psi(x,y) = \frac{1}{y}\,\Phi(x,y)$$

is dominated by the inverse of any polynomial. Thus (5.17.1) which equals

$$\int \Psi(x,y)\,|x|^{s+1}\,|y|^{s+1}\,\eta(y)\,d^\times x\,d^\times y$$

is absolutely convergent for $\mathrm{Re}\,s > -1$. In the general case we set

$$\begin{aligned}\Phi(x,y) &= \{\Phi(x,y) - \Phi(x,0)e^{-y^2}\} + \Phi(x,0)e^{-y^2}\\ &= \Phi_1(x,y) + \Phi_2(x,y).\end{aligned}$$

Since $\Phi_1(x,0) = 0$ we need only consider

$$\int \Phi_2(x,0)e^{-y^2}\,|x|^{s+1}\,|y|^s\,d^\times x\,d^\times y$$

which is the product of a constant and

$$\Gamma\Big(\frac{s}{2}\Big)\int \Phi_2(x,0)\,|x|^s\,dx.$$

The integral defines a function which is holomorphic for $\mathrm{Re}\,s > -1$ and, when the assumptions are satisfied, vanishes at $s = 0$.

We have to show that if $0 \le j \le m+n-1$ and $j - m_2$ is even then $Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$ is holomorphic at $-j$. Under these circumstances the function $Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$ is equal to

$$\int \eta(x)^{m_1}\,\eta(y)^{m_2}\,|x|^{m+n}\,|x|^s|y|^s\,\Phi(x,y)\,d^\times x\,d^\times y$$

which equals

$$\frac{(-1)^j}{\prod_{i=0}^{j-1}(s_i)}\int \eta(x)^{m_1}\,|x|^{s+m+n}\,|y|^{s+j}\,\frac{\partial^j\Phi}{\partial y^j}(x,y)\,d^\times x\,d^\times y.$$

The factor in front is holomorphic at $s = -j$. If

$$\Psi(x,y) = x^{m+n-j-1}\,\frac{\partial^j\Phi}{\partial y^j}(x,y)$$

the integral itself is equal to

$$\int |x|^{s+j+1}|y|^{s+j}\,\Psi(x,y)\,d^\times x\,d^\times y.$$

Since, by assumption,

$$\int \Psi(x,0)\,dx = 0,$$

it is holomorphic at $s = -j$.

We observe that if $m+n$ is even

$$\Phi(x,y) = e^{-\pi(x^2+y^2)}xy^{m+n}$$

satisfies the conditions of the lemma and, if $r = 0$ and $m_2 = 0$, $Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$ is equal to

$$\int e^{-\pi(x^2+y^2)}\,|x|^{m+n+s+1}\,|y|^{m+n+s}\,d^\times x\,d^\times y$$

which differs by an exponential from $\Gamma(s+m+n)$ and $L\big(s,\pi(\omega)\big)$. If $m_2 = 1$ we take $\Phi(x,y) = e^{-\pi(x^2+y^2)}\,y^{m+n+1}$ to obtain the same result. If $m+n$ is odd and $m_2 = 0$ the polynomial factor will be y^{m+n+1} but if $m+n$ is odd and $m_2 = 1$ it will again be xy^{m+n}.

Proposition 5.18 *Suppose π and π' are two infinite-dimensional irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$ such that, for some quasi-character ω of $F^\times$,*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right) = \omega(a)I.$$

If

$$\frac{L(1-s,\chi^{-1}\otimes\tilde{\pi})}{L(s,\chi\otimes\pi)}\,\varepsilon(s,\chi\otimes\pi,\psi) = \frac{L(1-s,\chi^{-1}\otimes\tilde{\pi}')}{L(s,\chi\otimes\pi')}\,\varepsilon(s,\chi\otimes\pi',\psi)$$

for all quasi-characters χ and π and π' are equivalent.

Suppose $\pi = \pi(\mu_1,\mu_2)$ or $\sigma(\mu_1,\mu_2)$. From Lemma 5.16 and the definitions the expression on the left is equal to

$$(i\,\mathrm{sgn}\,u)^{m_1+m_2}\,|u|^{2s+s_1+s_2-1}\pi^{2s+s_1+s_2-1}\frac{\Gamma\left(\frac{1-s-r_1+m_1}{2}\right)\Gamma\left(\frac{1-s-r_2+m_2}{2}\right)}{\Gamma\left(\frac{s+r_1+m_1}{2}\right)\Gamma\left(\frac{s+r_2+m_2}{2}\right)}$$

if χ is trivial and $\mu_i(x) = |x|^{r_i}(\mathrm{sgn}\,x)^{m_1}$. If $\chi(x) = \mathrm{sgn}\,x$ and n_i is 0 or 1 while $m_i + n_1 = 1 \pmod 2$ the quotient is

$$(i\,\mathrm{sgn}\,u)^{m_1+m_2}\,|u|^{2s+s_1+s_2-1}\pi^{2s+s_1+s_2-1}\frac{\Gamma\left(\frac{1-s-r_1-n_1}{2}\right)\Gamma\left(\frac{1-s-r_2+n_2}{2}\right)}{\Gamma\left(\frac{s+r_1+n_1}{2}\right)\Gamma\left(\frac{s+r_2+n_2}{2}\right)}.$$

If we let π' be $\pi(\mu_1',\mu_2')$ or $\sigma(\mu_1',\mu_2')$ we obtain similar formulae with r_i replaced by r_i' and m_i by m_i'.

Consider first the quotients for π. The first has an infinite number of zeros of the form $-r_1 - m_1 - 2p$ where p is a non-negative integer and an infinite number of the form $-r_2 - m_2 - 2p$ where p is a non-negative integer, but no other zeros. Similarly the zeros of the second are at points $-r_1 - n_1 - 2p$ or $-r_2 - n_2 - 2p$. Thus if the quotients are equal $r_1 + m_1 \equiv r_2 + n_2 \equiv r_2 + m_2 + 1 \pmod 2$. Moreover if $r_1 + m_1 = r_2 + m_2 + 1 \pmod 2$ then $\pi = \sigma(\mu_1,\mu_2)$ and, as we saw in Theorem 5.11, $\sigma(\mu_1\eta,\mu_2\eta) = \sigma(\mu_1,\mu_2)$ so that the two quotients are equal. As a result either $r_1 + m_1 = r_2 + m_2 + 1 \pmod 2$ and $r_1' + m_1' = r_2' + m_2' + 1 \pmod 2$ or neither of these congruences hold.

Suppose first that $\pi = \pi(\mu_1,\mu_2)$ and $\pi'(\mu_1',\mu_2')$. Then the first quotient for π has zeros at the points $-r_1 - m_1$, $-r_1 - m_1 - 2,\cdots$ and $-r_2 - m_2$, $-r_2 - m_2 - 2,\cdots$ while that for π' has zeros at $-r_1' - m_1'$, $-r_1' - m_1' - 2,\cdots$ and $-r_2' - m_2'$, $-r_2' - m_2' - 2,\cdots$. Thus either $r_1 + m_1 = r_1' + m_1'$ or $r_1 + m_1 = r_2' + m_2'$. Interchanging μ_1' and μ_2' if necessary we may assume that the first of these two alternatives hold. Then $r_2 + m_2 = r_2' + m_2'$. Moreover $r_1 + r_2 = r_1' + r_2'$ and $|m_1 - m_2| = |m_1' - m_2'|$. If $m_1 = m_1'$ it follows immediately that $\mu_1 = \mu_1'$ and $\mu_2 = \mu_2'$. Suppose that $m_1 \neq m_1'$. Examining the second quotient we see that either $r_1 + n_1 = r_1' + n_1'$ or $r_1 + n_1 = r_2' + n_2'$. The first equality is incompatible with the relations $r_1 + m_1 = r_1' + m_1'$ and $m_1 \neq m_1'$. Thus $r_1 + n_1 = r_2' + n_2'$. For the same reason $r_2 + n_2 = r_1' + n_1'$. Interchanging the roles of μ_1,μ_2 and μ_1',μ_2' if necessary we may suppose that $m_1 = 0$ and $m_1' = 1$. Then $r_1 = r_1' + 1$. Since $r_1 + r_2 = r_1' + r_2'$ we have $r_2 = r_2' - 1$ so that $m_2 = 1$, $m_2' = 0$. Thus $n_1 = n_2' = 1$ and $r_1 = r_2'$ so that $r_2 = r_1'$. It follows that $\mu_1 = \mu_2'$ and $\mu_2 = \mu_1'$.

Finally we suppose that $\pi = \sigma(\mu_1,\mu_2)$ and $\pi' = \sigma(\mu_1',\mu_2')$. Then there are quasi-characters ω_1 and ω_1' of $\mathbb{C}^\times$ such that $\pi = \pi(\omega_1)$ and $\pi' = \pi(\omega_1')$. Replacing ω_1 by the quasi-character $z \to \omega_1(\bar{z})$ does not change $\pi(\omega_1)$ so we may suppose that $\omega_1(z) = (z\bar{z})^r z^m$ while $\omega_1'(z) = (z\bar{z})^{r'} z^{m'}$. Since

ω_1 and ω_1' must have the same restriction to $\mathbb{R}^\times$ the numbers $2r+m$ and $2r'+m'$ are equal while $m \equiv m' \pmod 2$. Apart from a constant and an exponential factor the quotient

$$\varepsilon(s,\pi,\psi)\,\frac{L(1-s,\tilde{\pi})}{L(s,\pi)}$$

is given by

$$\frac{\Gamma(1-s-r)}{\Gamma(s+r+m)}$$

whose pole furthest to the left is at $1-r$. Consequently $r=r'$ and $m=m'$.

Corollary 5.19 *Suppose π and π' are two irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$. Suppose there is a quasi-character ω of $\mathbb{R}^\times$ such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

If for all quasi-characters χ, $L(s,\chi\otimes\pi) = L(s,\chi\otimes\pi')$, $L(s,\chi^{-1}\otimes\pi) = L(s,\chi^{-1}\otimes\pi')$, and $\varepsilon(s,\chi\otimes\pi,\psi) = \varepsilon(s,\chi\otimes\pi',\psi)$ then π and π' are equivalent.

Combining Lemma 5.16 with the previous propositon we infer that there is a pair of quasi-characters μ_1 and μ_2 such that both π and π' are one of the representations $\pi(\mu_1,\mu_2)$ or $\sigma(\mu_1,\mu_2)$. However the computations made during the proof of Lemma 5.16 show that $L\big(s,\chi\otimes\pi(\mu_1,\mu_2)\big)$ differs from $L\big(s,\chi\otimes\sigma(\mu_1,\mu_2)\big)$ for a suitable choice of χ.

Let K be the quaternion algebra of $\mathbb{R}$. We could proceed along the lies of the fourth paragraph and associate to every finite-dimensional irreducible representation Ω of $K^\times$ a representation $\pi(\Omega)$ of $G_{\mathbb{R}}$. Since we have just classified the representations of $G_{\mathbb{R}}$ we can actually proceed in a more direct manner.

We identify K with the algebra of 2×2 complex matrices of the form

$$z = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

Then

$$z^\iota = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$$

and $\nu(z) = z^\iota$ is the scalar matrix $(|a|^2+|b|^2)I$ while $\tau(z)$ is the scalar matrix $(a+\bar{a}+b+\bar{b})I$. Let ρ_1 be the two dimensional representation of $K^\times$ associated to this identification and let ρ_n be the nth symmetric power of ρ_1. Any irreducible representation is equivalent to a representation of the form $\chi\otimes\rho_n$ where χ is a quasi- character of $\mathbb{R}^\times$. Thus

$$(\chi\otimes\rho_n)(h) = \chi\big(\nu(h)\big)\,\rho_n(h)$$

Since $\nu(h)$ is always positive we may suppose that χ is of the form $\chi(x) = |x|^r$.

Let Ω be a finite dimensional representation and let Ω act on U. In the first paragraph we introduced the space $\mathcal{S}(K,U)$. It is clear that if Φ is in $\mathcal{S}(K,U)$ the integrals

$$Z(\alpha_{\mathbb{R}}^s\otimes\Omega,\Phi) = \int_{K^\times}\Omega(h)\,|\nu(h)|^s\,\Phi(h)\,d^\times h$$

and

$$Z(\alpha_{\mathbb{R}}^s\otimes\Omega^{-1},\Phi) = \int_{K^\times}\Omega^{-1}(h)\,|\nu(h)|^s\,\Phi(h)\,d^\times h$$

converge absolutely in some right half-plane.

Proposition 5.20 *Suppose $\chi(x) = |x|^r$ and $\Omega = \chi \otimes \rho_n$. Let ω be the character of $\mathbb{C}^\times$ defined by $\omega(z) = (z\bar{z})^{r-1/2} z^{n+1}$. Set $L(s,\Omega) = L(s,\omega)$ and*

$$\varepsilon(s,\Omega,\psi_{\mathbb{R}}) = \lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})\,\varepsilon(s,\omega,\psi_{\mathbb{C}/\mathbb{R}})$$

The quotient

$$\frac{Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi)}{L(s,\Omega)}$$

can be analytically contained to the whole complex plane as a holomorphic function. Given u in U there exists a Φ in $\mathcal{S}(K,U)$ such that

$$\frac{Z(\alpha_{\mathbb{R}}^{s+1/2},\Phi)}{L(s,\Omega)} = a^s u.$$

For all Φ the two functions

$$\frac{Z(\alpha_{\mathbb{R}}^{3/2-s}\otimes\Omega^{-1},\Phi')}{L(1-s,\widetilde{\Omega})}$$

and

$$-\varepsilon(s,\Omega,\psi_{\mathbb{R}})\,\frac{Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi)}{L(s,\Omega)}$$

are equal. Finally $Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi)$ is bounded in any region obtained by removing discs about the poles $L(s,\Omega)$ from a vertical strip of finite width.

Suppose K_1 is the subgroup of $K^\times$ formed by the elements of reduced norm one. Let Φ_1 be the function on $\mathbb{R}$ defined by

$$\Phi_1(t) = \int_{K_1} \Omega(h)\,\Phi(th)\,dh$$

Φ_1 belongs to $\mathcal{S}(\mathbb{R})$ and if ω_0 is the quasi-character of $\mathbb{R}^\times$ defined by $\Omega(t) = \omega_0(t)I$ the function $\omega_0(t)\Phi_1(t)$ is even. Moreover if the multiplcative Haar measures are suitably normalized

$$Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi) = Z(\alpha_{\mathbb{R}}^{2s+1}\omega_0,\Phi_1).$$

Since $\omega_0(t) = |t|^{2r}t^n$ we can integrate by parts as in the proof of Lemma 5.17 to see that for any non-negative integer m

$$Z(\alpha_{\mathbb{R}}^{2s+1}\omega_0,\Phi_1) = \frac{(-1)^m}{\prod_{j=0}^{m-1}(2s+2r+n+j+1)} \int \eta(t)^{m+n}\,|t|^{2s+2r+m+n+1}\,\frac{\partial^m\Phi_1}{\partial t^m}\,d^\times t.$$

The integral is holomorphic for $\mathrm{Re}(2s+2r+m+n) > -1$ and, if $\frac{\partial^m\Phi_1}{\partial t^m}$ vanishes at $t=0$, for $\mathrm{Re}(2s+2r+m+n) > -2$. Thus the function on the left has an analytic continuation to the whole complex plane as a meromorphic function with simple poles. Since

$$L(s,\Omega) = 2(2\pi)^{-(s+r+n+1/2)}\Gamma\Big(s+r+n+\frac{1}{2}\Big)$$

we have to show that its poles occur at the points $s+r+n+\frac{1}{2}+j=0$ with $j=0,1,2,\cdots$. Since $\frac{\partial^m\Phi_1}{\partial t^m}$ vanishes at 0 if $m+n$ is odd its only poles are at the points $2s+2r+2n+2j+1=0$ with $n+2j\geq 0$. To exclude the remaining unwanted poles we have to show that $\frac{\partial^m\Phi_1}{\partial t^m}=0$ at 0 if $m<n$. If we expand Φ in a Taylor's series about 0 we see that $\frac{\partial^m\Phi_1}{\partial t^m}=0$ at 0 unless the restriction of ρ_n to K_1 is contained in the representation on the polynomials of degree m on K. This can happen only if $m\geq n$.

Since $\widetilde{\Omega}$ is equivalent to the representation $h\to\Omega^{-1}(h^\iota)$ the quotient

$$\frac{Z(\alpha_{\mathbb{R}}^{3/2-s}\otimes\Omega^{-1},\Phi')}{L(1-s,\widetilde{\Omega})}$$

is also holomorphic. The argument used to prove Lemma 5.15.1 shows that there is a scalar $\lambda(s)$ such that, for all Φ,

$$\frac{Z(\alpha_{\mathbb{R}}^{3/2-s}\otimes\Omega^{-1},\Phi')}{L(1-s,\widetilde{\Omega})}=\lambda(s)\,\frac{Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi)}{L(s,\Omega)}.$$

We shall used the following lemma to evaluate $\lambda(s)$.

Lemma 5.20.1 *Let φ be a function in $\mathcal{S}(\mathbb{C})$ of the form*

$$\varphi(x)=e^{-2\pi x\bar{x}}P(x,\bar{x})$$

where P is a polynomial in x and $\bar{x}$. Suppose $\varphi(xu)=\varphi(x)\,\omega^{-1}(u)$ if $u\bar{u}=1$. Define the function Φ in $K^\times$ by

$$\Phi(z)=\varphi(\alpha)\,\omega(\alpha)\,(\alpha\bar{\alpha})^{-\frac{1}{2}}\langle u,\widetilde{\Omega}(z)\widetilde{u}\rangle$$

if $\nu(z)=\alpha\bar{\alpha}$. Then Φ extends to a function in $\mathcal{S}(K)$ and its Fourier transform is given by

$$\Phi'(z)=-\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})\,\varphi'(\alpha)\,\omega^{-1}(\alpha)\,(\alpha\bar{\alpha})^{-\frac{1}{2}}\langle\Omega(z)u,\widetilde{u}\rangle$$

if φ' is the Fourier transform of φ.

By linearity we may assume that φ is of the form

$$\varphi(x)=e^{-2\pi x\bar{x}}(x\bar{x})^p\bar{x}^{n+1}$$

where p is a non-negative integer. We may suppose that the restriction of ρ_n to the elements of norm one is orthogonal and identify the space U on which its acts with its dual $\widetilde{U}$. Then $\widetilde{\Omega}=\alpha_{\mathbb{R}}^{-r-n}\otimes\rho_n$. Thus if

$$z=\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$$

the value of Φ at z is

$$e^{-2\pi(a\bar{a}+b\bar{b})}(a\bar{a}+b\bar{b})^{r+n+p}\langle u,\widetilde{\Omega}(z)\widetilde{u}\rangle=e^{-2\pi(a\bar{a}+b\bar{b})}(a\bar{a}+b\bar{b})^{p}\langle u,\rho_n(z)\widetilde{u}\rangle$$

The expression on the right certainly defines a function in $\mathcal{S}(K)$.

We are trying to show that if

$$F(z)=\varphi(\alpha)\,\omega(\alpha)\,(\alpha\bar{\alpha})^{-\frac{1}{2}}\,\Omega^{-1}(z)$$

when $z = \alpha\bar{\alpha}$ then the Fourier transform of F is given by

$$F'(z) = -\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})\,\varphi'(\alpha)\,\omega^{-1}(\alpha)\,(\alpha\bar{\alpha})^{-\frac{1}{2}}\,\Omega(z). \tag{5.20.2}$$

If h_1 and h_2 have norm one

$$F(h_1 z h_2) = \Omega(h_2^{-1})\,F(z)\,\Omega(h_1^{-1})$$

and therefore

$$F'(h_1 z h_2) = \Omega(h_1)\,F'(z)\,\Omega(h_2)$$

In particular if z is a scalar in K the operator $F'(z)$ commutes with the elements of norm one and is therefore a scalar operator. The expression $F_1(z)$ on the right of (5.20.2) has the same properties so that all we need do is show that for some pair of vectors u and $\widetilde{u}$ which are not orthogonal

$$\langle F'(z)u, \widetilde{u}\rangle = \langle F_1(z)u, \widetilde{u}\rangle$$

for all positive scalars z.

If we only wanted to show that $F'(z) = c\,F_1(z)$ where c is a positive constant it would be enough to show that

$$\langle F'(z)u, \widetilde{u}\rangle = c\langle F_1(z)u, \widetilde{u}\rangle. \tag{5.20.3}$$

Once this was done we could interchange the roles of φ and φ' and Φ and Φ' to show that $c^2 = 1$. To obviate any fuss with Haar measures we prove (5.20.3).

Recall that if

$$a(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ \theta & e^{-i\theta} \end{pmatrix}$$

then, apart from a positive constant,

$$\int_{K_1} \langle u, \widetilde{\Omega}(k)\widetilde{u}\rangle\, f(k)\, dk$$

is equal to

$$\langle u, \tilde{u}\rangle \int_0^{\pi} \sin(n+1)\theta\, \sin\theta\, f\big(a(\theta)\big)\, d\theta$$

if f is a class function on K_1, the group of elements of norm one. The equality is of course a consequence of the Weyl character formula and the Schur orthogonality relations.

If x is a positive scalar in K then, apart from a positive constant, $\Phi'(x)$ is given by

$$\int_{K^\times} \Phi(z)\,\psi_{\mathbb{R}}\big(\tau(xz)\big)\,|\nu(z)|^2\, d^\times z$$

which is a positive multiple of

$$\int_0^\infty t^3\varphi(t)\left\{\int_{K_1}\langle u, \widetilde{\Omega}(k)\widetilde{u}\rangle\,\psi_{\mathbb{R}}\big(xt\,\tau(k)\big)\,dk\right\} d^\times t.$$

Since $\tau(k)$ is a class function this expression is a positive multiple of

$$\langle u, \widetilde{u}\rangle \int_0^\infty t^3\varphi(t)\left\{\int_0^\pi \sin(n+1)\theta\sin\theta\,\psi_{\mathbb{R}}(2xt\,\cos\theta)\,d\theta\right\} d^\times t$$

Integrating the inner integral by parts we obtain

$$\langle u, \widetilde{u}\rangle \frac{n+1}{4\pi iux} \int_0^\infty t^2 \varphi(t) \left\{ \int_0^\pi \cos(n+1)\theta \, \psi_{\mathbb{R}}(2xt \, \cos\theta) \, d\theta \right\} d^\times t.$$

On the other hand if x, which is a positive real number, is regarded as an element of $\mathbb{C}$ then $\varphi'(x)$ is a positive multiple of

$$\int_{C^\times} \varphi(z) \, \psi_{\mathbb{R}}\big(\tau(xz)\big) z\bar{z} \, d^\times z$$

or of

$$\int_0^\infty t^2 \varphi(t) \left\{ \int_0^{2\pi} e^{-i(n+1)\theta} \psi_{\mathbb{R}}(xt \, \cos\theta) \, d\theta) \right\} d^\times t.$$

Since

$$\int_0^{2\pi} e^{-i(n+1)\theta} \psi_{\mathbb{R}}(xt\cos\theta) \, d\theta = 2 \int_0^\pi \cos(n+1)\theta \, \psi_{\mathbb{R}}(xt\cos\theta) \, d\theta$$

and $\lambda(\mathbb{C}/\mathbb{R}) = i \operatorname{sgn} u$ the identity (5.20.3) follows for any choice of u and $\widetilde{u}$.

To evaluate $\lambda(s)$ we choose Φ as in the lemma and compute

$$\langle Z(\alpha_{\mathbb{R}}^{s+\frac{1}{2}} \otimes \Omega, \Phi v), \widetilde{v}\rangle = \int \Phi(z) \, |\nu(z)|^{s+\frac{1}{2}} \langle \Omega(z)v, \widetilde{v}\rangle \, d^\times z$$

and

$$\langle Z(\alpha_{\mathbb{R}}^{\frac{3}{2}-s} \otimes \Omega^{-1}, \Phi v), \widetilde{v}\rangle = \int \Phi(z) \, |\nu(z)|^{\frac{3}{2}-s} \langle v, \widetilde{\Omega}(z)\widetilde{v}\rangle \, d^\times z.$$

The first is equal to

$$\int_{K^\times/K_1} |\nu(z)|^{s+\frac{1}{2}} \left\{ \int_{K_1} \Phi(zk) \langle \Omega(zk)v, \widetilde{v}\rangle \, dk \right\} d^\times z.$$

Since

$$\int_{K_1} \langle \Omega(zk)v, \widetilde{v}\rangle \langle u, \widetilde{\Omega}(zk)\widetilde{u}\rangle \, dk$$

is, by the Schur orthogonality relations, equal to

$$\frac{1}{\deg\Omega} \langle v, \widetilde{u}\rangle \langle u, \widetilde{v}\rangle$$

the double integral is equal to

$$\frac{1}{\deg\Omega} \langle v, \widetilde{u}\rangle \langle u, \widetilde{v}\rangle \int_{K^\times} \varphi(\alpha) \, \omega(\alpha) \, (\alpha\bar{\alpha})^s \, d^\times z$$

where $\alpha\bar{\alpha} = \nu(z)$. If the Haar measure on $\mathbb{C}^\times$ is suitably chosen the integral here is equal to $Z(\omega\alpha_{\mathbb{C}}^s, \varphi)$. The same choice of Haar measures lead to the relation

$$\langle Z(\alpha_{\mathbb{R}}^{\frac{3}{2}-s} \otimes \Omega^{-1}, \Phi v), \widetilde{v}\rangle = \frac{-\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})}{\deg\Omega} \langle v, \widetilde{u}\rangle \langle u, \widetilde{v}\rangle Z(\omega^{-1}\alpha_{\mathbb{C}}^{1-s}, \varphi').$$

Since $L(s,\Omega) = L(s,\omega)$ and $L(s,\widetilde{\Omega}) = L(s,\omega^{-1})$ we can compare the functional equation for $Z(\omega\alpha_{\mathbb{C}}^{s},\varphi)$ with that for $Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi v)$ to see that

$$\lambda(s) = -\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})\,\varepsilon(s,\omega,\psi_{\mathbb{C}/\mathbb{R}})$$

as asserted.

If

$$\varphi(x) = e^{-2\pi x\bar{x}}$$

then $Z(\alpha_{\mathbb{C}}^{s}\omega,\varphi)$ is an exponential times $L(s,\omega)$ so that $Z(\alpha_{\mathbb{R}}^{s+1/2}\otimes\Omega,\Phi v)$ is, with a suitable choice of v and $\widetilde{u}$, a non-zero scalar times an exponential times $L(s,\omega)u$. The last assertion of the proposition is proved in the same way as Lemma 5.15.2.

We end this paragraph with the observation that the space $W(\pi,\psi)$ of Theorem 5.13 cannot exist when π is finite-dimensional. If $W = W(\pi,\psi)$ did exist the contragredient representation $\tilde{\pi}$ on the dual space $\widetilde{W}$ would also be finite dimensional and $\tilde{\pi}(X_+)$ would be nilpotent. However if λ is the linear functional $\varphi \to \varphi(e)$ then $\tilde{\pi}(X_+)\lambda = -2i\pi a$ if $\psi(x) = e^{2i\pi ax}$.

§**6. Representations of** $GL(2,\mathbb{C})$**.** In this paragraph we have to review the representation theory of $G_{\mathbb{C}} = GL(2,\mathbb{C})$ and prove the local functional equation for the complex field. Many of the definitions and results of the previous paragraph are applicable, after simple modifications which we do not always make explicit, to the present situation.

The standard maximal compact subgroup of $GL(2,\mathbb{C})$ is the group $U(2,\mathbb{C})$ of unitary matrices. $\mathcal{H}_1$ will be the space of infinitely differentiable compactly supported functions on $G_{\mathbb{C}}$. $\mathcal{H}_2$ will be the space of functions on $U(2,\mathbb{C})$ which are finite linear combinations of the matrix elements of finite dimensional representations. $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ can be regarded as a space of measures. Under convolution it forms an algebra called the Hecke algebra. The notion of an elementary idempotent and the notion of an admissible representation of $\mathcal{H}_{\mathbb{C}}$ are defined more or less as before.

Let $\mathfrak{g}$ be the Lie algebra of the real Lie group of $GL(2,\mathbb{C})$ and let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. $\mathfrak{A}$ will be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. A representation of $\mathfrak{A}$ will be said to be admissible if its restriction to the Lie algebra of $U(2,\mathbb{C})$ decomposes into a direct sum of irreducible finite dimensional representations each occurring with finite multiplicity. There is a one-to-one correspondence between classes of irreducible admissible representations of $\mathcal{H}_{\mathbb{C}}$ and those of $\mathfrak{A}$. We do not usually distinguish between the two. The representation $\tilde{\pi}$ contragredient to π and the tensor product of π with a quasi-character of $\mathbb{C}^\times$ are defined as before.

If μ_1 and μ_2 are two quasi-characters of $\mathbb{C}^\times$ we can introduce the space $\mathcal{B}(\mu_1,\mu_2)$ and the representation $\rho(\mu_1,\mu_2)$ of $\mathcal{H}_{\mathbb{C}}$ or of $\mathfrak{A}$ on $\mathcal{B}(\mu_1,\mu_2)$. In order to study this representation we identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_\ell(2,\mathbb{C}) \oplus \mathfrak{g}_\ell(2,\mathbb{C})$ in such a way that $\mathfrak{g}$ corresponds to the elements of $X \oplus \bar{X}$. If $\mathfrak{A}_1$ is the universal enveloping algebra of $\mathfrak{g}_\ell(2,\mathbb{C})$ we may then identify $\mathfrak{A}$ with $\mathfrak{A}_1 \otimes \mathfrak{A}_1$.

In the previous paragraph we introduced the elements D and J of $\mathfrak{A}_1$. Set $D_1 = D \otimes 1$, $D_2 = 1 \otimes D$, $J_1 = J \otimes 1$, and $J_2 = 1 \otimes J$. These four elements lie in the centre of $\mathfrak{A}$. A representation of $\mathfrak{A}$ is admissible if its restriction to the Lie algebra of the group $SU(2,\mathbb{C})$ of unitary matrices of determinant one decomposes into the direct sum of irreducible finite dimensional representations each occurring with finite multiplicity.

The first part of the next lemma is verified by calculations like those used in the proof of Lemma 5.6. The second is a consequence of the Frobenius reciprocity law applied to the pair $SU(2,\mathbb{C})$ and its subgroup of diagonal matrices.

Lemma 6.1. *Let*

$$\mu_1(z) = (z\bar{z})^{s_i - \frac{1}{2}(a_i+b_i)} z^{a_i} \bar{z}^{b_i}$$

and

$$\mu_1\mu_2^{-1}(z) = \mu(z) = (z\bar{z})^{s-\frac{1}{2}(a+b)} z^a \bar{z}^b$$

where a_i, b_i, a, and b are non-negative integers and $a_ib_i = ab = 0$.

(i) *On $\mathcal{B}(\mu_1,\mu_2)$ we have the following four relations*

$$\rho(D_1) = \frac{1}{2}\left\{\left(s + \frac{a-b}{2}\right)^2 - 1\right\} I$$

$$\rho(D_2) = \frac{1}{2}\left\{\left(s + \frac{b-a}{2}\right)^2 - 1\right\} I$$

$$\rho(J_1) = \left\{(s_1+s_2) + \frac{a_1 - b_1 + a_2 - b_2}{2}\right\} I$$

$$\rho(J_2) = \left\{(s_1+s_2) + \frac{b_1 - a_1 + b_2 - a_2}{2}\right\} I$$

(ii) $\rho(\mu_1,\mu_2)$ *is admissible and contains the representation* ρ_n *of the Lie algebra of* $SU(2,\mathbb{C})$ *if and only if* $n \geq a+b$ *and* $n \equiv a+b \pmod 2$ *and then it contains it just once.*

ρ_n is the unique irreducible representation of $SU(2,\mathbb{C})$ of degree $n+1$. Let $\mathcal{B}(\mu_1,\mu_2,\rho_n)$ be the space of functions in $\mathcal{B}(\mu_1,\mu_2)$ transforming according to ρ_n.

Theorem 6.2. (i) *If* μ *is not of the form* $z \to z^p\bar{z}^q$ *or* $z \to z^{-p}\bar{z}^{-q}$ *with* $p \geq 1$ *and* $q \geq 1$ *then* $\rho(\mu_2,\mu_2)$ *is irreducible. A representation equivalent to* $\rho(\mu_1,\mu_2)$ *will be denoted by* $\pi(\mu_1,\mu_2)$*,*

(ii) *If* $\mu(z) = z^p\bar{z}^q$ *with* $p \geq 1$, $q \geq 1$ *then*

$$\mathcal{B}_s(\mu_1,\mu_2) = \sum_{\substack{n\geq p+q \\ n\equiv p+q \pmod 2}} \mathcal{B}(\mu_1,\mu_2,\rho_n)$$

is the only proper invariant subspace of $\mathcal{B}(\mu_1,\mu_2)$. $\sigma(\mu_1,\mu_2)$ *will be any representation equivalent to the representation on* $\mathcal{B}_s(\mu_1,\mu_2)$ *and* $\pi(\mu_1,\mu_2)$ *will be any representation equivalent to the representation on the quotient space*

$$\mathcal{B}_f(\mu_1,\mu_2) = \mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_s(\mu_1,\mu_2)$$

(iii) *If* $\mu(z) = z^{-p}\bar{z}^{-q}$ *with* $p \geq 1$, $q \geq 1$ *then*

$$\mathcal{B}_f(\mu_1,\mu_2) = \sum_{\substack{|p-q|\leq n<p+q \\ n\equiv p+q \pmod 2}} \mathcal{B}(\mu_1,\mu_2,\rho_n)$$

is the only proper invariant subspace of $\mathcal{B}(\mu_1,\mu_2)$. $\pi(\mu_1,\mu_2)$ *will be any representation equivalent to the representation* $\mathcal{B}_f(\mu_1,\mu_2)$ *and* $\sigma(\mu_1,\mu_2)$ *will be any representation equivalent to the representation on the quotient space*

$$\mathcal{B}_s(\mu_1,\mu_2) = \mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_f(\mu_1,\mu_2).$$

(iv) $\pi(\mu_1,\mu_2)$ *is equivalent to* $\pi(\mu_1',\mu_2')$ *if and only if* $(\mu_1,\mu_2) = (\mu_1',\mu_2')$ *or* $(\mu_1,\mu_2) = (\mu_2',\mu_1')$.

(v) *If* $\sigma(\mu_1,\mu_2)$ *and* $\sigma(\mu_1',\mu_2')$ *are defined they are equivalent if and only if* $(\mu_1,\mu_2) = (\mu_1',\mu_2')$ *or* $(\mu_1,\mu_2) = (\mu_2',\mu_1')$.

(vi) *If* $\mu(z) = z^p\bar{z}^q$ *with* $p \geq 1$, $q \geq 1$ *there is a pair of characters* ν_1,ν_2 *such that* $\mu_1\mu_2 = \nu_1\nu_2$ *and* $\nu_1\nu_2^{-1} = z^p\bar{z}^{-q}$ *and* $\sigma(\mu_1,\mu_2)$ *is equivalent to* $\pi(\nu_1,\nu_2)$.

(vii) *Every irreducible admissible representation of* $\mathcal{H}_{\mathbb{C}}$ *or* $\mathfrak{A}$ *is a* $\pi(\mu_1,\mu_2)$ *for some choice of* μ_1 *and* μ_2.

The proofs of the first three assertions will be based on two lemmas.

Lemma 6.2.1. *If there exists a proper invariant subspace* V *of* $\mathcal{B}(\mu_1,\mu_2)$ *which is finite dimensional then* $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$ *with* $p \geq 1$, $q \geq 1$ *and* $V = \mathcal{B}_f(\mu_1,\mu_2)$.

Lemma 6.2.2. *Let* V *be a proper invariant subspace of* $\mathcal{B}(\mu_1,\mu_2)$ *and let* n_0 *be the smallest integer such that some subspace of* V *transforms according to the representation* ρ_{n_0} *of the Lie algebra of* $SU(2,\mathbb{C})$. *Either*

$$V = \sum_{n\geq n_0} \mathcal{B}(\mu_1,\mu_2,\rho_n)$$

or V contains a finite-dimensional invariant subspace.

Grant these lemmas for a moment and let V be a proper invariant subspae of $\mathcal{B}(\mu_1,\mu_2)$. As in the case of the non-archimedean and real fields there is an invariant non-degenerate bilinear form on $\mathcal{B}(\mu_1,\mu_2)\times\mathcal{B}(\mu_1^{-1},\mu_2^{-1})$. The orthogonal complement $V^\perp$ of V in $\mathcal{B}(\mu_1^{-1},\mu_2^{-1})$ is a proper invariant subspace. By Lemma 6.2.1 they cannot both contain an invariant finite dimensional subspace. Therefore by Lemma 6.2.2 one of them is of finite codimension. The other must be of finite dimension. If V is finite-dimensional then $\mu_1\mu_2^{-1}(z)=z^{-p}\bar z^{-q}$ and $V=\mathcal{B}_f(\mu_1,\mu_2)$. If $V^\perp$ is finite dimensional then $\mu_1\mu_2^{-1}(z)=z^p\bar z^q$. Since the orthogonal complement of $\mathcal{B}_f(\mu_1,\mu_2)$ is $\mathcal{B}_s(\mu_1,\mu_2)$ we must have $V=\mathcal{B}_s(\mu_1,\mu_2)$.

We shall now show that $\mathcal{B}_f(\mu_1,\mu_2)$ is invariant when $\mu_1\mu_2^{-1}(z)=z^{-p}\bar z^{-1}$. It will follow from duality that $\mathcal{B}_s(\mu_1,\mu_2)$ is invariant when $\mu_1\mu_2^{-1}(z)=z^p\bar z^q$. Every irreducible finite-dimensional representation π of $\mathfrak{A}$ determines a representation π of $G_{\mathbb{C}}$. If π acts on X there is a nonzero vector v_0 in X such that

$$\pi\left(\begin{pmatrix} z & x\\ 0 & z^{-1}\end{pmatrix}\right)v_0=z^m\bar z^n v_0$$

for all z in $\mathbb{C}^\times$ and all x in $\mathbb{C}$. v_0 is determined up to a scalar factor and m and n are non-negative integers. Moreover there is a quasi-character ω_0 of $\mathbb{C}^\times$ such that

$$\pi\left(\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\right)=\omega_0(a)I$$

Thus

$$\pi\left(\begin{pmatrix} z_1 & x\\ 0 & z_2\end{pmatrix}\right)v_0=\omega_1(z_1)\,\omega_2(z_2)\,v_0$$

where $\omega_1\omega_2^{-1}(z)=z^m\bar z^n$. π is determined up to equivalence by ω_1 and ω_2 so we write $\pi=\kappa(\omega_1,\omega_2)$. As long as $\omega_1\omega_2^{-1}(z)=z^m\bar z^n$ with non-negative integers m and n the representation $\kappa(\omega_1,\omega_2)$ exists. By the Clebsch–Gordan formula the restriction of $\kappa(\omega_1,\omega_2)$ or its contragredient to $SU(2,\mathbb{C})$ breaks up into the direct sum of the representations ρ_i with $|m-n|\le i\le m+n$ and $1\equiv m+n\ (\mathrm{mod}\,2)$. Let π be $\kappa(\omega_1,\omega_2)$ and let $\tilde\pi$, the contragredient representation, act on $\widetilde X$. To each vector $\widetilde v$ in $\widetilde X$ we associate the function

$$\varphi(g)=\langle v_0,\tilde\pi(g)\widetilde v\rangle$$

on $G_{\mathbb{C}}$. The map $\widetilde v\to\varphi$ is linear and injective. Moreover $\tilde\pi(g)\widetilde v\to\rho(g)\varphi$ while

$$\varphi\left(\begin{pmatrix} z_1 & x\\ 0 & z_2\end{pmatrix}g\right)=\omega_1^{-1}(z_1)\,\omega_2^{-1}(z_2)\,\varphi(g)$$

so that if $\mu_1=\omega_1^{-1}\,\alpha_{\mathbb{C}}^{-1/2}$ and $\mu_2=\omega_2^{-1}\,\alpha_{\mathbb{C}}^{1/2}$ the function φ belongs to $\mathcal{B}(\mu_1,\mu_2)$. As we vary ω_1 and ω_2 the quasi-characters μ_1 and μ_2 vary over all pairs such that $\mu_1\mu_2^{-1}(z)=z^{-p}\bar z^{-q}$ with $p\ge 1$ and $q\ge 1$.

We have still to prove the two lemmas. Suppose V is a proper finite-dimensional subspace of $\mathcal{B}(\mu_1,\mu_2)$. The representation of $\mathfrak{A}$ on V is certainly a direct sum of irreducible representations each occurring with multiplicity one. Let V' be an irreducible subspace of V and let $\widetilde V'$ be the dual space of V'. Let λ be the linear functional $\lambda:\varphi\to\varphi(e)$ on V'. If π is the representation of $\mathfrak{A}$ or of $G_{\mathbb{C}}$ on V' then

$$\tilde\pi\left(\begin{pmatrix} z_1 & x\\ 0 & z_2\end{pmatrix}\right)\lambda=\mu_1^{-1}(z_1)\,\mu_2^{-1}(z_2)\,(z_1\bar z_1 z_2^{-1}z_2^{-1})^{-\frac12}\,\lambda$$

Thus if $\omega_1 = \mu_1^{-1}\,\alpha_{\mathbb{C}}^{-1/2}$ and $\omega_2 = \mu_2^{-1}\,\alpha_{\mathbb{C}}^{1/2}$ the representation $\tilde{\pi}$ is $\kappa(\omega_1, \omega_2)$. It follows immediately that $\mu_1\mu_2^{-1}$ is of the form $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$ with $p \geq 1$ and $q \geq 1$ and that V' and therefore V is $\mathcal{B}_f(\mu_1, \mu_2)$.

To prove the second lemma we regard $\mathfrak{g}$ as the real Lie algebra of 2×2 complex matrices. Then

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

is the centre of $\mathfrak{g}$ and

$$\mathfrak{u} = \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \mid a \in \mathbb{R},\ b \in \mathbb{C} \right\}$$

is the Lie algebra of $SU(2, \mathbb{C})$. If

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix} \mid a \in \mathbb{R},\ b \in \mathbb{C} \right\}$$

then $\mathfrak{u} \oplus \mathfrak{g}$ is the Cartan decomposition of the Lie algebra of the special linear group. The space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is invariant under the adjoint action of $\mathfrak{u}$ on $\mathfrak{g}_{\mathbb{C}}$. Moreover $\mathfrak{u}$ acts on $\mathfrak{g}_{\mathbb{C}}$ according to the representation ρ_2. One knows that $\rho_2 \otimes \rho_n$ is equivalent to $\rho_{n+2} \oplus \rho_n \oplus \rho_{n-2}$ if $n \geq 2$, that $\rho_2 \otimes \rho_1$ is equivalent to $\rho_3 \oplus \rho_1$ and, of course, that $\rho_2 \otimes \rho_0$ is equivalent to ρ_2. The map of $\mathfrak{g}_{\mathbb{C}} \otimes \mathcal{B}(\mu_1, \mu_2, \rho_n)$ into $\mathcal{B}(\mu_1, \mu_2)$ which sends $X \otimes f$ to $\rho(X)f$ commutes with the action of $\mathfrak{u}$. Thus $\rho(X)f$ is contained in

$$\mathcal{B}(\mu_1, \mu_2, \rho_{n+2}) \oplus \mathcal{B}(\mu_1, \mu_2, \rho_n) \oplus \mathcal{B}(\mu_1, \mu_2, \rho_{n-2}).$$

It is understood that $\mathcal{B}(\mu_1, \mu_2, \rho_\ell) = 0$ if $\ell < 0$.

Now let V be a proper invariant subspace of $\mathcal{B}(\mu_1, \mu_2)$. Let n_0 be the smallest non-negative integer n for which V contains $\mathcal{B}(\mu_1, \mu_2, \rho_n)$. If $n \geq n_0$ set

$$V(n) = \sum_{\substack{n \geq k \geq n_0 \\ k \equiv n_0 \pmod 2}} \mathcal{B}(\mu_1, \mu_2, \rho_k)$$

If V contains every $V(n)$ there is nothing to prove so assume that there is a largest integer n_1 for which V contains $V(n_1)$. All we need do is show that $V(n_1)$ is invariant under $\mathfrak{g}$. It is invariant under $\mathfrak{a}$ and $\mathfrak{u}$ by construction so we need only verify that if X lies in $\mathfrak{g}_{\mathbb{C}}$ then $\rho(X)$ takes $V(n_1)$ into itself. It is clear that $\rho(X)$ takes $V(n_1 - 2)$ into $V(n_1)$ so we have only to show that it takes $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1})$ into $V(n_1)$. Take f in $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1})$ and let $\rho(X)f = f_1 + f_2$ with f_1 in $V(n_1)$ and f_2 in $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$. Certainly f_2 lies in V. Since

$$V \cap \mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$$

is either 0 or $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$ and since, by construction, it is not $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$ the function f_2 is 0.

The first three assertions of the theorem are now proved and we consider the remaining ones. We make use of the fact that D_1, D_2, J_1 and J_2 generate the centre of $\mathfrak{A}$ as well as a result of Harish-Chandra to be quoted later. Suppose π and π' are two irreducible representations of $\mathfrak{A}$ which are constituents of $\rho(\mu_1, \mu_2)$ and $\rho(\mu_1', \mu_2')$ respectively. Assume π and π' contain the same representations of the Lie algebra of $SU(2, \mathbb{C})$ and are associated to the same homomorphism of the centre of $\mathfrak{A}$ into $\mathbb{C}$. Comparing the scalars $\pi(J_1)$ and $\pi(J_1)$ with $\pi'(J_1)$ and $\pi'(J_2)$ we find that $\mu_1\mu_2 = \mu_1'\mu_2'$. Let

$\mu_1\mu_2^{-1}(z) = (z\bar{z})^{s-\frac{a+b}{2}} z^a \bar{z}^b$ and let $\mu_1'\mu_2'^{-1}(z) = (z\bar{z})^{s'-\frac{a'+b'}{2}} z^{a'} \bar{z}^{b'}$. Comparing $\pi(D_1)$ and $\pi(\nu_2)$ with $\pi'(D_1)$ and $\pi'(D_2)$ we see that

$$\left(s + \frac{a-b}{2}\right)^2 = \left(s' + \frac{a'-b'}{2}\right)^2$$

and

$$\left(s + \frac{b-a}{2}\right)^2 = \left(s' + \frac{b'-a'}{2}\right)^2.$$

These relations will hold if $\mu_1\mu_2^{-1} = \mu_1'\mu_2'^{-1}$ or $\mu_1^{-1}\mu_2 = \mu_1'\mu_2'^{-1}$ and therefore, when $\mu_1\mu_2 = \mu_1'\mu_2'$, $(\mu_1, \mu_2) = (\mu_1', \mu_2')$ or $(\mu_1, \mu_2) = (\mu_2', \mu_1')$. If neither of these alternatives hold we must have

$$s = \frac{a'-b'}{2}, \qquad s' = \frac{a-b}{2},$$

or

$$s = \frac{b'-a'}{2}, \qquad s' = \frac{b-a}{2}.$$

Since $\mu_1\mu_2 = \mu_1'\mu_2'$ the integers $a+b$ and $a'+b'$ must have the same parity. Let $\mu = \mu_1\mu_2^{-1}$ and $\mu' = \mu_1'\mu_2'^{-1}$. In the first case $\mu\mu'$ is of the form $\mu\mu'(z) = z^{2p}$ and $\mu\mu'^{-1}$ is of the form $\bar{z}^{2q}$ and in the second $\mu\mu'(z) = \bar{z}^{2p}$ while $\mu\mu'^{-1}(z) = z^{2q}$. Since $\{\mu_1, \mu_2\}$ is not $\{\mu_1', \mu_2'\}$ neither p nor q is 0. In the first case $\mu = z^p\bar{z}^q$ and $\mu' = z^p\bar{z}^{-q}$ and in the second $\mu = z^q\bar{z}^p$ while $\mu' = z^{-q}\bar{z}^p$.

In conclusion we see that π and π' contain the same representations of the Lie algebra of $SU(2,\mathbb{C})$ and are associated to the same homomorphism of the centre of $\mathfrak{A}$ into $\mathbb{C}$ if and only if one of the following alternatives holds.

(i) For some pair of quasi-characters ν_1 and ν_2 we have $\{\pi, \pi'\} = \{\pi(\nu_1, \nu_2), \pi(\nu_1, \nu_2)\}$ or $\{\pi, \pi'\} = \{\pi(\nu_1, \nu_2), \pi(\nu_2, \nu_1)\}$.

(ii) For some pair of quasi-characters ν_1 and ν_2 we have $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \sigma(\nu_1, \nu_2)\}$ or $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \sigma(\nu_2, \nu_1)\}$.

(iii) For some pair of quasi-characters ν_1 and ν_2 with $\nu_1\nu_2^{-1}(z) = z^p\bar{z}^q$ where $p \geq 1, q \geq 1$ we have $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \pi(\nu_1', \nu_2')\}$ where $\nu_1\nu_2 = \nu_1'\nu_2'$ and $\nu'\nu_2'^{-1}(z)$ is either $z^p\bar{z}^{-1}$ or $z^{-p}\bar{z}^q$.

(iv) For some pair of quasi-characters ν_1 and ν_2 with $\nu_1\nu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$ where $p \geq 1, q \geq 1$ we have $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \pi(\nu_1', \nu_2')\}$ where $\nu_1\nu_2 = \nu_1'\nu_2'$ and $\nu_1'\nu_2'^{-1}(z)$ is either $z^p\bar{z}^{-q}$ or $z^{-p}\bar{z}^q$.

The remaining assertions are now all consequences of a theorem of Harish-Chandra which, in the special case of interest to us, we may state in the following manner.

Lemma 6.2.3. *If π is an irreducible admissible representation of $\mathfrak{A}$ there exists a pair of quasi-characters μ_1 and μ_2 such that $\rho(\mu_1, \mu_2)$ and π contain at least one irreducible representation of the Lie algebra of $SU(2,\mathbb{C})$ in common and are associated to the same homomorphism of the centre of $\mathfrak{A}$ into $\mathbb{C}$. When this is so π is a constituent of $\rho(\mu_1, \mu_2)$.*

As before $\chi \otimes \pi(\mu_1, \mu_2)$ is $\pi(\chi\mu_1, \chi\mu_2)$ and $\chi \otimes \sigma(\mu_1, \mu_2)$ is $\sigma(\chi\mu_1, \chi\mu_2)$. If

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)I$$

then $\tilde{\pi} = \omega_0^{-1} \otimes \pi$.

Theorem 6.3. *Let π be an infinite dimensional irreducible admissible representation of $\mathcal{H}_{\mathbb{C}}$ and let ψ be a non-trivial additive character of $\mathbb{C}$. There is exactly one space $W(\pi,\psi)$ of functions on $G_{\mathbb{C}}$ which satisfies the following three conditions.*

(i) *Every function W in $W(\pi,\psi)$ satisifies*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\, W(g).$$

(ii) *The functions in $W(\pi,\psi)$ are continuous and $W(\pi,\psi)$ is invariant under the operators $\rho(f)$ for f in $\mathcal{H}_{\mathbb{C}}$. Moreover the representation of $\mathcal{H}_{\mathbb{C}}$ on $W(\pi,\psi)$ is equivalent to π.*

(iii) *If W is in $W(\pi,\psi)$ there is a positive number N such that*

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|t|^N)$$

as $|t| \to \infty$.

Since every π is of the form $\pi = \pi(\mu_1,\mu_2)$ the existence is rather easy to prove. If Φ is in $\mathcal{S}(\mathbb{C}^2)$ let

$$\theta(\mu_1,\mu_2,\Phi) = \int_{\mathbb{C}^\times} \Phi(t,t^{-1})\, \mu_1(t)\, \mu_2^{-1}(t)\, d^\times t$$

We let $W(\mu_1,\mu_2,\psi)$ be the space of functions on $G_{\mathbb{C}}$ of the form

$$W(g) = W_\Phi(g) = \mu_1(\det g)\, |\det g|_{\mathbb{C}}^{1/2}\, \theta(\mu_1,\mu_2,r(g)\Phi)$$

where Φ in $\mathcal{S}(\mathbb{C}^2)$ is $SU(2,\mathbb{C})$-finite under the action defined by r. It is clear that $W(\mu_1,\mu_2,\psi) = W(\mu_2,\mu_1,\psi)$ and that $W(\mu_1,\mu_2,\psi)$ is invariant under right translations by elements of $\mathcal{H}_{\mathbb{C}}$ and of $\mathfrak{A}$.

The existence of $W(\pi,\psi)$ will, as before, be a consequence of the following analogue of Lemma 5.13.1.

Lemma 6.3.1 *Suppose $\mu_1\mu_2^{-1}(t) = (t\bar{t})^{s-\frac{a+b}{2}} t^a \bar{t}^b$ with $\operatorname{Re} s > -1$. Then there is a bijection A of $W(\mu_1,\mu_2,\psi)$ with $\mathcal{B}(\mu_1,\mu_2)$ which commutes with the action of $\mathcal{H}_{\mathbb{C}}$.*

As before A associates to W_Φ the function

$$f_{\Phi^\sim}(g) = \mu_1(\det g)\, |\det g|_{\mathbb{C}}^{1/2}\, z\big(\mu_1\mu_2^{-1}\alpha_{\mathbb{C}}, \rho(g)\Phi^\sim\big)$$

The proof of course proceeds as before. However we should check that A is surjective. Theorem 6.2 shows that, under the present circumstances, there is no proper invariant subspace of $\mathcal{B}(\mu_1,\mu_2)$ containing $\mathcal{B}(\mu_1,\mu_2,\rho_{a+b})$ so that we need only show that at least one nonzero function in $\mathcal{B}(\mu_1,\mu_2,\rho_{a+b})$ is of the form f_Φ where Φ is in $\mathcal{S}(\mathbb{C}^2)$ and $SU(2,\mathbb{C})$-finite under right translations.

If

$$\Phi(x,y) = e^{-2\pi(x\bar{x}+y\bar{y})}\bar{y}^a y^b$$

then, since $a+b=0$, Φ transforms under right translations by $SU(2,\mathbb{C})$ according to ρ_{a+b} so we need only check that f_Φ is not 0. Proceeding according to the definition we see that

$$\begin{aligned} f_\Phi(e) &= \int_{\mathbb{C}^\times} \Phi(0,t)(t\bar{t})^{s-\frac{a+b}{2}} t^a \bar{t}^b\, d^\times t \\ &= \int_{\mathbb{C}^\times} e^{-2\pi t\bar{t}}(t\bar{t})^{1+s+\frac{a+b}{2}}\, d^\times t. \end{aligned}$$

Apart from a constant which depends on the choice of Haar measure this is

$$(2\pi)^{-s-\frac{a+b}{2}}\Gamma\Big(1+s+\frac{a+b}{2}\Big)$$

and is thus not 0.

Just as in the previous paragraph $W(\mu_1,\mu_2,\psi)$ is spanned by functions W_Φ where Φ is of the form

$$\Phi(x,y)=e^{-2\pi(x\bar{x}+u\bar{u}y\bar{y})}x^p\bar{x}^q y^m\bar{y}^n$$

where p, q, m, and n are integers. u is determined by the relation $\psi(z)=e^{4\pi i\operatorname{Re}uz}$. We can show that

$$W_\Phi\left(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}\right)$$

decreases exponentially as $|t|\to\infty$.

To prove the uniqueness we will use a differential equation as in the previous chapter. This time the equations are a little more complicated. Suppose $W_1(\pi,\psi)$ is a space of functions satisfying the first two conditions of the theorem. We regard ρ_n as acting on the space V_n of binary forms of degree n according to the rule

$$\rho_n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)\varphi(x,y)=\varphi(ax+cy,bx+dy)$$

If

$$\varphi(x,y)=\sum_{\substack{|k|\le n\\ \frac{n}{2}-k\in\mathbb{Z}}}\varphi^k x^{\frac{n}{2}+k}x^{\frac{n}{2}-k}$$

then φ^k is called the k^{th} coordinate of φ. On the dual space $\widetilde{V}_n$ we introduce the dual coordinates.

If ρ_n is contained in π there is an injection A of V_n into $W_1(\pi,\psi)$ which commutes with the action of $SU(2,\mathbb{C})$. Let $\Phi(g)$ be the function on $G_{\mathbb{C}}$ with values in $\widetilde{V}_n$ defined by

$$\langle\varphi,\Phi(g)\rangle=A\varphi(g).$$

It is clear that $W_1(\pi,\psi)$ is determined by Φ which is in turn determined by $W_1(\pi,\psi)$ up to a scalar factor. The function $\Phi(g)$ is determined by the function

$$\varphi(t)=\Phi\left(\begin{pmatrix} t^{\frac{1}{2}} & 0\\ 0 & t^{-\frac{1}{2}}\end{pmatrix}\right)$$

on the positive real numbers. If $\varphi^k(t)$ is the k^{th} coordinate of $\varphi(t)$ and if π is a constituent of $\rho(\mu_1,\mu_2)$ the differential equations

$$\rho(D_1)\Phi=\frac{1}{2}\Big\{\Big(s+\frac{a-b}{2}\Big)^2-1\Big\}\Phi$$
$$\rho(D_2)\Phi=\frac{1}{2}\Big\{\Big(s+\frac{b-a}{2}\Big)^2-1\Big\}\Phi$$

may, if our calculations are correct, be written as

$$\frac{1}{2}\Big[t\frac{d}{dt}+k-1\Big]^2\varphi^k-t^2\frac{|u|^2}{2}\varphi^k+\Big(\frac{n}{2}+k\Big)tiu\varphi^{k-1}=\frac{1}{2}\Big(s+\frac{a-b}{2}\Big)^2\varphi^k$$
$$\frac{1}{2}\Big[t\frac{d}{dt}-k-1\Big]^2\varphi^k-t^2\frac{|u|^2}{2}\varphi^k-\Big(\frac{n}{2}-k\Big)ti\bar{u}\varphi^{k+1}=\frac{1}{2}\Big(s+\frac{b-a}{2}\Big)^2\varphi^k.$$

We have set $\varphi^k = 0$ if $|k| \geq n/2$. Recall that $\psi(z) = e^{4\pi i \operatorname{Re} uz}$. These equations allow one to solve for all φ^k in terms of $\varphi^{n/2}$ or $\varphi^{-n/2}$.

For $k = \frac{n}{2}$ the second equation may be written as

$$\frac{1}{2}\frac{d^2\varphi^{n/2}}{dt^2} + \left(-\frac{1}{2} - \frac{n}{2}\right)\frac{1}{t}\frac{d\varphi}{dt}^{n/2} + \left\{-\frac{|u|^2}{2} + \frac{(\frac{n}{2}+1)^2}{2t^2}\right\}\varphi^{n/2} = \left(s - \frac{b-a}{2}\right)^2\varphi^{n/2}. \qquad (*)$$

If we have two independent solutions of this equation their Wronskian $W(t)$ is a non-trivial solution of the equation

$$\frac{dW}{dt} = \frac{(n+1)}{t}W$$

and therefore a non-zero multiple of t^{n+1}. Since we already have shown the existence of a solution of $(*)$ which decreases exponentially we see that there cannot be another solution which is bounded by a power of t as $t \to \infty$. The uniqueness of the space $W(\pi, \psi)$ follows

Every irreducible admissible representation of $\mathcal{H}_{\mathbb{C}}$ is of the form $\pi = \pi(\mu_1, \mu_2)$. Moreover $\pi(\mu_1, \mu_2) = \pi(\mu_1', \mu_2')$ if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$. Thus we may set

$$L(s, \pi) = L(s, \mu_1)\, L(s, \mu_2)$$

and

$$\epsilon(s, \pi, \psi) = \epsilon(s, \mu_1, \psi)\, \epsilon(s, \mu_2, \psi).$$

Then

$$L(s, \tilde{\pi}) = L(s, \mu_1^{-1})\, L(s, \mu_2^{-1}).$$

The local functional equation which is proved just as in the real case read as follows.

Theorem 6.4. *Let π be an infinite-dimensional irreducible admissible representation of $\mathcal{H}_{\mathbb{C}}$. Let ω be the quasi-character of $\mathbb{C}^\times$ defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a) I$$

for a in $\mathbb{C}^\times$. If W is in $W(\pi, \psi)$ the integrals

$$\Psi(g, s, W) = \int_{\mathbb{C}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|_{\mathbb{C}}^{s-1/2}\, d^\times a,$$

$$\widetilde{\Psi}(g, s, W) = \int_{\mathbb{C}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|_{\mathbb{C}}^{s-1/2} \omega^{-1}(a)\, d^\times a$$

converge absolutely in some right half-plane. Set

$$\Psi(g, s, W) = L(s, \pi)\, \Phi(g, s, W),$$

$$\widetilde{\Psi}(g, s, W) = L(s, \tilde{\pi})\, \widetilde{\Phi}(g, s, W).$$

The functions $\Phi(g, s, W)$ and $\widetilde{\Phi}(g, s, W)$ can be analytically continued to the whole complex plane as holomorphic functions of s. For a suitable choice of W the function $\Phi(e, s, W)$ is an exponential function of s. The functional equation

$$\widetilde{\Phi}(wg, 1-s, W) = \epsilon(s, \pi, \psi)\, \Phi(g, s, W)$$

is satisfied. Moreover, if W is fixed $|\Psi(g, s, W)|$ remains bounded as g varies over a compact subset of $G_{\mathbb{C}}$ and s varies in a vertical strip of finite width from which discs about the poles of $L(s, \pi)$ have been removed.

The following lemma can be verified by an explicit computation. The first assertion may also be proved by the method of Lemma 5.16.

Lemma 6.5 *If $\sigma = \sigma(\mu_1, \mu_2)$ and $\pi = \pi(\mu_1, \mu_2)$ are defined then*

$$\frac{L(1-s, \widetilde{\sigma})\,\epsilon(s, \sigma, \psi)}{L(s, \sigma)} = \frac{L(1-s, \tilde{\pi})\,\epsilon(s, \pi, \psi)}{L(s, \pi)}$$

and the quotient

$$\frac{L(s, \chi \otimes \sigma)}{L(s, \chi \otimes \pi)}$$

is the product of a constant, a polynomial, and an exponential. Moreover the polynomial is of positive degree for some choice of the quasi-character χ.

We verify the last assertion. There is no harm in supposing that $\sigma = \pi(\nu_1, \nu_2)$ and that $\chi\mu_1(z) = z^{a+p}\bar{z}^{b+q}$, $\chi\mu_2(z) = z^a\bar{z}^b$, $\chi\nu_1(z) = z^{a+p}\bar{z}^b$, and $\chi\nu_2(z) = z^a z^{b+q}$. $p \geq 1$ and $q \geq 1$ are integers. Varying χ is equivalent to varying a and b through all the integers. If m_1 is the largest of $a+p$ and $b+q$ and m_2 is the largest of a and b while n_1 is the largest of $a+p$ and b and n_2 is the largest of a and $b+q$ the quotient

$$\frac{L(s, \chi \otimes \sigma)}{L(s, \chi \otimes \pi)}$$

differs from

$$\frac{\Gamma(s+n_1)\,\Gamma(s+n_2)}{\Gamma(s+m_1)\,\Gamma(s+m_2)}$$

by a constant times an exponential. It is clear that n_1 and n_2 are both greater than or equal to m_2 and that either n_1 or n_2 is greater than or equal to m_1. Thus the quotient is a polynomial. If $p \geq q$ choose a and b so that $b+q > a \geq b$. Then $n_1 = m+1$ and $n_2 > m_2$ so that the quotient is of positive degree. If $q \geq p$ choose a and b so that $a+p > b \geq a$. Then $n_2 = m_1$ and $n_1 > m_2$.

Lemma 6.6. *Let π and π' be two infinite-dimensional irreducible representations of $\mathcal{H}_{\mathbb{C}}$. Suppose there is a quasi-character ω of $\mathbb{C}^\times$ such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

and

$$\pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for all a in $\mathbb{C}^\times$. If

$$\epsilon(s, \chi \otimes \pi', \psi)\frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}')}{L(s, \chi \otimes \pi')} = \epsilon(s, \chi \otimes \pi, \psi)\frac{L(1-s, \chi^{-1} \otimes \tilde{\pi})}{L(s, \chi \otimes \pi)}$$

for all quasi-characters χ then π and π' are equivalent.

Let $\pi = \pi(\mu_1, \mu_2)$ and let $\pi' = (\mu_1', \mu_2')$. We let

$$\mu_i(z) = (z\bar{z})^{s_i}\left\{\frac{z}{(z\bar{z})^{1/2}}\right\}^{a_i}$$

and

$$\mu_i'(z) = (z\bar{z})^{s_i'} \left\{ \frac{z}{(z\bar{z})^{1/2}} \right\}^{a_i}$$

with a_i and a_i' in $\mathbb{Z}$. By assumptions $s_1 + s_2 = s_1' + s_2'$ and $a_1 + a_2 = a_1' + a_2'$. Choose

$$\chi(z) = \left\{ \frac{z}{(z\bar{z})^{1/2}} \right\}^{n}$$

with n in $\mathbb{Z}$. The quotient on the right has the same zeros and poles as

$$\frac{\Gamma(1 - s - s_1 + |\frac{n+a_1}{2}|)}{\Gamma(s + s_1 + |\frac{n+a_1}{2}|)} \cdot \frac{\Gamma(1 - s - s_2 + |\frac{n+a_2}{2}|)}{\Gamma(s + s_2 + |\frac{n+a_2}{2}|)}.$$

A pole of the numerator can cancel a pole of the denominator if and only if there are two non-negative integers ℓ and m such that

$$s_1 - s_2 = 1 + \ell + m + |\frac{n + a_1}{2}| + |\frac{n + a_2}{2}|$$

or

$$s_2 - s_1 = 1 + \ell + m + |\frac{n + a_1}{2}| + |\frac{n + a_2}{2}|.$$

This can happen only if $\mu_1\mu_2^{-1}$ is of the form $\mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$ or $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$ where $p \geq 1$ and $q \geq 1$ are integers. Since $\pi(\mu_1, \mu_2)$ is infinite-dimensional it cannot be of either these forms and no poles cancel.

Consequently for every integer n, $\{s_1 + |\frac{n+a_1}{2}|, s_2 + |\frac{n+a_2}{2}|\} = \{s_1' + |\frac{n+a_1'}{2}|, s_2' + |\frac{n+a_2'}{2}|\}$. This can happen only if $s_1 = s_1'$, $a_1 = a_1'$, $s_2 = s_2'$, and $a_2 = a_2'$ or $s_1 = s_2'$, $a_1 = a_2'$, $s_2 = s_1'$, and $a_2 = a_1'$. Thus π and π' are equivalent.

The following proposition is an easy consequence of these two lemmas.

Proposition 6.7. *Suppose π and π' are two irreducible admissible representations of $\mathcal{H}_{\mathbb{C}}$. Suppose there is a quasi-character ω of $\mathbb{C}^*$ such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

and

$$\pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I.$$

If $L(s, \chi \otimes \pi) = L(s, \chi \otimes \pi')$, $L(s, \chi^{-1} \otimes \tilde{\pi}) = L(s, \chi^{-1} \otimes \tilde{\pi}')$ and

$$\epsilon(s, \chi \otimes \pi, \psi) = \epsilon(s, \chi \otimes \pi', \psi)$$

for all quasi-characters χ the representations π and π' are equivalent.

§**7. Characters.** If F is a non-archimedean local field and π is an admissible representation of G_F the operator $\pi(f)$ is of finite rank for every f in $\mathcal{H}_F$ and therefore ha a trace $\operatorname{Tr}\pi(f)$. In this paragraph we prove that if π is irreducible there is a locally integrable function χ_π on G_f such that

$$\operatorname{Tr}\pi(f) = \int_{G_F} f(g)\,\chi_\pi(g)\,dg.$$

Although $\operatorname{Tr}\pi(f)$ depends on the choice of the Haar measure the function χ_π does not.

The following simple lemma shows that χ_π determines the class of π.

Lemma 7.1 *If* $\{\pi_1,\cdots,\pi_p\}$ *is a set of inequivalent irreducible admissible representations of* $\mathcal{H}_F$ *the set of linear forms* $\operatorname{Tr}\pi_1, \operatorname{Tr}\pi_2,\cdots,\operatorname{Tr}\pi_p$ *is linearly independent.*

Let π_i act on V_i and let ξ be an elementary idempotent such that none of the spaces $\pi_i(\xi)V_i$, $1\le i\le p$, are 0. Let $\bar\pi_i$ be the representation of $\xi\mathcal{H}_F\xi$ on the finite dimensional space $\pi_i(\xi)V_i = V_i(\xi)$. Suppose $\bar\pi_i$ and $\bar\pi_j$ are equivalent. Then there is an invertible linear map A from $V_i(\xi)$ to $V_j(\xi)$ which commutes with the action of $\xi\mathcal{H}_F\xi$. Choose a non-zero vector v_i in $V_i(\xi)$ and let $v_j = Av_i$. We are going to show that π_i and π_j are equivalent. It is enough to show that, for any f in $\mathcal{H}_F$, $\pi_i(f)v_i = 0$ if and only if $\pi_j(f)v_j = 0$. But $\pi_i(f)v_i = 0$ if and only if $\pi_i(\xi * h)\pi_i(f)v_i = 0$ for all h in $\mathcal{H}_F$. Since $\pi_i(\xi * h)\pi_i(f)v_i = \pi_i(\xi * h * f * \xi)v_i$ and $\xi * h * f * \xi$ is in $\xi\mathcal{H}_F\xi$ the assertion follows.

Thus the representations $\bar\pi_1,\cdots,\bar\pi_p$ are inequivalent. Using this we shall show that the linear forms $\operatorname{Tr}\bar\pi_1,\cdots,\operatorname{Tr}\bar\pi_p$ on $\xi\mathcal{H}_F\xi$ are linearly independent. The lemma will then be proved. Take h in $\xi\mathcal{H}_F\xi$. Since $\bar\pi_i$ is irreducible and finite-dimensional $\operatorname{Tr}\bar\pi_i(hf) = 0$ for all f in $\xi\mathcal{H}_F\xi$ if and only if $\bar\pi_i(h) = 0$. Suppose we had $h_1,\cdots,\mathfrak{h}_p$ in $\xi\mathcal{H}_F\xi$ so that for at least one i the operator $\bar\pi_i(h_i)$ was not 0 while

$$\sum_{i=1}^{p}\operatorname{Tr}\bar\pi_i(h_i f) = 0$$

for all f in $\xi\mathcal{H}_F\xi$. There must then be at least two integers j and k such that $\bar\pi_j(h_j)\neq 0$ and $\bar\pi_k(h_k)\neq 0$. Since $\bar\pi_j$ and $\bar\pi_k$ are not equivalent we can find an h in $\xi\mathcal{H}_F\xi$ such that $\bar\pi_j(h) = 0$ while $\bar\pi_k(h)$ is invertible. Replacing h_i by h_ih we obtain a relation of the same type in which the number of i for which $\bar\pi_i(h_i) = 0$ has been increased. By induction we see that no such relation is possible. Since $\xi\mathcal{H}_F\xi$ contains a unit the required independence follows.

For most of these notes the existence of χ_π is irrelevant. It is used only toward the end. The reader who is more interested in automorphic forms than in group representations will probably want to take the existence of χ_π for granted and, for the moment at least, skip this paragraph. To do so will cause no harm. However he will eventually have to turn back to read the first few pages in order to review the definition of the Tamagawa measure.

Choose a non-trivial additive character ψ of F. If X is an analytic manifold over F and ω is a differential form of highest degree on X we can associate to ω a measure on X which is denoted by $|\omega|_F$ or sometimes simply by ω. If $X = F$ and $\omega = dx$ is the differential of the identity application the measure $|\omega|_F = dx$ is by definition the Haar measure on F which is self-dual with respect to ψ. In general if p belongs to X and $x^1,\cdots,x^n$ are local coordinates near p so that

$$\omega = a(x^1,\cdots,x^n)\,dx^1\wedge\cdots\wedge dx^n$$

then, if f is a continuous real-valued functions with support in a small neighborhood of p,

$$\int_X f|\omega|_F = \int f(x^1,\cdots,x^b)\,|a(x^1,\cdots,x^b)|\,dx^1\cdots dx^n.$$

The absolute value $|a(x^1, \cdots, x^n)|$ is the normalized absolute value in the field F. To prove the existence of the measure ω one has to estabish the usual formula for a change of variable in a multiple integral. For this and other facts about these measures we refer to the notes of Weil [12].

If G is an algebraic group over F then G_F is an analytic space. If ω is a left-invariant form of highest degree on G_F the measure $|\omega|_F$ is a Haar measure on G_F. It is called the Tamagawa measure. It depends on ω and ψ.

If M is the algebra of 2×2 matrices over F the additive group of M is an algebraic group. If a typical element of M is

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\mu = da \wedge db \wedge dc \wedge dd$$

is an invariant form of highest degree and $|\mu| = dx$ is the additive Haar measure which is self-dual with respect to the character $\psi_M(x) = \psi_F\big(\tau(x)\big)$ if τ is the trace of x.

On the multiplicative group G of M we take the form $\omega(x) = (\det x)^{-2}\mu(x)$. The associated Haar measure is

$$|\omega(x)| = |\det x|_F^{-2}\, dx = |x|_M^{-1}\, dx.$$

An element of G is said to be regular if its eigenvalues are distinct. The centralizer in G_F of a regular element in G_F is a Cartan subgroup of G_F. Such a Cartan subgroup B_F is of course abelian. There seems to be no canonical choice for the invariant forms on B_F. However the centralizer of B_F in M_F is an algebra E of degree two over F. It is either isomorphic to the direct sum of F with itself or it is a separable quadratic of F. B_F is the multiplicative group of E. In the first paragraph we introduced a map ν from E to F. Once a form μ_E on E which is invariant for the additive group has been chosen we can set $\mu_B(x) = \nu(x)^{-1}\mu_E(x)$. μ_b is an invariant form on B_F. The associated measure is invariant under all automorphisms of E over F. We should also recall at this point that two Cartan subgroups B_F and B'_F are conjugate in G_F if and only if the corresponding algebras are isomorphic.

Once μ_E and therefore μ_B has been chosen we can introduce on $B_F \setminus G_F$ which is also an analytic manifold the form ω_B which is the quotient of ω by μ_B. Then

$$\int_{G_F} f(g)\,\omega(g) = \int_{B_F\setminus G_F} \left\{ \int_{B_F} f(bg)\,\mu_B(b) \right\} \omega_B(g).$$

The centre of the algebra of M_F is isomorphic to F and the centre Z_F of G_F is isomorphic to $F^\times$. On $F^\times$ we have the form $x^{-1}\,dx$. We take μ_Z to be the corresponding form on Z_F. μ_B^0 will be the quotient of μ_B by μ_Z and ω^0 will be the quotient of ω by μ_Z. The corresponding integration formulae are

$$\int_{B_F} f(b)\,\mu_B(b) = \int_{Z_f\setminus B_F} \left\{ \int_{Z_F} f(zb)\,\mu_Z(z) \right\} \mu_B^0(b)$$

and

$$\int_{G_F} f(G)\,\omega(g) = \int_{Z_F\setminus G_F} \left\{ \int_{Z_F} f(zg)\,\mu_Z(z) \right\} \omega^0(g).$$

If g belongs to G_F its eigenvalues α_1 and α_2 are the roots of the equation

$$X^2 - \tau(g)X + \nu(g) = 0$$

and

$$\frac{(\alpha_1 - \alpha_2)^2}{\alpha_1\alpha_2} = \frac{\{\tau(g)\}^2 - 4\nu(g)}{\nu(g)}$$

belongs to F. Set

$$\delta(g) = \left|\frac{(\alpha_1 - \alpha_2)^2}{\alpha_1\alpha_2}\right|_F.$$

Since g is regular if and only if $\delta(g) \neq 0$ the set $\widehat{G}_F$ of regular elements is open in G_F and its complement has measure zero.

There are two more integration formulae that we shall need. Their proof proceeds as for archimedean fields. Choose a system S of representatives of the conjugacy classes of Cartan subgroups of G_F. Then

$$\int_{G_F} f(g)\,\omega(g) = \sum_{B_F \in S} \frac{1}{2}\int_{B_F} \delta(b)\left\{\int_{B_F\backslash G_F} f(g^{-1}bg)\,\omega_B(g)\right\}\mu_B(b) \tag{7.2.1}$$

$$\int_{Z_F\backslash G_F} f(g)\,\omega^0(g) = \sum_{B_F \in S} \frac{1}{2}\int_{Z_F\backslash B_F} \delta(b)\left\{\int_{B_F\backslash G_F} f(g^{-1}bg)\,\omega_B(g)\right\}\mu_B^0(b) \tag{7.2.2}$$

if f is an integrable function on G_F or $Z_F \setminus G_F$. Notice that the sum on the right is not necessarily finite. Let $\widehat{B}_F = B_F \cap \widehat{G}_F$ and let

$$\widehat{B}_F^G = \{g^{-1}bg \,\big|\, b \in \widehat{B}_F,\, g \in G_F\}.$$

Then $\widehat{G}_F$ is the disjoint union

$$\bigcup_{B_F \in S} \widehat{B}_F^G.$$

There is a simple lemma to be verified.

Lemma 7.2 (i) *For any Cartan subgroup B_F the set $\widehat{B}_F^G$ is open.*
(ii) *The set $\widehat{G}_F$ is open.*
(iii) *The set $\widetilde{G}_F$ of g in G_F whose eigenvalues do not belong to F is open.*

The second statement is a consequence of the first. If B_F corresponds to the separable quadratic extension E then $\widehat{B}_F^G$ is the set of matrices with distinct eigenvalues in E and if B_F splits and therefore corresponds to the direct sum of F with itself, $\widehat{B}_F^G$ is the set of matrices with distinct eigenvalues in F. Thus the first assertion is a consequence of the following lemma which is a form of Hensel's lemma or of the implicit function theorem.

Lemma 7.2.1 *Let E be a separable extension of F. Assume the equation*

$$X^p + a_1X^{p-1} + \cdots + a_p = 0$$

with coeficients in F has a simple root λ in E. Given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $b_1, \cdots, b_p$ are in F and $|b_i - a_i|_F < \delta$ for $1 \leq i \leq p$ the equation

$$X^p + b_1X^{p-1} + \cdots + b_p = 0$$

has a root μ in E for which $|\lambda - \mu|_E < \varepsilon$.

There is no need to prove this lemma. To prove the third assertion we have to show that the set of matrices with eigenvalues in F is closed. Suppose $g_n \to g$ and g_n has eigenvalues λ_n and μ_n in F. Then $\lambda_n + \mu_n \to \tau(g)$ and $\lambda_n\mu_n \to \nu(g)$. If λ_n and μ_n did not remain in a compact subset of $F^\times$ then, since their product does, we would have, after passing to a subsequence, $|\lambda_n| \to 0$, $|\mu_n| \to \infty$ or $|\lambda_n| \to \infty$, $|\mu_n| \to 0$. In either case $\lambda_n + \mu_n$ could not converge. Thus, again passing to a subsequence, we have $\lambda_n \to \lambda$ and $\mu_n \to \mu$. λ and μ are the eigenvalues of g.

If the characteristic of F is not two the sets $\widehat{G}_F$ and $\widetilde{G}_F$ are the same. We now introduce a function on G_F which plays an important role in the discussion of characters. If B_F is a split Cartan subgroup we set $c(B_F) = 1$ but if B_F is not split and corresponds to the quadratic extension E we set

$$c(B_F) = |\varpi|_F^{\frac{t+1}{2}}$$

ϖ is a generator of $\mathfrak{p}_F$ and $\mathfrak{p}_F^{t+1}$ is the discriminant of E over F. If g in $\widehat{G}_F$ belongs to the Cartan subgroup B_F we set

$$\xi(g) = c(B_F)\delta^{-1/2}(g).$$

If g is singular we set $\xi(g) = \infty$. The factor $c(B_F)$ is important only in characteristic two when there are an infinite number of conjugacy classes of Cartan subgroups.

Lemma 7.3 *The function ξ is locally constant on $\widehat{G}_F$ and bounded away from zero on any compact subset of G_F. It is locally integrable on $Z_F \setminus G_F$ and on G_F.*

It is of course implicit in the satement of the lemma that ξ is constant on cosets of Z_F. The two previous lemmas show that ξ is locally constant on $\widehat{G}_F$. To prove the remaining assertions we recall some facts about orders and modules in separable quadratic extensions of non-archimedean fields.

If E is a separable quadratic extension of F an order R of E is a subring of O_E which contains O_F and a basis of E. A module I in E is a finitely generated O_F submodule of E which contains a basis of E. If I is a module the set

$$\{\alpha \in E \,|\, \alpha I \subseteq I\}$$

is an order R_I. It is clear that an order is a module and that $R_R = R$. Two modules I and J are said to be equivalent if there is an α in $E^\times$ so that $J = \alpha I$. Then $R_I = R_J$.

Suppose the module I is contained on O_E and contains 1. Since I/O_F is a torsion-free O_F module the module I has a basis of the form $\{1, \delta\}$. Since δ is integral δ^2 belongs to I. Therefore I is an order and $R_I = I$. Since any module is equivalent to a module which contains 1 and lies in O_E the collection of modules I for which $R_I = R$ forms, for a given order R, a single equivalence class.

As observed any order has a basis, over O_F, of the form $\{1, \delta\}$. The absolute values of the numbers δ occurring in such bases are bounded below. A basis $\{1, \delta\}$ is said to be normal if δ has the smallest possible absolute value. It is easily seen, by considering the ramified and unramified extensions separately, that if $\{1, \delta\}$ is normal

$$R = O_F + \delta O_E.$$

Thus R determines and is determined by $|\delta|_E$. It is easily seen that if E/F is unramified $|\delta|_E$ is any number of the form $|\varpi_E|_E^n$ with $n \geq 0$. ϖ_E is a generator of $\mathfrak{p}_E$. We set $n = \omega(R)$. If E/F is ramified, $|\delta|_E$ is any number of the form $|\varpi_E|_E^{2n+1}$ with $n \geq 0$. We set $\omega(R) = n$. In the ramified case

$$[E^\times : F^\times(U_E \cap R)] = 2|\varpi_F|_F^{-\omega(R)}.$$

In the unramified case
$$[E^\times : F^\times(U_E \cap R)] = |\varpi_F|_F^{-\omega(R)}(1 + |\varpi_F|_F)$$
unless $\omega(R) = 0$ and then
$$[E^\times : F^\times(U_E \cap R)] = 1.$$

It is clear that R' contains R if and only if $\omega(R') \le \omega(R)$. Thus $\omega(R) + 1$ is the number of orders which contain R. If γ belongs to O_E but not to O_F let $R(\gamma)$ be the order with basis $\{1, \gamma\}$ and let $\omega(\gamma) = \omega\big(R(\gamma)\big)$.

Lemma 7.3.1 *Let $\bar\gamma$ be the conjugate of γ in E and let*
$$|(\gamma - \bar\gamma)^2|_F^{1/2} = |\varpi_F|_F^{m(\gamma)}.$$
If $\mathfrak{p}_E^{t+1}$ is the discriminant of E and γ belongs to O_K but not to O_F then
$$m(\gamma) = \omega(\gamma) + \frac{t+1}{2}.$$

Let $\{1, \delta\}$ be a normal basis of $R(\gamma)$. Then $\gamma = a + b\delta$ with a and b in O_F. Moreover $\delta = c + d\gamma$ with c and d in O_F. Thus $\gamma = (a + bc) + bd\gamma$ so that $a + bc = 0.$ and $bd = 1$. Therefore b is a unit and $|\gamma - \bar\gamma| = |\delta - \bar\delta|$. We can thus replace γ by δ. Suppose first that E/F is unramified so that $t + 1 = 0$. We take $\delta = \varepsilon\varpi_F^n$ where $n = \omega\big(R(\gamma)\big)$ and ε is a unit of O_E. Since
$$\delta - \bar\delta = (\varepsilon - \bar\varepsilon)\varpi_F^n$$
we have only to show that $\varepsilon - \bar\varepsilon$ is a unit. ε is not congruent to an element of O_F modulo $\mathfrak{p}_E$ and therefore $\{1, \varepsilon\}$ determines a basis of $O_E/\mathfrak{p}_E$. Since the Galois group acts faithfully on $O_E/\mathfrak{p}_E$ the number $\varepsilon - \bar\varepsilon$ is not in $\mathfrak{p}_E$.

If E/F is ramified we may take $\delta = \varpi_F^n\varpi_E$ with $n = \omega(\delta)$. It is well-known that
$$|\varpi_E - \overline{\varpi}_E| = |\varpi_E|_E^{t+1}$$
Thus
$$|(\delta - \bar\delta)^2|_F^{1/2} = |\delta - \bar\delta|_E^{1/2} = |\varpi_F|_F^n\,|\varpi_K|_K^{\frac{t+1}{2}} = |\varpi_F|_F^{n+\frac{t+1}{2}}$$
The lemma follows.

There are two more lemmas to be proved before we return to the proof of Lemma 7.3.

Lemma 7.3.2 *Let C be a compact subset of $Z_F \setminus G_F$ and let χ_C be the characteristic function of C and of its inverse image in G_F. There is a constant c such that for every b in G_F which is contained in an anisotropic Cartan subgroup*
$$\int_{Z_F\backslash G_F} \chi_C(g^{-1}bg)\,\omega^0(g) \le c\xi(b).$$

The assertion is trivial unless b is regular. Then the assumption is that its eigenvalues are distinct and do not lie in F. Any h in G_F can be written as
$$g_1 \begin{pmatrix} \varpi_F^p & \\ & \varpi_F^q \end{pmatrix} g_2$$

where g_1 and g_2 belongs to $GL(2,O_F)$ and $p \le q$. The numbers ϖ_F^p and ϖ_F^q are the elementary divisors of h. Let T_r be the set of all those h for which $q-p \le r$. T_r is the inverse image of a compact subset T_r' of $Z_F \setminus G_F$. If r is sufficiently large C is contained in T_r'. Thus we may replace χ_C and χ_r the characteristic function of T_r'.

If h belongs to $GL(2,O_F)$ then $h^{-1}g^{-1}bgh$ belongs to T_r if and only if $g^{-1}bg$ belongs to T_r. Thus the integral is the product of the measure of $GL(2,O_F) \cap Z_F \setminus GL(2,O_F)$ by the number of right cosets of $Z_F\,GL(2,O_F)$ whose elements g are such that $g^{-1}bg$ belong to T_r. If H is such a coset and B_F is the Cartan subgroup containing b then for any b' in B_F the coset $b'H$ has the same property. Thus the integral equals

$$\text{measure}\big(GL(2,O_F) \cap Z_F \setminus GL(2,O_F)\big) \sum [B_F g\, GL(2,O_F) : Z_F\, GL(2,O_F)].$$

The sum is over a set of representatives of the cosets in $B_F \setminus G_F / GL(2,O_F)$.

Let B_F correspond to the separable quadratic extension E. Choose a basis of O_E over O_F. It will also be a basis of E over F. By means of this basis we identify G_F with the group of invertible linear transformations of E over F. $GL(2,O_F)$ is the stabilizer of O_E. Every γ in $E^\times$ determines a linear transformation $b_\gamma : x \to \gamma x$ of E. The set of all such linear transformations is a Cartan subgroup conjugate to B_F and with no loss of generality we may assume that it is B_F. Choose γ so that $b = b_\gamma$.

Every module is of the form gO_E with g in G_F. Moreover g_1O_E and g_2O_E are equivalent if and only if g_1 and g_2 belong to same double coset in $B_F \setminus G_F/GL(2,O_F)$. Thus there is a one-to-one correspondence between the collection of double cosets and the collection of orders of E. Let $B_F g\, GL(2,O_F)$ correspond to the order R. The index

$$[B_F\, g\, GL(2,O_F) : Z_F\, GL(2,O_F)]$$

is equal to

$$[B_F : B_F \cap Z_F\, g\, GL(2,O_F)g^{-1}]$$

Two elements b_1 and b_2 in B_F belong to the same coset of $B_F \cap Z_F\, g\, GL(2,O_F)g^{-1}$ if and only if there is a z in Z_F and an h in $GL(2,O_F)$ such that

$$b_1 g = b_2 z g h$$

This can happen if and only if

$$b_1 g O_E = b_2 z g O_E.$$

Let $I = gO_E$ and let $b_i = b_{\gamma_i}$. If we identify Z_F and $F^\times$ so that z may be regarded as an element of $F^\times$ the last relation is equivalent to

$$\gamma_1 I = \gamma_2 z I$$

or $\gamma_1^{-1}\gamma_2 z \in R \cap U_E$. Thus

$$[B_F\, g\, GL(2,O_F) : Z_F GL(2,O_F)] = [E^\times : F^\times (R \cap U_E)].$$

Let $|\det b|_F = |\gamma|_K = |\varpi_F|_F^m$. Let ϖ_F^p and ϖ_F^q with $p \le q$ be the elementary divisors of $g^{-1}bg$. Certainly $p+q=m$. The matrix $g^{-1}bg$ belongs to T_r if and only if $q-p = m-2p \le r$. If s is the integral part of $\frac{r-m}{2}$ this is so if and only if $\varpi_F^s g^{-1}bg$ has integral coefficients, that is if and only if

$$\varpi_F^s\, g^{-1} bg\, O_E \subseteq O_E$$

or $\varpi_F^s\gamma \in R$.

Our integral is therefore equal to

$$\text{measure}\big(GL(2,O_F)\cap Z_F\setminus GL(2,O_F)\big)\sum_{\varpi_F^s\gamma\in R}[E^\times : F^\times(R\cap U_E)]. \tag{$*$}$$

The sum is over all orders of E which contains $\varpi_F^s\gamma$. $\varpi_F^s\gamma$ does not lie in F. If it does not lie in O_K the sume is zero. If it lies in O_K then $\varpi_F^s\gamma$ belongs to R if and only if $\omega(R)\le\omega(\varpi_F^s\gamma)$. In this case the expression $(*)$ is bounded by

$$2\ \text{measure}\big(GL(2,O_F)\cap Z_F\setminus GL(2,O_F)\big)\sum_{0\le k\le\omega(\varpi_F^s\gamma)}|\varpi_F|_F^{-k}.$$

This in turn is bounded by a constant, which is independent of B_F and r, times

$$|\varpi_F|_F^{-\omega(\varpi_F^s\gamma)}$$

We have $c(B_F)=|\varpi_F|_F^{\frac{t+1}{2}}$, $m(\varpi_F^s\gamma)=s+m(\gamma)\le\frac{r-m}{2}+m(\gamma)$, and

$$\delta(b)^{1/2}=\frac{|(\gamma-\bar\gamma)^2|_F^{1/2}}{|\gamma\bar\gamma|_F^{1/2}}=|\varpi_F|_F^{-m/2}\,|\varpi_F|_F^{m(\gamma)}.$$

To prove the lemma we have only to show that

$$-m(\gamma)+\frac{m}{2}+\frac{t+1}{2}+\omega(\varpi_F^s\gamma)$$

is bounded above by a constant which depends only on r. By the previous lemma

$$\omega(\varpi_F^s\gamma)=m(\varpi_F^s\gamma)-\frac{t+1}{2}$$

so that

$$-m(\gamma)+\frac{m}{2}+\frac{t+1}{2}+\omega(\varpi_F^s\gamma)\le\frac{r-m}{2}+\frac{m}{2}=\frac{r}{2}.$$

Suppose the Cartan subalgebra B_F corresponds to the algebra E. Once the measure μ_E on E has been chosen we can form the measure μ_B on B_F and the measure ω_B on $B_F\setminus G_F$. Once μ_E and therefore μ_B and ω_B are chosen we let $n(B_F)$ be that constant which makes $n(B_F)\mu_E$ self-dual with respect to the character $x\to\psi\big(\tau(x)\big)$ on E.

Lemma 7.3.3 *If r is a non-negative integer there is a constant d_r such that for any Cartan subgroup B_F and any b in B_F*

$$\int_{B_F\backslash G_F} \chi_r(g^{-1}bg)\,\omega_B(g) \le d_r n(B_F)\,\delta(b)^{-1/2}.$$

We may again suppose that b belongs to $\widehat{B}_F$. If B_F is anisotropic the left side is equal to

$$\frac{1}{\text{measure}(Z_F\setminus B_F)}\int_{Z_F\backslash G_F} \chi_r(g^{-1}bg)\,\omega^0(g).$$

Suppose B_F corresponds to the quadratic extension E. If E/F is unramified

$$\text{measure}(Z_f\setminus B_F) = \frac{1}{n(B_F)}(1+|\varpi_F|)$$

because $n(B_F)\mu_E$ assigns the measure 1 to O_E. If E/F is ramified $n(B_F)\mu_E$ assigns the measure $|\varpi_F|^{\frac{t+1}{2}}$ to O_E and

$$\text{measure}(Z_F\setminus B_F) = \frac{2}{n(B_F)}|\varpi_F|^{\frac{t+1}{2}} = \frac{2}{n(B_F)}c(B_F)$$

In these cases the assertion is therefore a consequence of the previous lemma.

If the inequality of the lemma is true for one Cartan subgroup it is true for all conjugate subgroups. To complete the proof we have to verify it when B_F is the group A_F of diagonal matrices. Since we are now dealing with a fixed Cartan subgroup the choice of Haar measure on $B_F\setminus G_F$ is not important. Moreover $GL(2,O_F)\,T_r\,GL(2,O_F) = T_r$ so that, using the Iwasawa decomposition and the associated decomposition of measures, we may take the itnegral to be

$$\int_F \chi_r\left(\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx$$

if

$$b = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

The argument in the integrand is

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & (1-\frac{\beta}{\alpha})x\\ 0 & 1\end{pmatrix}$$

Changing the variables in the integral we obtain

$$\frac{1}{|1-\frac{\beta}{\alpha}|}\int_F \chi_r\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx.$$

Let $|\alpha| = \varpi_F|^{\ell}$, $|\beta| = |\varpi_F|^m$, and $|x| = |\varpi_F|^n$. With no loss of generality we may suppose $|\alpha| \ge |\beta|$. If $n \ge 0$ the elementary divisors of

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

are ϖ_F^ℓ and ϖ_F^m so that it is in T_r if and only if $m-\ell \le r$. If $n<0$ its elementary divisors are $\varpi_F^{\ell+n}$ and ϖ_F^{m-n} so that it is in T_r if and only if $m-\ell-2n\le r$. Thus the integral is at most

$$\text{measure}\{x \mid |x| \le |\varpi_F|^{\frac{m-\ell-r}{2}}\}$$

which is, apart from a factor depending on the choice of the Haar measure, $|\varpi_F|^{\frac{m-\ell-r}{2}}$. Since

$$|\varpi_F|^{\frac{m-\ell-r}{2}} = \left|\frac{\beta}{\alpha}\right|^{\frac{1}{2}} |\varpi_F|^{-\frac{r}{2}}$$

and

$$\frac{\left|\frac{\beta}{\alpha}\right|^{\frac{1}{2}}}{\left|1-\frac{\beta}{\alpha}\right|} = \delta(b)^{-\frac{1}{2}}$$

the lemma follows.

We return to Lemma 7.3 and prove first that ξ is bounded away from zero on each compact subset C. In other words we show that there is a positive constant c such that $\xi(h)\ge c$ on C. There is a z in Z_F such that every matrix in zC has integral entries. Since $\xi(zh)=\xi(h)$ we may as well assume that every matrix in C itself has integral entires. There is a constant $c_1>0$ such that

$$|\det h|_F^{1/2} \ge c_1$$

on C and a constant c_2 such that

$$|\tau(h)^2 - 4\nu(h)|^{1/2} \le c_2$$

on C. τ and ν are the trace and determinant of h. Thus

$$\delta^{-1/2}(h) \ge \frac{c_1}{c_2}$$

on C. $\xi(h)$ is certainly bounded away from 0 on the singular elements and the preceding inequality shows that it is bounded away from 0 on the regular elements in C which lie in a split Cartan subalgebra. Suppose h is regular and lies in the anisotropic Cartan subgroup B_F. Let B_F correspond to the field E and let h have eigenvalues γ and $\bar\gamma$ in E. Then

$$|(\gamma-\bar\gamma)^2|_F^{-\frac{1}{2}}\, c(B_F) = |\varpi_F|^{-m(\gamma)}\, |\varpi_F|^{\frac{t+1}{2}} = |\varpi_F|^{-\omega(\gamma)}$$

Since $\omega(\gamma)\ge 0$ we have $\xi(h)\ge c_1$.

The function ξ is certainly measurable. It is locally integrable in G_F if and only if it is locally integrable on $Z_F\setminus G_F$. Let C be a compact set in $Z_F\setminus G_F$. We have to show

$$\int_{Z_F\setminus G_F} \chi_C(g)\,\xi(g)\,\omega^0(g)$$

is finite. As usual it will be enough to show that

$$\int_{Z_F\setminus G_F} \chi_r(g)\,\xi(g)\,\omega^0(g)$$

is finite for every non-negative integer r. According to formula (7.2.2) this integral is the sum of

$$\frac{1}{2}\int_{Z_F\backslash A_F} \xi(a)\,\delta(a)\left\{\int_{A_F\backslash G_F} \chi_r(g^{-1}ag)\,\omega_A(g)\right\}\,\mu_A^0(a)$$

and

$$\frac{1}{2}\sum_{B_F\in S'}\int_{Z_F\backslash B_F} \xi(b)\,\delta(b)\left\{\int_{B_F\backslash G_F} \chi_r(g^{-1}bg)\,\omega_B(g)\right\}\,\mu_B^0(b).$$

It is easy to see that there is a compact set C_0 in $Z_F\setminus A_F$ such that $\chi_r(g^{-1}ag)=0$ for all g unless the projection of a lies in C_0. Thus the first integral need only be taken over C_0. The inner integral is at most $d_r n(A_F)\,\delta(a)^{-1/2}$. Since $\xi(a)\,\delta(a)\,\delta(a)^{-1/2}=1$ on A_F the first integral causes no trouble. We can also use Lemma 7.3.3 to see that the sum over S', which is by the way a set of representatives for the conjugacy classes of anisotropic Cartan subgroups, is less than or equal to

$$\frac{1}{2}\sum_{B_F\in S'} d_r n(B_F)c(B_F)\int_{Z_F\backslash B_F}\mu_B^0(b).$$

If the characteristic is not two this sum is finite and there is no problem.

In general if B_F corresponds to the field E and $\mathfrak{p}_F^{t_E+1}$ is the discriminant of E we have $c(B_F)=|\varpi_F|^{\frac{t_E+1}{2}}$ and

$$n(B_F)\int_{Z_F\backslash B_F}\mu_B^0(b)\le 2|\varpi_F|^{(t_E+1)/2}$$

To complete the proof we have to show that

$$\sum_E |\varpi_F|^{t_E+1}$$

is finite if F has characteristic 2. The sum is over all separable quadratic extensions of F. Let $M(t)$ be the number of extensions E for which $t_E=t$. Associated to any such E is a quadratic character of $F^\times$ with conductor $\mathfrak{p}_F^{t+1}$. Thus

$$M(t)\le[F^\times:(F^\times)^2(1+\mathfrak{p}_F^{t+1})]=2[U_F:U_F^2(1+\mathfrak{p}_F^{t+1})]$$

if $t\ge 0$. Of course $M(-1)=1$. Any element of U_F is congruent modulo $1+\mathfrak{p}_F^{t+1}$ to an element of the form

$$a_0+a_1\varpi_F+\cdots+a_t\varpi_F^t.$$

Such a number is a square if $a_i=0$ for i odd. Thus

$$M(t)=0(|\varpi_F|^{-\frac{t+1}{2}})$$

and the series converges.

We can now begin the study of characters.

Proposition 7.4 *The character of an absolutely cuspidal representation exists as a locally integrable function whose absolute value is bounded by a multiple of ξ. It is continuous on $\widehat{G}_F \cup \widetilde{G}_F$.*

If the character χ_π of π exists and χ is a quasi-character of $F^\times$ then the character of $\pi' = \chi \otimes \pi$ also exists and $\chi_{\pi'}(g) = \chi(\det g)\,\chi_\pi(g)$. Thus the proposition has only to be proved for unitary representations π. π is then square integrable and we can make use of the following lemma for which, although it is well-known, we provide a proof.

Lemma 7.4.1 *Let f belong to $\mathcal{H}_F$ and let u be a vector of length one in the space on which the absolutely cuspidal unitary representation π acts. Then*

$$\operatorname{Tr}\pi(f) = d(\pi)\int_{Z_F\backslash G_F}\left\{\int_{G_F} f(h)\big(\pi(g^{-1}hg)u,u\big)\,dh\right\}dg$$

if $d(\pi)$ is the formal degree of π.

Let Q be the operator

$$\pi(f) = \int_{G_F} f(h)\,\pi(h)\,dh.$$

Let $\{v_i\}$ be an orthonormal basis of the space on which π acts. All but a finite number of the coefficients

$$Q_{ij} = (Qv_i, v_j)$$

are zero. We have

$$\big(\pi(g^{-1})\,Q\pi(g)u,u\big) = \big(Q\pi(g)u,\pi(g)u\big)$$

The right side equals

$$\sum_i \big(Q\pi(g)u,v_i\big)\big(v_i,\pi(g)u\big)$$

which in turn equals

$$\sum_i\sum_j \big(\pi(g)u,v_j\big)\,Q_{ji}\big(v_i,\pi(g)u\big)$$

In both series there are only a finite number of non-zero terms. Thus

$$\int_{Z_F\backslash G_F}\big(\pi(g^{-1})\,Q\pi(g)u,u\big)\,dg = \sum_{i,j} Q_{ji}\int_{Z_F\backslash G_F}\big(\pi(g)u,v_j\big)\big(v_i,\pi(g)u\big)\,dg$$

The integrals on the right exist because the representation is square-integrable. Applying the Schur orthogonality relations we see that the right side is equal to

$$\frac{1}{d(\pi)}\sum_{i,j} Q_{ij}(v_i,v_j) = \frac{1}{d(\pi)}\sum_i Q_{ii} = \frac{1}{d(\pi)}\operatorname{Tr}\pi(f).$$

Since

$$\big(\pi(g^{-1})\,Q\pi(g)u,u\big) = \int_{G_F} f(h)\,\big(\pi(g^{-1})\,\pi(h)\,\pi(g)u,u\big)\,dh$$

the lemma follows.

Observe that the integral of the lemma is an iterated and not a double integral. It is the limit as r approaches infinity of

$$\int_{T'_r}\left\{\int_{G_F} f(h)\left(\pi(g^{-1}hg)u,u\right)dh\right\}dg$$

Since T'_r is compact this integral is absolutely convergent and equals

$$\int_{G_F} f(h)\left\{\int_{T'_r}\left(\pi(g^{-1}hg)u,u\right)dg\right\}dh.$$

To prove the first part of the proposition we show that the sequence of functions

$$\varphi_r(h)=\int_{T'_r}(\pi(g^{-1}hg)u,u)\,dg$$

is dominated locally by a multiple of ξ and converges almost everywhere on G_F. We shall set

$$\chi_\pi(h)=d(\pi)\lim_{r\to\infty}\varphi_r(h)$$

whenever the limit exists.

When proving the second part of the proposition we shall make use of the following lemma.

Lemma 7.4.2 *Let C_1 be a compact subset of $\widetilde{G}_F$ and let C_2 be a compact set in G_F. The image in $Z_F\setminus G_F$ of*

$$\{g\in G_F \mid g^{-1}C_1g\cap Z_FC_2\neq 0\}$$

is compact.

The set is clearly closed so we have only to show that it is contained in some compact set. We may suppose that $GL(2,O_F)\,C_2\,GL(2,O_F)=C_2$. Let

$$g=\begin{pmatrix}\alpha & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}h$$

with h in $GL(2,O_F)$. Then

$$g^{-1}C_1g\cap Z_FC_2\neq 0$$

if and only if

$$\begin{pmatrix}\alpha & x\\ 0 & 1\end{pmatrix}^{-1}C_1\begin{pmatrix}\alpha & x\\ 0 & 1\end{pmatrix}\cap Z_FC_2\neq 0.$$

We have to show that this condition forces α to lie in a compact subset of $F^\times$ and x to lie in a compact subset of F. Since

$$\det(g^{-1}cg)=\det c$$

we may replace $Z_F\,C_2$ by the compact set

$$C_3=\{h\in Z_FC_2 \mid \det h\in \det C_1\}.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a typical element of C_1. The entry c is never 0 on C_1 and therefore its absolute value is bounded below,

$$\begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - xc & y \\ c\alpha & cx + d \end{pmatrix}.$$

The number y is of no interest. The matrix on the right cannot lie in C_3 unless $|cx + d|$ is bounded above by some number depending on C_3. Since $|d|$ is bounded above and $|c|$ is bounded below x is forced to lie in some compact set Ω of F. If C_4 is the compact set

$$\{\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,|x \in \Omega,\, h \in C_1\}$$

we have finally to show that if

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} C_4 \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cap C_3 \neq 0$$

then α is forced to lie in a compact subset of $F^\times$. We now let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a typical element of C_4. On C_4 both $|b|$ and $|c|$ are bounded blow. Since

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/\alpha \\ c\alpha & d \end{pmatrix}$$

and all matrix entries are bounded above in absolute value on C_3 the absolute value $|\alpha|$ must indeed be bounded above and below.

If π acts on V then for any u in V the support of the function $(\pi(g)u, u)$ has been shown, in the second paragraph, to be compact modulo Z_F. Let C be its compact image in $Z_F \setminus G_F$. Let C_1 be a compact subset of G_F. By the previous lemma the set of g in G_F such that

$$\big(\pi(g^{-1}hg)u, u\big) \neq 0$$

for some h in C_1 has an image in $Z_F \setminus G_F$ which is contained in a compact set C_2. Therefore the integral

$$\int_{Z_F \setminus G_F} \big(\pi(g^{-1}hg)u, u\big)\, dg = \int_{C_2} \big(\pi(g^{-1}hg)u, u\big)\, dg$$

is convergent for h in C_1. Moreover if r is large enough T'_r contains C_2 and

$$\varphi_r(h) = \int_{Z_f \setminus G_F} \big(\pi(g^{-1}hg)u, u\big)\, dg.$$

Therefore the sequence $\{\varphi_r\}$ converges uniformly on any compact subset of $\widetilde{G}_F$ and its limit $d^{-1}(\pi)\, \chi_\pi(h)$ is continuous on $\widetilde{G}_F$. We may state the following proposition.

Proposition 7.5 *If h belongs to $\widetilde{G}_F$ then*

$$\int_{Z_F\backslash G_F} \big(\pi(g^{-1}hg)u,u\big)\,dg$$

exists and is equal to $d^{-1}(\pi)\,\chi_\pi(h)$.

Since

$$|\big(\pi(g)u,u\big)| \le \chi_C(g)$$

it follows from Lemma 7.3.2 that, for some constant c,

$$|\varphi_r(h)| \le c\xi(h)$$

on $\widetilde{G}_F$. The set $\widehat{G}_F - \widetilde{G}_F$ is $\widehat{A}_F^G$ which is open. To complete the proof of Proposition 7.4 we show that on the intersection of $\widehat{A}_F^G$ with a compact subset of G_F the sequence $\{\varphi_r\}$ is dominated by a multiple of ξ and that it converges uniformly in a compact subset of $\widehat{A}_F^G$.

Let C_3 be a compact subset of G_F. Any h in $\widehat{A}_F$ may be written in the form

$$h = h_1^{-1}\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h_1$$

where h_1 belongs to $GL(2,O_F)$ and $\alpha \ne \beta$. In $C_3 \cap \widehat{A}_F^G$ the absolute values of α and β are bounded above and below. If C_3 is contained in $\widehat{A}_F^G$ the absolute value of $1-\frac{\beta}{\alpha}$ is also bounded above and below on C_3. Since

$$\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & (1-\frac{\beta}{\alpha})x\\ 0 & 1\end{pmatrix}$$

the absolute value of x will be bounded above.

Since $GL(2,O_F)\,T_r\,GL(2,O_F) = T_r$ the integral which defines $\varphi_r(h)$ is equal to

$$\int_{T_r'} \big(\pi(g^{-1}h'g)u,u\big)\,dg$$

if

$$h' = \begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

and we may as well assume that h itself is of this form. We are going to show that there is a constant c such that

$$|\varphi_r(h)| \le c\big|1-\frac{\beta}{\alpha}\big|^{-1}$$

for all r and all such h and that the sequence $\{\varphi_r\}$ converges uniformly if x remains in a compact subset of F and α, β and $1-\frac{\beta}{\alpha}$ remain in a compact subset of $F^\times$. Then the proof of the proposition will be complete.

The stabilizer of u is some open subgroup U of $GL(2,O_F)$. Let $h_1,\cdots,h_p$ be a set of coset representatives for $GL(2,O_F)/U$ and let $u_i = \pi(h_i)u$. apart from an unimportant factor coming from the Haar measure $\varphi_r(h)$ is given by

$$\sum_{i=1}^{p} \varphi_r^i(h)$$

with
$$\varphi_r^i(h) = \int \left(\pi \left(\begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) \, dx_1 \, d^\times \gamma.$$
The integral is taken over the set of all those γ and x_1 for which
$$\begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix}$$
belongs to T_r. Since
$$\begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix}^{-1} h \begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & (1 - \frac{\beta}{\alpha})(\gamma^{-1} x + x_1) \\ 0 & 1 \end{pmatrix}$$
we can change variables in the integral to obtain
$$|1 - \frac{\beta}{\alpha}|^{-1} \int \left(\pi \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}(1 - \frac{\beta}{\alpha})x + x_1 \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) \, dx_1 \, d^\times \gamma. \tag{7.4.3}$$
The integration is now taken over all those x_1 and γ for which
$$\begin{pmatrix} \gamma & \gamma(1 - \frac{\beta}{\alpha})^{-1} x_1 \\ 0 & 1 \end{pmatrix} \tag{7.4.4}$$
is in T_r.

Let $|1 - \frac{\beta}{\alpha}| = |\varpi_F|^t$, $|\gamma| = |\varpi_F|^m$, and $|x| = |\varpi_F|^n$. Let ϖ_F^p and ϖ_F^q be the elementary divisors of the matrix (7.4.4). We now list the possibilities for p and q together with the condition that the matrix belong to T_r, that is that $q - p$ be at most r.

(a) $m \geq 0, -t + m + n \geq 0, p = 0, q = m : 0 \leq m \leq r$
(b) $m \geq 0, -t + m + n \leq 0, p = -t + m + n, q = n - t : -r \leq m + 2n - 2t$
(c) $m \leq 0, -t + m + n \leq m, p = -t + m + n, q = n - t : -r \leq m + 2n - 2t$
(d) $m \leq 0, -t + m + n \geq m, p = m, q = 0 : -r \leq m \leq 0.$

These conditions amount to the demand $-r \leq m \leq r$ and that $2n \geq 2t - r - m$. On the other hand we know that there is an integer s such that
$$\int_{|x| \leq |\varpi_F|^j} \pi \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \right) u_i \, dx = 0$$
for $1 \leq i \leq p$ if $j \leq s$.

Thus if $|\gamma| = |\varpi_F|^m$ the integral
$$\int \left(\pi \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}(1 - \frac{\beta}{\alpha})^{x + x_1}_{1} \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) \, dx_1$$
taken over all x_1 for which
$$\begin{pmatrix} \gamma & \gamma(1 - \frac{\beta}{\alpha}) x_1 \\ 0 & 1 \end{pmatrix}$$
is in T_r is zero if $2t - r - m \leq 2s$. Therefore in (7.4.3) we need only take the integral over those γ and x for which $|\gamma| = |\varpi_F|^m$ with $0 \leq m_r \leq 2(t - s)$ and $|x| \leq |\varpi_F|^{t - \frac{m+r}{2}}$. We should also have $m \leq r$ but

since we are about to replace the integrand by its absolute value that does not matter. For each such γ the integration with respect to x gives a result which is bounded in absolute value by a constant times $|\varpi_F|^{t-\frac{m+r}{2}}$. Integrating with respect to γ we obtain a result which is bounded in absolute value by a constant times

$$|\varpi_F|^t \sum_{k=0}^{2(t-s)-1} |\varpi_F|^{-\frac{k}{2}} \le |\varpi_f|^s \sum_{k=0}^{\infty} |\varpi_F|^{\frac{k}{2}}$$

The right side depends on neither r nor t.

The value of $|1-\frac{\beta}{\alpha}|\varphi_r^i(h)$ is

$$\int \left(\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)\pi\left(\begin{pmatrix}1 & \varpi_F^r\gamma^{-1}(1-\frac{\beta}{\alpha})x+x_1\\ 0 & 1\end{pmatrix}\right)u_i,u_i\right)dx_1\,d^\times\gamma.$$

The integration is taken over those γ and x_1 for which $|\gamma| = |\varpi_F|^m$ with $0 \le m < 2(t-s)$ and $|x| \le |\varpi_F|^{t-\frac{m}{2}}$. Of course $|1-\frac{\beta}{\alpha}| = |\varpi_F|^t$. Since we are now interested in a set of α and β on which t takes only a finite number of values we may as well assume it is constant. Then the integral is taken over a fixed compact subset of $F \times F^\times$. The integrand converges uniformly on this set uniformly in the α, β and x under consideration as r approaches infinity.

We have still to prove the existence of the character of a representation which is not absolutely cuspidal. Most of them are taken care of by the next proposition.

Proposition 7.6 *Let μ_1 and μ_2 be a pair of quasi-characters of $F^\times$. Let χ_{μ_1,μ_2} be the function which is 0 on $\widehat{G}_F \cap \widetilde{G}_F$, undefined on the singular elements, and equal to*

$$\{\mu_1(\alpha)\,\mu_2(\beta)+\mu_2(\beta)\,\mu_1(\alpha)\}\left|\frac{\alpha\beta}{(\alpha-\beta)^2}\right|^{1/2}$$

at an element of g of $\widehat{A}_F^G$ with eigenvalues α and β. Then χ_{μ_1,μ_2} is continuous on $\widehat{G}_F$ and is dominated in absolute value by some multiple of ξ. Moreover if $\pi = \rho(\mu_1,\mu_2)$

$$\operatorname{Tr}\pi(f) = \int_{G_F} \chi_{\mu_1,\mu_2}(g)\,f(g)\,dg$$

for all f in $\mathcal{H}_F$.

Only the last assertion requires verification. Since the absolute value of χ_{μ_1,μ_2} is bounded by a multiple of ξ the function χ_{μ_1,μ_2} is locally integrable. Suppose f belongs to $\mathcal{H}_F$. When applied to the function $\chi_{\mu_1,\mu_2}f$ the relation (7.2.1) shows that

$$\int_{G_F} \chi_{\mu_1,\mu_2}(g)\,f(g)\,dg \tag{7.6.1}$$

is equal to

$$\frac{1}{2}\int_{A_F}\delta(a)\left\{\int_{A_F\backslash G_F}\chi_{\mu_1,\mu_2}(g^{-1}ag)\,f(g^{-1}ag)\,dg\right\}da.$$

Since χ_{μ_1,μ_2} is a class function this may be written as

$$\frac{1}{2}\int_{A_F}\{\mu_1(\alpha)\,\mu_2(\beta)+\mu_2(\alpha)\,\mu_1(\beta)\}\left|\frac{(\alpha-\beta)^2}{\alpha\beta}\right|^{1/2}\left\{\int_{A_F\backslash G_F} f(g^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} g)\,dg\right\}da$$

if

$$a=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}.$$

Since a is conjugate to

$$\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}$$

we have

$$\int_{A_F\backslash G_F} f\left(g^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} g\right)\,dg=\int_{A_F\backslash G_F} f\left(g^{-1}\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix} g\right)\,dg.$$

Thus (7.6.1) is equal to

$$\int_{A_F}\mu_1(\alpha)\,\mu_2(\beta)\left|\frac{(\alpha-\beta)^2}{\alpha\beta}\right|^{1/2}\left\{\int_{A_F\backslash G_F} f\left(g^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} g\right)\,dg\right\}da. \tag{7.6.2}$$

As long as the measure on $A_F\setminus G_F$ is the quotient of the measure on G_F by that on A_F the choice of Haar measure on A_F and G_F is not relevant. Thus we may write (7.6.2) as

$$\int_{A_F}\mu_1(\alpha)\,\mu_2(\beta)\left|\frac{(\alpha-\beta)^2}{\alpha\beta}\right|^{1/2}\left(f\left(k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} nk\right)\,dk\,dn\right\}da.$$

The inner integral is taken over $GL(2,O_F)\times N_F$. If

$$n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

then

$$n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} n=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & (1-\frac{\beta}{\alpha})x\\ 0 & 1\end{pmatrix}$$

Changing variables in the last integral we obtain

$$\int_{A_F}\mu_1(\alpha)\,\mu_2(\beta)\,|\frac{\alpha}{\beta}|^{1/2})\left\{\int f\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} nk\right)\,dk\,dn\right\}da. \tag{7.6.3}$$

To evaluate $\operatorname{Tr}\pi(f)$ we observe that if φ belongs to $\mathcal{B}(\mu_1,\mu_2)$ then, if k_1 is in $GL(2,O_F)$

$$\pi(f)\,\varphi(k_1)=\int_{G_F}\varphi(k_1 g)\,f(g)\,dg.$$

Replacing g by $k_1^{-1}g$ and writing the integral out in terms of the Haar measure we have chosen we obtain

$$\int_{GL(2,O_F)}\varphi(k_1)\left\{\int f(k_1^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} nk_2)\mu_1(\alpha)\,\mu_2(\beta)\,|\frac{\alpha}{\beta}|^{1/2}\,da\,dn\right\}dk_2.$$

The inner integral is taken over $A_F \times N_F$. We have of course used the relation

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} nk_2\right) = \mu_1(\alpha)\mu_2(\beta)\,|\frac{\alpha}{\beta}|^{1/2}\varphi(k_2).$$

If

$$K(k_1,k_2) = \int f\left(k_1^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} nk_2\right)\,\mu_1(\alpha)\,\mu_2(\beta)\,|\frac{\alpha}{\beta}|^{1/2}\,da\,dn$$

then

$$\pi(f)\,\varphi(k_1) = \int_{GL(2,O_F)} K(k_1,k_2)\,\varphi(k_2)\,dk_2.$$

$\mathcal{B}(\mu_1,\mu_2)$ may be regarded as a space of functions on $GL(2,O_F)$. Then $\pi(f)$ is the integral operator with kernel $K(k_1,k_2)$. It is easily seen that this operator, when allowed to act on the space of all $GL(2,O_F)$-finite functions on $GL(2,O_F)$, has range in $\mathcal{B}(\mu_1,\mu_2)$. Thus the trace of $\pi(f)$ is the same as the trace of the integral operator which is of course

$$\int_{GL(2,O_F)} K(k,k)\,dk.$$

When written out in full this integral becomes (7.6.3).

Theorem 7.7 *Let π be an irreducible admissible representation of $\mathcal{H}_F$. There is a function χ_π which is continuous on G_F and locally bounded in absolute value of $\widehat{G}_F$ by a multiple of ξ such that*

$$\mathrm{Tr}\,\pi(f) = \int_{G_F} \chi_\pi(g)\,f(g)\,dg$$

for all f in $\mathcal{H}_F$.

The theorem has only to be verified for the one-dimensional and the special representations. If π is a one-dimensional representation associated to the quasi-character χ we may take $\chi_\pi(g) = \chi(\det g)$. χ_π is locally bounded and therefore, by Lemma 7.3, locally bounded by a multiple of ξ.

Suppose π_1, π_2 and π_3 are three admissible representations of F on space V_1, V_2, and V_3 respectively. Suppose also that there is an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

of $\mathcal{H}_F$-modules. If f is in $\mathcal{H}_F$ all the operators $\pi_1(f)$, $\pi_2(f)$ and $\pi_3(f)$ are of finite rank so that

$$\mathrm{Tr}\,\pi_2(f) = \mathrm{Tr}\,\pi_1(f) + \mathrm{Tr}\,\pi_3(f).$$

Thus if χ_{π_1} and χ_{π_2} exist so does χ_{π_3}. Applying this observation to $\pi_3 = \sigma(\mu_1,\mu_2)$, $\pi_2 = \rho(\mu_1,\mu_2)$, and $\pi_1 = \pi(\mu_1,\mu_2)$ we obtain the theorem.

If F is taken to be the real or complex field Theorem 7.7 is a special case of a general and difficult theorem of Harish-Chandra. The special case is proved rather easily however. In fact Proposition 7.6 is clearly valid for archimedean fields and Theorem 7.7 is clearly valid for archimedean fields if π is finite-dimensional. There remains only the special representations and these are taken care of as before.

§**8. Odds and ends** In this paragraph various facts which will be used in the discussion of the constant term in the Fourier expansion of an automorphic form are collected together. If H is a locally compact abelian group a continuous complex-valued function f on H will be called H-finite or simply finite if the space spanned by the translates of f is finite dimensional.

Let H be a group of the form

$$H = H_0 \times \mathbb{Z}^m \times \mathbb{R}^n$$

where H_0 is compact. We regard $\mathbb{Z}^m \times \mathbb{R}^n$ as a subgroup of $\mathbb{R}^{m+n}$. The projection

$$\xi_i : h = (h_0, x_1, \cdots, x_{m+n}) \to x_i$$

may be regarded as a function on H with values in $\mathbb{R}$. If $p_1, \cdots, p_{m+n}$ is a sequence of non-negative integers and χ is a quasi-character we may introduce the function

$$\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$$

on H.

Lemma 8.1 *For any sequence $p_1, \cdots, p_{m+n}$ and any quasi-character χ the function $\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$ is continuous and finite. These functions form a basis of the space of continuous finite functions on H.*

If χ is a fixed quasi-character of H and p is a non-negative integer let $V(\chi, p)$ be the space spanned by the functions $\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$ with $0 \leq p_i \leq p$. Since it is finite-dimensional and invariant under translations the first assertion of the lemma is clear.

To show that these functions are linearly independent we shall use the following simple lemma.

Lemma 8.1.1 *Suppose $E_1, \cdots, E_r$ are r sets and $\mathcal{F}_1, \cdots \mathcal{F}_r$ are linearly independent sets of complex-valued functions on $E_1, \cdots, E_r$ respectively. Let $\mathcal{F}$ be the set of functions*

$$(x_1, \cdots, x_r) \to f_1(x_1)\, f_2(x_2) \cdots f_r(x_r)$$

on $E_1 \times \cdots \times E_r$. Here f_i belongs to $\mathcal{F}_i$. Then $\mathcal{F}$ is also linearly independent.

Any relation

$$\sum_{f_1, \cdots, f_r} a(f_1, \cdots, f_r)\, f_1(x_1) \cdots f_r(x_r) \equiv 0$$

leads to

$$\sum_{f_r} \left\{ \sum_{f_1, \cdots, f_{r-1}} a(f_1, \cdots, f_r)\, f_1(x_1) \cdots f_{r-1}(x_{r-1}) \right\} f_r(x_r) \equiv 0$$

As $\mathcal{F}_r$ is linearly independent this implies that

$$\sum_{f_1, \cdots f_{r-1}} a(f_1, \cdots, f_r)\, f_1(x_1) \cdots f_{r-1}(x_{r-1}) \equiv 0$$

and the lemma follows by induction.

To show that the functions $\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$ span the space of continuous finite functions we use another simple lemma.

Lemma 8.1.2. *Let H_1 and H_2 be two locally compact abelian groups and let $H = H_1 \times H_2$. Then every continuous finite function f on H is a finite linear combination of the form*

$$f(x,y) = \sum_i \lambda_i \, \varphi_i(x) \, \psi_i(y)$$

where the φ_i and ψ_i are continuous functions on H_1 and H_2 respectively.

Let V be any finite dimensional space of continuous functions on H. We associate to any point ξ in H the linear functional $f \to f(\xi)$ on V. Since no function but zero is annihilated by all these functionals we can choose $\xi_1, \cdots, \xi_p$ so that the corresponding functionals form a basis of the dual of V. Then we can choose a basis $f_1, \cdots, f_p$ of V so that $f_i(\xi_j) = \delta_{ij}$.

Now suppose V is invariant under translations. It could for example be the space spanned by the translates of a single finite continuous function. The space V_1 of functions φ on H_1 defined by $\varphi(x) = f(x, 0)$ with f in V is finite dimensional and translation invariant. Therefore the functions in it are finite and of course continuous. We define V_2 in a similar manner. If f is in V the function $h \to f(g + h)$ is, for any g in H, also in V. Thus

$$f(g+h) = \sum_i \lambda_i(g) \, f_i(h).$$

Since

$$\lambda_i(g) = f(g + \xi_i)$$

the function λ_i belongs to V. If $\varphi_i(x) = \lambda_i(x, 0)$ and $\psi_i(y) = f_i(0, y)$ then

$$f(x,y) = \sum \varphi_i(x) \, \psi_i(y)$$

as required.

These two lemmas show that we need prove the final assertions of Lemma 8.1 only for H compact, $H = \mathbb{Z}$, or $H = \mathbb{R}$.

Suppose H is compact. If we have a non-trivial relation

$$\sum_{i=1}^r a_i \, \chi_i(h) \equiv 0$$

we may replace h be $g + h$ to obtain

$$\sum_{i=1}^r a_i \, \chi_i(g) \, \chi_i(h) \equiv 0.$$

If such a relation holds we must have $r \geq 2$ and at least two coefficients say a_1 and a_2 must be different from zero. Choose g so that $\chi_1(g) \neq \chi_2(g)$. Multiplying the first relation by $\chi_1(g)$ and subtracting the second relation from the result we obtain a relation

$$\sum_{i=2}^r b_i \, \chi_i(h) \equiv 0.$$

Since $b_2 = \{\chi_1(g) - \chi_2(g)\}a_2$ the new relation is non-trivial. The independence of the quasi-characters can therefore be proved by induction on r.

To prove that when H is compact the quasi-characters span the space of finite continuous functions we have just to show that any finite-dimensional space V of continuous functions which is translation invariant is spanned by the quasi-characters it contains. Choose a basis $\{f_i\}$ of V as before and let

$$\rho(g)\, f_i = \sum \lambda_{ij}(g)\, f_j.$$

We saw that the functions $\lambda_{ij}(g)$ are continuous. Thus the action of H on V by right translations is continuous and V is the direct sum of one-dimensional translation invariant spaces. Each such space is easily seen to contain a character.

When applied to a locally compact abelian group the argument of the previous paragraph leads to weaker conclusions. We can then find subspaces $V_1, \cdots, V_r$ of V and quasi-characters $\chi_1, \cdots, \chi_r$ of H such that

$$V = \sum_{i=1}^{r} \oplus V_i$$

and, for every h in H,

$$\{\rho(h) - \chi_i(h)\}^{\dim V_i}$$

annihilates V_i. Now we want to take H equal to $\mathbb{Z}$ or $\mathbb{R}$. Then H is not the union of a finite number of proper closed subgroups. Suppose $\mu_1, \cdots, \mu_2$ are quasi-characters of H and for every h in H the operator

$$\prod_{i=1}^{s} \{\rho(h) - \mu_i(h)\} \tag{8.1.3}$$

on V is singular. Then for every h in H there is an i and a j such that $\mu_i(h) = \chi_j(h)$. If

$$H_{ij} = \{h \mid \mu_i(h) = \chi_j(h)\}$$

then H_{ij} is a closed subgroup of H. Since the union of these closed subgroups is H there must be an i and a j such that $H_{ij} = H$ and $\mu_i = \chi_j$. If the operator (8.1.3) were zero the same argument would show that for every j there is an i such that $\mu_i = \chi_j$.

If μ is a quasi-character of H, now taken to be $\mathbb{Z}$ or $\mathbb{R}$, we let $V(\mu, p)$ be the space spanned by the functions $\mu\,\xi^i$, with $0 \le i \le p$. ξ is the coordinate function on H. It is clear that $V(\mu, p)$ is annihilated by $\{\rho(h) - \mu(h)\}^{p+1}$ for all h in H. Suppose $\mu, \mu_1, \cdots, \mu_2$ are distinct and

$$V = V(\mu, p) \sum_{i=1}^{s} V(\mu_i, p_i)$$

is not zero. Decomposing V as above we see that $\chi_1, \cdots, \chi_r$ must all be equal to μ on one hand and on the other that every μ_i is a χ_j. This is a contradiction. Thus if there is any non-trivial relation at all between the functions $\chi\,\xi^i$ where χ is any quasi-character and i is a non-negative integer there is one of the form

$$\sum_{i=0}^{p} a_i\, \mu\, \xi^i = 0.$$

Since the polynomial $\sum_{i=0}^{p} a_i\, \xi^i$ would then have an infinite number of zeros this is impossible.

To prove the functions $\chi\,\xi^i$ span the space of finite continuous functions we have only to show that if χ is a given quasi-character and V is a finite-dimensional space of continuous functions which is invariant under translations and annihilated by $\{\rho(h) - \chi(h)\}^{\dim V}$ for all h in H then every function in V is the product of χ and a polynomial. Since we can always multiply the functions in V by χ^{-1} we may as well suppose that χ is trivial. We have only to observe that any function f annihilated by the operator $\{\rho(h) - 1\}^n$ for all h in H is a polynomial of degree at most n. This is clear if $n = 1$ so by induction we can assume that $\rho(h)f - f$ is a polynomial $\sum_{i=0}^{n-1} a_i(h)\,\xi^i$. We can certainly find a polynomial f' of degree n such that

$$\rho(1)f' - f' = \sum_{i=0}^{n-1} a_i(1)\,\xi^i$$

and we may as well replace f by $f - f'$. The new f satisfies $\rho(1)f = f$. It is therefore bounded. Moreover $\rho(h)f - f$ is a bounded polynomial function and therefore a constant $c(h)$. $c(h)$ is a bounded function of h and satisfies $c(h_1 + h_2) = c(h_1) + c(h_2)$. It is therefore zero and the new f is a constant.

Lemma 8.1 is now completely proved. Although it is trivial it is important to the notes and we thought it best to provide a proof. We might as well prove Lemma 2.16.4 at the same time. Let B be the space of all functions f on $\mathbb{Z}$ such that for some n_0 depending on f we have $f(n) = 0$ for $n < n_0$. Let A_0 be the space of functions on $\mathbb{Z}$ which vanish outside a finite set. $\mathbb{Z}$ acts on B and on A_0 by right translations and therefore it also acts on $\bar{B} = B/A_0$. In particular let $D = \rho(1)$. We have merely to show that if P is a polynomial with leading coefficient 1 then the null space of $P(D)$ in $\bar{B}$ is finite dimensional. If

$$P(X) = \prod_{i=1}^{r} (X - \alpha_i)^{p_i}$$

the null space of $P(D)$ is the direct sum of the null spaces of the operators $(D - \alpha_i)^{p_i}$. The null space of $(D - \alpha)^p$ is the image in $\bar{B}$ of the functions in B which are zero to the left of 0 and of the form

$$n \to \alpha^n\, Q(n)$$

to the right of 0. Q is a polynomial of degree at most p.

Lemma 8.1 is certainly applicable to the direct product of a finite number of copies of the multiplicative group of a local field F. If $H = (F^\times)^n$ any finite continuous function on H is a linear combination of functions of the form

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{n} \{\chi_i(x_i)\,(\log|x_i|_F)^{n_i}\}.$$

Let $\mathcal{B} = \mathcal{B}_F$ be the space of continuous functions f on G_F which satisfy the following three conditions.

(i) f is finite on the right under the standard maximal compact subgroup K of G_F.

(ii) f is invariant on the left under N_F.

(iii) f is A_F-finite on the left.

$\mathcal{B}_F$ is invariant under left translations by elements of A_F. If f is in $\mathcal{B}_F$ let V be the finite-dimensional space generated by these left translates. Choose $g_1, \cdots, g_p$ in G_F so that the linear functions $\varphi \to \varphi(g_i)$ are a basis of the dual of V and let $f_1, \cdots, f_p$ be the dual basis. If a is A_F we may write

$$f(a, g) = \sum_{i=1}^{p} \theta_i(a)\, f_i(g).$$

Then

$$\theta_i(a) = f(ag_i)$$

so that

$$\theta_i(ab) = \sum_{j=1}^{p} \theta_j(a)\, f_j(bg_i).$$

Thus the functions θ_i are continuous and finite. We may write them in the form

$$\theta_i(a) = \sum c^i_{m,n,\mu,\nu}\, \mu(\alpha_1)\, \nu(\alpha_2)\, (\log|\alpha_1|)^m\, (\log|\alpha_2|)^n$$

if

$$a = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

The sum is over all quasi-characters μ and ν of $F^\times$ and all non-negative integers m and n. Of course only a finite number of the coefficients $c^i_{m,n,\mu,\nu}$ are different from zero.

We may replace μ by $\alpha_F^{1/2}\mu$ and ν by $\alpha_F^{-1/2}\nu$ in the sum. Thus if

$$f_{m,n,\mu,\nu} = \sum_{i=1}^{p} c^i_{m,n,\mu,\nu}\, f_i$$

we have

$$f(ag) = \left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \sum \mu(\alpha_1)\, \nu(\alpha_2)\, (\log|\alpha_1|)^m\, (\log|\alpha_2|)^n\, f_{m,n,\mu,\nu}(g). \tag{8.2}$$

Let M be a non-negative integer and S a finite set of pairs of quasi-characters of $F^\times$. $\mathcal{B}(S,M)$ will be the sset of f in for which the sum in (8.2) need only be taken over those m, n, μ, ν for which $m+n \le M$ and (μ,ν) belong to S. Observe that the functions $f_{m,n,\mu,\nu}$ are determined by f. $\mathcal{B}$ is the union of the spaces $\mathcal{B}(S,M)$; if S consists of the single pair (μ_1,μ_2) we write $\mathcal{B}(\mu_1,\mu_2,M)$ instead of $\mathcal{B}(S,M)$. If f is in (μ_1,μ_2,M)

$$f(ag) = \left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \mu_1(\alpha_1)\, \mu_2(\alpha_2)\, \sum (\log|a_1|)^m\, (\log|a_2|)^n\, f_{m,n}(g).$$

The space $\mathcal{B}(\mu_1,\mu_2,0)$ is just $\mathcal{B}(\mu_1,\mu_2)$.

The functions $f_{m,n,\mu,\nu}$ are uniquely determined and by their construction belong to the space spanned by left translates of f by elements of A_F. Thus if f belongs to $\mathcal{B}(S,M)$ so do the functions $f_{m,n,\mu,\nu}$. We want to verify that $f_{0,0,\mu,\nu}$ belongs to $\mathcal{B}(\mu,\nu,M)$. If

$$b = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

and we replace a by ab in the relation (8.2) we find that

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \sum \mu(\alpha_1)\, \nu(\alpha_2)\, (\log|\alpha_1|)^m\, (\log|\alpha_2|)^n)\, f_{m,n,\mu,\nu}$$

is equal to

$$\left|\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right|^{1/2}\sum \mu(\alpha_1\beta_1)\,\nu(\alpha_2\beta_2)(\log|\alpha_1|+\log|\beta_2|)^m\,(\log|\alpha_2|+\log|\beta_2|)^n\,f_{m,n,\mu,\nu}(g).$$

Fix b and g are regard this equality as an identity in the variable a. Because of Lemma 8.1 we can compare the coefficients of the basic finite functions. The coefficient of $\mu(\alpha_1)\nu(\alpha_2)$ on one side is $f_{0,0,\mu,\nu}(bg)$. On the other it is

$$\left|\frac{\beta_1}{\beta_2}\right|^{1/2}\sum_{m+n\le M}\mu(\beta_1)\,\nu(\beta_2)\,(\log|\beta_1|)^m\,(\log|\beta_2|)^n\,f_{m,n,\mu,\nu}(g).$$

The resulting identity is the one we wanted to verify.

Taking $a=1$ in (8.2) we see that

$$f(g)=\sum_{(\mu,\nu)\in S} f_{0,0,\mu,\nu}(g).$$

Therefore

$$\mathcal{B}(S,M)=\sum_{(\mu,\nu)\in S}\mathcal{B}(\mu,\nu,M).$$

The sum is direct.

It is fortunately possible to give a simple characterization of $\mathcal{B}$.

Proposition 8.3 *Let φ be a continuous function on G_F. Assume φ is K-finite on the right and invariant under N_F on the left. Then φ belongs to $\mathcal{B}$ if and only if the space*

$$\{\rho(\xi f)\,\varphi \mid f\in\mathcal{H}_F\}$$

is finite dimensional for every elementary idempotent in $\mathcal{H}_F$.

We have first to show that if φ belongs to $\mathcal{B}$

$$\{\rho(\xi f)\,\varphi \mid f\in\mathcal{H}_F\}$$

is finite dimensional. φ belongs to some $\mathcal{B}(S,M)$. Both $\mathcal{B}$ and $\mathcal{B}(S,M)$ are invariant under right translations by elements of $\mathcal{H}_F$. Thus we have only to show that the range of $\rho(\xi)$ as an operator on $\mathcal{B}(S,M)$ is finite dimensional. This is tantamount to showing that any irreducible representation of K occurs with finite multiplicity in the representation of $\mathcal{B}(S,M)$.

Let σ be such a representation and let V be the space of continuous functions on K which transform according to σ under right translations. V is finite dimensional. If f is in $\mathcal{B}(S,M)$ we may write

$$f(ag)\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\sum\mu(\alpha_1)\,\nu(\alpha_2)\,(\log|\alpha_1|)^m)\,(\log|\alpha_2|)^n\,f_{m,n,\mu,\nu}(g)$$

The restriction of $f_{m,n,\mu,\nu}$ to K lies in V. Call this restriction $\bar f_{m,n,\mu,\nu}$. Moreover f is determined by its restriction to A_FK. Thus

$$f\to\sum_{\substack{(\mu,\nu)\in S\\ m+n\le M}}\oplus\bar f_{m,n,\mu,\nu}$$

is an injection of the space of functions under consideration into the direct sum of a finite number of copies of V.

The converse is more complicated. Suppose φ is K-finite on the right, invariant under N_F on the left, and the space

$$\{\rho(\xi f)\,\varphi \,\big|\, f \in \mathcal{H}_F\}$$

is finite dimensional for every elementary idempotent ξ. Choose ξ so that $\rho(\varepsilon)\varphi = \varphi$. There is actually a function f in $\xi\mathcal{H}_F\xi$ such that $\rho(f)\varphi = \varphi$. If F is non-archimedean ξ is itself a function so this is clear. If F is archimedean we observe that if f_1 is an approximation to the δ-function then $\rho(f_1)\varphi$ is close to φ. Then if $f_1' = \xi * f_1 * \xi$ the function f_1' is in $\xi\mathcal{H}_F\xi$ and $\rho(f_1')\varphi$ is also close to φ. The existence of f then follows from the fact that $\rho(\xi\mathcal{H}_F\xi)\varphi$ is finite dimensional. This argument was used before in Paragraph 5.

Take F to be archimedean. Then φ must be an infinitely differentiable function on G_F. Let $\mathfrak{Z}$ be the centre of the universal enveloping algebra of the Lie algebra of G_F. If Z is in $\mathfrak{Z}$ then

$$\rho(Z)\varphi = \rho(Z)\,\rho(f)\varphi = \rho(Z * f)\varphi$$

and $Z * f$ is still in $\xi\mathcal{H}_F\xi$. Thus φ is also $\mathfrak{Z}$-finite. For the rest of the proof in the archimedean case we refer to Chapter I of [11].

Now take F non-archimedean. We may replace ξ by any elementary idempotent ξ' for which $\xi'\xi = \xi$. In particular if we choose n to be a sufficiently large positive integer and let K' be the elements of K which are congruent to the identity modulo $\mathfrak{p}^n$ we may take

$$\xi = \sum \xi_i$$

where the sum is over all elementary idempotents corresponding to irreducible representations of K whose kernel contains K'. Notice that n is at least 1. Then $\xi\mathcal{H}_F\xi$ is the space of functions on G_F which are constant on double cosets of K'.

Let V be the space spanned by the functions $\rho(k)\varphi$ with k in K. It is finite dimensional and all the functions in V satisfy the same conditions as φ. Let φ_i, $1 \le i \le p$, be a basis of V. If k belongs to K we may write

$$\varphi(gk) = \sum_{i=1}^{p} \theta_i(k)\,\varphi_k(g)$$

and φ is determined by the functions θ_i and the restrictions of the functions φ_i to A_F. To show that φ is A_F-finite on the left we have merely to show that the restriction of each φ_i to A_F is finite. We may as well just show that the restriction of φ to A_F is finite.

Suppose f is in $\xi\mathcal{H}_F\xi$ and $\rho(f)\varphi = \varphi$. If a is in Z_F then

$$\lambda(a)\varphi = \rho(a^{-1})\varphi = \rho(\delta_{a^{-1}} * f)\varphi$$

if $\delta_{a^{-1}}$ is the δ-function at a^{-1}. Since $\delta_{a^{-1}} * f$ is still in $\xi\mathcal{H}_F\xi$ the function φ is certainly Z_F-finite and so is its restriction $\bar\varphi$ to A_F. If α and β are units and $\alpha \equiv \beta \equiv 1 (\operatorname{mod} \mathfrak{p}^n)$ then

$$\lambda\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right)\bar\varphi = \bar\varphi.$$

Thus the translate of $\bar{\varphi}$ by the elements of $A_f \cap K$ span a finite dimensional space and if ϖ is a generator of $\mathfrak{p}$ we have only to show that the translates of $\bar{\varphi}$ by the group

$$H = \left\{ \begin{pmatrix} \varpi p & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, p \in \mathbb{Z} \right\}$$

span a finite-dimensional space. Suppose the span W of

$$\left\{ \lambda\left(\begin{pmatrix} \varpi p & 0 \\ 0 & 1 \end{pmatrix} \right) \bar{\varphi} \,\middle|\, p \leq 0 \right\}$$

is finite dimensional. Then

$$\lambda\left(\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

maps W into itself and annihilates no vector but zero so that it has an inverse on W which must be

$$\lambda\left(\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Thus W is invariant under H and $\bar{\varphi}$ is finite.

To show that W is finite dimensional we show that if

$$a = \begin{pmatrix} \varpi^{-p} & 0 \\ 0 & 1 \end{pmatrix}$$

with $p > 0$ there is a function f_a in $\xi \mathcal{H}_F \xi$ such that

$$\lambda(a)\bar{\varphi} = \bar{\varphi}'$$

if $\varphi' = \rho(f_a)\varphi$. There is an f in $\xi \mathcal{H}_F \xi$ such that

$$\varphi(g) = \int_{G_F} \varphi(gh)\, f(h)\, dh$$

for all g in G_F. Thus if b belongs to A_F

$$\lambda(a)\, \bar{\varphi}(b) = \varphi(a^{-1}b) = \int_{G_F} \varphi(ba^{-1}h)\, f(h)\, dh.$$

If $f_1(h) = f(ah)$ the integral is equal to

$$\int_{G_F} \varphi(bh)\, f_1(h)\, dh.$$

If f_1 were in $\xi \mathcal{H}_F \xi$ we would be done. Unfortunately this may not be so. However $f_1(hk) = f_1(h)$ if k belongs to K'. If

$$k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then
$$f_1(kh) = f\left(\begin{pmatrix} \alpha & \varpi^{-p}\beta \\ \varpi^p\gamma & \delta \end{pmatrix} ah\right).$$
Thus $f_1(kh) = f_1(h)$ if $\alpha \equiv \delta \equiv 1 \,(\mathrm{mod}\,\mathfrak{p}^n)$, $\gamma \equiv 0 \,(\mathrm{mod}\,\mathfrak{p}^n)$, and $\beta \equiv 0 \,(\mathrm{mod}\,\mathfrak{p}^{n+p})$. Set
$$f_2(h) = \int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) dx$$
where the Haar measure is so chosen that the measure of the underlying space $\mathfrak{p}^n/\mathfrak{p}^{n+p}$ is 1. Since $\varphi(bnh) = \varphi(bh)$ for all n in N_F
$$\lambda(a)\,\bar{\varphi}(b) = \int_{G_F} \varphi(bh)\, f_2(h)\, dh.$$
We show that f_2 lies in $\xi\mathcal{H}_F\xi$.

Certainly $f_2(hk) = f_2(h)$ if k is in K'. Moreover, because of its construction, $f_2(kh) = f_2(h)$ if
$$k = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$
with $\alpha \equiv \delta \equiv 1 \,(\mathrm{mod}\,\mathfrak{p}^n)$ and $\beta \equiv 0 \,(\mathrm{mod}\,\mathfrak{p}^n)$. Since every element of K' is a product
$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$
where both terms lie in K' we have only to show that f_2 is invariant under the first factor. If
$$k = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$
with $\gamma \equiv 0 \,(\mathrm{mod}\,\mathfrak{p}^n)$ and
$$k_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1+x\gamma \end{pmatrix}\begin{pmatrix} \frac{1}{1+x\gamma} & 0 \\ \frac{-\gamma}{1+x\gamma} & 1 \end{pmatrix}$$
then
$$k_1(x)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k = \begin{pmatrix} 1 & \frac{x}{1+x\gamma} \\ 0 & 1 \end{pmatrix}.$$
Moreover if x is in O_F
$$f_1\big(k_1(x)g\big) = f_1(g).$$
Thus $f_2(kg)$ which is given by
$$\int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} kh\right) dx$$
is equal to
$$\int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1\left(\begin{pmatrix} 1 & \frac{x}{1+x\gamma} \\ 0 & 1 \end{pmatrix} h\right) dx.$$

Since the map $x \to \frac{x}{1+x\gamma}$ is a one-to-one map of the finite set $\mathfrak{p}^n/\mathfrak{p}^{n+p}$ onto itself it is measure preserving and the above integral is equal to $f_2(h)$.

Analyzing the above proof one sees that in the non-archimedean case the left translates of $\bar{\varphi}$ are contained in the space X obtained by restricting the functions in $\rho(\xi\mathcal{H}_F\xi)\varphi$ to A_F. Thus if Y is the space of the functions on K/K' the left translates of φ by elements of A_F are contained in the space of functions on $N_F \setminus G_F$ of the form

$$\varphi'(ak) = \sum \theta_i(k)\,\varphi_i(a)$$

with θ_i in Y and φ_i in X.

In the archimedean case Y is the space of continuous functions θ on K for which $\theta * \xi = \xi * \theta = \theta$. It is again finite dimensional. X is defined in the same way. In this case there are a finite number of invariant differential operators $D_1, \cdots, D_r$ on A_F such that the left translates of φ by elements of A_F are contained in the space of functions $N_F \setminus G_F$ of the form

$$\varphi'(ak) = \sum \theta_i(k)\,\varphi_i(a)$$

with θ_i in Y and φ_i in $\sum_{j=1}^r D_j X$.

There is a corollary of these observations. Let $F_1, \cdots, F_n$ be a finite collection of local fields. Let $G_i = G_{F_i}$, $N_i = N_{F_i}$, $A_i = A_{F_i}$, and let K_i be the standard maximal compact subgroup of G_i. We set $G = \prod_{i=1}^n G_i$, $N = \prod_{i=1}^n N_i$ and so on. If $\mathcal{H}_i = \mathcal{H}_{F_i}$ we let $\mathcal{H} = \otimes_i \mathcal{H}_i$. $\mathcal{H}$ may be regarded as an algebra of measures on G.

Corollary 8.4 *Let φ be a continuous function on $N \setminus G$ which is K-finite on the right. If for every elementary idempotent ξ in $\mathcal{H}$ the space*

$$\{\rho(\xi f)\varphi \mid f \in \mathcal{H}\}$$

is finite dimensional φ is A-finite on the left.

If φ satisfies the conditions of the lemma so does any left translate by an element of A. Thus we need only show that φ is A_i-finite on the left for each i. If g is in G we write $g = (g_i, \widehat{g}_i)$ where g_i is in G_i and $\widehat{g}_i$ is in $\widehat{G}_i = \prod_{j \neq i} G_j$. We may suppose that there is a ξ' of the form $\xi' = \otimes_i \xi_i'$ where ξ_i' is an elementary idempotent of $\mathcal{H}_i$ such that $\rho(\xi')\varphi = \varphi$. By means of the imbedding $f \to f \otimes \prod_{j \neq i} \xi_j'$ the algebra $\mathcal{H}_i$ becomes a subalgebra of $\mathcal{H}$. The left translates of φ by A_i all lie in the space of functions of the form

$$\varphi(a_i k_i, \widehat{g}_1) = \sum_j \theta_j(k_i)\,\varphi(a_i, \widehat{g}_1)$$

where θ_j lie in a certain finite dimensional space determined by ξ_i' and the φ_j lie in the space obtained by restricting the functions in $\rho(\xi_i \mathcal{H}_i)\varphi$ to $A_i \times \widehat{G}_i$ or, in the archimedean case, the space obtained from this space by applying certain invariant differential operators. ξ_i is a certain elementary idempotent which may be different from ξ_i'.

With the odds taken care of we come to the ends.

Proposition 8.5 *Let $\mathcal{B}(\mu,\nu,\infty) = \bigcup_{M\ge 0} \mathcal{B}(\mu,\nu,M)$. If an irreducible admissible representation π of $\mathcal{H}_F$ is a constituent of the representation $\rho(\mu,\nu,\infty)$ on $\mathcal{B}(\mu,\nu,\infty)$ it is a constituent of $\rho(\mu,\nu)$.*

There are two invariant subspaces V_1 and V_2 of $\mathcal{B}(\mu,\nu,\infty)$ such that V_1 contains V_2 and π is equivalent to the representation on $\mathcal{H}_F$ on V_1/V_2. Choose M so that $V_1 \cap \mathcal{B}(\mu,\nu,M)$ is not contained in V_2. Since π is irreducible

$$V_1 = V_2 + \big(V_1 \cap \mathcal{B}(\mu,\nu,M)\big)$$

and

$$V_1/V_2 = \{V_2 + \big(V_1 \cap \mathcal{B}(\mu,\nu,M)\big)\}/V_2$$

is isomorphic as an $\mathcal{H}_F$ module to

$$V_1 \cap \mathcal{B}(\mu,\nu,M)/V_2 \cap \mathcal{B}(\mu,\nu,M)$$

so that we may as well suppose that V_1 is contained in $\mathcal{B}(\mu,\nu,M)$.

Given π we choose M as small as possible. If $M = 0$ there is nothing to prove so assume M is positive. If φ is in $\mathcal{B}(\mu,\nu,M)$ we can express

$$\varphi\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} g\right)$$

as

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \mu(\alpha_1)\,\nu(\alpha_2) \sum_{m+n\le M} (\log|\alpha_1|)^m\,(\log|\alpha_2|)^n\,\varphi_{m,n}(g)$$

We can express

$$\varphi\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} g\right)$$

in two ways because the second factor can be absorbed into the first or the third. One way we obtain

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \mu(\alpha_1)\,\nu(\alpha_2) \sum_{m+n\le M} (\log|\alpha_1|)^m(\log|\alpha_2|)^n\,\varphi_{m,n}\left(\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} g\right)$$

and the other way we obtain

$$\left|\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right|^{1/2} \mu(\alpha_1\beta_1)\,\nu(\alpha_2\beta_2) \sum_{m+n\le M} (\log|\alpha_1| + \log|\beta_1|)^m\,(\log|\alpha_2| + \log|\beta_2|)^n\varphi_{m,n}(g).$$

On comparing coefficients we see that if $m + n = M$

$$\varphi_{m,n}\left(\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} g\right) = \left|\frac{\beta_1}{\beta_2}\right|^{1/2} \mu(\beta_1)\,\nu(\beta_2)\,\varphi_{m,n}(g)$$

so that $\varphi_{m,n}$ is in $\mathcal{B}(\mu,\nu)$. Consider the map

$$\varphi \to \bigoplus_{m+n=M} \varphi_{m,n}$$

of V_1 into

$$\bigoplus_{m+n=M} \mathcal{B}(\mu,\nu).$$

Its kernel is $V_1 \cap \mathcal{B}(\mu,\nu,M-1)$. Since $V_2 + (V_1 \cap \mathcal{B}(\mu,\nu,M-1))$ cannot be V_1 the image of V_2 is not the same as the image of V_1. Since the map clearly commutes with the action of $\mathcal{H}_F$ the representation π is a constituent of $\bigoplus_{m+n=M} \rho(\mu,\nu)$.

Proposition 8.5 is now a consequence of the following simple lemma.

Lemma 8.6 *Suppose π is an irreducible representation of an algebra H. Suppose ρ is a representation of H of which π is a constituent and that ρ is the direct sum of the representations ρ_λ, $\lambda \in \Omega$. Then π is a constituent of at least one of the ρ_λ.*

Let ρ_λ act on X_λ and let ρ act on X the direct sum of X_λ. Supose that Y_1 and Y_2 are invariant subspaces of X and that the representation on the quotient Y_1/Y_2 is equivalent to π. There is a finite subset Λ_0 of Λ such that

$$Y_1 \cap (\sum_{\lambda\in\Lambda_0} X_\lambda)$$

is not contained in Y_2. We may as well replace Y_1 by $Y_1 \cap (\sum_{\lambda\in\Lambda_0} X_\lambda)$ and Y_2 by $Y_2 \cap (\sum_{\lambda\in\Lambda_0} X_\lambda)$ and suppose that Λ is finite. If $\Lambda = \{\lambda_1, \cdots, \lambda_p\}$ we have only to show that π is a constituent of ρ_{λ_1} or of $\rho_{\lambda_2} \oplus \cdots \oplus \rho_{\lambda_p}$ for we can then use induction. Thus we may as well take $p = 2$. If the projections of Y_1 and Y_2 on X_{λ_1} are not equal we can replace Y_1 and Y_2 by these projections to see that π is a constituent of ρ_{λ_1}. If they are equal $Y_1 = Y_2 + (Y_1 \cap X_{\lambda_2})$ and we can replace Y_1 and Y_2 by $Y_1 \cap X_{\lambda_2}$ and $Y_2 \cap X_{\lambda_2}$ to see that π is a constituent of ρ_{λ_2}.

References

for Chapter I

The Weil representation is constructed in:

1. Weil, A., *Sur certains groupes d'opérateurs unitaires*, Acta Math., t. 111, 1964.

One of the first to study representations of groups over non-archimedean local fields was F. Mautner in:

2. Mautner, F., *Spherical functions over $\mathfrak{p}$-adic fields*, I. Amer. Jour. Math., vol LXXX, 1958.

Absolutely cuspidal representations were first constructed by Gelfand and Graev. References to their work and that of Kirillov will be found in

3. Gelfand, I.M., M.I. Graev, and I.I. Pyatetskii–Shapiro, *Representation Theory and Automorphic Functions*, W.B. Saunders Co., 1966.

These representations were constructed in terms of the Weil representation by Shalika and by Tanaka.

4. Shalika, J. *Representations of the two-by-two unimodular group over local fields*, Notes, Institute for Advanced Study.
5. Tanaka, S., *On irreducible unitary representations of some special linear groups of the second order*, Osaka Jour. Math., 1966.

To classify the representations over an archimedean field we have used a theorem of Harish–Chandra which may be found in

6. Harish–Chandra, *Representations of semisimple Lie groups*, II, T.A.M.S., vol 76, 1954.

Our discussion of characters owes much to:

7. Sally, P.J. and J.A. Shalika, *Characters of the discrete series of representations of $SL(2)$ over a local field*, P.N.A.S., 1968.

Three standard references to the theory of L-functions are:

8. Lang, S., *Algebraic numbers*, Addison–Wesley, 1964.
9. Tate, J., *Fourier analysis in number fields and Hecke's Zeta-functions in algebraic number theory*, Thompson Book Co., 1967.
10. Weil, A., *Basic number theory*, Springer–Verlag, 1967.

In Paragraph 8 we have used a result from:

11. Harish–Chandra, *Automorphic forms on semisimple Lie groups*, Springer–Verlag, 1968.

Tamagawa measures are discussed in:

12. Weil, A., *Adèles and algebraic groups*, Institute for Advanced Study, 1961.

Chapter II: Global Theory

§**9. The global Hecke algebra.** Let F be a global field, that is, an algebraic number field of finite degree over the rationals or a function field in one variable over a finite field. $\mathbb{A}$ will be the adèle ring of F. Before studying the representations of $GL(2,\mathbb{A})$ or, more precisely, the representations of a suitable group algebra of $GL(2,\mathbb{A})$ we introduce some simple algebraic notions.

Let $\{V_\lambda \,|\, \lambda \in \Lambda\}$ be a family of complex vector spaces. Suppose that for all but a finite number of λ we are given a non-zero vector e_λ in V_λ. Let V^0 be the set of all $x = \prod_\lambda x_\lambda$ in $\prod_\lambda V_\lambda$ such that $x_\lambda = e_\lambda$ for al but a finite number of λ. Let C be the free vector space with complex coefficients over V^0 and let D be the subspace generated by vectors of the form

$$\big\{(aY_\mu + bZ_\mu) \times \prod_{\lambda\neq\mu} x_\lambda\big\} - a\,\big\{y_\mu \times \prod_{\lambda\neq\mu} x_\lambda\big\} - b\,\big\{z_\mu \times \prod_{\lambda\neq\mu} x_\lambda\big\}.$$

a and b belong to $\mathbb{C}$ and μ is any element of Λ. The quotient of C by D is called the tensor product of the V_λ with respect to the family e_λ and is written

$$V = \otimes_{e_\lambda} V_\lambda$$

or simply $\otimes V_\lambda$. It has an obvious universal property which characterizes it up to isomorphism. The image of $\prod x_\lambda$ in V is written $\otimes x_\lambda$.

If Λ' is a subset of Λ with finite complement we may form the ordinary tensor product

$$\otimes_{\lambda\in\Lambda-\Lambda'} V_\lambda$$

and we may form

$$\otimes_{\lambda\in\Lambda'} V_\lambda$$

with respect to the family e_λ. Then $\otimes_{\lambda\in\Lambda} V_\lambda$ is canonically isomorphic to

$$\big\{\otimes_{\lambda\in\Lambda-\Lambda'}\big\} \otimes \big\{\otimes_{\lambda\in\Lambda'} V_\lambda\big\}$$

If S is a finite subset of Λ let

$$V_S = \otimes_{\lambda\in S} V_\lambda$$

If S is so large that e_λ is defined for λ not in S let φ_S be the map of V_S into V which sends $\otimes_{\lambda\in S} x_\lambda$ to $\{\otimes_{\lambda\in S} x_\lambda\} \otimes \{\otimes_{\lambda\notin S} e_\lambda\}$. If S' contains S there is a unique map $\varphi_{S,S'}$ of V_S into $V_{S'}$ which makes

$$\begin{array}{ccc} V_S & \xrightarrow{\varphi_{S,S'}} & V'_S \\ & {\scriptstyle\varphi_S}\searrow \quad \swarrow{\scriptstyle\varphi_{S'}} & \\ & V & \end{array}$$

commutative. If we use these maps to form the inductive limit of the spaces V_S we obtain a space which the layman is unable to distinguish from V.

Suppose that for every λ we are given a linear map B_λ of V_λ into itself. If $B_\lambda\, e_\lambda = e_\lambda$ for all but a finite number of λ there is exactly one linear transformation B of $\otimes V_\lambda$ such that

$$B : \otimes x_\lambda \to \otimes B_\lambda\, x_\lambda$$

B is denoted by $\otimes B_\lambda$.

For example if A_λ, $\lambda \in \Lambda$ is a family of associative algebras, which may or may not have a unit, and if, for almost all λ, ξ_λ is a given idempotent of A_λ one may turn

$$A = \otimes_{\xi_\lambda} A_\lambda$$

into an algebra in such a way that

$$(\otimes a_\lambda)(\otimes b_\lambda) = \otimes(a_\lambda\, b_\lambda).$$

Let V_λ, $\lambda \in \Lambda$, be an A_λ module. If for almost all λ a vector e_λ such that $\xi_\lambda e_\lambda = e_\lambda$ is given we may turn $V = \otimes_{e_\lambda} V_\lambda$ into an $A = \otimes_{\xi_\lambda} A_\lambda$ module in such a way that

$$(\otimes a_\lambda)(\otimes x_\lambda) = \otimes(a_\lambda\, x_\lambda)$$

Suppose the family $\{e_\lambda\}$ is replaced by a family $\{e'_\lambda\}$ but that, for all but a finite number of λ, $e'_\lambda = \alpha_\lambda\, e_\lambda$ where α_λ is a non-zero scalar. Suppose for example that $e'_\lambda = \alpha_\lambda\, e_\lambda$ if λ is not in the finite set S. There is a unique map of $\otimes_{e_\lambda} V_\lambda$ to $\otimes_{e'_\lambda} V_\lambda$ which sends

$$\big\{ \otimes_{\lambda \in S}\, x_\lambda \big\} \otimes \big\{ \otimes_{\lambda \notin S}\, x_\lambda \big\}$$

to

$$\big\{ \otimes_{\lambda \in S}\, x_\lambda \big\} \otimes \big\{ \otimes_{\lambda \notin S}\, \alpha_\lambda\, x_\lambda \big\}$$

It is invertible and commutes with the action of A. Moreover apart from a scalar factor it is independent of S.

Now suppose F is a global field. A place of F is an equivalence class of injections, with dense image, of F into a local field. If λ_1 takes F into F_1 and λ_2 takes F into F_2 they are equivalent if there is a topological isomorphism φ of F_1 with F_2 such that $\lambda_2 = \varphi \circ \lambda_1$. The symbol for a place will be v. If v contains the imbedding λ_1 and a belongs to F we set $|a|_v = |\lambda_1(a)|$. To be definite we let F_v be the completion of F with respect to the absolute value $a \to |a|_v$. v is archimedean or non-archimedean according to the nature of F_v. Non-archimedean places will sometimes be denoted by $\mathfrak{p}$.

If $G_F = GL(2, F)$ we set

$$G_v = G_{F_v} = GL(2, F_v).$$

K_v will be the standard maximal compact subgroup of G_v. then $G_{\mathbb{A}} = GL(2, \mathbb{A})$ is the restricted direct product of the groups G_v with respect to the subgroups K_v.

If v is non-archimedean we set $O_v = O_{F_v}$ and $U_v = U_{F_v}$. O_v is the ring of integers of F_v and U_v is the group of units of O_v. Suppose M' is a quaternion algebra over F. Let $M'_v = M'_{F_v} = M' \otimes_F F_v$. For almost all v the algebra M'_v is split, that is, there is an isomorphism

$$\theta_v : M'_v \to M(2, F_v)$$

where $M(2, F_v)$ is the algebra of 2×2 matrices over F_v. For every place v at which M'_v is split we want to fix such an isomorphism θ_v. Let B be a basis of M over F and let L_v be the O_v module generated in M_v by B. We may and do choose θ_v so that for almost all v

$$\theta_v(L_v) = M(2, O_v).$$

If B' is another basis and $\{\theta'_v\}$ a family of isomorphism associated to B' then for every place v at which M'_v splits there is a g_v in $GL(2, F_v)$ such that

$$\theta'_v\, \theta_v^{-1} = g_v\, a\, g_v^{-1}$$

for all a in $M(2, F_v)$. Moreover g_v belongs to K_v for all but a finite number of v.

Suppose the family of isomorphisms θ_v has been chosen. If M'_v is split we define a maximal compact subgroup K'_v of G'_v, the group of invertible elements of M'_v, by the condition

$$\theta_v(K'_v) = K_v.$$

If M'_v is not split we set

$$K'_v = \{x \in M'_v \,\big|\, |\nu(x)|_v = 1\}.$$

This group is compact. In any case K'_v is defined for all v. Since many of the constructions to be made depend on the family K'_v which in turn depends on the family of θ_v it is very unfortunate that the family of θ_v is not unique. We should really check at every stage of the discussion that the constructions are, apart from some kind of equivalence, independent of the initial choice of θ_v. We prefer to pretend that the difficulty does not exist. As a matter of fact for anyone lucky enough not to have been indoctrinated in the functorial point of view it doesn't. We do however remark that any two choices of the family of K'_v lead to the same result for almost all v. $G'_{\mathbb{A}}$ is the restricted direct product of the groups G'_v with respect to the subgroups K'_v.

We have now to introduce the Hecke algebras $\mathcal{H}$ and $\mathcal{H}'$ of $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$. Let $\mathcal{H}_v$ be $\mathcal{H}_{F_v}$. If M'_v is split G'_v isomorphic, by means of θ_v, to G_v and we let $\mathcal{H}'_v$ be the algebra of measures on G'_v corresponding to $\mathcal{H}_v$. Suppose M'_v is not split. If v is non-archimedean $\mathcal{H}'_v$ is the algebra of measures defined by the locally constant compactly supported functions on G'_v. If v is archimedean $\mathcal{H}'_v$ will be the sum of two subspaces, the space of measuures defined by infinitely differentiable compactly supported functions on G'_v which are K'_v-finite on both sides and the space of measures on K'_v defined by the matrix coeffients of finite-dimensional representations of K'_v.

Let ε_v and ε'_v be the normalized Haar measures on K_v and K'_v. ε_v is an elementary idempotent of $\mathcal{H}_v$ and ε'_v is an elementary idempotent of $\mathcal{H}'_v$. We set

$$\mathcal{H} = \otimes_{\varepsilon_v} \mathcal{H}_v$$

and

$$\mathcal{H}' = \otimes_{\varepsilon'_v} \mathcal{H}'_v$$

If S is the finite set of places at which M'_v does not split we may write

$$\mathcal{H} = \big\{ \otimes_{v \in S} \mathcal{H}_v \big\} \otimes \big\{ \otimes_{v \notin S} \mathcal{H}_v \big\} = \mathcal{H}_s \otimes \widehat{\mathcal{H}}_s$$

and

$$\mathcal{H}' = \big\{ \otimes_{v \in S} \mathcal{H}'_v \big\} \otimes \big\{ \otimes_{v \notin S} \mathcal{H}'_v \big\} = \mathcal{H}'_s \otimes \widehat{\mathcal{H}}'_s$$

By construction, if M'_v is split, $\mathcal{H}_v$ and $\mathcal{H}'_v$ are isomorphic in such a way that ε_v and ε'_v correspond. Using these isomorphism we may construct an isomorphism of $\widehat{\mathcal{H}}_s$ and $\widehat{\mathcal{H}}'_S$. We may also write

$$G_{\mathbb{A}} = \big\{ \prod_{v \in S} G_v \big\} \times \big\{ \prod_{v \notin S} G_v \big\} = G_S \times \widehat{G}_S$$

and

$$G'_{\mathbb{A}} = \{ \prod_{v \in S} G'_v \} \times \{ \prod_{v \notin S} G'_v \} = G'_S \times \widehat{G}'_S.$$

The second factor is in both cases a restricted direct product. There is an isomorphism $\theta : \widehat{G}'_S \to \widehat{G}_s$ defined by

$$\theta(\prod_{v \notin S} g'_v) = \prod_{v \notin S} \theta_v(g'_v)$$

We will interpret $\widehat{\mathcal{H}}_S$ and $\widehat{\mathcal{H}}'_S$ as algebras of measures on $\widehat{G}_S$ and $\widehat{G}'_S$ and then the isomorphism between them will be that associated to θ.

We can also interpret the elements of $\mathcal{H}$ and $\mathcal{H}'$ as measures on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$. For example any element of $\mathcal{H}$ is a linear combination of elements of the form $f = \otimes_v f_v$. Let T be a finite set of places and suppose that $f_v = \varepsilon_v$ for v not in T. If T' contains T, on the group

$$G_{\mathbb{A}(T')} = \{ \prod_{v \in T'} G_v \} \times \{ \prod_{v \notin T'} K_v \}$$

we can introduce the product of the measures f_v. Since $G_{\mathbb{A}}$ is the union of these groups and the measures on them are consistent we can put the measure together to form a measure f on $G_{\mathbb{A}}$. If each f_v is the measure associated to a function then f is also. Such measures form a subalgebra $\mathcal{H}_1$ and $\mathcal{H}$.

The notion of an elementary idempotent of $\mathcal{H}$ or $\mathcal{H}'$ is defined in the obvious way. If ξ is an elementary idempotent of $\mathcal{H}$ there is another elementary idempotent ξ_1 of the form $\xi_1 = \otimes_v \xi_v$ where ξ_v is an elementary idempotent of $\mathcal{H}_v$ and $\xi_v = \varepsilon_v$ for almost all v so that $\xi_1 \xi = \xi$.

We shall now discuss the representations of $\mathcal{H}$. A representation π of $\mathcal{H}$ on the vector space V over $\mathbb{C}$ will be called admissible if the following conditions are satisifed

(i) Every w in V is a linear combination of the form $\sum \pi(f_i)\, w_i$ with f_i in $\mathcal{H}_1$.

(ii) If ξ is an elementary idempotent the range of $\pi(\xi)$ is finite dimensional.

(iii) Let v_0 be an archimedean place. Suppose that for each v an elementary idempotent ξ_v is given and that $\xi_v = \varepsilon_v$ for almost all v. Let $\xi = \otimes_v \xi_v$. If w is in V the map

$$f_{v_0} \to \pi\Big(f_{v_0} \otimes \{ \otimes_{v \neq v_0} \xi_v \} \Big) w$$

of $\xi_{v_0} \mathcal{H}_{v_0} \xi_{v_0}$ into the finite dimensional space $\pi(\xi) V$ is continuous.

Suppose that an admissible representation π_v of $\mathcal{H}_V$ on V_v is given for each v. Assume that for almost all v the range of $\pi_v(\varepsilon_v)$ is not zero. Assume also that the range of $\pi_v(\varepsilon_v)$ has dimension one when it is not zero. As we saw in the first chapter this supplementary condition is satisified if the representations π_v are irreducible. Choosing for almost all v a vector e_v such that $\pi_v(\varepsilon_v)\, e_v = e_v$ we may form $V = \otimes_{e_v} V_v$. Let π be the representation $\otimes_v \pi_v$ on V. Because of the supplementary condition it is, apart from equivalence, independent of the choice of the e_v.

π will be admissible. To see this observe first of all that condition (i) has only to be verified for vectors of the form $w = \otimes_v w_v$. Suppose $w_v = e_v$ when v is not in the finite set T which we suppose contains all archimedean places. If v is not in T let $f_v = \varepsilon_v$ so that $w_v = \pi(f_v)\, w_v$. If v is in T let

$$w_v = \sum \pi_v(f_v^i)\, w_v^i.$$

Then

$$w = \{ \otimes_{v \in T} \sum \pi_v(f_v^i)\, w_v^i \} \otimes \{ \otimes_{v \notin T} \pi(f_v)\, w_v \}.$$

Expanding the right hand side we obtain the desired relation. The second condition has only to be verified for elementary idempotents of the form $\xi = \otimes_v \xi_v$. Then

$$\pi(\xi)\, V = \otimes \pi(\xi_v)\, V_v$$

Since $\pi(\xi_v)V_v$ is finite dimensional for all v and $\pi(\xi_v)V_v = \pi(\varepsilon_v)V_v$, which has dimension one, for almost all v the right side is finite dimensional. The last condition results from the admissibility of π_{v_0}.

π cannot be irreducible unless each π_v is. Suppose however that each π_v is irreducible. If ξ_v is an elementary idempotent of $\mathcal{H}_v$ and if $\pi_v(\xi_v) \neq 0$ we have a representation π_{ξ_v} of $\xi_v\, \mathcal{H}_v\, \xi_v$ on $\pi_v(\xi_v)\, V_v$. Since it is irreducible π_{ξ_v} determines a surjective map

$$\pi_{\xi_v} : \xi_v\, \mathcal{H}_v\, \xi_v \to L(\xi_v)$$

if $L(\xi_v)$ is the ring of linear transformations of $V(\xi_v) = \pi_v(\xi_v)V_v$. To show that π is irreducible we have only to show that for every elementary idempotent of the form $\xi = \otimes_v \xi_v$ the representation of $\xi\mathcal{H}\xi$ on $V(\xi) = \pi(\xi)V$ is irreducible. Suppose that $\xi_v = \varepsilon_v$ if v is not in T. Then

$$V(\xi) = \otimes_v V(\xi_v)$$

is isomorphic to $\otimes_{v\in T} V(\xi_v)$. The full ring of linear transformations of this space is

$$\otimes_{v\in T} L(\xi_v)$$

and therefore the full ring of linear transformations of $V(\xi)$ is

$$\big\{ \otimes_{v\in T}\, L(\xi_v) \big\} \otimes \big\{ \otimes_{v\notin T}\, \pi_v(\varepsilon_v) \big\}.$$

This is the image under π of

$$\big\{ \otimes_{v\in T}\, \xi_v\, \mathcal{H}_v\, \xi_v \big\} \otimes \big\{ \otimes_{v\notin T}\, \varepsilon_v \big\}$$

which is contained in $\xi\mathcal{H}\xi$.

An admissible representation equivalent to one constructed by tensor products is said to be factorizable.

Proposition 9.1 *Every irreducible admissible representation of $\mathcal{H}$ is factorizable. The factors are unique up to equivalence.*

Suppose π is such a representation. Let I be the set of elementary idempotents of the form $\xi = \otimes \xi_v$ for which $\pi(\xi)$ is not 0. I is certainly not empty. Let $V(\xi) = \pi(\xi)V$ if V is the space on which π acts. If ξ and ξ' are elementary idempotents we write $\xi \le \xi'$ if $\xi'\xi = \xi$. Then $\xi\,\xi'$ will also equal ξ. If $\xi = \otimes \xi_v$ and $\xi' = \otimes \xi'_v$ then $\xi \le \xi'$ if and only if $\xi_v\, \xi'_v = \xi'_v\, \xi_v = \xi_v$ for all v. If $\xi \le \xi'$ and ξ belongs to I so does ξ'. $\xi\mathcal{H}\xi$ is a subalgebra of $\xi'\mathcal{H}\xi'$. Let $\iota(\xi', \xi)$ be the corresponding injection and let $L(\xi)$ and $L(\xi')$ be the spaces of linear transformations of $V(\xi)$ and $V(\xi')$. There is exactly one map

$$\varphi(\xi', \xi) : L(\xi) \to L(\xi')$$

which makes

$$\begin{tikzcd}
\xi\mathcal{H}\xi \arrow[r, "\iota(\xi',\xi)"] \arrow[d, "\pi_\xi"'] & \xi'\,\mathcal{H}\,\xi' \arrow[d, "\pi_{\xi'}"'] \\
L(\xi) \arrow[r, "\varphi(\xi',\xi)"] & L(\xi')
\end{tikzcd}$$

commutative.

There is a map of $\xi_v \mathcal{H}_v \xi_v$ into $\xi \mathcal{H} \xi$ which sends f_v to $f_v \otimes \{\otimes_{w \neq v} \xi_w\}$. Composing this map with π_ξ we obtain a map π_ξ^v of $\xi_v \mathcal{H}_v \xi_v$ onto a subalgebra $L_v(\xi)$ of $L(\xi)$. $L(\xi)$ and $L_v(\xi)$ have the same unit, namely $\pi_\xi(\xi)$. If $v \neq w$ the elements of $L_v(\xi)$ commute with those of $L_w(\xi)$. If we form the tensor product of the algebras $L_v(\xi)$ with respect to the family of units there is a map from $\otimes_v L_v(\xi)$ to $L(\xi)$ which sends $\otimes_v \lambda_v$ to $\prod_v \lambda_v$. Moreover we may identify $\otimes_v \xi_v \mathcal{H}_v \xi_v$ and $\xi \mathcal{H} \xi$. Since the diagram

$$\begin{array}{ccc} \otimes_v \, \xi_v \, \mathcal{H}_v \xi_v & \longrightarrow & \xi \, \mathcal{H} \, \xi \\ {\scriptstyle \otimes_v \pi_\xi^v} \big\downarrow & & {\scriptstyle \pi_\xi} \big\downarrow \\ \otimes_v \, L_v(\xi) & \longrightarrow & L(\xi) \end{array}$$

is commutative the bottom arrow is surjective.

Lemma 9.1.1 *The algebras $L_v(\xi)$ are simple and the map $\otimes_v L_v(\xi) \to L(\xi)$ is an isomorphism.*

To show that $L_v(\xi)$ is simple we need only show that the faithful $L_v(\xi)$-module $V(\xi)$ is spanned by a family of equivalent irreducible submodules. Let M be any irreducible submodule. Then the family $\{TM\}$ where T runs over the image of $1_v \otimes \{\otimes_{w \neq v} L_w(\xi)\}$ spans $V(\xi)$ and each TM is 0 or equivalent to M because T commutes with the elements of $L_v(\xi)$. 1_v is the unit of $L_v(\xi)$. We have only to show that $\otimes_v L_v(\xi) \hookrightarrow L(\xi)$. Since $\otimes_v L_v(\xi)$ is the inductive limit of $\otimes_{v \in T} L_v(\xi)$, where T is a finite set, we have only to show that the map is injective on these subalgebras. As they are tensor products of simple algebras they are simple and the map is certainly injective on them.

If $\xi \leq \xi'$ there is a commutative diagram

$$\begin{array}{ccc} \otimes_v \, \xi_v \, \mathcal{H}_v \, \xi_v & \xrightarrow{\iota(\xi', \xi)} & \otimes_v \, \xi'_v \, \mathcal{H}_v \, \xi'_v \\ \big\downarrow & & \big\downarrow \\ \otimes_v \, L_v(\xi) & & \otimes_v \, L_v(\xi') \\ \big\downarrow & & \big\downarrow \\ L(\xi) & \xrightarrow{\varphi(\xi', \xi)} & L(\xi') \end{array}$$

Moreover if $\iota_v(\xi', \xi)$ is the imbedding of $\xi_v \mathcal{H}_v \xi_v$ into $\xi'_v \mathcal{H}_v \xi'_v$ then $\iota(\xi', \xi) = \otimes_v \iota_v(\xi', \xi)$. We want to verify that a horizontal arrow $\otimes_v \varphi_v(\xi', \xi)$ can be inserted in the middle without destroying the commutativity. To do this we have only to show that if f_v is in $\xi_v \mathcal{H}_v \xi_v$ and therefore in $\xi'_v \mathcal{H}_v \xi'_v$ then $\pi_\xi^v(f_v) = 0$ if and only if $\pi_{\xi'}^v(f_v) = 0$. Let $U = \pi_\xi^v(f_v)$ and let $T = \pi_{\xi'}^v(f_v)$. If

$$E = \pi_{\xi'} \Big(\xi'_v \otimes \big\{ \otimes_{w \neq v} \xi_w \big\} \Big)$$

then

$$TE = \pi_{\xi'} \Big(f_v \otimes \big\{ \otimes_{w \neq v} \xi_w \big\} \Big)$$

is determined by its restriction to $V(\xi)$ and that restriction is U.

It is clear that if S is a sufficiently large finite set the map $\otimes_{w \in S} L_w(\xi') \to L(\xi')$ is an isomorphism. We suppose that S contains v. E belongs to the image M of $1_v \otimes \{\otimes_{w \neq v} L_w(\xi')\}$. Since M is simple and E is not 0 there are A_i, B_i $1 \leq i \leq r$ in M such that

$$\sum_{i=1}^{r} A_i \, E \, B_i = 1$$

Thus

$$T = \sum_i T\, A_i\, E\, B_i = \sum_i A_i\, T\, E\, B_i$$

and $T = 0$ if and only if $U = 0$.

Since the necessary compatibility conditions are satisifed we can take inductive limits, over I, to the left and right. The inductive limit of the $\xi\mathcal{H}\xi$ is $\mathcal{H}$ and that of the $\xi_v\mathcal{H}_v\xi_v$ is $\mathcal{H}_v$. Let L_v be that of $L_v(\xi)$ and L that of $L(\xi)$. There is a map $\pi^v : \mathcal{H}_v \to L_v$ and, for almost all v, $\pi^v(\varepsilon_v) = \mu_v$ is not zero. We have a commutative diagram

$$\begin{array}{ccc} \otimes\,\mathcal{H}_v & \longrightarrow & \mathcal{H} \\ {\scriptstyle \oplus\pi^v}\big\downarrow & & \big\downarrow \\ \otimes\, L_v & \longrightarrow & L \end{array}$$

in which the rows are isomophisms. Moreover L acts faithfully on V and the representation of $\mathcal{H}$ on V can be factored through L.

If A is an algebra with a minimal left ideal J then any faithful irreducible representation of A on a vector space X is equivalent to the representation on J. In fact we can choose x_0 in X so that $Jx_0 \neq 0$. The map $j \to jx_0$ of J to X gives the equivalence. Thus to prove that π is factorizable it will be enough to show that L has a minimal left ideal, that the representation of L on this minimal left ideal is a tensor product of representations σ_v of L_v, and that $\sigma_v \circ \pi^v$ is admissible.

Suppose A is a simple algebra and J is a left ideal in A. If a in A is not 0 and $aJ = 0$ then $Aa\, AJ = AJ = 0$. If J is not 0 this is impossible. Suppose e is an idempotent of A and $A_1 = eAe$. Let J_1 be a minimal left ideal of A_1 and let $J = AJ_1$. If J were not minimal it would properly contain a non-zero ideal J'. $J' \cap A_1$ would have to be 0. Since $Je = J$ we must have $eJ = eJe = 0$. Since this is a contradiction J is minimal. Suppose for example that A is the union of a family $\{A_\lambda\}$ of matrix algebras. Suppose that for each λ there is an idempotent e_λ in A such that $A_\lambda = e_\lambda A e_\lambda$ and that given λ_1 and λ_2 there is a λ_3 such that A_{λ_3} contains A_{λ_1} and A_{λ_2}. Then A is certainly simple and, by the preceding discussion, contains a minimal left ideal.

The algebras L and L_v satisfy these conditions. In fact, speaking a little loosely, L is the union of the $L(\xi)$ and L_v is the union of $L_v(\xi)$. Choose ξ so that $V(\xi) \neq 0$ and let J_v be a minimal left ideal in $L_v(\xi)$. Since $L_v(\xi)$ is one-dimensional for almost all v the ideal $J_v = L_v(\xi)$ for almost all v. Thus $J = \otimes J_v$ exists and is a minimal left ideal of $L(\xi)$. Thus $LJ = \otimes L_v J_v$. LJ is a minimal left ideal of L and $L_v J_V$ is a minimal left ideal of L_v. The representation of L on LJ is clearly the tensor product of the representations σ_v of L_v on $L_v J_V$.

Thus π is equivalent to the tensor product of the representations $\pi_v = \sigma_v \circ \pi^v$. The representations π_v are irreducible. Since it is easily seen that a tensor product $\otimes\pi_v$ is admissible only if each factor is admissible we may regard the first assertion of the proposition as proved.

If π is an admissible representation of $\mathcal{H}$ on V and v is a place we may also introduce a representation of $\mathcal{H}_v$ on V which we still call π. If u is in V we choose $\xi = \otimes_w \xi_w$ so that $\pi(\xi)u = u$. Then if f belongs to $\mathcal{H}_v$ we set

$$\pi(f)u = \pi\Big(f\,\xi_v \otimes \big\{\otimes_{w\neq v}\,\xi_w\big\}\Big)u$$

The second part of the proposition is a consequence of the following lemma whose proof is immediate.

Lemma 9.1.2 *Suppose $\pi = \otimes_w \pi_w$. Then the representation π of $\mathcal{H}_v$ is the direct sum of representations equivalent to π_v.*

Let S_a be the set of archimedean primes. One can also associate to an admissible representation π of $\mathcal{H}$ on V a representation of $\widehat{G}_{S_a}$, the group formed by the elements of G_A whose components at every archimedean place are 1, on V. If v is archimedean one can associate to π a representation of $\mathfrak{A}_v$, the universal enveloping algebra of the Lie algebra of G_v, on V. Finally π determines a representation of the group $Z_{\mathbb{A}}$ of scalar matrices in $GL(2,\mathbb{A})$. If π is irreducible there is a quasi-character η of I the group of idèles such that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

for all a in I. If π_v is associated to η_v and $\pi = \otimes_v \pi_v$ then π is associated to the quasi-character η defined by

$$\eta(a) = \prod_v \eta_v(a_v).$$

One may define the contragredient of π and the tensor product of π with a quasi-character of I. All the expected formal relations hold. In particular $\tilde{\pi}$ is equivalent to $\eta^{-1} \otimes \pi$ if π is irreducible.

The above discussion applies, *mutatis mutandis*, to the algebra $\mathcal{H}'$. The next proposition, which brings us a step closer to the theory of automorphic forms, applies to $\mathcal{H}$ alone.

Proposition 9.2 *Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of $\mathcal{H}$. Suppose that π_v is infinite dimensional for all v. Let ψ be a non-trivial character $\mathbb{A}/F$. There is exactly one space $W(\pi,\psi)$ of continuous functions on $G_{\mathbb{A}}$ with the following properties:*

(i) *If W is in $W(\pi,\psi)$ then for all g in $G_{\mathbb{A}}$ and all x in $\mathbb{A}$*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\, W(g)$$

(ii) *$W(\pi,\psi)$ is invariant under the operators $\rho(f)$, $f \in \mathcal{H}$, and transforms according to the representation π of $\mathcal{H}$. In particular it is irreducible under the action of $\mathcal{H}$.*

(iii) *If F is a number field and v an archimedean place then for each W in $W(\pi,\psi)$ there is a real number N such that*

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = O\big(|a|^N\big)$$

as $a \to \infty$ in $F_v^\times$.

In the last assertion $F_v^\times$ is regarded as a subgroup of I. F_v is a subgroup of $\mathbb{A}$ and the restriction ψ_v of ψ to F_v is non-trivial. Thus for each place v the space $W(\pi_v,\psi_v)$ is defined and we may suppose that π_v acts on it. Moreover for almost all v the largest ideal of F_v on which ψ_v if trivial is O_v and π_v contains the trivial representation of K_v. Thus by Proposition 3.5 there is a unique function φ_v^0 in $W(\pi_v,\psi_v)$ such that $\varphi_v^0(g_v\, k_v) = \varphi_v^0(g_v)$ for all k_v in K_v and $\varphi_v^0(I) = 1$. Then $\varphi_v^0(k_v) = 1$ for all k_v in K_v. π acts on

$$\otimes_{\varphi_v^0} W(\pi_v,\psi_v)$$

If g is in $G_{\mathbb{A}}$ and $\otimes\varphi_v$ belongs to this space then $\varphi_v(g_v) = 1$ for almost all v so that we can define a function φ on $G_{\mathbb{A}}$ by

$$\varphi(g) = \prod_v \varphi_v(g_v).$$

The map $\otimes\varphi_v \to \varphi$ extends to a map of $\otimes W(\pi_v,\psi_v)$ into a space $W(\pi,\psi)$ of functions on $G_{\mathbb{A}}$. $W(\pi,\psi)$ certainly has the required properties. We have to show that it is characterized by these properties.

Suppose $\mathfrak{M}$ is another space with these properties. There is an isomorphism T of $\otimes W(\pi_v,\psi_v)$ and $\mathfrak{M}$ which commutes with the action of $\mathcal{H}$. All we have to do is show that there is a constant c such that if $\varphi = \otimes\varphi_v$ then

$$T\varphi(g) = c\prod_v \varphi_v(g_v).$$

Let S be a finite set of places and let

$$W_S = \otimes_{v\in S} W(\pi_v,\psi_v)$$

and

$$\widehat{W}_S = \otimes_{v\neq S} W(\pi_v,\psi_v).$$

Then

$$\otimes\, W(\pi_v,\psi_v) = W_S \otimes \widehat{W}_S.$$

We first show that if S is given there is a function c_S on $\widehat{G}_S \times \widehat{W}_S$ such that if

$$f = T\Big(\big\{\otimes_{v\in S}\,\varphi_v\big\}\otimes\varphi\Big)$$

with φ in $\widehat{W}_S$ then

$$f(gh) = c_S(h,\varphi)\prod_{v\in S}\varphi_v(g_v)$$

if g is in G_S and h is in $\widehat{G}_S$.

Suppose that S consists of the single place v. If φ belongs to $\widehat{W}_S$ and h belongs to $\widehat{G}_S$ associate to every function φ_v in $W(\pi_v,\psi_v)$ the function

$$\varphi'_v(g_v) = f(g_v h)$$

on G_v. f is $T(\varphi_v\otimes\varphi)$. By construction, if φ_v is replaced by $\rho(f_v)\varphi_v$ with f_v in $\mathcal{H}_v$ the function φ'_v is replaced by $\rho(f_v)\varphi'_v$. Moreover if x is in F_v

$$\varphi'_v\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g_v\right) = \psi_v(x)\,\varphi'_v(g_v).$$

Since any conditions on rates of growth can easily be verified we see that the functions φ'_v are either all zero or they fill up the space $W(\pi_v,\psi_v)$. In both cases the map $\varphi_v \to \varphi'_v$ is a map of $W(\pi_v,\psi_v)$ into itself which commutes with the action of $\mathcal{H}_v$ and therefore consists merely of multiplication by a scalar $c_S(h,\varphi)$.

Now suppose that S' is obtained by adjoining the place w to S and that our assertion is true for S. Take h in $\widehat{G}_{S'}$ and φ in $\widehat{W}_{S'}$. If

$$f = T\Big(\big\{\otimes_{v\in S'}\,\varphi_v\big\}\otimes\varphi\Big)$$

then, for g in G_S, and g_w in G_w,

$$f(g\,g_w h) = c_S(g_w h,\varphi_w\otimes\varphi)\prod_{v\in S}\varphi_v(g_v).$$

The argument used before shows that for a given h and φ the function

$$g_w \to c_S(g_w h, \varphi_w \otimes \varphi)$$

is a multiple $c_{S'}(h,\varphi)$ of φ_w.

To prove the existence of c we obseve first that if S is the disjoint union of S_1 and S_2 we may write any h_1 in $\widehat{G}_{S_1}$ as $h_1 = h\prod_{v\in S_2} h_v$ with h in $\widehat{G}_S$. Suppose $\varphi_1 = \{\otimes_{v\in S_2}\varphi_v\}\otimes\varphi$ with φ in $\widehat{W}_S$ is in $\widehat{W}_{S_1}$. Then

$$c_{S_1}(h_1,\varphi_1) = \Big\{\prod_{v\in S_2}\varphi_v(h_v)\Big\}\,c_S(h,\varphi) \tag{9.2.1}$$

because the right hand side has all the properties demanded of the left. If S_1 is large enough that φ_v^0 exists for v not in S_1 then, by its definition, $c_{S_1}(h, \otimes_{v\notin S_1}\varphi_v^0)$ has a constant $c(S_1)$ on

$$\prod_{v\notin S_1} K_v$$

The formula (9.2.1) shows that $c(S) = c(S_1)$ if S contains S_1. We take c to be the common value of these constants. Given $\varphi = \otimes\,\varphi_v$ and $g = \prod g_v$ we choose S so that $\varphi_v = \varphi_v^0$ and $g_v \in K_v$ for v not in S. Then

$$\begin{aligned} T\,\varphi(g) &= c\Big(\prod_{v\notin S} g_v, \otimes_{v\notin S}\varphi_v\Big)\prod_{v\in S}\varphi_v(g_v) \\ &= c\prod_v \varphi_v(g_v). \end{aligned}$$

We observed that if π_v is finite dimensional the space $W(\pi_v,\psi_v)$ cannot exist if v is non-archimedean or real. Although we neglected to mention it, the argument used for real field also shows that $W(\pi_v,\psi_v)$ cannot exist if v is complex. The proof of Proposition 9.2 can therefore be used, with minor changes, to verify the next proposition.

Proposition 9.3 *If $\pi = \otimes\pi_v$ is given and if one of the representations π_v is finite dimensional there can exist no space $W(\pi,\psi)$ satisfying the first two conditions of the previous proposition.*

An admissible representation π of $\mathcal{H}$ on the space V is said to be unitary if there is a positive definite hermitian form (v_1,v_2) on V such that, if $f^*(g) = \bar{f}(g^{-1})$,

$$\big(\pi(f)v_1, v_2\big) = \big(v_1, \pi(f^*)v_2\big)$$

for all f in $\mathcal{H}$.

Lemma 9.4 *If π is unitary and admissible then V is the direct sum of mutually orthogonal invariant irreducible subspaces.*

The direct sum of the lemma is to be taken in the algebraic sense. We first verify that if V_1 is an invariant subspace and V_2 is its orthogonal complement then $V = V_1\oplus V_2$. Certainly $V_1\cap V_2 = 0$. Let ξ be an elementary idempotent and let $V(\xi)$, $V_1(\xi)$, $V_2(\xi)$ be the ranges of $\pi(\xi)$ in V, V_1, and V_2. let $V_1^\perp(\xi)$ be the range of $1-\pi(\xi)$ acting on V_1. Then $V(\xi)$ and $V_1^\perp(\xi)$ are orthogonal and

$$V_1 = V_1(\xi)\oplus V_1^\perp(\xi).$$

Thus $V_2(\xi)$ is just the orthogonal complement of $V_1(\xi)$ in $V(\xi)$. Since $V(\xi)$ is finite dimensional

$$V(\xi) = V_1(\xi)\oplus V_2(\xi).$$

Since every element of V is contained in some $V(\xi)$ we have $V = V_1 + V_2$.

To complete the proof we shall use the following lemma.

Lemma 9.4.1 *If π is a unitary admissible representation of $\mathcal{H}$ on the space V then V contains a minimal non-zero invariant subspace.*

Choose an idempotent ξ so that $V(\xi) = \pi(\xi)V \neq 0$. Since $V(\xi)$ is finite dimensional amongst all the non-zero subspaces of it obtained by intersecting it with an invariant subspace of V there is a minimal one N. Let M be the intersection of all invariant subspaces containing N. If M is not irreducible it is the direct sum of two orthogonal invariant subspaces M_1 and M_2. Then

$$N = M \cap V(\xi) = \pi(\xi)\, M = \pi(\xi)\, M_1 \oplus \pi(\xi)\, M_2$$

The right side is

$$\{M_1 \cap V(\xi)\} \oplus \{M_2 \cap V(\xi)\}$$

so that one of $M_1 \cap V(\xi)$ and $M_2 \cap V(\xi)$ is N. Then M_1 or M_2 contains M. This is a contradiction.

Let A be the set consisting of families of mutually orthogonal invariant, and irreducible subspaces of V. Each member of the family is to be non-zero. Let $\{V_\lambda\}$ be a maximal family. Then $V = \oplus_\lambda V_\lambda$. If not let $V_1 = \oplus_\lambda V_\lambda$. The orthogonal complement of V_1 would be different from zero and therfore would contain a minimal non-zero invariant subspace which when added to the family $\{V_\lambda\}$ would make it larger.

If T is a finite set of places most of the results of this paragraph are valid for representations π of $\widehat{\mathcal{H}}_T$. For example π is factorizable and $W(\pi, \psi)$ exists as a space of functions on $\widehat{G}_T$.

§**10. Automorphic Forms.** In this paragraph F is still a global field. We shall begin by recalling a simple result from reduction theory. If v is a place of $\mathbb{A}$ and a is in $\mathbb{A}$ then $|a|_v$ is the absolute value of a_v the vth component of a. If a is in I

$$|a| = \prod_v |a|_v$$

Lemma 10.1 *There is a constant c_0 such that if g belongs to $G_{\mathbb{A}}$ there is a γ in G_F for which*

$$\prod_v \max\{|c|_v, |d|_v\} \le c_0 |\det g|^{\frac{1}{2}}$$

if

$$\gamma\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If F is a number field let O_F be the ring of integers in F and if F is a function field take any transcendental element x of F over which F is separable and let O_F be the integral closure in F of the ring generated by 1 and x. A place v will be called finite if $|a|_v \le 1$ for all a in O_F; otherwise it will be called infinite. If S is a finite set of places which contains all the infinite places let

$$\mathbb{A}(S) = \{a \in \mathbb{A} \,\big|\, |a|_v \le 1 \text{ if } v \ne S\}$$
$$I(S) = \{a \in I \,\big|\, |a|_v = 1 \text{ if } v \ne S\}$$

Then $\mathbb{A} = F + \mathbb{A}(S)$ and if S is sufficiently large $I = F^\times I(S)$. We first verify that if $I = F^\times I(S)$ then

$$G_{\mathbb{A}} = G_F\, G_{\mathbb{A}(S)}$$

where $G_{\mathbb{A}(S)} = GL\big(2, \mathbb{A}(S)\big)$. If v is not in S then v is non-archimedean and we can speak of ideals of F_v. Any element of $G_{\mathbb{A}}$ may be written as a product

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in which the second factor belongs to

$$K = \prod_v K_v$$

and therefore to $G_{\mathbb{A}(S)}$. It will be sufficient to show that the first factor is in $G_F\, G_{\mathbb{A}(S)}$. If $\alpha = \alpha_1\alpha_2$ and $\gamma = \gamma_1\gamma_2$ with α_1 and γ_1 in $F^\times$ and α_2 and γ_2 in $I(S)$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha_1\gamma_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

The first factor is in G_F and the third in $G_{\mathbb{A}(S)}$. Since $\frac{\beta}{\alpha_1\gamma_2}$ belongs to $F + \mathbb{A}(S)$ the second factor is in $G_F\, G_{\mathbb{A}(S)}$ and the assertion follows.

There is certainly a u in O_F such that $|u|_v < 1$ at all finite places in S. Enlarging S if necessary we may assume that a finite place v belongs to S if and only if $|u|_v < 1$. Then

$$F \cap \mathbb{A}(S) = \big\{\frac{x}{u^m} \,\big|\, x \in O_F,\, m \in \mathbb{Z}\big\}.$$

We identify the prime ideals of O_F with the places corresponding to them. By the theory of rings of quotients the proper ideals of $F \cap \mathbb{A}(S)$ are the ideals of the form

$$\big(F \cap \mathbb{A}(S)\big) \prod_{\mathfrak{p} \notin S} \mathfrak{p}^{m_\mathfrak{p}}$$

Since $I = F^\times I(S)$ every such ideal is principal. Thus $F \cap \mathbb{A}(S)$ is a principal ideal domain.

To prove the lemma we show that there is a constant c_0 such that if g belongs to $G_{\mathbb{A}(S)}$ there is a γ in $G_{F\cap\mathbb{A}(S)}$ such that

$$\prod_{v\in S} \max\{|c|_v, |d|_v\} \le c_0 |\det g|^{\frac{1}{2}}$$

if

$$\gamma\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Fix a Haar measure on the additive group $\mathbb{A}(S)$. This determines a measure on $\mathbb{A}(S) \oplus \mathbb{A}(S)$. The group $L = \big(F \cap \mathbb{A}(S)\big) \oplus \big(F \cap \mathbb{A}(S)\big)$ is a discrete subgroup of $\mathbb{A}(S) \oplus \mathbb{A}(S)$ and the quotient $\mathbb{A}(S) \oplus \mathbb{A}(S)/L$ is compact and has finite measure c_1. If g belongs to $G_{\mathbb{A}(S)}$ the lattice Lg is also discrete and the quotient $\mathbb{A}(S) \oplus \mathbb{A}(S)/Lg$ has measure $c_1|\det g|$.

Suppose $(m, n) = (\mu, \nu)g$ belongs to Lg. If $a \ne 0$ belongs to $F \cap \mathbb{A}(S)$ then

$$\prod_{v\in S} \max\{|am|_v, |an|_v\} = \big(\prod_{v\in S} |a|_v\big)\big(\prod_{v\in S} \max\{|c|_v, |d|_v\}\big).$$

Since

$$1 = \prod_v |a|_v = \big(\prod_{v\in S} |a|_v\big)\big(\prod_{v\ne S} |a|_v\big)$$

the product $\prod_{v\in S} |a|_v$ is at least 1 and

$$\prod_{v\in S} \max\{|am|_v, |an|_v\} \ge \prod_{v\in S} \max\{|m|_v, |n|_v\}.$$

Let R be a positive number and consider the set

$$E = \big\{(m, n) \in Lg \mid \prod_{v\in S} \max\{|m|_v, |n|_v\} \le R\big\}.$$

The previous inequality shows that if I contains a non-zero element of Lg it contains one $(m, n) = (\mu, \nu)g$ for which μ and ν are relatively prime. Then we may choose κ and λ in $F \cap \mathbb{A}(S)$ so that $\kappa\nu - \lambda\mu = 1$. If

$$\gamma = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$$

then γ belongs to $G_F \cap \mathbb{A}(S)$ and if

$$\gamma g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $c = m$ and $d = n$ so that

$$\prod_{v\in S} \max\{|c|_v, |d|_v\} \le R.$$

To prove the lemma we have to show that there is a constant c_0 such that if g is in $G_{\mathbb{A}(S)}$ and $R = c_0|\det g|^{\frac{1}{2}}$ the set E is not reduced to $\{0\}$. We will show in fact that there is a constant c_2 such that for all g there is a non-zero vector (m,n) in Lg with

$$\sup_{v\in S} \max\{|m|_v, |n|_v\} \le c_2|\det g|^{\frac{1}{2s}}$$

if s is the number of elements in S. There is certainly a positive constant c_3 such that the measure of

$$\big\{(m,n) \in \mathbb{A}(S) \oplus \mathbb{A}(S) \,\big|\, \sup_{v\in S} \max\{|m|_v, |n|_v\} \le R\big\}$$

is, for any choice of R, at least c_3R^{2s}. Choose c_2 so that

$$c_2 > 2\big(\frac{c_1}{c_3}\big)^{\frac{1}{2s}}.$$

If Lg contained no non-zero vector satisfying the desired inequality the set

$$\big\{(m,n) \in \mathbb{A}_s \oplus \mathbb{A}_S \,\big|\, \sup_{v\in S} \max\{|m|_v, |n|_v\} \le \frac{c_2}{2}|\det g|^{\frac{1}{2s}}\big\}$$

would intersect none of its translates by the elements of Lg. Therefore its measure would not be changed by projection on $\mathbb{A}(S) \oplus \mathbb{A}(S)/Lg$ and we would have

$$c_1 \le c_3\big(\frac{c_2}{2}\big)^{2s}$$

which is impossible.

Choose some place v of F which is to be archimedean if F is a number field. If c is any positive constant there is a compact set C in I such that

$$\{a \in I \,\big|\, |a| \ge c\}$$

is contained in

$$\{ab \,\big|\, a \in F_v^\times,\ |a| \ge c,\ b \in C\}$$

If ω_1 is a compact subset of $\mathbb{A}$, ω_2 a compact subset of I, and c a positive constant we may introduce the Siegel domain $\mathfrak{S} = \mathfrak{S}(\omega_1, \omega_2, c, v)$ consisting of all

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k$$

with x in ω_1, a in I, b in ω_2, b_1 in $F_v^\times$ with $|b_1| \ge c$, and k in K. Then $Z_{\mathbb{A}}\mathfrak{S} = \mathfrak{S}$. If we use the Iwasawa decomposition of $G_{\mathbb{A}}$ to calculate integrals we easily see that the projection of $\mathfrak{S}$ on $Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$ has finite measure. Moreover it follows readily from the previous lemma that, for a suitable choice of ω_1, ω_2, and c,

$$G_{\mathbb{A}} = G_F\,\mathfrak{S}.$$

Thus $Z_{\mathbb{A}}\, G_F \setminus G_{\mathbb{A}}$ has finite measure.

Let φ be a continuous function on $G_F \setminus G_{\mathbb{A}}$. If it is $Z_{\mathbb{A}}$-finite the space V spanned by the functions $\rho(a)\,\varphi$, $a \in Z_{\mathbb{A}}$, is finite dimensional. We may choose a finite set of points $g_1, \cdots, g_p$ and a basis $\varphi_1, \cdots, \varphi_p$ of V so that $\varphi_i(g_j) = \delta_{ij}$. Then

$$\rho(a)\,\varphi = \sum_{i=1}^{p} \lambda_i(a)\,\varphi_i.$$

Since $\lambda_i(a) = \varphi(ag_i)$ the function λ_i are continuous and finite as functions on $Z_{\mathbb{A}}$ or $Z_F \setminus Z_{\mathbb{A}}$. Since $Z_F \setminus Z_{\mathbb{A}}$ is isomorphic to $F^\times \setminus I$ it satisfies the hypothesis of Lemma 8.1 and λ_i is a finite linear combination of functions of the form

$$\lambda_i\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right) = \chi(a)\,(\log|\alpha|)^m$$

where χ is a quasi-character of $F^\times \setminus I$.

A continuous function φ on $G_F \setminus G_{\mathbb{A}}$ which is $Z_{\mathbb{A}}$-finite will be called slowly increasing if for any compact set Ω in $G_{\mathbb{A}}$ and any $c > 0$ there are constants M_1 and M_2 such that

$$|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)| \le M_2\,|a|^{M_1}$$

for g in Ω, a in I, and $|a| \ge c$. If such an inequality is valid, with suitable choice of M_2, for any M_1 we will say, for lack of a better terminology, that φ is rapidly decreasing.

Suppose φ is a continuous function on $G_F \setminus G_{\mathbb{A}}$. Assume it is K-finite on the right and that for every elementary idempotent ξ in $\mathcal{H}$ the space

$$\{\rho(\xi f)\varphi \,\big|\, f \in \mathcal{H}\}$$

is finite dimensional. An argument used more than once already shows that there is a ξ and an f in $\xi\mathcal{H}_1\xi$ such that $\rho(f)\varphi = \varphi$. If a belongs to $Z_{\mathbb{A}}$

$$\rho(a)\,\varphi = \rho(\delta_a * f)\,\varphi$$

so that φ is $Z_{\mathbb{A}}$-finite. Thus we can make the following definition.

Definition 10.2 A continuous function φ on $G_F \setminus G_{\mathbb{A}}$ is said to be an automorphic form if

(i) It is K-finite on the right

(ii) For every elementary idempotent ξ in $\mathcal{H}$ the space

$$\{\rho(\xi f)\varphi \,\big|\, f \in \mathcal{H}\}$$

is finite dimensional.

(iii) If F is a number field ξ is slowly increasing.

We observe, with regret, in passing that there has been a tendency of late to confuse the terms automorphic form and automorphic function. If not the result it is certainly the cause of much misunderstanding and is to be deplored.

Let $\mathcal{A}$ be the vector space of automorphic forms. If φ is in $\mathcal{A}$ and f is in $\mathcal{H}$ then $\rho(f)\varphi$ is in $\mathcal{A}$ so that $\mathcal{H}$ operates on $\mathcal{A}$. A continuous function on φ on $G_F \setminus G_{\mathbb{A}}$ is said to be cuspidal if

$$\int_{F\setminus\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

for all g in$G_{\mathbb{A}}$. An automorphic form which is cuspidal is called a cusp form. The space $\mathcal{A}_0$ of cusp forms is stable under the action of $\mathcal{H}$.

Proposition 10.3 *Let F be a function field and let φ be a function on $G_F \setminus G_{\mathbb{A}}$. If φ satisfies the following three conditions it is a cusp form.*

(i) *φ is K-finite on the right.*

(ii) *φ is cuspidal.*

(iii) *There is a quasi-character η of $F^\times \setminus I$ such that*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I.

If ξ is an elementary idempotent of $\mathcal{H}$ there is an open subgroup K' of K such that ξ is invariant under translations on either side by the elements of K'. Therefore the functions $\rho(\xi f)\varphi$ are invariant under right translations. To prove the proposition we show that if K' is a given open subgroup of K and η is a given quasi-character of $F^\times \setminus I$ then the space V of all continuous functions φ on $G_F \setminus G_{\mathbb{A}}$ which are cuspidal and satisfy $\varphi(gk) = \varphi(g)$ for all k in K' as well as

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in $F^\times \setminus I$ is finite-dimensional.

We shall show that there is a compact set C in $G_{\mathbb{A}}$ such that the support of every φ in V is contained in $G_F Z_{\mathbb{Z}} C$. Then the functions in V will be determined by their restrictions to C. Since C is contained in the union of a finite number of left translates of K' they will actually be determined by their values on a finite set and V will be finite dimensional.

Choose a Siegel domain $\mathfrak{S} = \mathfrak{S}(\omega_1, \omega_2, c, v)$ so that $G_{\mathbb{A}} = G_F \mathfrak{S}$. If

$$\mathfrak{S}' = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k \,\middle|\, x \in \omega_1,\ b \in \omega_2,\ b_1 \in F_v^\times,\ |b_1| \geq c,\ k \in K \right\}$$

we have just to show that the support in $\mathfrak{S}'$ of every φ in V is contained in a certain compact set which is independent of φ. In fact we have to show the existence of a constant c_1 such that φ vanishes on

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k$$

as soon as $|b_1| \geq c_1$. Let $k_1, \cdots, k_n$ be a set of representatives of the cosets of K/K' and let $\varphi_i(g) = \varphi(gk_i)$. If k belongs to $k_i K'$ then $\varphi(gk) = \varphi_i(g)$ and it will be enough to show that there is a constant c_2 such that, for $1 \leq i \leq n$,

$$\varphi_i\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

if x belongs to $\mathbb{A}$ and $|a| > c_2$. It is enough to show this for a single, but arbitrary, φ_i. Since φ_i satisifes the same hypothesis as φ, perhaps with a different group K', we just prove the corresponding fact for φ.

We use the following lemma which is an immediate consequnce of the theorem of Riemann–Roch as described in reference [10] of Chapter I.

Lemma 10.3.1 *Let X be an open subgroup of $\mathbb{A}$. There is a constant c_2 such that $\mathbb{A} = F + aX$ if a belongs to I and $|a| > c_2$.*

Let X be the set of all y for which

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

belongs to K'. Since

$$\varphi\left(\begin{pmatrix} 1 & ay \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)$$

we have

$$\varphi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)$$

if z is in aX. The equation also holds for z in F and therefore for all z in $\mathbb{A}$ if $|a| > c_2$. Then

$$\varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \frac{1}{\text{measure}(F \backslash \mathbb{A})} \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) dz$$

which by assumption is zero.

There is a corollary.

Proposition 10.4 *Suppose φ is a cusp form and for some quasi-character η of $F^\times \backslash I$*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I. Then φ is compactly supported modulo $G_F \backslash Z_{\mathbb{A}}$. Moreover the function

$$a \to \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

on $F^\times \backslash I$ is compactly supported.

The first assertion has just been verified. We know moreover that there is a constant c such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is 0 for $|a| \geq c$. If

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\varphi'(g) = \varphi(gw)$ then φ' is also a cusp form. Since

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(w^{-1}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} w\right) = \eta(a)\,\varphi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

there is also a constant c_1 such that it vanishes for $|a| \leq c_1$.

Proposition 10.5 *Let F be a function field and η a quasi-character of $F^\times \backslash I$. Let $\mathcal{A}_0(\eta)$ be the space of cusp forms φ for which*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I. The representation of $\mathcal{H}$ on $\mathcal{A}_0(\eta)$ is the direct sum of irreducible admissible representations each occurring with finite multiplicity.

The proof of Proposition 10.3 showed that the representation π of $\mathcal{H}$ on $\mathcal{A}_0(\eta)$ is admissible. Let $\eta'(\alpha) = |\eta(\alpha)|^{-1}\eta(\alpha)$. The map $\varphi \to \varphi'$ is an isomorphism of $\mathcal{A}_0(\eta)$ with $\mathcal{A}_0(\eta')$ which replaces π by $\eta_1 \otimes \pi$ if $\eta_1(\alpha) = |\eta(\alpha)|^{-1/2}$. Thus we may as well suppose that η is a character. Then if φ_1 and φ_2 belong to $\mathcal{A}_0(\eta)$ the function $\varphi_1\overline{\varphi_2}$ is a function on $G_F\, Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Since it has compact support we may set

$$(\varphi_1, \varphi_2) = \int_{G_F\, Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi_1(g)\,\overline{\varphi_2(g)}\,dg.$$

It is easily seen that

$$\big(\rho(f)\varphi_1, \varphi_2\big) = \big(\varphi_1, \rho(f^*)\varphi_2\big)$$

so that, by Lemma 9.4, π is the direct sum of irreducible admissible representations. Since π is admissible the range of $\pi(\xi)$ is finite dimensional for all ξ so that no irreducible representation occurs an infinite number of times.

The analogue of this proposition for a number field is somewhat more complicated. If φ is a continuous function on $G_{\mathbb{A}}$, if v is a place of F, and if f_v belongs to $\mathcal{H}_v$ we set

$$\rho(f_v)\varphi = \int_{G_v} \varphi(gh_v)\, f_v(h_v)\, dh_v.$$

Since f_v may be a measure the expression on the right is not always to be taken literally. If v is archimedean and if the function $\varphi(hg_v)$ on G_v is infinitely differentiable for any h in $G_{\mathbb{A}}$ then for any X in $\mathfrak{A}_v$ the universal enveloping algebra of G_v, we can also define $\rho(X)\varphi$. If S is a finite set of places we can in a similar fashion let the elements of

$$\mathcal{H}_s = \otimes_{v \in S} \mathcal{H}_v$$

or, if every place in S is archimedean,

$$\mathfrak{A}_S = \otimes_{v \in S} \mathfrak{A}_v$$

act on φ. It is clear what an elementary idempotent in $\mathcal{H}_S$ is to be. If $S = S_a$ is the set of archimedean places we set $\mathcal{H}_a = \mathcal{H}_S$.

Proposition 10.6 *Suppose F is a number field. A continuous function φ on $G_F \backslash G_{\mathbb{A}}$ is a cusp form if it satisfies the following five conditions.*

(i) *φ is K-finite on the right.*

(ii) *φ is cuspidal.*

(iii) *There is a quasi-character η of $F^\times \backslash I$ such that*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I.

(iv) *For any elementary idempotent ξ in $\mathcal{H}_a$ the space*

$$\{\rho(\xi f)\,\varphi \,\big|\, f \in \mathcal{H}_a\}$$

is finite dimensional.

(v) *φ is slowly increasing.*

There is a ξ in $\mathcal{H}_a$ such that $\rho(\xi)\varphi = \varphi$. Because of the fourth condition φ transforms according to a finite dimensional representation of $\xi\mathcal{H}_a\xi$ and the usual argument shows that there is a *function* f in $\mathcal{H}_a$ such that $\rho(f)\,\varphi = \varphi$.

Since φ is invariant under right translations by the elements of an open subgroup of $\prod_{v\notin S_a} K_v$ this implies in turn the existence of another function f in $\mathcal{H}$ such that $\rho(f)\varphi = \varphi$. From Theorem 2 of [14] one infers that φ is rapidly decreasing.

As before we may assume that η is a character. Then φ is bounded and therefore its absolute value is square integrable on $G_F\,Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$ which has finite measure. Let $L^2(\eta)$ be the space of measurable functions h on $G_F \setminus G_{\mathbb{A}}$ such that

$$h\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,h(g)$$

for all g in $G_{\mathbb{A}}$ and all a in I and

$$\int_{G_F Z_{\mathbb{A}}\setminus G_{\mathbb{A}}} |h(g)|^2\,dg < \infty.$$

According to a theorem of Godement (see reference [11] to Chapter I) any closed subspace of $L^2(\eta)$ which consists entirely of bounded functions is finite dimensional.

What we show now is that if ξ is an elementary idempotent of $\mathcal{H}$ the space

$$V = \{\rho(\xi f)\,\varphi \,\big|\, f \in \mathcal{H}\}$$

is contained in such a closed subspace. The functions in V itself certainly satisfy the five conditions of the proposition and therefore are bounded and in $L^2(\eta)$. Replacing ξ by a larger idempotent if necessary we may suppose that $\xi = \xi_a \otimes \hat{\xi}_a$ where ξ_a is an elementary idempotent in $\mathcal{H}_a$. There is a two-sided ideal $\mathfrak{a}$ in $\xi_a\mathcal{H}_a\xi_a$ such that $\rho(f)\varphi = 0$ if f belongs to $\mathfrak{a}$. The elements of $\mathfrak{a}$ continue to annihilate V and its closure in $L^2(\eta)$. Approximating the δ-function as usual we see that there is a function f_1 in $\mathcal{H}_a$ and a polynomial P with non-zero constant term such that $P(f_1)$ belongs to $\mathfrak{a}$. Therefore there is a function f_2 in $\mathcal{H}_a$ such that $f_2 - 1$ belongs to $\mathfrak{a}$. To complete the proof of the proposition we have merely to refer to Theorem 2 of [14] once again.

For a number field the analogue to Proposition 10.4 is the following.

Propositon 10.7 *Suppose φ is a cusp form and for some quasi-character η of $F^\times \setminus I$*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I. Then for any real number M_1 there is a real number M_2 such that

$$\left|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\right| \le M_2\,|a|^{M_1}$$

for all a in I. Moreover the absolute value of φ is a square integrable on $G_F Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$.

We need another corollary of Proposition 10.6. To prove it one has just to explain the relation between automorphic forms on $G_{\mathbb{A}}$ and $G_{\mathbb{R}}$, which is usually assumed to be universally known, and then refer to the first chapter of reference [11] to Chapter I. It is perhaps best to dispense with any pretence of a proof and to rely entirely on the reader's initiative. We do not however go so far as to leave the proposition itself unstated.

Proposition 10.8 *Let $\mathfrak{Z}_v$ be the centre of $\mathfrak{A}_v$ and let $\mathfrak{I}$ be an ideal of finite codimension in $\mathfrak{Z} = \otimes_{v \in S_a} \mathfrak{Z}_v$. Let ξ be an elementary idempotent of $\mathcal{H}$ and η a quasi-character of $F^\times \setminus I$. Then the space of infinitely differentiable functions φ on $G_F \setminus G_{\mathbb{A}}$ which satisfy the following five conditions is finite dimensional.*

(i) *φ is cuspidal.*

(ii) *$\rho(\xi)\varphi = \varphi$.*

(iii) *If a is in I then*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g).$$

(iv) *$\rho(X)\varphi = 0$ for all X in $\mathfrak{I}$*

(v) *φ is slowly increasing.*

Proposition 10.9 *Let η be a quasi-charcter of $F^\times \setminus I$ and let $\mathcal{A}_0(\eta)$ be the space of cusp forms φ for which*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \varphi(g)$$

for all a in I. The representation of $\mathcal{H}$ on $\mathcal{A}_0(\eta)$ is the direct sum of irreducible admissible representations each occurring with finite multiplicity.

Every element of $\mathcal{A}_0(\eta)$ is annihilated by some ideal of finite codimension in $\mathfrak{Z}$. If $\mathfrak{I}$ is such an ideal let $\mathcal{A}_0(\eta, \mathfrak{I})$ be the space of functions in $\mathcal{A}_0(\eta)$ annihilated by $\mathfrak{I}$. It is enough to prove the first part of the proposition for the space $\mathcal{A}_0(\eta, \mathfrak{I})$. Then one may use the previous proposition and argue as in the proof of Proposition 10.5. To show that every representation occurs with finite multiplicity one combines the previous proposition with the observation that two functions transforming under the same representation of $\mathcal{H}$ are annihilated by the same ideal in $\mathfrak{Z}$.

$\mathcal{H}$ acts on the space $\mathcal{A}$. An irreducible admissible representation π of $\mathcal{H}$ is a constituent of the representation on $\mathcal{A}$ or, more briefly, a constituent of $\mathcal{A}$ if there are two invariant subspaces U and V of $\mathcal{A}$ such that U contains V and the action on the quotient space U/V is equivalent to π. A constituent of $\mathcal{A}_0$ is defined in a similar fashion. The constituents of $\mathcal{A}_0$ are more interesting than the constituents of $\mathcal{A}$ which are not constituents of $\mathcal{A}_0$.

Theorem 10.10 *Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of $\mathcal{H}$ which is a constituent of $\mathcal{A}$ but not of $\mathcal{A}_0$. Then there are two quasi-characters μ and ν of $F^\times \setminus I$ such that for each place v the representation π_v is a constituent of $\rho(\mu_v, \nu_v)$.*

μ_v is the restriction of μ to $F_v^\times$. Let $\mathcal{B}$ be the space of all continuous functions φ on $G_{\mathbb{A}}$ satisfying the following conditions.

(i) For all x in $\mathbb{A}$

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \varphi(g).$$

(ii) For all α and β in $F^\times$

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) = \varphi(g).$$

(iii) φ is K-finite on the right.

(iv) For every elementary idempotent ξ in $\mathcal{H}$ the space

$$\{\rho(\xi f)\varphi \,\big|\, f \in \mathcal{H}\}$$

is finite dimensional.

Lemma 10.10.1 *A continuous function φ on $G_{\mathbb{A}}$ which satisfies the first three of these conditions satisfies the fourth if and only if it is $A_{\mathbb{A}}$-finite on the left.*

A is the group of diagonal matrices. Since φ is a function on $A_F \setminus G_{\mathbb{A}}$ it is $A_{\mathbb{A}}$ finite if and only if it is $A_F \setminus A_{\mathbb{A}}$ finite. If it is $A_F \setminus A_{\mathbb{A}}$ finite there is a relation of the form

$$\varphi(ag) = \sum_i \lambda_i(a)\,\varphi_i(g)$$

where the λ_i are finite continuous functions on $A_F \setminus A_{\mathbb{A}}$. Since $A_F \setminus A_{\mathbb{A}}$ is isomorphic to the direct product of $F^\times \setminus I$ with itself it is a group to which Lemma 8.1 can be applied. Thus there is a unique family $\varphi_{m,n,\mu,\nu}$ of functions on $G_{\mathbb{A}}$ such that

$$\varphi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| \sum \mu(a_1)\,\nu(a_2)\,(\log|a_1|)^m\,(\log|a_2|)^n\,\varphi_{m,n,\mu,\nu}(g)$$

The functions $\varphi_{m,n,\mu,\nu}$ also satisfy the first three conditions. Moreover there is a finite set S of pairs (μ,ν) and a non-negative integer M such that $\varphi_{m,n,\mu,\nu}$ is 0 if (μ,ν) does not belong to S or $m+n > M$.

Given S and M let $\mathcal{B}(S,M)$ be the space of continuous functions f on $G_{\mathbb{A}}$ which satisfy the first three conditions and for which

$$f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} g\right)$$

can be expanded in the form

$$\left|\frac{a_1}{a_2}\right|^{\frac{1}{2}} \sum \mu(a_1)\,\nu(a_2)\,(\log|a_1|)^m\,(\log|a_2|)^n\,f_{m,n,\mu,\nu}(g)$$

where the sum is taken only over the pairs (μ,ν) in S the pairs (m,n) for which $m+n \le M$. $\mathcal{B}(S,M)$ is invariant under $\mathcal{H}$. To show that if φ is $A_F \setminus A_{\mathbb{A}}$ finite it satisfies the fourth condition we show that the range of $\rho(\xi)$ on $\mathcal{B}(S,M)$ is finite dimensional.

A function f in $\mathcal{B}(S,M)$ is determined by the restriction of the finitely many functions $f_{m,n,\mu,\nu}$ to K. If f is in the range of $\rho(\xi)$ these resrictions lie in the range of $\rho(\xi)$ acting on the continuous functions on K. That range is finite dimensional.

We have also to show that if φ satisifes the fourth condition it is $A_{\mathbb{A}}$ finite. The space V spanned by the right translates of φ by the elements of K is finite dimensional and each element in it satisfies all four conditions. Let $\varphi_1, \cdots, \varphi_p$ be a basis of V. We can express $\varphi(gk)$ as

$$\sum_{i=1}^p \lambda_i(k)\,\varphi_i(g).$$

Because of the Iwasawa decomposition $G_{\mathbb{A}} = N_{\mathbb{A}} A_{\mathbb{A}} K$ it is enough to show that the restriction of each φ_i to $A_{\mathbb{A}}$ is finite. Since φ_i satisfies the same conditions as φ we need only consider the restriction of φ.

Since φ is K finite there is a finite set S of places such that φ is invariant under right translations by the elements of $\prod_{v \notin S} K_v$. Let

$$I_S = \prod_{v \in S} F_v^\times.$$

We regard I_S as a subgroup of I. If we choose S so large that $I = F^\times I(S)$ then every element α of I is a product of $\alpha = \alpha_1\alpha_2\alpha_3$ with α_1 in $F^\times$, α_2 in I_S, and α_3 in $I(S)$ such that its component at any place in S is 1. If β in I is factored in a similar fashion

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}\alpha_2 & 0\\ 0 & \beta_2\end{pmatrix}\right).$$

Thus we need only show that the restriction of φ to

$$A_S = \left\{ \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \mid \alpha,\, \beta \in I_S\right\}$$

is finite. This is a consequence of Corollary 8.4 since the restriction of φ to G_S clearly satisfies the conditions of the corollary.

The next lemma explains the introduction of $\mathcal{B}$.

Lemma 10.10.2 *If π is a constituent of $\mathcal{A}$ but not of $\mathcal{A}_0$ then it is a constituent of $\mathcal{B}$.*

If φ belongs to $\mathcal{A}$ the functions

$$\varphi_0(g) = \frac{1}{\operatorname{measure}(F\setminus\mathbb{A})}\int_{F\setminus\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\, dx$$

belongs to $\mathcal{B}$. The map $\varphi \to \varphi_0$ commutes with the action of $\mathcal{H}$ and its kernel is $\mathcal{A}_0$. Suppose U and V are two invariant subspaces of $\mathcal{A}$ and π occurs on the quotient of U by V. Let U_0 be the image of U and V_0 be the image of V in $\mathcal{B}$. Since π is irreducible there are two possibilities. Either $U_0 \neq V_0$ in which case π is equivalent to the representation on U_0/V_0 and is a constituent of $\mathcal{B}$ or $U_0 = V_0$. In the latter case

$$U = V + U \cap \mathcal{A}_0$$

and π is equivalent to the representation on

$$U \cap \mathcal{A}_0 / V \cap \mathcal{A}_0$$

which is precisely the possibility we have excluded.

Lemma 10.10.3 *If π is a constituent of $\mathcal{B}$ then there is a pair of quasi-characters μ, ν and a non-negative integer M such that π is a constituent of $\mathcal{B}(\mu,\nu,M)$.*

If S consists of the single pair (μ,ν) then, by definition, $\mathcal{B}(\mu,\nu,M) = \mathcal{B}(S,M)$. Suppose π occurs on the quotient of U by V. Choose the finite set S of pairs of quasi-characters and the non-negative integer M so that $U \cap \mathcal{B}(S,M)$ is different from $V \cap \mathcal{B}(S,M)$. Then π occurs on the quotient of $U \cap \mathcal{B}(S,M)$ by $V \cap \mathcal{B}(S,M)$ and we may as well assume that U is contained in $\mathcal{B}(S,M)$. The argument used in the eighth paragraph in an almost identical context shows that

$$\mathcal{B}(S,M) = \oplus_{(\mu,\nu)\in S}\mathcal{B}(\mu,\nu,M)$$

so that the lemma is a consequence of Lemma 8.6.

The next lemma is proved in exactly the same way as Proposition 8.5

Lemma 10.10.4 *If π is a constituent of $\mathcal{B}(\mu,\nu,M)$ for some M then it is a constituent of $\mathcal{B}(\mu,\nu) = \mathcal{B}(\mu,\nu,0)$.*

Let μ_v and ν_v be the restrictions of μ and ν to $F_v^\times$. For almost all v the quasi-characters μ_v and ν_v are unramifed and there is a unique function φ_v^0 in $\mathcal{B}(\mu_v,\nu_v)$ such that $\varphi_v^0(g_vk_v) = \varphi_v^0(g_v)$ for all k_v in K_v while $\varphi_v^0(e) = 1$. We can form

$$\otimes_{\varphi_v^0}\mathcal{B}(\mu_v,\nu_v)$$

There is clearly a linear map of this space into $\mathcal{B}(\mu,\nu)$ which sends $\otimes\varphi_v$ to the function

$$\varphi(g) = \prod_v \varphi_v(g_v)$$

It is easily seen to be surjective and is in fact, although this is irrelevant to our purposes, an isomorphism. In any case an irreducible constituent of $\mathcal{B}(\mu,\nu)$ is a constituent of $\otimes_v\rho(\mu_v,\nu_v)$.

With the following lemma the proof of Theorem 10.10 is complete.

Lemma 10.10.5 *If the irreducible admissible representation $\pi = \otimes_v\pi_v$ is a constituent of $\rho = \otimes\rho_v$, the tensor product of admissible representations which are not necessarily irreducible, then, for each v, π_v is a constituent of ρ_v.*

As in the ninth paragraph π and ρ determine representations π and ρ of $\mathcal{H}_v$. The new π will be a constituent of the new ρ. By Lemma 9.12 the representation π of $\mathcal{H}_v$ is the direct sum of representations equivalent to π_v. Thus π_v is a constituent of π and therefore of ρ. Since ρ is the direct sum of representations equivalent to ρ_v, Lemma 8.6 shows that π_v is a constituent of ρ_v.

The considerations which led to Proposition 8.5 and its proof will also prove the following proposition.

Proposition 10.11 *If π is an irreducible constituent of the space $\mathcal{A}_0$ then for some quasi-character η it is a constituent of $\mathcal{A}_0(\eta)$.*

Observe that if π is a constituent of $\mathcal{A}_0(\eta)$ then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

for all a in I. There are two more lemmas to be proved to complete the preparations for the Hecke theory.

Lemma 10.12 *Suppose there is a continuous function φ on $G_{\mathbb{A}}$ with the following properties.*

(i) *φ is K finite on the right.*

(ii) *For all α and β in $F^\times$ and all x in $\mathbb{A}$*

$$\varphi\left(\begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix} g\right) = \varphi(g).$$

(iii) *There is a quasi-character η of $F^\times \backslash I$ such that*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I.

(iv) *There is a finite set S of non-archimedean places such that the space*

$$V = \rho(\widehat{\mathcal{H}}_S)\varphi$$

transforms under $\widehat{\mathcal{H}}_S$ according to the irreducible admissible representation $\pi = \otimes_{v \notin S}\pi_v$.

Then V is a subspace of $\mathcal{B}$ and there are two quasi-characters μ and ν of $F^\times \setminus I$ such that π_v is a constituent of $\rho(\mu_v, \nu_v)$ for all v not in S.

If one observes that there is a finite set T of places which is disjoint from S such that $I = F^\times I_T$ one can proceed as in Lemma 10.10.1 to show that φ is $\mathbb{A}$-finite on the right. Thus there is a finite set R of pairs of quasi-characters and a non-negative integer M such that V is contained in $\mathcal{B}(R, M)$. The same reduction as before shows that π is a constituent of the representation of $\widehat{\mathcal{H}}_S$ on some $\mathcal{B}(\mu, \nu)$ and that π_v is a constituent of $\rho(\mu_v, \nu_v)$ if v is not in S.

Lemma 10.13 *Let φ be a continuous function on $G_F\backslash G_{\mathbb{A}}$. If φ satisfies the four following conditions it is an automorphic form.*

(i) *φ is K finite on the right.*

(ii) *There is a quasi-character η of $F^\times \setminus I$ such that*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I.

(iii) *There is a finite set S of non-archimedean places such that $\rho(\widehat{\mathcal{H}}_S)\varphi$ transforms according to an irreducible admissible representation of $\widehat{\mathcal{H}}_S$.*

(iv) *If F is a number field φ is slowly increasing.*

We have to show that for every elementary idempotent ξ in $\mathcal{H}$ the space $\rho(\xi\mathcal{H})\varphi$ is finite dimensional. If f is a continuous function on $G_F \setminus G_{\mathbb{A}}$ let

$$f_0(g) = \frac{1}{\text{measure}(F \setminus \mathbb{A})} \int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx.$$

The map $f \to f_0$ commutes with the action of $\mathcal{H}$ or of $\widehat{\mathcal{H}}_S$. Consquently φ_0 satisfies the conditions of the previous lemma and belongs to a space $\mathcal{B}(R, M)$ invariant under $\mathcal{H}$ on which $\rho(\xi)$ has a finite dimensional range.

We need only show that

$$V = \{f \in \rho(\xi\mathcal{H})\,\varphi \mid f_0 = 0\}$$

is finite dimensional. If F is a function field then, by Proposition 10.3, V is contained in $\mathcal{A}_0(\eta)$. More precisely it is contained in the range of $\rho(\xi)$, as an operator on $\mathcal{A}_0(\eta)$, which we know is finite dimensional. Suppose F is a number field. Since every place of S is non-archimedean the third condition guarantees that φ is an eigenfunction of every element of $\mathfrak{Z}$. In particular there is an ideal $\mathfrak{I}$ of finite codimension in $\mathfrak{Z}$ which annihilates φ and therefore every element of $\rho(\xi\mathcal{H})\varphi$. By Proposition 10.6 the space V is contained in $\mathcal{A}_0(\eta)$ and therefore in $\mathcal{A}_0(\eta, \mathfrak{I})$. By Proposition 10.8 the range of $\rho(\xi)$ in $\mathcal{A}_0(\eta, \mathfrak{I})$ is finite dimensional.

§**11. Hecke theory.** The preliminaries are now complete and we can broach the central topic of these notes. Let ψ be a non-trivial character of $F \setminus \mathbb{A}$. For each place v the restriction ψ_v of ψ to F_v is non-trivial. Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $\mathcal{H}$. The local L-functions $L(s, \pi_v)$ and the factors $\varepsilon(s, \pi_v, \psi_v)$ have all been defined. Since for almost all v the representation π_v contains the trivial representation of K_v and O_v is the largest ideal on which ψ_v is trivial almost all of the factor $\varepsilon(s, \pi_v, \psi_v)$ are identically 1 and we can form the product

$$\varepsilon(s, \pi) = \prod_v \varepsilon(s, \pi_v, \psi_v).$$

In general it depends on ψ. Suppose however that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a) I$$

and that η is trivial on $F^\times$. If ψ is replaced by the character $x \to \psi(\alpha x)$ with α in $F^\times$ then $\varepsilon(s, \pi_v, \psi_v)$ is multiplied b $\eta_v(\alpha) |\alpha|_v^{2s-1}$ so that $\varepsilon(s, \pi)$ is multiplied by

$$\prod_v \eta_v(\alpha)\, |\alpha|_v^{2s-1} = \eta(\alpha) |\alpha|^{2s-1} = 1$$

The product

$$\prod_v L(s, \pi_v)$$

does not converge and define a function $L(s, \pi)$ unless π satisfies some further conditions.

Theorem 11.1 *Suppose the irreducible admissible representation $\pi = \otimes \pi_v$ is a constituent of $\mathcal{A}$. Then the infinite products defining $L(s, \pi)$ and $L(s, \tilde{\pi})$ converge absolutely in a right half-plane and the functions $L(s, \pi)$ and $L(s, \tilde{\pi})$ themselves can be analytically continued to the whole complex plane as meromorphic functions of s. If π is a constituent of $\mathcal{A}_0$ they are entire. If F is a number field they have only a finite number of poles and are bounded at infinity in any vertical strip of finite width. If F is a function field with field of constants $\mathbb{F}_q$ they are rational functions of q^{-s}. Finally they satisfy the functional equation*

$$L(s, \pi) = \varepsilon(s, \pi)\, L(1 - s, \tilde{\pi}).$$

Observe that if $\pi = \otimes_v \pi_v$ then $\tilde{\pi} = \otimes_v \tilde{\pi}_v$. Consider first a representation π which is a constituent of $\mathcal{A}$ but not of $\mathcal{A}_0$. There are quasi-characters μ and ν of $F^\times \setminus I$ such that π_v is a constituent of $\rho(\mu_v, \mu_v)$ for all v. Since π_v has to contain the trivial representation of K_v for all but a finite number of v it is equal to $\pi(\mu_v, \nu_v)$ for almost all v.

Consider first the representation $\pi' = \otimes_v \pi(\mu_v, \nu_v)$. Recall that

$$L\big(s, \pi(\mu_v, \nu_v)\big) = L(s, \mu_v)\, L(s, \nu_v)$$
$$L\big(s, \tilde{\pi}(\mu_v, \nu_v)\big) = L(s, \mu_v^{-1})\, L(s, \nu_v^{-1})$$

and

$$\varepsilon\big(s, \pi(\mu_v, \nu_v), \psi_v\big) = \varepsilon(s, \mu_v, \psi_v)\, \varepsilon(s, \nu_v, \psi_v)$$

If χ is any quasi-character of $F^\times \setminus I$ the product

$$\prod_v L(s,\chi_v)$$

is known to converge in a right half plane and the funtion $L(s,\chi)$ it defines is known to be analytically continuable to the whole plane as a meromorphic function. Moreover if

$$\varepsilon(s\chi) = \prod_v \varepsilon(s,\chi_v,\psi_v)$$

the functional equation

$$L(s,\chi) = \varepsilon(s,\chi)\, L(1-s,\chi^{-1})$$

is satisifed. Since

$$L(s,\pi') = L(s,\mu)\, L(s,\nu)$$

and

$$L(s,\tilde{\pi}') = L(s,\mu^{-1})\, L(s,\nu^{-1})$$

they too are defined and meromorphic in the whole plane and satisfy the functional equation

$$L(s,\pi') = \varepsilon(s,\pi')\, L(1-s,\tilde{\pi}').$$

The other properties of $L(s,\pi')$ demanded by the lemma, at least when π' is a constituent of $\mathcal{A}$, can be inferred from the corresponding properties of $L(s,\mu)$ and $L(s,\nu)$ which are well known.

When π_v is not $\pi(\mu_v,\mu_v)$ it is $\sigma(\mu_v,\nu_v)$. We saw in the first chapter that

$$\frac{L\big(s,\sigma(\mu_v,\nu_v)\big)}{L\big(s,\pi(\mu_v,\nu_v)\big)}$$

is the product of a polynomial and an exponential. In particular it is entire. If we replace $\pi(\mu_v,\nu_v)$ by π_v we change only a finite number of the local factors and do not disturb the converge of the infinite product. If S is the finite set of places v at which $\pi_v = \sigma(\mu_v,\nu_v)$ then

$$L(s,\pi) = L(s,\pi') \prod_{v\in S} \frac{L\big(s,\sigma(\mu_v,\nu_v)\big)}{L\big(s,\pi(\mu_v,\nu_v)\big)}$$

and therefore is meromorphic with no more poles that $L(s,\pi')$. For $L(s,\tilde{\pi})$ the corresponding equation is

$$L(s,\tilde{\pi}) = L(s,\tilde{\pi}') \prod_{v\in S} \frac{L\big(s,\sigma(\mu_v^{-1},\nu_v^{-1})\big)}{L\big(s,\pi(\mu_v^{-1},\nu_v^{-1})\big)}.$$

The functional equation of $L(s,\pi)$ is a consequence of the relations

$$\frac{L\big(s,\sigma(\mu_v,\nu_v)\big)}{L\big(s,\pi(\mu_v,\nu_v)\big)} = \frac{\varepsilon\big(s,\sigma(\mu_v,\nu_v),\psi_v\big)\, L\big(1-s,\sigma(\mu_v^{-1},\nu_v^{-1})\big)}{\varepsilon\big(s,\pi(\mu_v,\nu_v),\psi_v\big)\, L\big(1-s,\pi(\mu_v^{-1},\nu_v^{-1})\big)}$$

which were verified in the first chapter. It also follows from the form of the local factors that $L(s,\pi)$ and $L(s,\tilde{\pi})$ are rational functions of q^{-s} when F is a function field. If F is a number field $L(s,\pi)$ is bounded in vertical strips of finite width in a right half-plane and, because of the functional equation, in vertical strips in a left half-plane. Its expression in terms of $L(s,\pi')$ prevents it from growing very fast at infinity in any vertical strip of finite width. The Phrágmen–Lindelöf principle implies that it is bounded at infinity in any such strip.

Now suppose π is a constituent of $\mathcal{A}_0$. It is then a constituent of $\mathcal{A}_0(\eta)$ if

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

for a in I. Since the representation of $\mathcal{H}$ in $\mathcal{A}_0(\eta)$ is the direct sum of invariant irreducible subspaces there is an invariant subspace U of $\mathcal{A}_0(\eta)$ which transforms according ot π. Let φ belong to U. If g is in $G_{\mathbb{A}}$

$$\varphi_g(x) = \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$$

is a function on $F \setminus \mathbb{A}$. Since φ_g is continuous it is determined by its Fourier series. The constant term is

$$\frac{1}{\operatorname{measure} F \setminus \mathbb{A}} \int_{F\setminus\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx$$

which is 0 because φ is a cusp form. If ψ is a given non-trivial character of $F \setminus \mathbb{A}$ the other non-trivial characters are the functions $x \to \psi(\alpha x)$ with α in $F^\times$. Set

$$\varphi_1(g) = \frac{1}{\operatorname{measure} F \setminus \mathbb{A}} \int_{F\setminus\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, \psi(-x)\, dx.$$

Since φ is a function on $G_F \setminus G_{\mathbb{A}}$.

$$\varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) = \frac{1}{\operatorname{measure} F \setminus \mathbb{A}} \int_{F\setminus\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, \psi(-\alpha x)\, dx$$

if α belongs to $F^\times$. Thus, formally at least,

$$\varphi(g) = \varphi_g(e) = \sum_{\alpha \in F^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

In any case it is clear that φ_1 is not 0 unless φ is.

Let

$$U_1 = \{\varphi_1 \,|\, \varphi \in U\}.$$

Since the map $\varphi \to \varphi_1$ commutes with the action of $\mathcal{H}$ the space U_1 is invariant and transforms according to π under right translation by $\mathcal{H}$. Moreover

$$\varphi_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\, \varphi_1(g)$$

if x is in $\mathbb{A}$. If F is a number field φ is slowly increasing. Therefore if Ω is a compact subset of $G_{\mathbb{A}}$ there is a real number M such that

$$\varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) = O(|a|^M)$$

as $|a| \to \infty$ for all g in Ω. Propositions 9.2 and 9.3 imply that all π_v are infinite dimensional and that U_1 is $W(\pi,\psi)$. Therefore U_1 is completely determined by π and ψ and U is completely determined by π. We have therefore proved the following curious proposition.

Proposition 11.1.1 *If an irreducible represntation of $\mathcal{H}$ is contained in $\mathcal{A}_0(\eta)$ it is contained with multiplicity one.*

For almost all v there is in $W(\pi_v,\psi_v)$ a function φ_v^0 such that $\varphi_v^0(g_vk_v)=\varphi_v^0(g_v)$ for all k_v in K_v while $\varphi_v^0(e)=1$. $W(\pi,\psi)$ is spanned by functions of the form

$$\varphi_1(g)=\prod_v \varphi_v(g_v) \tag{11.1.2}$$

where φ_v is in $W(\pi_v,\psi_v)$ for all v and equal to φ_v^0 for almost all v.

Suppose φ corresponds to a function φ_1 of the form (11.1.2). Suppose $\varphi=\varphi_v^0$ so that π_v contains the trivial representation of K_v. If ε_v is the normalized Haar measure on K_v let λ_v be the homomorphism of $\varepsilon_v\mathcal{H}_v\varepsilon_v$ into $\mathbb{C}$ associated to π_v. If f_v is in $\varepsilon_v\mathcal{H}_v\varepsilon_v$ then

$$\lambda_v(f_v)\,\varphi(g)=\int_{G_v}\varphi(gh)\,f_v(h)\,dh$$

and if λ_v' is the homomorphism associated to $|\eta_v|^{-1/2}\otimes\pi_v$

$$\lambda_v'(f_v)\,|\eta(\det g)|^{-\frac{1}{2}}\,\varphi(g)$$

is equal to

$$\int_{G_v}|\eta(\det gh)|^{-\frac{1}{2}}\,\varphi(gh)\,f_v(h)\,dh.$$

Since φ is a cusp form the function $|\eta(\det g)|^{-1/2}\,\varphi(g)$ is bounded and λ_v' satisifies the conditions of Lemma 3.10. Thus if $\pi_v=\pi(\mu_v,\nu_v)$ both μ_v and ν_v are unramified and

$$|\eta(\varpi_v)|^{\frac{1}{2}}\,|\varpi_v|^{\frac{1}{2}}\le|\mu_v(\varpi_v)|\le|\eta(\varpi_v)|^{\frac{1}{2}}\,|\varpi_v|^{-\frac{1}{2}}$$
$$|\eta(\varpi_v)|^{\frac{1}{2}}\,|\varpi_v|^{\frac{1}{2}}\le|\nu_v(\varpi_v)|\le|\eta(\varpi_v)|^{\frac{1}{2}}\,|\varpi_v|^{-\frac{1}{2}}$$

if ϖ_v is the generator of the maximal ideal of O_v. Consequently the infinite products defining $L(s,\pi)$ and $L(s,\tilde{\pi})$ converge absolutely for $\operatorname{Re} s$ sufficiently large.

We know that for any v and any φ_v in $W(\pi_v,\psi_v)$ the integral

$$\int_{F_v^\times}\varphi_v\left(\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}g_v\right)|a_v|^{s-\frac{1}{2}}\,d^\times a_v$$

converges absolutely for $\operatorname{Re} s$ large enough. Suppose that, for all a in I, $|\eta(a)|=|a|^r$ with r real. Applying Lemma 3.11 we see that if $s+r>\frac{1}{2}$ and φ_v^0 is defined

$$\int_{F_v^\times}\left|\varphi_v^0\left(\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}g_v\right)\right||a_v|^{s-\frac{1}{2}}\,d^\times a_v$$

is, for g_v in K_v, at most

$$\frac{1}{\left(1-|\varpi_v|^{s+r-\frac{1}{2}}\right)^2}.$$

Thus if φ_1 is of the form (11.1.2) the integral

$$\Psi(g,s,\varphi_1)=\int_I \varphi\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right)\,|a|^{s-\frac{1}{2}}\,d^\times a$$

is absolutely convergent and equal to

$$\prod_v \Psi(g_v,s,\varphi_v)$$

for $\operatorname{Re} s$ sufficiently large. Since $\Phi(g_v,s,\varphi_v)$ is, by Proposition 3.5, equal to 1 for almost all v we can set

$$\Phi(g,s,\varphi_1)=\prod_v \Phi(g_v,s,\varphi_v)$$

so that

$$\Psi(g,s,\varphi_1)=L(s,\pi)\,\Phi(g,s,\varphi_1).$$

We can also introduce

$$\widetilde{\Psi}(g,s,\varphi_1)=\int_I \varphi_1\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right)\,\eta^{-1}(a)\,|a|^{s-\frac{1}{2}}\,d^\times a$$

and show that

$$\widetilde{\Psi}(g,s,\varphi_1)=L(s,\tilde{\pi})\,\widetilde{\Phi}(g,s,\varphi_1)$$

if

$$\widetilde{\Phi}(g,s,\varphi_1)=\prod_v \widetilde{\Phi}(g_v,s,\varphi_v).$$

Lemma 11.1.3 *There is a real number s_0 such that for all φ_1 in $W(\pi,\psi)$ the integrals*

$$\Psi(g,s,\varphi_1)=\int_I \varphi_1\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right)\,|a|^{s-\frac{1}{2}}\,d^\times a$$
$$\widetilde{\Psi}(g,s,\varphi_1)=\int_I \varphi_1\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right)\,\eta^{-1}(a)\,|a|^{s-\frac{1}{2}}\,d^\times a$$

are absolutely convergent for $\operatorname{Re} s > s_0$. The functions $\Psi(g,s,\varphi_1)$ and $\widetilde{\Psi}(g,s,\varphi_1)$ can both be extended to entire functions of s. If F is a number field they are bounded in vertical strips and if F is function field they are rational functions of q^{-s}. Moreover

$$\widetilde{\Psi}(wg,1-s,\varphi_1)=\Psi(g,s,\varphi_1).$$

We have seen that the first assertion is true for functions of the form (11.1.2). Since they form a basis of $W(\pi,\psi)$ it is true in general. To show that

$$\varphi(g)=\sum_{\alpha\in F^\times}\varphi_1\left(\begin{pmatrix} \alpha & 0\\ 0 & 1\end{pmatrix} g\right)$$

we need only show that the series on the right is absolutely convergent. We will do this later on in this paragraph. At the moment we take the equality for granted. Then, for all φ_1, $\Psi(g,s,\varphi_1)$ which equals

$$\int_{F^\times\backslash I} \Big\{ \sum_{\alpha\in F^\times} \varphi_1\left(\begin{pmatrix}\alpha a & 0\\ 0 & 1\end{pmatrix} g\right) \Big\} |a|^{s-\frac{1}{2}}\, d^\times a$$

is equal to

$$\int_{F^\times\backslash I} \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-\frac{1}{2}}\, d^\times a$$

for $\operatorname{Re} s$ sufficiently large. Also $\widetilde{\Psi}(g,s,\varphi_1)$ is equal to

$$\int_{F^\times\backslash I} \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) \eta^{-1}(a)\, |a|^{s-\frac{1}{2}}\, d^\times a.$$

We saw in the previous paragraph that, for a given g and any real number M,

$$\left|\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)\right| = O(|a|^M)$$

as $|a|$ approaches 0 or ∞. Thus the two intregals define entire functions of s which are bounded in vertical strips. If F is a function field the function

$$a \to \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)$$

has compact support on $F^\times \setminus I$ so that the integral can be expressed as a finite Laurent series in q^{-s}.

The function $\widetilde{\Psi}(wg, 1-s, \varphi_1)$ is equal to

$$\int \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} wg\right) \eta^{-1}(a)\, |a|^{\frac{1}{2}-s}\, d^\times a.$$

Since w is in G_F the equality $\varphi(wh) = \varphi(h)$ holds for all h in $G_{\mathbb{A}}$ and this integral is equal to

$$\int \varphi\left(\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix} g\right) \eta^{-1}(a)\, |a|^{\frac{1}{2}-s}\, d^\times a.$$

Since

$$\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}$$

we can change variables in the integral to obtain

$$\int \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-\frac{1}{2}}\, d^\times a$$

which is $\Psi(g,s,\varphi_1)$.

If we choose φ_1 of the form (11.1.2) we see that $L(s,\pi)\,\Phi(g,s,\varphi_1)$ is entire and bounded in vertical strips of finite width. For almost all v the value of $\Phi(g_v, s, \varphi_v^0)$ at the identity e is 1 and for such v we choose $\varphi_v = \varphi_v^0$. At the other places we choose φ_v so that $\Phi(e, s, \varphi_v)$ is an exponential $e^{a_v s}$ with real a_v. Then $\Phi(e, s, \varphi_1)$ is an exponential. Consequently $L(s, \pi)$ is also entire and bounded in vertical strips of finite width. If F is a number field $\Phi(e, s, \varphi_1)$ will be a power of q^{-s} so that $L(s, \pi)$ will be a finite Laurent series in q^{-s}. Similar considerations apply to $L(s, \tilde{\pi})$.

To prove the functional equation we start with the relation

$$L(s, \pi) \prod_v \Phi(e, s, \varphi_v) = L(1 - s, \tilde{\pi}) \prod_v \widetilde{\Phi}(w, 1 - s, \varphi_v).$$

By the local functional equation the right hand side is

$$L(1 - s, \tilde{\pi}) \prod_v \big\{\varepsilon(s, \pi_v, \psi_v)\, \Phi(e, s, \varphi_v)\big\}.$$

Cancelling the term $\prod_v \Phi(e, s, \varphi_v)$ we obtain

$$L(s, \pi) = \varepsilon(s, \pi)\, L(1 - s, \tilde{\pi}).$$

Corollary 11.2 *Suppose $\pi = \otimes_v \pi_v$ is a constituent of $\mathcal{A}$. For any quasi-character ω of $F^\times \setminus I$ the products*

$$\prod_v L(s, \omega_v \otimes \pi_v)$$

and

$$\prod_v L(s, \omega_v^{-1} \otimes \tilde{\pi}_v)$$

are absolutely convergent for $\operatorname{Re} s$ sufficiently large. The functions $L(s, \omega \otimes \pi)$ and $L(s, \omega^{-1} \otimes \tilde{\pi})$ they define can be analytically continued to the whole complex plane as meromorphic functions which are bounded at infinity in vertical strips of finite width and have only a finite number of poles. If F is a function field they are rational functions of q^{-s}. If π is a constituent of $\mathcal{A}_0$ they are entire. In all cases they satisfy the functional equation

$$L(s, \omega \otimes \pi) = \varepsilon(s, \omega \otimes \pi)\, L(1 - s, \omega^{-1} \otimes \tilde{\pi})$$

if

$$\varepsilon(s, \omega \otimes \pi) = \prod_v \varepsilon(s, \omega_v \otimes \pi_v, \psi_v).$$

If $\pi = \otimes_v \pi_v$ is a constituent of $\mathcal{A}$ or $\mathcal{A}_0$ and ω is a quasi-character of $F^\times \setminus I$ so is $\omega \otimes \pi$. Moreover $\omega \otimes \pi = \otimes_v (\omega_v \otimes \pi_v)$.

The converses to the corollary can take various forms. We consider only the simplest of these. In particular, as far as possible, we restrict ourselves to cusp forms.

Theorem 11.3 *Let $\pi = \otimes\pi_v$ be a given irreducible representation of $\mathcal{H}$. Suppose that the quasi-character η of I defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

is trivial on $F^\times$. Suppose there is a real number r such that whenever $\pi_v = \pi(\mu_v, \nu_v)$ the inequalities

$$|\varpi_v|^{-r} \leq |\mu_v(\varpi_v)| \leq |\varpi_v|^r$$

and

$$|\varpi_v|^{-r} \leq |\nu_v(\varpi_v)| \leq \varpi_v|^r$$

are satisfied. Then for any quasi-character ω of $F^\times \backslash I$ the infinite products

$$L(s, \omega \otimes \pi) = \prod_v L(s, \omega_v \otimes \pi_v)$$

and

$$L(s, \omega^{-1} \otimes \tilde{\pi}) = \prod_v L(s, \omega_v^{-1} \otimes \tilde{\pi}_v)$$

are absolutely convergent for $\mathrm{Re}\, s$ large enough. Suppose $L(s, \omega \otimes \pi)$ and $L(s, \omega^{-1} \otimes \tilde{\pi})$ are, for all ω, entire functions of s which are bounded in vertical strips and satisfy the functional equation

$$L(s, \omega \otimes \pi) = \varepsilon(s, \omega \otimes \pi)\, L(1 - s, \omega^{-1} \otimes \tilde{\pi})$$

If the π_v are all infinite dimensional π is a constituent of $\mathcal{A}_0$.

The absolute convergence of the infinite products is clear. We have to construct a subspace U of $\mathcal{A}_0$ which is invariant under $\mathcal{H}$ and transforms according to the representation π. $W(\pi, \psi)$ transforms according to π. If φ_1 belongs to $W(\pi, \psi)$ set

$$\varphi(g) = \sum_{\alpha \in F^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

We shall see later that this series converges absolutely and uniformly on compact subsets of $G_{\mathbb{A}}$. Thus φ is a continuous function on $G_{\mathbb{A}}$. Since the map $\varphi_1 \to \varphi$ commutes with right translations by elements of $\mathcal{H}$ we have to show that, for all φ_1, φ is in $\mathcal{A}_0$ and that φ is not zero unless φ_1 is.

Since ψ is a character of $F \backslash \mathbb{A}$

$$\varphi\left(\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g\right) = \varphi(g)$$

for all ξ in F. Thus, for each g,

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & a \end{pmatrix} g\right)$$

is a function on $F \backslash \mathbb{A}$. The constant term of its Fourier expansion is

$$\frac{1}{\text{measure}\, F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx.$$

The integral is equal to

$$\sum_{\alpha} \int_{F\backslash\mathbb{A}} \varphi_1\left(\begin{pmatrix}1 & \alpha x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)\, dx.$$

A typical term of this sum is

$$\varphi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)\int_{F\backslash\mathbb{A}} \psi(\alpha x)\, dx = 0.$$

In particular φ is cuspidal. Another simple calculation shows that if β belongs to $F^\times$

$$\frac{1}{\operatorname{measure} F\setminus\mathbb{A}}\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\,\psi(-\beta x)\, dx$$

is equal to

$$\varphi_1\left(\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} g\right).$$

Thus φ_1 is zero if φ is.

By construction

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right) = \varphi(g)$$

if α is in $F^\times$. Moreover, for all a in I,

$$\varphi\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix} g\right) = \eta(a)\,\varphi(g).$$

If a is in $F^\times$ the right side is just $\varphi(g)$. Thus φ is invariant under left translations by elements of P_F, the group of super-triangular matrices in G_F. Since G_F is generated by P_F and $w = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ all we need do to show that φ is a function on $G_F\setminus G_{\mathbb{A}}$ is to show that

$$\varphi(wg) = \varphi(g).$$

By linearity we need only establish this when φ_1 has the form (11.1.2). The hypothesis implies as in the direct theorem that the integrals

$$\Psi(g,s,\varphi_1) = \int_I \varphi_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)\,|a|^{s-\frac{1}{2}}\, d^\times a$$

and

$$\widetilde{\Psi}(g,s,\varphi_1) = \int_I \varphi_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)\,\eta^{-1}(a)\,|a|^{s-\frac{1}{2}}\, d^\times a$$

converge absolutely for $\operatorname{Re} s$ sufficiently large. Moreover

$$\Psi(g,s,\varphi_1) = \prod_v \Psi(g_v,s,\varphi_v) = L(s,\pi)\prod_v \Phi(g_v,s,\varphi_v).$$

Almost all factors in the product on the right are identically one so that the product, and therefore $\Psi(g,s,\varphi_1)$, is an entire function of s. In the same way

$$\widetilde{\Psi}(g,s,\varphi_1) = L(s,\tilde{\pi}) \prod_v \widetilde{\Phi}(g_v,s,\varphi_v)$$

and is entire. Since

$$\widetilde{\Phi}(wg_v,1-s,\varphi_v) = \varepsilon(s,\pi_v,\psi_v)\,\Phi(g_v,s,\varphi_v)$$

the function $\widetilde{\Psi}(wg,1-s,\varphi_1)$ is equal to

$$L(1-s,\tilde{\pi})\,\varepsilon(s,\pi) \prod_v \Phi(g_v,s,\varphi_v)$$

which, because of the functional equation assumed for $L(s,\pi)$, is equal to $\Psi(g,s,\varphi_1)$.

>From its integral representation the function $\Psi(g,s,\varphi_1)$ is bounded in any vertical strip of finite width contained in a certain right half-plane. The equation just established shows that it is also bounded in vertical strips of a left half-plane. To verify that it is bounded in any vertical strip we just have to check that it grows sufficiently slowly that the Phrágmen–Lindelöf principle can be applied,

$$\Psi(g,s,\varphi_1) = L(s,\pi) \prod_v \Phi(g_v,s,\varphi_v).$$

The first term is bounded in any vertical strip by hypothesis. Almost all factors in the infinite product are identically 1. If v is non-archimedean $\Phi(g_v,s,\varphi_v)$ is a function of $|\varpi_v|^s$ and is therefore bounded in any vertical strip. If v is archimedean

$$\Phi(g_v,s,\varphi_v) = \frac{\Psi(g_v,s,\varphi_v)}{L(s,\pi_v)}$$

We have shown that the numerator is bounded at infinity in vertical strips. The denominator is, apart from an exponential factor, a Γ-function. Stirling's formula shows that it goes to 0 sufficiently slowly at infinity.

If $\operatorname{Re} s$ is sufficiently large

$$\Psi(g,s,\varphi_1) = \int_{F^\times\backslash I} \sum_{\alpha\in F^\times} \varphi_1\left(\begin{pmatrix}\alpha a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-\frac{1}{2}}\,d^\times a$$

which is

$$\int_{F^\times\backslash I} \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-\frac{1}{2}}\,d^\times a.$$

This integral converges absolutely when $\operatorname{Re} s$ is sufficiently large. If $\operatorname{Re} s$ is large and negative

$$\widetilde{\Psi}(wg,1-s,\varphi_1) = \int_{F^\times\backslash I} \varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} wg\right) \eta^{-1}(a)\,|a|^{\frac{1}{2}-s}\,d^\times a$$

which equals

$$\int_{F^\times\backslash I} \varphi\left(w\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix} g\right) \eta^{-1}(a)\,|a|^{\frac{1}{2}-s}\,d^\times a.$$

Using the relation

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and changing variables we see that this integral is equal to

$$\int_{F^\times \backslash I} \varphi\left(w \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-\frac{1}{2}}\, d^\times a.$$

Set

$$f_1(a) = \varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right)$$

and

$$f_2(s) = \varphi\left(w \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right).$$

We are trying to show that for any g the functions f_1 and f_2 are equal. The previous discussion applies to $\omega \otimes \pi$ as well as to π. If φ_1 is in $W(\pi, \psi)$ the function

$$\varphi_1'(g) = \omega(\det g)\, \varphi_1(g)$$

is in $W(\omega \otimes \pi, \psi)$. When φ_1 is replaced by φ_1' the function φ is replaced by

$$\varphi'(g) = \omega(\det g)\, \varphi(g)$$

and f_i is replaced by

$$f_i'(a) = \omega(\det g)\, \omega(a)\, f_i(a).$$

Thus for any quasi-character ω of $F^\times \backslash I$ the integral

$$\int_{F^\times \backslash I} f_1(a)\, \omega(a)\, |a|^{s-\frac{1}{2}}\, d^\times a$$

is absolutely convergent for $\operatorname{Re} s$ sufficiently large and the integral

$$\int_{F^\times \backslash I} f_2(s)\, \omega(a)\, |a|^{s-\frac{1}{2}}\, d^\times a$$

is absolutely convergent for $\operatorname{Re} s$ large and negative. Both integrals represent functions which can be analytically continued to the same entire function. This entire function is bounded in vertical strips of finite width.

The equality of f_1 and f_2 is a result of the following lemma.

Lemma 11.3.1 *Let f_1 and f_2 be two continuous functions on $F^\times \backslash I$. Assume that there is a constant c such that for all characters of ω of $F^\times \backslash I$ the integral*

$$\int_{F^\times \backslash I} f_1(a)\,\omega(a)\,|a|^s\,d^\times a$$

is absolutely convergent for $\operatorname{Re} s > c$ *and the integral*

$$\int_{F^\times \backslash I} f_2(a)\,\omega(a)\,|a|^s\,d^\times a$$

is absolutely convergent for $\operatorname{Re} s < -c$. *Assume that the functions represented by these integrals can be analytically continued to the same entire function and that this entire function is bounded in vertical strips of finite width. Then f_1 and f_2 are equal.*

Let I_0 be the group of idèles of norm 1. Then $F^\times \backslash I_0$ is compact. It will be enough to show that for each b in I the functions $f_1(ab)$ and $f_2(ab)$ on $F^\times \backslash I_0$ are equal. They are equal if they have the same Fourier expansions. Since any character of $F^\times \backslash I_0$ can be extended to a character of $F^\times \backslash I$ we have just to show that for every character ω of $F^\times \backslash I$

$$\hat{f}_1(\omega, b) = \omega(b) \int_{F^\times \backslash I_0} f_1(ab)\,\omega(a)\,d^\times a$$

is equal to

$$\hat{f}_2(\omega, b) = \omega(b) \int_{F^\times \backslash I_0} f_2(ab)\,\omega(a)\,d^\times a.$$

These two functions are functions on $I_0 \backslash I$ which is isomorphic to $\mathbb{Z}$ if F is a number field and to $\mathbb{R}$ if F is a function field.

If F is a function field we have only to verify the following lemma.

Lemma 11.3.2 *Suppose $\{a_1(n) \,\big|\, n \in \mathbb{Z}\}$ and $\{a_2(n) \,\big|\, n \in \mathbb{Z}\}$ are two sequences and $q > 1$ is a real number. Suppose*

$$\sum_n a_1(n)\,q^{-ns}$$

converges for $\operatorname{Re} s$ *sufficiently large and*

$$\sum_n a_2(n)\,q^{-ns}$$

converges absolutely for $\operatorname{Re} s$ *large and negative. If the functions they represent can be analytically continued to the same entire function of s the two sequences are equal.*

Once stated the lemma is seen to amount to the uniqueness of the Laurent expansion. If F is a number field the lemma to be proved is a little more complicated.

Lemma 11.3.3 *Suppose g_1 and g_2 are two continuous functions on $\mathbb{R}$. Suppose there is a constant c such that*

$$\hat{g}_1(s) = \int_{\mathbb{R}} g_1(x)\, e^{sx}\, dx$$

converges absolutely for $\operatorname{Re} s > c$ *and*

$$\hat{g}_2(s) = \int_{\mathbb{R}} g_2(x)\, e^{sx}\, dx$$

converges absolutely for $\operatorname{Re} s < -c$. *If $\hat{g}_1$ and $\hat{g}_2$ represent the same entire function and this function is bounded in vertical strips then $g_1 = g_2$.*

All we need do is show that for every compactly supported infinitely differentiable function g the functions $g * g_1$ and $g * g_2$ are equal. If

$$\hat{g}(s) = \int_{\mathbb{R}} g(x)\, e^{sx}\, dx$$

is the Laplace transform of g the Laplace transform of $g * g_i$ is $\hat{g}(s)\,\hat{g}_i(s)$. By the inversion formula

$$g * g_i(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \hat{g}(s)\,\hat{g}_i(s)\, e^{-xs}\, ds$$

where $b > c$ if $i = 1$ and $b < -c$ if $i = 2$. The integral converges because $\hat{g}$ goes to 0 faster than the inverse of any polynomial in a vertical strip. Cauchy's integral theorem implies that the integral is independent of b. The lemma follows.

To complete the proof of Theorem 11.3, and Theorem 11.1, we have to show that for any φ_1 in $W(\pi, \psi)$ the series

$$\sum_{\alpha \in F^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is uniformly absolutely convergent for g in a compact subset of $G_{\mathbb{A}}$ and that if $\varphi(g)$ is its sum then, if F is a number field, for any compact subset Ω of $G_{\mathbb{A}}$ and any $c > 0$ there are constants M_1 and M_2 such that

$$\left|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \le M_1\, |a|^{M_2}$$

for g in Ω and $|a| \ge c$. We prefer to prove these facts in a more general context which will now be described.

For us a divisor is just a formal product of the form

$$D = \prod \mathfrak{p}^{m_{\mathfrak{p}}}.$$

It is taken over all non-archimedean places. The integers $m_{\mathfrak{p}}$ are non-negative and all but a finite number of them are 0. Let S be a finite set of non-archimedean places containing all the divisors of D, that is, all places $\mathfrak{p}$ for which $m_{\mathfrak{p}} > 0$.

If a belongs to I we can write a in a unique manner as a product $a_S \hat{a}_S$ where the components of a_S outisde S are 1 and those of $\hat{a}_S$ inside S are 1. a_S belongs to $I_s = \prod_{v \in S} F_v^\times$. Let I_D^S be the set of idèles a such that, for any $\mathfrak{p}$ in S, $a_\mathfrak{p}$ is a unit which satisfies $a_\mathfrak{p} \equiv 1(\operatorname{mod} \mathfrak{p}^{m_\mathfrak{p}})$. Then $I = F^\times I_D^S$ and $F^\times \backslash I$ is isomorphic to $F^\times \cap I_D^S \backslash I_D^S$.

If $\mathfrak{p}$ is in S let $K_\mathfrak{p}^D$ be the subgroup of all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $K_\mathfrak{p}$ for which $c \equiv 0 \pmod{\mathfrak{p}^{m_\mathfrak{p}}}$. Let $\widehat{K}_\mathfrak{p}^D$ be the subgroup of such matrices for which $a \equiv d \equiv 1(\operatorname{mod} \mathfrak{p}^{m_\mathfrak{p}})$. Set

$$K_S^D = \prod_{\mathfrak{p} \in S} K_\mathfrak{p}^D$$

and set

$$\widehat{K}_S^D = \prod_{\mathfrak{p} \in S} \widehat{K}_\mathfrak{p}^D$$

$\widehat{K}_S^D$ is a normal subgroup of K_S^D and the quotient $K_S^D / \widehat{K}_S^D$ is abelian.

Let G_D^S be the set for all g in $G_\mathbb{A}$ such that $g_\mathfrak{p}$ is in the group $K_\mathfrak{p}^D$ for all $\mathfrak{p}$ in S. Any g in $G_\mathbb{A}$ may be written as a product $g_S \hat{g}_S$ where g_S has component 1 outside of S and $\hat{g}_S$ has component 1 inside S. G_S is the set of g_S and $\widehat{G}_S$ is the set of $\hat{g}_S$. In particular

$$G_D^S = K_S^D \cdot \widehat{G}_S.$$

It is easily seen that

$$G_\mathbb{A} = G_F \, G_D^S.$$

In addition to D and S we suppose we are given a non-trivial character ψ of $F \backslash \mathbb{A}$, two characters ε and $\hat{\varepsilon}$ of $K_S^D / \widehat{K}_S^D$, two complex valued functions $\alpha \to a_\alpha$ and $\alpha \to \hat{a}_\alpha$ on $F^\times$, an irreducible representation π of $\widehat{\mathcal{H}}_S = \otimes_{v \notin S} \mathcal{H}_v$, and a quasi-character η of $F^\times \backslash I$.

There are a number of conditions to be satisifed. If

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

belongs to K_S^D then

$$\hat{\varepsilon}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \varepsilon\left(\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\right).$$

If α belongs to $F^\times$ and β belongs to $F^\times \cap I_D^S$

$$a_{\alpha\beta} = \varepsilon\left(\begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix}\right) a_\alpha$$

and

$$\hat{a}_{\alpha\beta} = \hat{\varepsilon}\left(\begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix}\right) \hat{a}_\alpha.$$

The functions $\alpha \to a_\alpha$ and $\alpha \to \hat{a}_\alpha$ are bounded. Moreover $a_\alpha = \hat{a}_\alpha = 0$ if for some v in S the number α regarded as an element of F_v does not lie in the largest ideal on which ψ_v is trivial. If v belongs to S and a is a unit in O_v

$$\varepsilon\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta_v(a).$$

Let $\pi = \otimes_{v \notin S} \pi_v$. Then for a in $F_v^\times$

$$\pi_v\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta_v(a)\, I.$$

Because of these two conditions η is determined by π and ε. There is a real number r such that if $\pi_v = \pi(\mu_v, \nu_v)$

$$|\varpi_v|^r \le |\mu_v(\varpi_v)| \le |\varpi_v|^{-r}$$

and

$$|\varpi_v|^r \le |\nu_v(\varpi_v)| \le |\varpi_v|^{-r}.$$

Finally we suppose that π_v is infinite dimensional for all v not in S.

These conditions are rather complicated. Nonetheless in the next paragraph we shall find ourselves in a situation in which they are satisifed. When S is empty, $D = 0$, $a_\alpha = \hat{a}_\alpha = 1$ for all α they reduce to those of Theorem 11.3. In particular with the next lemma the proof of that theorem will be complete. We shall use the conditions to construct a space U of automorphic forms on $G_{\mathbb{A}}$ such that U transforms under $\widehat{\mathcal{H}}_S$ according to π while each φ in U satisifes

$$\varphi(gh) = \varepsilon(h)\, \varphi(g)$$

for h in K_S^D. If U is such a space then for any φ in U and any a in I

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\, \varphi(g).$$

This is clear if a belongs ot I_D^S and follows in general from the relation $I = F^\times\, I_D^S$.

Recall that $W(\pi, \psi)$ is the space of functions on $\widehat{G}_S$ spanned by functions of the form

$$\varphi_1(g) = \prod_{v \notin S} \varphi_v(g_v)$$

where φ_v belongs to $W(\pi_v, \psi_v)$ for all v and is equal to φ_v^0 for almost all v.

Lemma 11.4 *Suppose φ_1 belongs to $W(\pi, \psi)$.*

(i) *For any g in G_D^S the series*

$$\varphi(g) = \sum_{\alpha \in F^\times} a_\alpha\, \varepsilon(g_S)\, \varphi_1\left(\begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right)$$

converges absolutely. The convergence is uniform on compact subsets of G_D^S.

(ii) *The function φ defined by this series is invariant under left translation by the matrices in $G_F \cap G_D^S$ of the form*

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

(iii) *Suppose F is a number field. Let Ω be a compact subset of G_D^S. Then there are positive constants M_1 and M_2 such that*

$$|\varphi(g)| \le M_1 \left\{ |a| + |a|^{-1} \right\}^{M_2}$$

if

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h$$

with h in Ω, a in I_D^S, and $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ in G_D^S.

It is enough to prove these assertions when φ_1 has the form

$$\varphi_1(g) = \prod_{v \notin S} \varphi_v(g_v).$$

To establish the first and third assertions we need only consider the series

$$\sum_{\alpha \in F^\times} \delta(\alpha) \prod_{v \notin S} \left| \varphi_v \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g_v \right) \right| \tag{11.4.1}$$

where $\delta(\alpha) = 0$ if for some v in S the number α regarded as an element of F_v is not the largest ideal on which ψ_v is trivial and $\delta(\alpha) = 1$ otherwise.

We need only consider compact sets Ω of the form

$$\Omega = K_S^D \prod_{v \notin S} \Omega_v \tag{11.4.2}$$

where Ω_v is a compact subset of G_v and $\Omega_v = K_v$ for almost all v.

Lemma 11.4.3 *Suppose Ω is of the form (11.4.2). There is a positive number ρ such that for each non-archimedean place v which is not in S there is a constant M_v such that*

$$\left| \varphi_v \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h \right) \right| \le M_v |a|^{-\rho}$$

for a in $F_v^\times$ and h in Ω_v and a constant c_v such that

$$\left| \varphi_v \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h \right) \right| = 0$$

if $|a| > c_v$ *and* h *is in* Ω_v. *Moroever one may take* $M_v = c_v = 1$ *for almost all* v.

Since φ_v is invariant under an open subgroup of K_v for all v and is invariant under K_v for almost all v while $\Omega_v = K_v$ for almost all v it is enough to prove the existence of M_v, c_v, and ρ such that these relations are satisifed when $h = 1$. Since the function

$$a \to \varphi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

belongs to the space of the Kirillov model the existence of c_v is clear. c_v can be taken to be 1 when O_v is the largest ideal of F_v on which ψ_v is trivial and $\varphi_v = \varphi_v^0$.

The existence of M_v, for a given v and sufficiently large ρ, is a result of the absolute convergence of the integral defining $\Psi(e, s, \varphi_v)$. Thus all we need do is show the existence of a fixed ρ such that the inequality

$$\left|\varphi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| \leq |a|^{-\rho}$$

is valid for almost all v. For almost all v the representation π_v is of the form $\pi(\mu_v, \nu_v)$ with μ_v and ν_v unramified, O_v is the largest ideal of F_v on which ψ_v is trivial, and $\varphi_v = \varphi_v^0$. Thus, for such v,

$$\varphi_v\left(\begin{pmatrix} \varepsilon a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

if ε is a unit in O_v and

$$\sum_n \varphi_v\left(\begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix}\right) |\varpi_v|^{n\left(s-\frac{1}{2}\right)} = L(s, \pi_v).$$

If $\rho_v = \mu_v(\varpi_v)$ and $\sigma_v = \nu_v(\varpi_v)$

$$L(s, \pi_v) = \frac{1}{\left(1 - \rho_v|\varpi_v|^s\right)} \frac{1}{\left(1 - \sigma_v|\varpi_v|^s\right)}.$$

Since $|\rho_v| \leq |\varpi_v|^{-r}$ and $|\sigma_v| \leq |\varpi_v|^{-r}$

$$\left|\varphi_v\left(\begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix}\right)\right| = \left|\frac{\rho_v^{n+1} - \sigma_v^{m+1}}{\rho_v - \sigma_v}\right| \leq (n+1)\,|\varpi_v|^{-rn}.$$

Since $|\varpi_v| \leq \frac{1}{2}$ there is a constant $\varepsilon > 0$ such that

$$(n+1) \leq |\varpi_v|^{-\varepsilon n}$$

for all v and all $n \geq 0$.

If v is archimedean the integral representations of the functions in $W(\pi_v, \psi_v)$ show that there are positive constants c_v, d_v, and M_v such that

$$\left|\varphi_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right)\right| \leq M_v\,|a|^{-c_v}\,\exp\left(-\,d_v\,|a|_v^{\varepsilon_v}\right)$$

for a in $F_v^\times$ and h in Ω_v. ε_v is 1 if v is real and $\frac{1}{2}$ if v is complex.

Since we want to prove not only the first assertion but also the third we consider the sum

$$f\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\alpha\in F^\times} \delta(\alpha) \prod_{v\notin S} \left|\varphi_v\left(\begin{pmatrix} b_v\alpha & 0 \\ 0 & 1 \end{pmatrix} g_v\right)\right|$$

where g lies in the set (11.4.2) and b is an idèle such that $b_v = 1$ for all non-archimedean v. We also suppose that there is a positive number t such that $b_v = t$ for all archimedean v. If Λ is a set of α in F for which $|\alpha|_v \le c_v$ for all non-archimedean v not in S and $\delta(\alpha) \ne 0$ then

$$f\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is bounded by

$$\sum_{\substack{\alpha\in\Lambda\\ \alpha\ne 0}} \Big\{ \prod_{v\in S_a} M_v |\alpha t|_v^{-c_v} \exp(-d_v t\,|\alpha|_v^{\varepsilon_v})\Big\}\Big\{ \prod_{v\notin S\cup S_a} M_v |\alpha_v|^{-\rho}\Big\}.$$

If F is a function field Λ is a finite set and there is nothing more to prove. If it is a number field choose for each v in S a constant c_v such that $\delta(\alpha) = 0$ unless $|\alpha|_v \le c_v$. Since

$$\prod_v |\alpha|_v = 1,$$

$$\prod_{v\notin S\cup S_a} |\alpha|_v^{-\rho} \le \Big\{ \prod_{v\in S} c_v^{\rho}\Big\}\Big\{ \prod_{v\in S_a} |\alpha|_v^{\rho}\Big\}.$$

Thus our sum is bounded by a constant times the product of $\prod_{v\in S_a} t^{-c_v/\varepsilon_v}$ and

$$\sum_{\substack{\alpha\in\Lambda\\ \alpha\ne 0}} \prod_{v\in S_a} \big\{|\alpha|_v^{\rho-c_v}\ \exp\big(-d_v t |\alpha|_v^{\varepsilon_v}\big)\big\}.$$

The product $\prod_{v\in S_a} |\alpha|_v$ is bounded below on $\Lambda - \{0\}$. Multiplying each term by the same sufficiently high power of $\prod_{v\in S_a} |\alpha|_v$ we dominate the series by another series

$$\sum_{\alpha\in\Lambda} \prod_{v\in S_a} \big\{|\alpha|_v^{\rho_v}\ \exp(-d_v t\,|\alpha|_v^{\varepsilon_v})\big\}$$

in which the exponents ρ_v are non-negative. This in turn is dominated by $\prod_{v\in S_a} t^{-\rho_v/\varepsilon_v}$ times

$$\sum_{\alpha\in\Lambda} \prod_{v\in S_a} \exp\big(\frac{-d_v}{2} t\,|\alpha|_v^{\varepsilon_v}\big).$$

Λ may be regarded as a lattice in $\prod_{v\in S_a} F_v$. if $\lambda_1, \cdots, \lambda_n$ is a basis of Λ there is a constant d such that if $\alpha = \sum a_i\,\lambda_i$

$$\sum_{v\in S_a} \frac{d_v}{2}\,|\alpha|_v^{\varepsilon_v} \ge d \sum |a_i|.$$

Thus

$$f\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is dominated by some power of t times a multiple of

$$\Big\{\sum_{a=-\infty}^{\infty} e^{-dt|a|}\Big\}^n$$

which is bounded by a multiple of $\left(1+\frac{1}{t}\right)^n$.

The first assertion is now proved and the third will now follow from the second and the observation that every element of I_D^S is the product of an element of $F^\times$, an idèle whose components are 1 at all non-archimedean places and equal to the same positive number at all archimedean places, and an idèle which lies in a certain compact set.

Suppose ξ is in F and

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

belong to G_D^S. Then ξ is integral at each prime of S and $\psi_v(\alpha\xi) = 1$ if $a_\alpha \neq 0$. If g belongs to G_D^S and

$$h = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g$$

then $\varepsilon(h_S) = \varepsilon(g_S)$ and if v is not in S

$$\varphi_v\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h_v\right) = \psi_v(\alpha\xi)\,\varphi_v\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g_v\right).$$

If $a_\alpha \neq 0$

$$\prod_{v\notin S} \psi_v(\alpha\xi) = \prod_v \psi_v(\alpha\xi) = 1.$$

Consequently

$$\varphi\left(\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g\right) = \varphi(g).$$

If b belongs to I_D^S then

$$\varepsilon\left(\begin{pmatrix} b_S & 0 \\ 0 & b_S \end{pmatrix} g_S\right) = \eta(b_S)\,\varepsilon(g_S)$$

and

$$\varphi_1\left(\begin{pmatrix} \hat{b}_S\hat{\alpha}_S & 0 \\ 0 & \hat{b}_S \end{pmatrix} \hat{g}_S\right) = \eta(\hat{b}_S)\,\varphi_1\left(\begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right)$$

so that

$$\varphi\left(\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} g\right) = \eta(b)\,\varphi(g).$$

In particular if β belongs to $F^\times \cap I_D^S$

$$\varphi\left(\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} g\right) = \varphi(g).$$

If β belongs to $F^\times \cap I_D^S$ and

$$h = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g$$

then

$$\varepsilon(h_S) = \varepsilon\left(\begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix}\right) \varepsilon(g_S)$$

and $\varphi(h)$ is equal to

$$\sum_\alpha a_\alpha \varepsilon\left(\begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix}\right) \varepsilon(g_S)\,\varphi_1\left(\begin{pmatrix} \hat\alpha_S\hat\beta_S & 0 \\ 0 & 1 \end{pmatrix} \hat g_S\right).$$

Since

$$a_{\alpha\beta} = \varepsilon\left(\begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix}\right) a_\alpha$$

we can change variables in the summation to see that $\varphi(h) = \varphi(g)$.

The lemma is now proved. The function

$$\hat\varphi(g) = \sum_{\alpha\in F^\times} \hat a_\alpha\, \hat\varepsilon(g_S)\,\varphi_1\left(\begin{pmatrix} \hat\alpha_S & 0 \\ 0 & 1 \end{pmatrix} \hat g_S\right)$$

can be treated in the same fashion.

Theorem 11.5 *If ω is a quasi-characters of $F^\times \setminus I$ such that*

$$\omega_v(a_v)\,\varepsilon\left(\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

for all units a_v of O_v set

$$\Lambda(s,\omega) = \Big\{ \sum_{F^\times\cap I_D^S\backslash F^\times} a_\alpha\, \omega(\alpha_S)\, |\alpha_S|^{s-\frac12} \Big\} \prod_{v\notin S} L(s, \omega_v\otimes\pi_v).$$

If

$$\omega_v(a_v)\,\hat\varepsilon\left(\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

for all units a_v in O_v set

$$\widehat\Lambda(s,\omega) = \Big\{ \sum_{F^\times\cap I_D^S\backslash F^\times} \hat a_\alpha\, \omega(\alpha_S)\, |\alpha_S|^{s-\frac12} \Big\} \prod_{v\notin S} L(s, \omega_v\otimes\pi_v).$$

Then $\Lambda(s,\omega)$ and $\widehat\Lambda(s,\omega)$ are defined for $\operatorname{Re} s$ sufficiently large. Suppose that whenever ω is such that $\Lambda(s,\omega)$ or $\widehat\Lambda(s,\omega)$ is defined they can be analytically continued to entire functions which are bounded in vertical strips. Assume also that there is an A in $F^\times$ such that $|A|_{\mathfrak p} = |\varpi_{\mathfrak p}|^{m_{\mathfrak p}}$ for any $\mathfrak p$ in S and

$$\Lambda(s,\omega) = \Big\{ \prod_{v\in S} \omega_v(-A)\, |A|_v^{s-1/2} \Big\}\Big\{ \prod_{v\notin S} \varepsilon(s, \omega_v\otimes\pi_v, \psi_v) \Big\}\, \widehat\Lambda(1-s, \eta^{-1}\omega^{-1})$$

whenever $\Lambda(s,\omega)$ is defined. Then for any φ_1 in $W(\pi,\psi)$ there is an automorphic form φ on $G_{\mathbb{A}}$ such that

$$\varphi(g) = \sum a_\alpha \, \varepsilon(g_S)\, \varphi_1\left(\begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right)$$

on G_D^S.

The infinite products occurring in the definition of $\Lambda(s,\omega)$ and $\widehat{\Lambda}(s,\omega)$ certainly converge for $\operatorname{Re} s$ sufficiently large. To check that the other factors converge one has to check that

$$\sum |\alpha_S|^{s-\frac{1}{2}}$$

converges for $\operatorname{Re} s$ sufficiently large if the sum is taken over those elements α of a system of coset representatives of $F^\times \cap I_D^S \setminus F^\times$ for which $|\alpha|_v \le c_v$ for v in S. This is easily done.

A_S is the idèle whose components are 1 outside of S and A in S. Since

$$\begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & cA_S^{-1} \\ A_S b & a \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix}$$

normalizes K_S^D. In particular if g belongs to G_D^S so does

$$\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}.$$

Lemma 11.5.1 *If φ_1 is in $W(\pi,\psi)$ and g is in G_D^S then, under the hypotheses of the theorem,*

$$\widehat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}\right) = \varphi(g).$$

Let $\varphi'(g)$ be the function on the left. As before all we need do is show that for every character ω of $F^\times \cap I_D^S \setminus I_D^S$ and every g in G_D^S the integral

$$\int_{F^\times \cap I_D^S \setminus I_D^S} \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \omega(a)\, |a|^{s-\frac{1}{2}}\, d^\times a \tag{11.5.2}$$

is absolutely convergent for $\operatorname{Re} s$ large and positive. The integral

$$\int_{F^\times \cap I_D^S \setminus I_D^S} \varphi'\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \omega(a)\, |a|^{s-\frac{1}{2}}\, d^\times a \tag{11.5.3}$$

is absolutely convergent for $\operatorname{Re} s$ large and negative, and they can be analytically continued to the same entire function which is bounded in vertical strips.

If for any v in S the character

$$a \to \omega_v(a)\, \varepsilon\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)$$

on the group of units of O_v is not trivial the integrals are 0 when they are convergent. We may thus assume that

$$\omega_v(a)\,\varepsilon\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

for all units in O_v if v is in S.

We discuss the first integral in a formal manner. The manipulations will be justified by the final result. The integrand may be written as a double sum

$$\sum\sum a_{\alpha\gamma}\,\varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S\right)\,\varphi_1\left(\begin{pmatrix} \hat{a}_S\hat{\alpha}_S\hat{\gamma}_S & 0 \\ 0 & 1 \end{pmatrix}\hat{g}_S\right)\,\omega(a)\,|a|^{s-\frac{1}{2}}.$$

The inner sum is over γ in $F^\times \cap I_D^S$ and the outer over a set of coset representatives α of $F^\times \cap I_D^S \setminus F^\times$. Since

$$a_{\alpha\gamma}\,\varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix}\right) = a_\alpha\,\varepsilon\left(\begin{pmatrix} \gamma_S a_S & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and

$$\omega(a)\,|a|^{s-\frac{1}{2}} = \omega(\gamma a)\,|\gamma a|^{s-\frac{1}{2}}$$

the integral is equal to $\varepsilon(g_S)$ times the sum over α of

$$a_\alpha \int_{I_D^S} \varphi_1\left(\begin{pmatrix} \hat{\alpha}_S\hat{a}_S & 0 \\ 0 & 1 \end{pmatrix}\hat{g}_S\right)\,\varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix}\right)\,\omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a.$$

Since I_D^S is the direct product of

$$\hat{I}_S = \{a \in I \mid a_S = 1\}$$

and a compact group under which the integrand is invariant the previous expression is equal to

$$a_\alpha \int_{\hat{I}_S} \varphi_1\left(\begin{pmatrix} \hat{\alpha}_S a & 0 \\ 0 & 1 \end{pmatrix}\hat{g}_S\right)\,\omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a.$$

Changing variables to rid ourselves of the $\hat{\alpha}_S$ in the integrand and taking into account the relation

$$1 = \omega(\alpha)\,|\alpha|^{s-\frac{1}{2}} = \omega(\alpha_S)\,\omega(\hat{\alpha}_S)\,|\alpha_S|^{s-\frac{1}{2}}\,|\hat{\alpha}_S|^{s-\frac{1}{2}}$$

we can see the original integral is equal to

$$\varepsilon(g_S)\sum a_\alpha\,\omega(\alpha_S)\,|\alpha_S|^{s-\frac{1}{2}}\int_{\hat{I}_S}\varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\hat{g}_S\right)\,\omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a.$$

There is no harm in supposing that φ_1 is of the form

$$\varphi_1(\hat{g}_S) = \prod_{v\notin S}\varphi_v(g_v).$$

We have already seen that, in this case,

$$\int_{\hat{I}_S} \varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right) \omega(a)\,|a|^{s-\frac{1}{2}}\,d^\times a$$

is convergent for $\operatorname{Re} s$ large and positive and is equal to

$$\prod_{v \notin S} \int_{F_v^\times} \varphi_v\left(\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} g_v\right) \omega_v(a_v)\,|a_v|^{s-\frac{1}{2}}\,d^\times a_v.$$

If φ_v' is the function

$$\varphi_v'(h) = \omega_v(h)\,\varphi_v(h)$$

in $W(\omega_v \otimes \pi_v, \psi_v)$ this product is

$$\prod_{v \notin S} \left\{L(s, \omega_v \otimes \pi_v)\,\Phi(g_v, s, \varphi_v')\,\omega_v^{-1}(\det g_v)\right\}.$$

Thus the integral (11.5.2) is absolutely convergent for $\operatorname{Re} s$ large and positive and is equal to

$$\varepsilon(g_S)\,\omega(\det \hat{g}_S)\,\Lambda(s,\omega) \prod_{v \notin S} \Phi(g_v, s, \varphi_v').$$

The argument used in the proof of Theorem 11.3 shows that this function is entire.

If

$$h = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}$$

then

$$\hat{\varepsilon}(h_S) = \varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S\right).$$

Thus the integrand in (11.5.3) is equal to

$$\sum \hat{a}_\alpha\,\varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S\right) \varphi_1\left(\begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right) \omega(a)\,|a|^{s-\frac{1}{2}}.$$

The sum can again be written as a double sum over γ and α. Since

$$\hat{a}_{\alpha\gamma} = \varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix}\right) = \hat{a}_\alpha\,\varepsilon\left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma_S \end{pmatrix}\right)\,\varepsilon\left(\begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix}\right)$$

which equals

$$\hat{a}_\alpha\,\eta(\gamma_S)\,\varepsilon\left(\begin{pmatrix} \gamma_S^{-1} a_S & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and

$$\varphi_1\left(\begin{pmatrix} \hat{\alpha}_S \hat{\gamma}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S\right)$$

is equal to

$$\eta(\hat\gamma_S)\,\varphi_1\left(\begin{pmatrix}\hat\alpha_S & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ \hat A_S & 0\end{pmatrix}\begin{pmatrix}\hat\gamma_S^{-1}\hat a_S & 0\\ 0 & 1\end{pmatrix}\hat g_S\right)$$

we can put the sum over $F^\times \cap I_D^S$ and the integration over $F^\times \cap I_D^S \setminus I_D^S$ together to obtain $\varepsilon(g_S)$ times the sum over $F^\times \cap I_D^S \setminus F^\times$ of

$$\hat a_\alpha \int_{I_D^S} \varepsilon\left(\begin{pmatrix}a_S & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1\left(\begin{pmatrix}\hat\alpha_S & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ \hat A_S & 0\end{pmatrix}\begin{pmatrix}\hat a_S & 0\\ 0 & 1\end{pmatrix}\hat g_S\right)\omega(a)\,|a|^{s-\frac12}\,d^\times a.$$

We write

$$\begin{pmatrix}\hat\alpha_S & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ \hat A_S & 0\end{pmatrix}\begin{pmatrix}\hat a_S & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat\alpha_S & 0\\ 0 & \hat\alpha_S\end{pmatrix}\begin{pmatrix}-\hat\alpha_S^{-1}\hat A_S\hat a_S & 0\\ 0 & 1\end{pmatrix}$$

and then change variables in the integration to obtain the product of $\omega(-A_S)\,|A_S|^{s-\frac12}$ and

$$\hat a_\alpha\,\eta^{-1}(\alpha_S)\,\omega^{-1}(\alpha_S)\,|\alpha_S|^{\frac12-s}\int_{\hat I_S}\varphi_1\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\hat g_S\right)\omega(a)\,|a|^{s-\frac12}\,d^\times a.$$

Replacing a by a^{-1} and making some simple changes we see that the integral is equal to

$$\int_{\hat I_S}\varphi_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\hat g_S\right)\eta^{-1}(s)\,\omega^{-1}(a)\,|a|^{\frac12-s}\,d^\times a$$

which converges for $\operatorname{Re} s$ large and negative and is equal to

$$\prod_{v\notin S}\left\{L(1-s,\eta_v^{-1}\,\omega_v^{-1}\otimes\pi_v)\,\Phi(wg_v,1-s,\varphi'_v)\,\omega_v(\det g_v)\right\}.$$

Thus the integral 11.5.3 is equal to

$$\varepsilon(g_S)\,\omega(\det\hat g_S)\,\omega(-A_S)\,|A_S|^{s-\frac12}\widehat\Lambda(1-s,\eta^{-1}\omega^{-1})\prod_{v\notin S}\Phi(wg,1-s,\varphi'_v)$$

which is entire.

Since

$$\Phi(wg_v,1-s,\varphi'_v)=\varepsilon(s,\omega_v\otimes\pi_v,\psi)\,\Phi(g_v,s,\varphi'_v)$$

the analytic continuations of (11.5.2) and (11.5.3) are equal. We show as in the proof of Theorem 11.3 that the resultant entire function is bounded in vertical strips of finite width.

There is now a simple lemma to be proved.

Lemma 11.5.3 *The group $G_F \cap G_D^S$ is generated by the matrices in it of the form*

$$\begin{pmatrix} a & \beta \\ 0 & \delta \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}.$$

This is clear if S is empty. Suppose that S is not empty. If

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

belongs to $G_F \cap G_D^S$ and $|\alpha|_v = 1$ for all v in S then

$$g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta - \frac{\beta\gamma}{\alpha} \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta}{\gamma} \\ 0 & 1 \end{pmatrix}$$

and both matrices belong to $G_F \cap G_D^S$. In general if g is in $G_F \cap G_D^S$ then, for each v in S, $|\alpha|_v \leq 1$, $|\gamma|_v \leq 1$ and either $|\alpha|_v$ or $|\gamma|_v$ is 1. Choose ξ in F so that, for every v in S, $|\xi_v| = 1$ if $|\alpha|_v < 1$ and $|\xi_v| < 1$ if $|\alpha|_v = 1$. Then

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + \xi\gamma & \beta + \xi\delta \\ \gamma & \delta \end{pmatrix}$$

and $|\alpha + \xi\gamma|_v = 1$ for all v in S. The lemma follows.

We know that if

$$h = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

belongs to $G_F \cap G_D^S$ then $\varphi(hg) = \varphi(g)$ and $\hat{\varphi}(hg) = \hat{\varphi}(g)$. Suppose

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

is in $G_F \cap G_D^S$. Then

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} g\right) = \hat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}\right).$$

Since the argument on the right can be written

$$\begin{pmatrix} \delta & \gamma A^{-1} \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}$$

and the first term of this product lies in $G_F \cap G_D^S$ the right side is equal to

$$\hat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}\right) = \varphi(g).$$

Thus φ is invariant under $G_F \cap G_D^S$. Since $G_{\mathbb{A}} = G_F\, G_D^S$ the function φ extends in a unique manner to a function, still denoted φ, on $G_F \setminus G_{\mathbb{A}}$. It is clear that φ is K-finite and continuous and that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I. It is not quite so clear that φ is slowly increasing. If Ω is a compact subset of $G_{\mathbb{A}}$ there is a finite set $\gamma_1, \cdots, \gamma_\ell$ in G_F such that

$$\Omega = \cup_{i=1}^{\ell} \Omega \cap \gamma_i^{-1}\, G_D^S.$$

What we have to show then is that if γ belongs to G_F and $c > 0$ is given there are constants M_1 and M_2 such that for all g in $\Omega \cap \gamma^{-1}\, G_D^S$ and all a in I for which $|a| \geq c$

$$\left|\gamma\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \leq M_1 |a|^{M_2}.$$

If v is a place of F, which is not in S and is archimedean if F is a number field, there is a compact set C in I such that

$$\{a \in I \,\big|\, |a| \geq c\} \subseteq F^\times \{a \in F_v^\times \,\big|\, |a| \geq c\} C$$

Thus the inequality has only to be verified for a in $F_v^\times$ — of course at the cost of enlarging Ω. If

$$\gamma = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

then

$$\gamma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \gamma g$$

with $x = (1-\alpha)\frac{\beta}{\delta}$ and the conclusion results from Lemma 11.4 and the relation

$$B_{\mathbb{A}} = (B_{\mathbb{A}} \cap G_F)\,(B_{\mathbb{A}} \cap G_D^S)$$

if

$$B_{\mathbb{A}} = \Big\{ \begin{pmatrix} b & y \\ 0 & 1 \end{pmatrix} \in G_{\mathbb{A}} \Big\}.$$

Otherwise we write

$$\gamma = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\gamma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} 1 & \frac{\beta_1}{\delta_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \delta_1 a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta_2}{a} \\ 0 & 1 \end{pmatrix} g.$$

The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta_2}{a} \\ 0 & 1 \end{pmatrix} g$$

lies in a certain compact set which depends on Ω, c, and γ. The required inequality again follows from Lemma 11.4

The space U of functions φ corresponding to φ_1 in $W(\pi, \psi)$ transforms under $\widehat{\mathcal{H}}_S$ according to π. Lemma 10.13 implies that every element of U is an automorphic form. If it is not contained in $\mathcal{A}_0$, Lemma 10.12 applied to the functions

$$\varphi_0(g) = \frac{1}{\text{measure}\, F \setminus \mathbb{A}} \int_{F \setminus \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$$

with φ in U shows that there are two quasi-characters μ and ν on $F^\times \setminus I$ such that $\pi_v = \pi(\mu_v, \nu_v)$ for almost all v.

Corollary 11.6 *Suppose there does not exist a pair μ, ν of quasi-characters of $F^\times \setminus I$ such that $\pi_v = \pi(\mu_v, \nu_v)$ for almost all v. Then there is a constituent $\pi' = \otimes \pi'_v$ of $\mathcal{A}_0$ such that $\pi_v = \pi'_v$ for all v not in S.*

Since U transforms under $\widehat{\mathcal{H}}_S$ according to π it is, if v is not in S, the direct sum of subspaces transforming under $\mathcal{H}_v$ according to π_v. By assumption U is contained in $\mathcal{A}_0$ and therefore in $\mathcal{A}_0(\eta)$. $\mathcal{A}_0(\eta)$ is the direct sum of subspaces invariant and irreducible under $\mathcal{H}$. Choose one of these summands V so that the projection of U on V is not 0. If $\pi' = \otimes \pi'_v$ is the representation of $\mathcal{H}$ on V it is clear that $\pi'_v = \pi_v$ if v is not in S.

Another way to guarantee that U lies in the space of cusp forms and therefore that the conclusion of the corollary holds is to assume that for at least one v not in S the representation π_v is absolutely cuspidal.

§**12. Some extraordinary representations.** In [18] Weil has introduced a generalization of the Artin L-functions. To define these it is necessary to introduce the Weil groups. These groups are discussed very clearly in the notes of Artin–Tate but we remind the reader of their most important properties. If F is a local field let C_F be the multiplicative group of F and if F is a global field let C_F be the idèle class group $F^\times \setminus I$. If K is a finite Galois extension of F the Weil group $W_{K/F}$ is an extension of $\mathfrak{G}(K/F)$, the Galois group of K/F, by C_K. Thus there is an exact sequence

$$1 \to C_K \to W_{K/F} \to \mathfrak{G}(K/F) \to 1.$$

If L/F is also Galois and L contains K there is a continous homomorphism $\tau_{L/F,K/F}$ of $W_{L/F}$ onto $W_{K/F}$. It is determined up to an inner automorphism of $W_{K/F}$ by an element of C_K. In particular $W_{F/F} = C_F$ and the kernel of $\tau_{K/F,F/F}$ is the commutator subgroup of $W_{K/F}$. Also if $F \subseteq E \subseteq K$ we may regard $W_{K/F}$ as a subgroup of $W_{K/F}$. If F is global and v a place of F we also denote by v any extension of v to K. There is a homomorphism α_v of W_{K_v/F_v} into $W_{K/F}$ which is determined up to an inner automorphism by an element of C_K.

A representation σ of $W_{K/F}$ is a continuous homomorphism of $W_{K/F}$ into the group of invertible linear transformations of a finite-dimensional complex vector space such that $\sigma(w)$ is diagonalizable for all w in $W_{K/F}$. If K is contained in L then $\sigma \circ \tau_{L/F,K/F}$ is a representation of $W_{L/F}$ whose equivalence class is determined by that of σ. In particular if ω is a generalized character of C_F then $\omega \circ \tau_{K/F,F/F}$ is a one-dimensional representation of $W_{K/F}$ which we also call ω. If σ is any other representation $\omega \otimes \sigma$ has the same dimension as σ. If $F \subseteq E \subseteq K$ and ρ is a representation of $W_{K/E}$ on X let Y be the space of functions φ on $W_{K/F}$ with values in X which satisfy

$$\varphi(uw) = \rho(u)\,\varphi(w)$$

for all u in $W_{K/E}$. If $v \in W_{K/F}$ and $\varphi \in Y$ let $\sigma(v)\varphi$ be the function

$$W \to \varphi(wv)$$

$\sigma(v)\varphi$ also belongs to Y and $v \to \sigma(v)$ is a representation of $W_{K/F}$. We write

$$\sigma = \operatorname{Ind}(W_{K/F}, W_{K/E}, \rho).$$

If F is global and σ is a representation of $W_{K/F}$ then, for any place v, $\sigma_v = \sigma \circ \alpha_v$ is a representation of W_{K_v/F_v} whose class is determined by that of σ.

Now we remind ourselves of the definition of the generalized Artin L-functions. Since we are going to need a substantial amound of detailed information about these functions the best reference is probably [19]. In fact to some extent the purpose of [19] is to provide the background for this chapter and the reader who wants to understand all details will need to be quite familiar with it. If F is a local field then to every representation σ of $W_{K/F}$ we can associate a local L-function $L(s,\sigma)$. Moreover if ψ_F is a non-trivial additive character of F we can define a local factor $\varepsilon(s,\sigma,\psi_F)$. The L-function and the factor $\varepsilon(s,\sigma,\psi_F)$ depend only on the equivalance class of σ.

If F is a global field we set

$$L(s,\sigma) = \prod_v L(s,\sigma_v)$$

The product converges in a right half-plane and $L(s,\sigma)$ can be analytically continued to a function meromorphic in the whole complex plane. If ψ_F is a non-trivial character of $F \setminus \mathbb{A}$ the functions $\varepsilon(s,\sigma_v,\psi_v)$ are identically 1 for all but a finite number of v. If

$$\varepsilon(s,\sigma) = \prod_v \varepsilon(s,\sigma_v,\psi_v)$$

and $\widetilde{\sigma}$ is the representation contragredient to σ the functional equation

$$L(s,\sigma) = \varepsilon(s,\sigma)\, L(1-s,\widetilde{\sigma})$$

is satisfied. For all but finitely many places v the representation σ_v is the direct sum of d, the dimension of σ, one-dimensional representations. Thus there are generalized characters $\mu_v^1, \cdots \mu_v^d$ of C_{F_v} such that σ_v is equivalent to the direct sum of the one dimensional representations

$$w \mapsto \mu_v^i\big(\tau_{K_v/F_v,F_v/F_v}(w)\big).$$

Moreover, for all but finitely many of these v, $\mu_v^1, \cdots, \mu_v^d$ are unramified and there is a constant r, which does not depend on v, such that

$$|\mu_v^i(\varpi_v)| \le |\varpi_v|^r \qquad 1 \le i \le d.$$

If F is a global or a local field and σ is a representation of $W_{K/F}$ then $w \to \det\sigma(w)$ is a one-dimensional representation and therefore corresponds to a generalized character of C_F. We denote this character by $\det\sigma$.

If F is a local field, σ is a two-dimensional representation of $W_{K/F}$, and ψ_F is a non-trivial additive character of F then, as we saw in the first chapter, there is at most one irreducible admissible representation π of $\mathcal{H}_F$ such that

$$\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right) = \det\sigma(\alpha)\, I$$

and, for all generalized characters ω of C_F,

$$\begin{aligned} L(s,\omega\otimes\pi) &= L(s,\omega\otimes\sigma)\\ L(s,\omega^{-1}\otimes\tilde{\pi}) &= L(s,\omega^{-1}\otimes\sigma)\\ \varepsilon(s,\omega\otimes\pi,\psi_F) &= \varepsilon(s,\omega\otimes\sigma,\psi_F).\end{aligned}$$

If $\psi'_F(x) = \psi_F(\beta x)$ then

$$\varepsilon(s,\omega\otimes\sigma,\psi'_F) = \det\omega\otimes\sigma(\beta)\,\varepsilon(s,\omega\otimes\sigma,\psi_F)$$

and, since

$$\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right) = \det\sigma(\alpha)\, I$$

one also has

$$\varepsilon(s,\omega\otimes\pi,\psi'_F) = \det\omega\otimes\sigma(\beta)\,\varepsilon(s,\omega\otimes\pi,\psi_F).$$

Thus π, if it exists at all, is independent of ψ_F. We write $\pi = \pi(\sigma)$.

There are a number of cases in which the existence of $\pi(\sigma)$ can be verified simply by comparing the definitions of the previous chapter with those of [19]. If μ and ν are two quasi-characters of C_F and σ is equivalent to the representation

$$w \to \begin{pmatrix} \mu\big(\tau_{K/F,F/F}(w)\big) & 0 \\ 0 & \nu\big(\tau_{K/F,F/F}(w)\big) \end{pmatrix}$$

then $\pi(\sigma) = \pi(\mu,\nu)$. If K/F is a separable quadratic extension, χ is a quasi-character of $C_K = W_{K/K}$, and

$$\sigma = \operatorname{Ind}(W_{K/F}, W_{K/K}, \chi)$$

then $\pi(\sigma) = \pi(\chi)$. Observe that $\pi(\chi)$ is alway infinite-dimensional.

Suppose F is a global field and K is a separable quadratic extension of F. Let χ be a quasi-character of C_K and let

$$\sigma = \operatorname{Ind}(W_{K/F}, W_{K/K}, \chi).$$

If v does not split in K

$$\sigma_v = \operatorname{Ind}(W_{K_v/F_v}, W_{K_v/K_v}, \chi_v),$$

but if v splits in K the representation σ_v is the direct sum of two one-dimensional representations corresponding to quasi-characters μ_v and ν_v such that $\mu_v\nu_v^{-1}$ is a character. Thus $\pi(\sigma_v)$ is defined and infinite-dimensional for all v.

Proposition 12.1 *If there is no quasi-character μ of C_F such that $\chi(\alpha) = \mu(N_{K/F}\alpha)$ for all α in C_K the representation $\otimes_v \pi(\sigma_v)$ is a constituent of $\mathcal{A}_0$.*

If ω is a generalized character of F then

$$(\omega \otimes \sigma)_v = \omega_v \otimes \sigma_v.$$

Define a generalized charcter $\omega_{K/F}$ of C_K by

$$\omega_{K/F}(\alpha) = \omega\big(N_{K/F}(\alpha)\big).$$

Then

$$\omega \otimes \sigma = \operatorname{Ind}(W_{K/F}, W_{K/K}, \omega_{K/F}\chi)$$

and

$$L(s, \omega \otimes \sigma) = L(s, \omega_{K/F}\chi).$$

The L-function on the right is the Hecke L-function associated to the generalized character $\omega_{K/F}\chi$ of C_K. It is entire and bounded in vertical strips unless there is a complex number r such that

$$\omega_{K/F}(\alpha)\,\chi(\alpha) = |\alpha|^r = |N_{K/F}\alpha|^r.$$

But then

$$\chi(\alpha) = \omega^{-1}(N_{K/F}\alpha)\,|N_{K/F}\alpha|^r$$

which is contrary to assumption. The function

$$L(s, \omega^{-1} \otimes \widetilde{\sigma}) = L(s, \omega_{K/F}^{-1}\chi^{-1})$$

is also entire and bounded in vertical strips. It follows immediately that the collection $\{\pi(\sigma_v)\}$ satisfies the conditions of Theorem 11.3.

This proposition has a generalization which is one of the principal results of these notes.

Theorem 12.2 *Suppose F is a global field and σ is a two-dimensional representation of $W_{K/F}$. Suppose also that for every generalized character ω of C_F both $L(s,\omega\otimes\sigma)$ and $L(s,\omega^{-1}\otimes\widetilde{\sigma})$ are entire functions which are bounded in vertical strips. Then $\pi(\sigma_v)$ exists for every place v and $\otimes_v\pi(\sigma_v)$ is a constituent of $\mathcal{A}_0$.*

We observe that the converse to this theorem is an immediate consequence of Theorem 11.1.

We are going to apply Corollary 11.6. There are a large number of conditions which must be verified. We know that $\pi(\sigma_v)$ is defined for all but a finite number of v. In particular it is defined for v archimedean for then σ_v is either induced from a quasi-character of a quadratic extension of F_v or is the direct sum of two one-dimensional representations. If σ_v is equivalent to the direct sum of two one-dimensional representations corresponding to quasi-charcters μ_v and ν_v then $\mu_v\nu_v^{-1}$ is a character so that $\pi(\sigma_v)$ is infinite-dimensional. Let S be the set of places for which $\pi(\sigma_v)$ is infinite-dimensional. Let S be the set of places for which $\pi(\sigma_v)$ is not defined or, since this is still conceivable, finite dimensional. We are going to show that S is empty but at the moment we are at least sure that it is finite. If v is not in S set $\pi_v=\pi(\sigma_v)$.

If v is in S the representation σ_v must be irreducible so that

$$L(s,\omega_v\otimes\sigma_v)=L(s,\omega_v^{-1}\otimes\widetilde{\sigma}_v)=1$$

for every generalized character ω_v of $F_v^\times$. The Artin conductor $\mathfrak{p}_v^{m_v}$ of σ_v is defined in the Appendix to [19]. There is a constant c_v, depending on σ_v, such that if ω_v is unramifed

$$\varepsilon(s,\omega_v\otimes\sigma_v,\psi_v)=c_v\,\omega_v(\varpi_v)^{m_v+2n}\,|\varpi_v|^{(m_v+2n_v)(s-\frac{1}{2})}$$

if $\mathfrak{p}_v^{-n_v}$ is the largest ideal on which ψ_v is trivial. ψ_v is the restriction to F_v of a given non-trivial character of $F\setminus\mathbb{A}$.

We take

$$D=\prod_{\mathfrak{p}\in S}\mathfrak{p}^{m_\mathfrak{p}}$$

and $\eta=\det\sigma$. We define ε and $\hat{\varepsilon}$ by

$$\varepsilon\left(\begin{pmatrix}a_v&0\\0&b_v\end{pmatrix}\right)=\det\sigma_v(b_v)$$

and

$$\hat{\varepsilon}\left(\begin{pmatrix}a_v&0\\0&b_v\end{pmatrix}\right)=\det\sigma_v(a_v)$$

if v belongs to S and a_v and b_v are units of O_v. If α belongs to $F^\times$ and $|\alpha|_v=|\varpi_v|^{-n_v}$ for every v in S we set $a_\alpha=1$ and $\hat{a}_\alpha=\prod_{v\in S}c_v\,\det\sigma_v(\alpha)$; otherwise we set $a_\alpha=\hat{a}_\alpha=0$.

The function $\Lambda(s,\omega)$ of Theorem 11.5 is defined only if ω_v is unramified at each place of S and then it equals

$$\Big\{\prod_{v\in S}\omega_v(\varpi_v^{-n_v})\,|\varpi_v|^{-n_v(s-\frac{1}{2})}\Big\}\Big\{\prod_{v\notin S}L(s,\omega_v\otimes\pi_v)\Big\}$$

which is

$$\Big\{\prod_{v\in S}\omega_v(\varpi_v^{-n_v})\,|\varpi_v|^{-n_v(s-\frac{1}{2})}\Big\}\,L(s,\omega\otimes\sigma).$$

$\widehat{\Lambda}(s,\omega^{-1}\eta^{-1})$ is also defined if ω_v is unramifed at each place of S and is equal to

$$\Big\{\prod_{v\in S} c_v\,\omega_v(\varpi_v^{n_v})\,|\varpi_v|^{-n_v(s-\frac{1}{2})}\Big\}\,L(s,\omega^{-1}\otimes\widetilde{\sigma}).$$

Choose A in $F^\times$ so that $|A_v| = |\varpi_v|^{m_v}$ for every v in S. Then

$$\prod_{v\in S}\omega_v(-A)\,|A|_v^{s-\frac{1}{2}} = \prod_{v\in S}\omega_v(\varpi_v)^{m_v}\,|\varpi_v|^{m_v(s-\frac{1}{2})}.$$

The functional equation asserts that $L(s,\omega\otimes\sigma)$ is equal to

$$\Big\{\prod_{v\in S}\varepsilon(s,\omega_v\otimes\sigma_v,\psi_v)\Big\}\Big\{\prod_{v\notin S}\varepsilon(s,\omega_v\otimes\sigma_v,\psi_v)\Big\}\,L(1-s,\omega^{-1}\otimes\widetilde{\sigma}).$$

The first factor is equal to

$$\Big\{\prod_{v\in S} c_v\,\omega_v(\varpi_v)^{2n_v}\,|\varpi_v|^{2n_v(s-\frac{1}{2})}\Big\}\Big\{\prod_{v\in S}\omega_v(-A)\,|A|_v^{s-\frac{1}{2}}\Big\}.$$

Therefore $\Lambda(s,\omega)$ is equal to

$$\Big\{\prod_{v\in S}\omega_v(-A)\,|A|_v^{s-\frac{1}{2}}\Big\}\Big\{\prod_{v\notin S}\varepsilon(s,\omega_v\otimes\sigma_v,\psi_v)\Big\}\widehat{\Lambda}(1-s,\omega^{-1}\eta^{-1}).$$

The assumptions of Theorem 11.5 are now verified. It remains to verify that of Corollary 11.6. It will be a consequence of the following lemma.

Lemma 12.3 *Suppose F is a global field, K is a Galois extension of F, and ρ and σ are two representations of the Weil group $W_{K/F}$. If for all but a finite number of places v of F the local representations ρ_v and σ_v are equivalent then ρ and σ are equivalent.*

We set

$$L_0(s,\sigma) = \prod_{\mathfrak{p}} L(s,\sigma_{\mathfrak{p}}).$$

The product is taken over all non-archimedean places. We first prove the following lemma.

Lemma 12.4 *If σ is unitary the order of the pole of $L_0(s,\sigma)$ at $s=1$ is equal to the multiplicity with which the trivial representation is contained in σ.*

There are fields $E_1,\cdots,E_r$ lying between F and K, characters $\chi_{E_1},\cdots,\chi_{E_r}$, and integers $m_1,\cdots,m_r$ such that σ is equivalent to

$$\oplus_{i=1}^r m_i\,\mathrm{Ind}(W_{K/F},W_{K/E},\chi_{E_i})$$

Let $\delta_i = 1$ if χ_{E_i} is trivial and 0 otherwise. Since

$$L_0(s,\sigma) = \prod_{i=1}^r L_0(s,\chi_{E_i})^{m_i}$$

the order of its pole at $s = 1$ is $\sum_{i=1}^{r} m_i\,\delta_i$. However

$$\mathrm{Ind}(W_{K/F}, W_{K/E}, \chi_{E_i})$$

contains the trivial representation if and only if χ_{E_i} is trivial and then it contains it exactly once. Thus $\sum_{i=1}^{r} m_i\,\delta_i$ is also the number of times the trivial representation occurs in σ.

Observe that if T is any finite set of non-archimedean primes the order of the pole of

$$\prod_{\mathfrak{p}\notin T} L(s, \sigma_{\mathfrak{p}})$$

at $s = 1$ is the same as that of $L_0(s, \sigma)$.

The first step of the proof of Lemma 12.3 is to reduce it to the case that both ρ and σ are unitary. Then ρ and σ certainly have the same degree d. Let ρ act on X and let σ act on Y. Under the restriction ρ to C_K the space X decomposes into the direct sum of invariant one-dimensional subspaces $X_1, \cdots, X_d$ which transform according to quasi-characters $\mu^1, \cdots, \mu^d$ of C_K. If a is a real number let

$$M(a) = \{i \mid |\mu^i(\alpha)| = |\alpha|^a \text{ for all } \alpha \text{ in } C_K\}$$

and let

$$X(a) = \sum_{i\in M(a)} X_i$$

$X(a)$ is invariant under $W_{K/F}$ and $X = \sum_a \oplus X(a)$. Let $\rho(a)$ be the restriction of ρ to $X(a)$. Replacing ρ by σ and X by Y we can define $\nu^1, \cdots, \nu^d$ and $Y(a)$ in a similar fashion.

We now claim that if ρ_v is equivalent to σ_v then $\rho_v(a)$ is equivalent $\sigma_v(a)$ for each a. To see this we need only verify that any linear transformation from X to Y which commutes with the action of W_{K_v/F_v} or even of C_{K_v} takes $X(a)$ to $Y(a)$. Observe that under the restriction of ρ_v to C_{K_v} the space X_i transforms according to the character μ_v^i and that $|\mu_v^i(\alpha)| = |\alpha|^a$ for all α in C_{K_v} if and only if $|\mu^i(\alpha)| = |\alpha|^a$ for all α in C_K. Thus $X(a)$ and $Y(a)$ can be defined in terms of ρ_v and σ_v alone. The assertion follows.

Thus we may as well assume that for some real number a

$$|\mu^i(\alpha)| = |\nu^i(\alpha)| = |\alpha|^a$$

for all i and all α in C_K. Replacing σ by $\alpha \to |\alpha|^{-a}\,\sigma(\alpha)$ and ρ by $\alpha \to |\alpha|^{-a}\,\rho(\alpha)$ if necessary we may even assume that $a = 0$. Then ρ and σ will be equivalent to unitary representations and we now suppose them to be unitary.

If τ is irreducible and $\rho \simeq \tau \oplus \rho'$ and $\sigma \simeq \tau \oplus \sigma'$ then ρ'_v is equivalent to σ'_v whenever ρ_v is equivalent to σ_v. Since we can use induction on d it is enough to show that if τ is irreducible and unitary and contained in ρ then it is contained in σ. Let $\widetilde{\rho}$ and $\widetilde{\sigma}$ be the representations contragredient to ρ and σ. Certainly $(\widetilde{\rho}\otimes\tau)_v = \widetilde{\rho}_v \otimes \tau_v$ is equivalent to $(\widetilde{\sigma}\otimes\tau)_v$ for all but a finite number of v. Moreover $\widetilde{\rho}\otimes\tau$ contains $\widetilde{\tau}\otimes\tau$ which contains the identity. If $\widetilde{\sigma}\otimes\tau$ contains the identity then, as is well-known and easily verified, σ contains τ. On the other hand the orders of the poles of $L_0(s, \widetilde{\rho}\otimes\tau)$ and $L_0(s, \widetilde{\sigma}\otimes\tau)$ at $s = 1$ are clearly equal so that, by Lemma 12.4, $\widetilde{\sigma}\otimes\tau$ contains the trivial representations if $\widetilde{\rho}\otimes\tau$ does.

We return to the proof of Theorem 12.2. It follows from Lemma 12.3 that if the assumptions of Corollary 11.6 are not satisfied σ is equivalent to the direct sum of two one-dimensional representations associated to quasi-characters μ and ν of C_F. Then

$$L(s,\omega\otimes\sigma)=L(s,\omega\mu)\,L(s,\omega\nu).$$

The two functions on the right are Hecke L-functions. The function on the left is entire for every choice of ω. Taking $\omega=\mu^{-1}$ and $\omega=\nu^{-1}$ we see that $L(s,\mu^{-1}\nu)$ and $L(s,\nu^{-1}\mu)$ have a zero at $s=1$. Let $\mu^{-1}\,\nu(\alpha)=|\alpha|^r\,\chi(\alpha)$ where χ is a character. Then

$$\begin{aligned}L(s,\mu^{-1}\nu)&=L(s+r,\chi)\\ L(s,\nu^{-1}\mu)&=L(s-r,\chi^{-1}).\end{aligned}$$

Now neither $L(s,\chi)$ nor $L(s,\chi^{-1})$ has a zero in the set $\operatorname{Re}s\geq 1$. Therefore $1+r<1$ and $1-r<1$. This is impossible.

We can now apply Corollary 11.6 to assert that there is a constituent $\pi'=\prod_v\otimes\pi'_v$ of $\mathcal{A}_0$ such that $\pi'_v=\pi(\sigma_v)$ for v not in S. To prove the theorem we need only show that $\pi'_v=\pi(\sigma_v)$ for v in S. Taking the quotient of the two functional equations

$$L(s,\omega\otimes\sigma)=\Big\{\prod_v\varepsilon(s,\omega_v\otimes\sigma_v,\psi_v\Big\}\,L(1-s,\omega^{-1}\otimes\widetilde{\sigma})$$

and

$$L(s,\omega\otimes\pi')=\Big\{\prod_v\varepsilon(s,\omega_v\otimes\pi'_v,\psi_v)\Big\}\,L(1-s,\omega^{-1}\otimes\tilde{\pi}'),$$

we find that

$$\prod_{v\in S}\frac{L(s,\omega_v\otimes\sigma_v)}{L(s,\omega_v\otimes\pi'_v)}$$

is equal to

$$\left\{\prod_{v\in S}\frac{\varepsilon(s,\omega_v\otimes\sigma_v\psi_v)}{\varepsilon(s,\omega_v\otimes\pi'_v,\psi_v)}\right\}\left\{\prod_{v\in S}\frac{L(1-s,\omega_v^{-1}\otimes\widetilde{\sigma}_v)}{L(1-s,\omega_v^{-1}\otimes\tilde{\pi}'_v)}\right\}.$$

We need one more lemma. If v is a non-archimedean place and ω_v is a quasi-character of $F_v^\times$ let $m(\omega_v)$ be the smallest non-negative integer such that ω_v is trivial on the units of O_v congruent to 1 modulo $\mathfrak{p}_v^{m(\omega_v)}$.

Lemma 12.5 *Suppose S is a finite set of non-archimedean places and $v_0\in S$. Suppose that we are given a quasi-character χ_{v_0} of $F_{v_0}^\times$ and for each $v\neq v_0$ in S a non-negative integer m_v. Then there is a quasi-character ω of C_F such that $\omega_{v_0}=\chi_{v_0}$ and $m(\omega_v)\geq m_v$ if $v\neq v_0$ belongs to S.*

Suppose $\chi_{v_0}(\alpha)=|\alpha|_{v_0}^a\,\chi'_{v_0}(\alpha)$ where χ'_{v_0} is a character. If ω' is a character of C_F and $\omega'_{v_0}=\chi'_{v_0}$ while $m(\omega'_v)\geq m_v$ for $v\neq v_0$ in S we may take ω to be the generalized character $\alpha\to|\alpha|^r\,\omega'(\alpha)$ of C_F. In other words we may assume initially that χ_{v_0} is a character. Let A be the group of idèles whose component at places not in S is 1, whose component of a place $v\notin v_0$ in S is congruent to 1 modulo $\mathfrak{p}_v^{m_v}$, and whose component at v_0 is arbitrary. Certainly $F^\times\cap A=\{1\}$. We claim that $F^\times A$ is closed in I. Indeed if $\alpha\in I$ there is a compact neighborhood X of α on which the norm is bounded above by $1/\varepsilon$ and below by ε where ε is a positive constant. If $\beta \in F^\times$ and $\gamma \in A$ then $|\beta\gamma| = |\gamma|$. Moreover

$$A_\varepsilon = \{\gamma \in A \,\big|\, \varepsilon \leq |\gamma| \leq \frac{1}{\varepsilon}\}$$

is compact. Since $F^\times$ is discrete $F^\times A_\varepsilon$ is closed. Since any point has a compact neighborhood whose intersection with $F^\times A$ is closed the set $F^\times A$ is itself closed.

We can certainly find a character of A which equals χv_0 on $F^\times_{v_0}$ and, for any $v \neq v_0$ in S, is non-trivial on the set of units in O_v congruent to 1 modulo $\mathfrak{p}_v^{m_v}$. Extend this character to $F^\times A$ by setting it equal to 1 on $F^\times$. The result can be extended to a character of I which is necessarily 1 on $F^\times$. We take ω to be this character.

Let $\pi'_v\left(\left(\begin{smallmatrix} \alpha_v & 0 \\ 0 & \alpha_v \end{smallmatrix}\right)\right) = \eta_v(\alpha_v)$. If $\eta(\alpha) = \prod_v \eta_v(\alpha_v)$ then η is a quasi-character of $F^\times \setminus I$. Since, by construction, $\eta = \det\sigma$ on I^s_D the quasi-characters η and $\det\sigma$ are equal. Therefore $\eta_v = \det\sigma_v$ for all v. We know that if $m(\omega_v)$ is sufficiently large

$$L(s, \omega_v \otimes \sigma_v) = L(s, \omega_v \otimes \pi'_v) = 1$$

and

$$L(1-s, \omega_v^{-1} \otimes \widetilde{\sigma}_v) = L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v) = 1.$$

Moreover, by Proposition 3.8

$$\varepsilon(s, \omega_v \otimes \pi'_v, \psi_v) = \varepsilon(s, \omega_v\eta_v, \psi_v)\, \varepsilon(s, \omega_v, \psi_v).$$

It is shown in the Appendix of [19] that if $m(\omega_v)$ is sufficiently large

$$\varepsilon(s, \omega_v \otimes \sigma_v, \psi_v) = \varepsilon(s, \omega_v \det\sigma_v, \psi_v)\, \varepsilon(s, \omega_v, \psi_v).$$

Applying Lemma 12.5 and the equality preceding it we see that if v is in S and ω_v is any quasi-character of $F_v^\times$

$$\frac{L(s, \omega_v \otimes \sigma_v)}{L(s, \omega_v \otimes \pi'_v)} = \left\{\frac{\varepsilon(s, \omega_v \otimes \sigma_v, \psi_v)}{\varepsilon(s, \omega_v \otimes \pi'_v, \psi_v)}\right\} \left\{\frac{L(1-s, \omega_v^{-1} \otimes \widetilde{\sigma})}{L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v)}\right\}.$$

Recalling that

$$L(s, \omega_v \otimes \sigma_v) = L(1-s, \omega_v^{-1} \otimes \widetilde{\sigma}_v) = 1$$

for v in S we see that

$$\frac{L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v)}{L(s, \omega_v \otimes \pi'_v)} = \frac{\varepsilon(s, \omega_v \otimes \sigma_v, \psi_v)}{\varepsilon(, \omega_v \otimes \pi'_v, \psi_v)}. \tag{12.5.1}$$

The theorem will follow if we show that

$$L(s, \omega_v \otimes \pi'_v) = L(1-s, \omega_v^{-1} \otimes \pi'_v) = 1$$

for all choices of ω_v.

If not, either π'_v is a special representation or there are two quasi-characters μ_v and ν_v of $F_v^\times$ such that $\pi'_v = \pi(\mu_v, \nu_v)$. According to (12.5.1) the quotient

$$\frac{L(1-s, \omega_v^{-1} \otimes \pi'_v)}{L(s, \omega_v \otimes \pi'_v)}$$

is an entire function of s for every choice of ω_v. If $\pi'_v = \pi(\mu_v, \nu_v)$ and $m(\mu_v^{-1}\nu_v)$ is positive

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \frac{1 - |\varpi_v|^s}{1 - |\varpi_v|^{1-s}}$$

which has a pole at $s = 1$. If $m(\mu_v^{-1}\nu_v) = 0$

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \left\{\frac{1 - |\varpi_v|^s}{1 - |\varpi_v|^{1-s}}\right\} \left\{\frac{1 - \mu_v^{-1}\,\nu_v(\varpi_v)\,|\varpi_v|^s}{1 - \mu_v\,\nu_v^{-1}(\varpi_v)\,|\varpi_v|^{1-s}}\right\}$$

which has a pole at $s = 1$ unless $\mu_v\,\nu_v^{-1}(\varpi_v) = |\varpi_v|$. But then it has a pole at $s = 2$. If π'_v is the special representation associated to the pair of quasi-characters

$$\alpha \to \mu_v(\alpha)\,|\alpha|^{\frac{1}{2}} \qquad \alpha \to \mu_v(\alpha)\,|\alpha|^{-\frac{1}{2}}$$

of $F_v^\times$ then

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \frac{1 - |\varpi_v|^{s+\frac{1}{2}}}{1 - |\varpi_v|^{\frac{1}{2}-s}}$$

which has a pole at $s = \frac{1}{2}$.

There is a consequence of the theorem which we want to observe.

Proposition 12.6 *Suppose E is a global field and that for every separable extension F of E, every Galois extension K of F, and every irreducible two-dimensional representation σ of $W_{K/F}$ the function $L(s,\sigma)$ is entire and bounded in vertical strips. Then if F_1 is the completion of E at some place, K_1 is a Galois extension of F_1, and σ_1 is a two-dimensional representation of W_{K_1/F_1} the representation $\pi(\sigma_1)$ exists.*

We begin with a simple remark. The restriction of σ_1 to C_{K_1} is the direct sum of two one-dimensional representations corresponding to generalized characters χ_1 and χ_2 of C_{K_1}. If τ belongs to $G = \mathfrak{G}(K_1/F_1)$ either $\chi_1\big(\tau(\alpha)\big) = \chi_1(\alpha)$ for all α in C_K or $\chi_1\big(\tau(\alpha)\big) = \chi_2(\alpha)$ for all α in C_K. If the representation σ_1 is irreducible there is at least one τ for which $\chi_1\big(\tau(\alpha)\big) = \chi_2(\alpha)$. If $\chi_1 \neq \chi_2$, the fixed field L_1 of

$$H = \big\{\tau \in G \,\big|\, \chi_1\big(\tau(\alpha)\big) \equiv \chi_1(\alpha)\big\}$$

is a quadratic extension of F. The restriction of σ_1 to W_{K_1/L_1} is the direct sum of two one-dimensional representations and therefore is trivial on the commutator subgroup $W^c_{K_1/L_1}$ which is the kernel of $\tau_{K_1/F_1,L_1/F_1}$. With no loss of generality we may suppose that K_1 equals L_1 and is therefore a quadratic extension of F_1. Then σ_1 is equivalent to the representation

$$\mathrm{Ind}(W_{K_1/F_,}, W_{K_1/K_1}, \chi_1).$$

If σ_1 is reducible $\pi(\sigma_1)$ is defined. The preceding remarks show that it is defined if σ_1 is irreducible and $\sigma_1(\alpha)$ is not a scalar matrix for some α in C_{K_1}. The proposition will therefore follow from Theorem 12.2 and the next lemma.

Lemma 12.7 *Suppose F_1 is the completion of the field E at some place, K_1 is a Galois extension of F_1, and σ_1 is an irreducible two-dimensional representation such that $\sigma_1(\alpha)$ is a scalar matrix for all α in C_{K_1}. Then there is a separable extension F of E, a Galois extension K of F, a place v of K, and isomorphism φ of K_v with K_1 which takes F_v to F_1, and an irreducible two-dimensional representation σ of $W_{K/F}$ such that σ_v is equivalent to $\sigma_1 \circ \varphi$.*

Observe that the existence of σ_1 forces F_1 to be non-archimedean. We establish a further sequence of lemmas.

Lemma 12.8 *Suppose V is a finite dimensional real vector space, G is a finite group of linear transformations of V, and L is a lattice in V invariant under G. If χ is a quasi-character of L invariant under G there is a quasi-character χ' of V invariant under G and a positive integer m such that the restrictions of χ' and χ to mL are equal.*

Let $\widehat{V}$ be the dual of V and $\widehat{V}_{\mathbb{C}}$ its complexification. There is a y in $\widehat{V}_{\mathbb{C}}$ such that $\chi(x) = e^{2\pi i\langle x,y\rangle}$ for all x in L. If z belongs to $\widehat{V}_C$ the generalized character $x \to e^{2\pi i\langle x,z\rangle}$ is trivial on L if and only if z belongs to $\frac{\widehat{L}}{m}$. $\widehat{L}$ is the lattice

$$\{v \in \widehat{V} \bigm| \langle x,v\rangle \in Z \text{ for all } x \text{ in } L\}.$$

Let $\widehat{G}$ be the group contragredient to G. We have to establish the existence of an m and a z in $\frac{\widehat{L}}{m}$ such that $y-z$ is fixed by $\widehat{G}$. If σ belongs to $\widehat{G}$ then $\sigma y - y = w_\sigma$ belongs to $\widehat{L}$. Clearly $\sigma w_\tau + w_\sigma = w_{\sigma\tau}$. Set

$$z = \frac{1}{[G:1]} \sum_\tau w_\tau.$$

If m is taken to be $[G:1]$ this is the required element.

Lemma 12.9 *Suppose F is a global field, K is a Galois extension of it, and v is a place of K. Suppose also that $[K_v : F_v] = [K : F]$ and let χ_v be a quasi-character of C_{K_v} invariant under $G = \mathfrak{G}(K_v/F_v) = \mathfrak{G}(K/F)$. There is a closed subgroup A of finite index in C_K which is invariant under G and contains C_{K_v} and a quasi-character χ of A invariant under G whose restriction to C_{K_v} is χ_v.*

Suppose first that the fields have positive characteristic. We can choose a set of non-negative integers n_w, $w \neq v$, all but a finite number of which are zero, so that the group

$$B = C_{K_v} \times \prod_{w\neq v} U_{K_w}^{n_w}$$

is invariant under G and contains no element of $K^\times$ except 1. $U_{K_w}^{n_w}$ is the group of units of O_{K_w} which are congruent to 1 modulo $\mathfrak{p}_{K_w}^{n_w}$. We extend χ_v to B by setting it equal to 1 on

$$\prod_{w\neq v} U_{K_w}^{n_w}$$

and then to $A = K^\times B/K^\times$ by setting it equal to 1 on $K^\times$.

Now let the fields have characteristic 0. Divide places of K different from v into two sets, S, consisting of the archimedean places, and T, consisting of the non-archimedean ones. Choose a collection of non-negative integers π'_w, $w \in T$, all but a finite number of which are zero, so that

$$B' = C_{K_v} \times \prod_{w \in S} C_{K_w} \times \prod_{w \in T} U_{K_w}^{n'_w}.$$

is invariant under G and contains no roots of unity in K except 1. If w is archimedean let U_{K_w} be the elements of norm 1 in K_w and set

$$B'_a = \prod_{w \in S} U_{K_w} \times \prod_{w \in T} U_{K_w}^{n'_w}.$$

B'/B'_a is isomorphic to the product of C_{K_v} and

$$V = \prod_{w \in S} C_{K_w}/U_{K_w}$$

which is a vector group. The projection L of

$$M = B'_1(B' \cap K^\times)/B'_1$$

on V is a lattice in V and the projection is an isomorphism. Define the quasi-character μ of L so that if m in M projects to m_1 in C_{K_v} and to m_2 in V then

$$\chi_v(m_1)\,\mu(m_2) = 1.$$

μ is invariant under G. Choose a quasi-character μ' of V and an integer n so that μ' and μ are equal on nL. Let ν' be the quasi-character obtained by lifting $\chi_v \times \mu'$ from $C_{K_v} \times V$ to B'. It follows from a theorem of Chevalley ([20] Theorem 1) that we can choose a collection of non-negative ingegers $\{n_w \bigm| w \in T\}$ all but a finite number of which are zero so that $n_w \geq n'_w$ for all w in T, so that

$$B = C_{K_v} \times \prod_{w \in S} C_{K_w} \times \prod_{w \in T} U_{K_w}^{n_w}$$

is invariant under G, and so that every element of $B \cap K^\times$ is an nth power of some element of $B' \cap K^\times$. The restriction ν of ν' to B is trivial on $B \cap K^\times$. We take $A = K^\times B/K^\times$ and let χ be the quasi-character which is 1 on $K^\times$ and ν on B.

Lemma 12.10 *Suppose F_1 is a completion of the global field E, K_1 is a finite Galois extension of F_1 with Galois group G_1, and χ_{K_1} is a quasi-character of C_{K_1} invariant under G_1. There is a separable extension F of E, a Galois extension K of F, a place v of K such that $[K_v : F_v] = [K : F]$, an isomorphism φ of K_v with K_1 which takes F_v to F_1, and a quasi-character χ of C_K invariant under $\mathfrak{g}(K/F)$ such that $\chi_v = \chi_{K_1} \circ \varphi$.*

We may as well suppose that $F_1 = E_w$, where w is some place of E. It is known ([8], p.31) that there is a polynomial with coefficients in E such that if θ is a root of this polynomial $E_w(\theta)/E_w$ is isomorphic to K_1/F_1. Let L be the splitting field of this polynomial and extend w to a place of L. The extended place we also call w. Replacing E by the fixed field of the decomposition group of w if necessary we may suppose that $F_1 = E_w$, $K_1 = L_w$ and $[L_w : E_w] = [L : E]$. Now set $\chi_w = \chi_{K_1}$ and extend χ_w to a quasi-character χ' of A as in the previous lemma.

Let K be the abelian extension of L associated to the subgroup A. Since A is invariant under $\mathfrak{G}(L/E)$ the extension K/E is Galois. Let v be a place of K dividing the place w of L. Since A contains C_{L_w} the fields K_v and L_w are equal. Let F be the fixed field of the image of $\mathfrak{G}(K_v/E_w)$ in $\mathfrak{G}(K/E)$. Let v also denote the restriction of v to F. The fields F_v and E_w are the same. The mapping $N_{K/L} : C_K \to C_L$ maps C_K into A. Let $\chi = \chi' \circ N_{K/L}$. Then χ is clearly invariant under $\mathfrak{G}(K/F)$. Since $N_{K/L}$ restricted to K_v is an isomorphism of K_v with L_w which takes F_v onto E_w the lemma is proved.

To prove Lemma 12.7 we need only show that if F is a global field, K is a Galois extension of F, χ is a quasi-character of C_K invariant under $\mathfrak{G}(K/F)$, v is a place of K such that $[K : F] = [K_v : F_v]$, and σ_1 is an irreducible two-dimensional representation of W_{K_v/F_v} such that $\sigma_v(\alpha) = \chi_v(\alpha)I$ for all α in C_{K_v} then there is a two-dimensional representation σ of $W_{K/F}$ such that σ_v is equivalent to σ_1. σ will be irreducible because σ_1 is.

Let σ_1 act on X. Let ρ_v be the right regular representation of W_{K_v/F_v} on the space V_v of functions f on W_{K_v/F_v} satisfying

$$f(\alpha w) = \chi_v(\alpha)\, f(w)$$

for all α in C_{K_v} and all w in W_{K_v/F_v}. If λ is a non-zero linear functional on X the map from x to the function $\lambda(\sigma_1(w)x)$ is a W_{K_v/F_v}-invariant isomorphism of X with a subspace Y of V_v.

Let V be the space of all functions f on $W_{K/F}$ satisfying

$$f(\alpha w) = \chi(\alpha)\, f(w)$$

for all α in C_K and all w in $W_{K/F}$. Since $[K : F] = [K_v : F_v]$ the groups $\mathfrak{G}(K/F)$ and $\mathfrak{G}(K_v/F_v)$ are equal. Therefore

$$W_{K/F} = C_K\, W_{K_v/F_v}$$

Moreover $C_{K_v} = C_K \cap W_{K_v/F_v}$. Thus the restriction of functions in V to W_{K_v/F_v} is an isomorphism of V with V_v. For simplicity we identify the two spaces. Let ρ be the right regular representation of $W_{K/F}$ on V. If α belongs to C_K then

$$f(w\alpha) = \chi(w\alpha w^{-1})\, f(w) = \chi(\alpha)\, f(w)$$

because χ is $\mathfrak{G}(K/F)$ invariant. Therefore $\rho(\alpha) = \chi(\alpha)\, I$ and a subspace V is invariant under $W_{K/F}$ if and only if it is invariant under W_{K_v/F_v}. If we take for σ the restriction of ρ to Y then σ_v will be equivalent to σ_1.

References

for Chapter II

Automorphic forms are discussed in terms of group representations in [3] and [11] as well as:

13. Godement, R., *Analyse spectrale des fonctions modulaires*, Seminaire Bourbaki, No. 278.

14. Godement, R., *Introduction à la theorie de Langlands*, Seminaire Bourbaki, No. 321.

As its name implies the Hecke theory is a creation of Hecke.

15. Hecke E., *Mathematische Werke*.

Maass seems to have been the first to consider it outside the classical context.

16. Maass, H., *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. 121 (1944).

It seems to have been Weil who first used several L-functions to prove a converse theorem.

17. Weil, A., *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. 168 (1967).

His generalizations of the Artin L-functions are introduced in:

18. Weil, A., *Sur la théorie du corps de classes*, Jour. Math. Soc. Japan, vol. 3 (1951).

For various technical facts used in the twelfth paragraph we refer to:

19. Langlands R., *On the functional equation of the Artin L-functions*, Notes, Yale University (in preparation).

We have also had occasion to refer to:

20. Chevalley, C., *Deux théorèmes d' arithmetique*, Jour. Math. Soc. Japan, vol. 3 (1951).

A result more or less the same as Proposition 12.1 is proved in:

21. Shalika, J.A. and S. Tanaka, *On an explicit construction of a certain class of automorphic forms*, Preprint.

Chapter III: Quaternion Algebras

§**13. Zeta-functions for** $M(2,F)$**.** In this paragraph F is again a local field and $A = M(2,F)$ is the algebra of 2×2 matrices with entries from F. The multiplicative group $A^\times$ of A is just $G_F = GL(2,F)$. If g is in G_F we set

$$|g|_A = \alpha_A(g) = |\mathrm{det} g|_F^2.$$

Let π be an admissible representation of $\mathcal{H}_F$ on the space V. Let the contragredient representation $\tilde\pi$ act on on $\widetilde V$. If v belongs to V and $\tilde v$ to $\widetilde V$ the function

$$\langle \pi(g)v, \tilde v\rangle = \langle v, \tilde\pi(g^{-1})\tilde v\rangle$$

is characterized by the relation

$$\int \langle \pi(gh), v, \tilde v\rangle\, f(h)\, dh = \langle \pi(g)\,\pi(f)v, \tilde v\rangle$$

for all f in $\mathcal{H}_F$.

If Φ belongs to the Schwartz space $\mathcal{S}(A)$ and v belongs to V and $\tilde v$ to $\widetilde V$ we set

$$Z(\pi, \Phi, v, \tilde v) = \int_{G_F} \Phi(g)\,\langle \pi(g)v, \tilde v\rangle\, d^\times g$$

and

$$Z(\tilde\pi, \Phi, v, \tilde v) = \int_{G_F} \Phi(g)\,\langle v, \tilde\pi(g)\tilde v\rangle\, d^\times g$$

The choice of Haar measure is not important provided that it is the same for both integrals.

If ω is a quasi-character of $F^\times$

$$Z(\omega\otimes\pi, \Phi, v, \tilde v) = \int_{G_F} \Phi(g)\,\omega(\mathrm{det} g)\langle \pi(g)v, \tilde v\rangle\, d^\times g$$

The purpose of this paragraph is to prove the following theorem.

Theorem 13.1 *Let π be an irreducible admissible representation of $\mathcal{H}_F$ and $\tilde\pi$ its contragredient. Let π act on V and $\tilde\pi$ on $\widetilde V$.*

(i) *For every v in V, $\tilde v$ in $\widetilde V$, and Φ in $\mathcal{S}(A)$ the integrals defining $Z(\alpha_F^s\otimes\pi, \Phi, v, \tilde v)$ and $Z(\alpha_F^s\otimes\tilde\pi, \Phi, v, \tilde v)$ converge absolutely for $\mathrm{Re}\, s$ sufficiently large.*

(ii) *Both functions can be analytically continued to functions which are meromorphic in the whole plane and bounded at infinity in vertical strips of finite width.*

(iii) *If*

$$Z(\alpha_F^{s+\frac12}\otimes\pi, \Phi, v, \tilde v) = L(s,\pi)\,\Xi(s, \Phi, v, \tilde v)$$

and

$$Z(\alpha_F^{s+\frac12}\otimes\tilde\pi, \Phi, v, \tilde v) = L(s,\tilde\pi)\,\widetilde\Xi(s, \Phi, v, \tilde v)$$

then $\Xi(s, \Phi, v, \tilde v)$ and $\widetilde\Xi(s, \Phi, v, \tilde v)$ are entire.

(iv) *There exist ϕ, $v_1, \cdots, v_n$ and $\tilde v_1, \cdots, \tilde v_n$ such that $\sum_{i=1}^n \Xi(s, \Phi, v_i, \tilde v_i)$ is of the form $a\, e^{bs}$ with $a \neq 0$.*

(v) *If Φ' is the Fourier transform of Φ with respect to the character $\psi_Z(x) = \psi_F(\mathrm{tr}\, x)$ then*

$$\widetilde\Xi(1-s, \Phi', v, \tilde v) = \varepsilon(s, \pi, \psi_F), \Xi(s, \Phi, v, \tilde v).$$

We suppose first that F is non-archimedean and π is absolutely cuspidal. Then we may take π in the Kirillov form so that V is just $\mathcal{S}(F^\times)$. Since an additive character $\psi_F = \psi$ is given we will of course want to take the Kirillov model with respect to it. The next lemma is, in the case under consideration, the key to the theorem.

Lemma 13.1.1 *If φ belongs to $\mathcal{S}(F^\times)$, v belongs to V, and $\tilde{v}$ belongs to $\widetilde{V}$ set*

$$\Phi(g) = \varphi(\det g)\langle v, \tilde{\pi}(g)\tilde{v}\rangle\,|\det g|_F^{-1}$$

if g belongs to G_F and set $\Phi(g) = 0$ if g in A is singular. Then Φ belongs to $\mathcal{S}(A)$ and its Fourier transform is given by

$$\Phi'(g) = \varphi'(\det g)\langle \pi(g)v, \tilde{v}\rangle\,|\det g|_F^{-1}\,\eta^{-1}(\det g)$$

if g belongs to G_F and

$$\Phi'(g) = 0$$

if g is singular. Here η is the quasi-character of $F^\times$ defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\,I$$

and

$$\varphi' = \pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi.$$

This lemma is more easily appreciated if it is compared with the next one which is simpler but which we do not really need.

Lemma 13.1.2 *Let $\mathcal{S}_0(A)$ be the space of all Φ in $\mathcal{S}(A)$ that vanish on the singular elements and satisfy*

$$\int \phi\left(g_1\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g_2\right)dx = 0$$

for g_1 and g_2 in G_F. If Φ is in $\mathcal{S}_0(A)$ so is its Fourier transform.

Since $\mathcal{S}_0(A)$ is stable under left and right translations by the elements of G_F it is enough to show that

$$\Phi'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$$

for a in F and that

$$\int_F \Phi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)dx = 0$$

To verify these relations we just calculate the left sides!

$$\Phi'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \int_A \Phi(g)\,\psi_A\left(g\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)dg$$

The right side is a positive multiple of

$$\int_{G_F} \Phi(g)\,\psi_A\left(g\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)|\det g|^2\,d^\times g$$

which equals

$$\int_{G_F/N_F} \psi_A\left(g\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)|\det g|^2\Big\{\int_F \Phi\left(g\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)dx\Big\}\,d^\times g$$

This is 0 because the inner integral vanishes identically.

$$\int_F \Phi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\, dx$$

is equal to

$$\int \Big\{ \int \Phi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)\, \psi_F(\alpha+\delta+\gamma x)\, d\alpha\, d\beta\, d\gamma\, d\delta \Big\}\, dx$$

which, by the Fourier inversion fomula, is equal to

$$\int \Phi\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right)\, \psi_F(\alpha+\delta)\, d\alpha\, d\delta\, d\beta$$

which equals

$$\int |\alpha|\, \psi_F(\alpha+\delta) \Big\{ \Phi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\right)\, d\beta \Big\}\, d\alpha\, d\delta$$

and this is 0.

We return to the proof of Lemma 12.1.1 for absolutely cuspidal π. Since $\langle v, \tilde{\pi}(g)\tilde{v}\rangle$ has compact support on G_F modulo Z_F the function $\Phi(g)$ belongs to $\mathcal{S}(A)$. Moreover

$$\int_F \Phi\left(g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right)\, dx$$

is equal to

$$\varphi(\det gh)\, |\det gh|_F^{-1} \int \langle \pi(g^{-1}v, \tilde{\pi}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\, \tilde{\pi}(h)\, v\rangle\, dx.$$

Since π is absolutely cuspidal this integral is 0. Thus Φ belongs to $\mathcal{S}_0(A)$ and, in particular, Φ' vanishes at the singular elements.

Suppose we can show that for all choice of φ, v, and $\tilde{v}$

$$\Phi'(e) = \varphi'(e)\, \langle v, \tilde{v}\rangle. \tag{13.1.3}$$

If h belongs to G_F set $\Phi_1(g) = \Phi(h^{-1}g)$. If $a = \det h$, $\varphi_1(x) = |a|\, \varphi(a^{-1}x)$, and $v_1 = \pi(h)\, v$,

$$\Phi_1(g) = \varphi_1(\det g)\langle v_1, \tilde{\pi}(g)\tilde{v}\rangle\, |\det g|_F^{-1}.$$

Then $\Phi_1'(e)$ is equal to

$$\varphi_1'(e)\, \langle v_1, v\rangle.$$

On the other hand

$$\varphi_1' = \pi(w)\, \varphi = |a|\, \pi(w)\, \pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi$$

which equals

$$|a|\, \pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\, \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\, \pi(w)\, \varphi.$$

Thus $\Phi'(h)$, which equals $\Phi_1'(e)\, |\det h|^{-2}$, is

$$\varphi'(\det h)\langle \pi(h)\, v, \tilde{v}\rangle\, \eta^{-1}(\det h)\, |\det h|^{-1}.$$

The formula (13.1.3) will be a consequence of the next lemma.

Lemma 13.1.4 *Let $d\varepsilon$ be the normalized Haar measure on the group $U = U_F$. If ν is a character of U set*

$$\eta(\nu, x) = \int_U \nu(\varepsilon)\, \psi(\varepsilon x)\, d\varepsilon$$

if x is in F. Let dx be the Haar measure on F which is self-dual with respect to ψ. Then

$$\int_F \eta(\nu, x\varpi^n)\, \psi(ax)\, dx = 0$$

unless $|a| = |\varpi|^n$ but if $a = \zeta\varpi^n$ with ζ in U

$$\int_F \eta(\nu, x\varpi^n)\, \psi(ax)\, dx = \nu(-\zeta)\, |\varpi|^{-n}\, c^{-1}$$

if c is the measure of U with respect to dx.

The general case results from the case $n = 0$ by a change of variable; so we suppose $n = 0$. In this case the formulae amount to a statement of the Fourier inversion formula for the function which is 0 outside of U and equal to $c^{-1}\nu(\varepsilon)$ on U.

Suppose we could show that there is a positive constant d which does not depend on π such that for all φ, v, and $\tilde{v}$

$$\Phi'(e) = d\varphi'(e)\, \langle v, \tilde{v}\rangle.$$

Then we would have

$$\Phi'(g) = d\varphi'(\det g)\, \langle \pi(g)v, \tilde{v}\rangle\, \eta^{-1}(\det g).$$

Exchanging π and $\tilde{\pi}$ and recalling that $\tilde{\pi} = \eta^{-1} \otimes \pi$ we see that Φ'', the Fourier transform of Φ', is given by

$$\Phi''(g) = d^2\varphi''(\det g)\langle v, \tilde{\pi}(g)\tilde{v}\rangle\, |\det g|_F^{-1}\, \eta(\det g),$$

where $\varphi'' = \tilde{\pi}(w)\varphi_1$ if $\varphi_1(a) = \varphi'(a)\,\eta^{-1}(a)$. According to the remarks preceding the statement of Theorem 2.18, φ'' is the product of $\pi(w)\,\varphi' = \eta(-1)\,\varphi$ and $\eta^{-1}(\det g)$. Thus

$$\Phi''(g) = \eta(-1)\, d^2\varphi(\det g)\, \langle v, \tilde{\pi}(g)\tilde{v}\rangle\, |\det g|_F^{-1}.$$

Since $\Phi'' = \Phi(-g) = \eta(-1)\,\Phi(g)$ the numbers d^2 and d are both equal to 1. The upshot is that in the proof of the formula (13.1.3) we may ignore all positive constants and in particular do not need to worry about the normalizaton of Haar measures.

Moreover it is enough to prove the formula for φ, v, $\tilde{v}$ in a basis of the spaces in which they are constrained to lie. Oddly enough the spaces are all the same and equal to $\mathcal{S}(F^\times)$. Assume $\varphi_1 = v$, $\varphi_2 = \tilde{v}$, and φ are supported respectively by $\varpi^{n_1}U$, $\varpi^{n_2}U$, and $\varpi^n U$ and that, for all ε in U, $\varphi_1(\varpi^{n_1}\varepsilon) = \nu_1^{-1}(\varepsilon)$, $\varphi_2(\varpi^{n_2}\varepsilon) = \nu_2^{-1}(\varepsilon)$ and $\varphi(\varpi^n\varepsilon) = \nu^{-1}(\varepsilon)$. ν, ν_1 and ν_2 are three characters of U.

The formal Mellin transforms of these three functions are $\widehat{\varphi}_1(\mu, t) = \delta(\mu\nu_1^{-1})t^{n_1}$, $\widehat{\varphi}_2(\mu, t) = \delta(\mu\nu_2^{-1})t^{n_2}$, and $\widehat{\varphi}(\mu, t) = \delta(\mu\nu^{-1})t^n$. Recall that, for example,

$$\widehat{\varphi}(\mu, t) = \sum_n t^n \int_U \varphi(\varpi^n\varepsilon)\, \mu(\varepsilon)\, d\varepsilon.$$

The scalar product $\langle \varphi_1, \varphi_2 \rangle$ is equal to

$$\int \varphi_1(a)\,\varphi_2(-a)\,d^\times a = \delta(\nu_1\nu_2)\,\delta(n_1 - n_2)\,\nu_2(-1).$$

If $\eta(\varepsilon\varpi^n) = \nu_0(\varepsilon)\,z_0^n$ then

$$\widehat{\varphi}'(\mu, t) = C(\mu, t)\,\widehat{\varphi}(\mu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1})$$

which equals

$$\delta(\nu\mu\nu_0)\sum_m C_m(\nu^{-1}\nu_0^{-1})t^{m-n}z_0^{-n}.$$

Consequently

$$\widehat{\varphi}'(1) = C_n(\nu^{-1}\nu_0^{-1})\,z_0^{-n}.$$

Thus the formula to be proved reads

$$\Phi'(e) = C_n(\nu^{-1}\nu_0^{-1})\,z_0^{-n}\,\nu_2(-1)\,\delta(\nu_1\nu_2)\,\delta(n_1 - n_2).$$

Almost all g in A can be written in the form

$$g = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

with a and b in $F^\times$ and x and y in F. The additive Haar measure dg on A may be written as

$$dg = |\mathrm{det} g|_F^2\,d^\times g = |b^4|\,d^\times b\,dx|a|\,d^\times a\;dy$$

and for any g of this form

$$\psi_A(g) = \psi_F\big(b(x - y)\big)$$

while $\Phi(g)$ is equal to

$$\eta^{-1}(b)\,|b^2a|^{-1}\,\varphi(b^2a)\langle\pi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi_1, \tilde{\pi}\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\varphi_2\rangle.$$

Let f_1 and f_2 be the two functions which appear in the scalar product. Their formal Mellin transforms can be calculated by the methods of the second paragraph,

$$\hat{f}_1(\mu, t) = \nu_0(-1)\,C(\mu, t)\,\eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x)\,\mu^{-1}\nu_0^{-1}(\zeta)\,z_0^{-r-n_1}t^{-r-n_2}$$

if $a = \zeta\varphi^r$ and

$$\hat{f}_2(\mu, t) = \eta(\mu\nu_2^{-1}, \varpi^{n_2}y)\,t^{n_2}.$$

The scalar product of f_1 and f_2 is equal to

$$\int f_1(a)\,f_2(-a)\,d^\times a$$

which, by the Plancherel theorem for $F^\times$, is equal to

$$\sum_\mu \mu(-1)\frac{1}{2\pi}\int_0^{2\pi} \hat f_1(\mu, e^{i\theta})\,\hat f_2(\mu^{-1}, e^{-i\theta})\,d\theta.$$

A typical integral is equal to the product of $\nu_0(-1)\,\mu^{-1}\nu_0^{-1}(\zeta)z_0^{-r-n_1}$ and

$$\int_0^{2\pi} C(\mu, e^{i\theta})\,e^{-i(r+n_1+n_2)\theta}\,\eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x)\,\eta(\mu^{-1}\nu_2^{-1}, \varpi^{n_2}y)\,d\theta$$

which equals

$$2\pi C_{r+n_1+n_2}(\mu)\,\eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x)\,\eta(\mu^{-1}\nu_2, \varpi^{n_2}y).$$

Also if $a = \zeta\varpi^r$

$$\eta^{-1}(b)\,|b^2a|^{-1}\,\varphi(b^2a) = \varphi(b^2\varpi^r)\,\nu^{-1}(\zeta)\,\eta^{-1}(b)\,|b^2|^{-1}|a|^{-1}.$$

If we put all this information together we get a rather complicated formula for $\Phi(g)$ which we have to use to compute $\Phi'(e)$. $\Phi'(e)$ is expressed as an integral with respect to a, b, x, and y. We will not try to write down the integrand. The integral with respect to a is an integration over ζ followed by a sum over r. The integrand is a sum over μ. The integration over ζ annihilates all but one term, that for which $\mu\nu\nu_0 = 1$. We can now attempt to write down the resulting integrand, which has to be integrated over b, x, and y, and summed over r. It is the product of

$$\eta^{-1}(b)\,|b|^2\,\nu(-1)\,z_0^{-r-n_1}\,\varphi(b^2\varpi^r)\,C_{r+n_1+n_2}(\nu^{-1}\nu_0^{-1})$$

and

$$\eta(\nu\nu_1^{-1}, \varpi^{n_1}x)\,\eta(\nu\nu_0\nu_2^{-1}, \varpi^{n_2}y)\,\psi_F\big(b(x-y)\big).$$

The second expression can be integrated with respect to x and y. Lemma 13.1.4 shows that the result is 0 unless $|b| = |\varpi|^{n_1} = |\varpi|^{n_2}$. In particular $\Phi'(e) = 0$ if $n_1 \neq n_2$. If $n_1 = n_2$ the integration over b need only be taken over $\varpi^{n_1}U$. Then the summation over r disappears and only the term for which $r + 2n_1 = n$ remains. Apart from positive constants which depend only on the choices of Haar measure $\Phi'(e)$ is equal to

$$z_0^{-n}\nu_1(-1)\,C_n(\nu^{-1}\nu_0^{-1})\int_U \nu_1^{-1}\nu_2^{-1}(\varepsilon)\,d\varepsilon.$$

Since

$$\int_U \nu_1^{-1}\nu_2^{-1}(\varepsilon)\,d\varepsilon = \delta(\nu_1\nu_2)$$

the proof of Lemma 13.1.1 is complete.

Since $L(s,\pi) = L(s\tilde\pi) = 1$ if π is absolutely cuspidal the first three assertions of the theorem are, for such π, consequences of the next lemma.

Lemma 13.1.5 *Suppose Φ belongs to $\mathcal{S}(A)$, v belongs to V, and $\tilde{v}$ belongs to $\widetilde{V}$. If π is absolutely cuspidal the integral*

$$\int \Phi(g)\langle\pi(g)v,\tilde{v}\rangle\,|\mathrm{det}g|^{s+\frac{1}{2}}\,d^{\times}g$$

is absolutely convergent for $\mathrm{Re}\,s$ sufficiently large and the functions it defines can be analytically continued to an entire function.

Suppose the integral is convergent for some s. If ξ is an elementary idempotent such that $\pi(\xi)v = v$ the integral is not changed if Φ is replaced by

$$\Phi_1(g) = \int_{GL(2,O_F)} \Phi(gh^{-1})\,\xi(h)\,dh.$$

Since π is absolutely cuspidal it does not contain the trivial representation of $GL(2, O_F)$ and we can choose ξ to be orthogonal to the constant functions on $GL(2, O_F)$. Then $\Phi_1(0) = 0$. Thus, when proving the second assertion of the lemma we can suppose that $\Phi(0) = 0$.

The support of $\langle\pi(g)v,\tilde{v}\rangle$ is contained in a set $Z_F C$ with C compact. Moreover there is an open subgroup K' of $GL(2, O_F)$ such that the functions $\Phi(g)$ and $\langle\pi(g)v,\tilde{v}\rangle$ are invariant under right translations by the elements of K'. If

$$C \subseteq U_{i=1}^{p} g_i K'$$

the integral is equal to

$$\sum_{i=1}^{p}\langle\pi(g_i)v,\tilde{v}\rangle\,|\mathrm{det}g_i|^{s+\frac{1}{2}}\int_{F^{\times}}\Phi\left(\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} g_i\right)\,\eta(a)\,|a|^{2s+1}\,d^{\times}a,$$

if each of the integrals in this sum converges. They are easily seen to converge if $\mathrm{Re}\,s$ is sufficiently large and if $\Phi(0) = 0$ they converge for all s. The lemma is proved.

Now we verify a special case of the fifth assertion.

Lemma 13.1.6 *Suppose φ is in $\mathcal{S}(F^{\times})$ and*

$$\Phi(g) = \varphi(\mathrm{det}g)\langle v,\tilde{\pi}(g)\tilde{v}\rangle\,|\mathrm{det}g|^{-1}.$$

Then for all u in V and all $\tilde{u}$ in $\widetilde{V}$

$$\widetilde{\Xi}(1-s,\Phi',u,\tilde{u}) = \varepsilon(s,\pi,\psi)\,\Xi(s,\Phi,u,\tilde{u}).$$

The expression $\Xi(s,\Phi,u,\tilde{u})$ is the integral over G_F of

$$|\mathrm{det}g|^{s-\frac{1}{2}}\varphi(\mathrm{det}g)\,\langle\pi(g)u,\tilde{u}\rangle\,\langle v,\tilde{\pi}(g)\tilde{v}\rangle.$$

The integral

$$\int_{SL(2,F)}\langle\pi(gh)u,\tilde{u}\rangle\,\langle v,\tilde{\pi}(gh)\tilde{v}\rangle\,dh$$

depends only on $\mathrm{det}g$. Set it equal to $F(\mathrm{det}g)$. Then $\Xi(s,\Phi,u,\tilde{u})$ is equal to

$$\int_{F^{\times}}\varphi(a)\,F(a)\,|a|^{s-\frac{1}{2}}\,d^{\times}a.$$

By Lemma 13.1.1
$$\Phi'(g) = \varphi'(\det g)\, |\det g|^{-1}\, \eta^{-1}(\det g)\, \langle \pi(g)v, \tilde{v} \rangle$$
so that $\Xi(s, \Phi', u, \tilde{u})$ is equal to
$$\int_{F^\times} \varphi'(a)\, \widetilde{F}(a)\, |a|^{s-\frac{1}{2}}\, \eta^{-1}(a)\, d^\times a$$
if
$$\widetilde{F}(a) \int_{SL(2,F)} \langle u, \tilde{\pi}(gh)\tilde{u} \rangle\, \langle \pi(gh)v, \tilde{v} \rangle\, dh$$
whenever $a = \det g$. Since the integrand is not changed when g is replaced by
$$\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} g$$
we have $\widetilde{F}(b^2 a) = \widetilde{F}(a)$ and $\widetilde{F}(a) = \widetilde{F}(a^{-1})$. The same relations are valid for F. Also $\widetilde{F}(a) = F(a^{-1})$ so that $F = \widetilde{F}$.

We remind ourselves that we are now trying to show that
$$\int_{F^\times} \varphi'(a)\, \widetilde{F}(a)\, \eta^{-1}(a)\, |a|^{\frac{1}{2}-s}\, d^\times a$$
is equal to
$$\varepsilon(s, \pi, \psi) \int_{F^\times} \varphi(a)\, F(a)\, |a|^{s-\frac{1}{2}}\, d^\times a.$$

If U' is an open subgroup of U_F such that
$$\tilde{\pi}\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\right) \tilde{u} = \tilde{u}$$
and
$$\pi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\right) v = v$$
for ε in U' then F and $\widetilde{F}$ are constant on cosets of $(F^\times)^2 U'$ which is of finite index in $F^\times$. Write
$$F(a) = \sum_{i=1}^{p} c_i \chi_i(a)$$
where χ_i are characters of $F^\times/(F^\times)^2 U'$. We may assume that all c_i are different from 0. Then
$$F(a^{-1}) = \sum_{i=1}^{p} c_i\, \chi_i(a^{-1}).$$

The factor $\varepsilon(s, \pi \otimes \chi_i, \psi)$ was defined so that
$$\int_{F^\times} \varphi'(a)\, \chi_i^{-1}\, \eta^{-1}(a)\, |a|^{\frac{1}{2}-s}\, d^\times a$$

would be equal to

$$\varepsilon(s,\chi_i\otimes\pi,\psi)\int_{F^\times}\varphi(a)\,\chi_i(a)\,|a|^{s-\frac{1}{2}}\,d^\times a.$$

All we need do is show that π and $\chi_i\otimes\pi$ are equivalent, so that

$$\varepsilon(s,\chi_i\otimes\pi,\psi)=\varepsilon(s,\pi,\psi).$$

A character χ is one of the χ_i if and only if χ is trivial on $(F^\times)^2$ and

$$\int_{F^\times/(F^\times)^2}F(a)\,\chi(a)\,d^\times a\neq 0.$$

This integral is equal to

$$\int_{G_F/Z_F}\chi(g)\langle\pi(g)u,\tilde u\rangle\,\langle v,\tilde\pi(g)\tilde v\rangle\,dg$$

which equals

$$\int_{G_F/Z_F}\langle\chi\otimes\pi(g)u,\tilde u\rangle\,\langle v,\tilde\pi(g)\tilde v\rangle\,dg.$$

The integral does not change if π is replaced by $\omega\otimes\pi$. Thus the Schur orthogonality relations imply that it is non-zero only if π and $\chi\otimes\pi$ are equivalent.

If Φ belongs to $\mathcal{S}_0(A)$ the functions $\Phi(g)\,|\mathrm{det}g|^{s+\frac{1}{2}}$ belongs to $\mathcal{H}_F$ and we can form the operator

$$T(s,\Phi)=\int_{G_F}\Phi(g)\,|\mathrm{det}g|^{s+\frac{1}{2}}\pi(g)\,d^\times g.$$

If Φ has the form of the previous lemma the functional equation may be written as

$$T(1-s,\Phi')=\varepsilon(s,\pi,\psi)\,T(s,\Phi).$$

Lemma 13.1.7 *Given a non-zero w in V, the set of all u in V such that for some Φ of the form*

$$\Phi(g)=\varphi(\mathrm{det}g)\,\langle v,\tilde\pi(g)\tilde v\rangle\,|\mathrm{det}g|_F^{-1}$$

the vector $T(s,\Phi)w$ is of the form $e^{bs}u$ is a set that spans V.

If the function Φ is of this form so is the function $\Phi'(g)=\Phi(hg)$ and

$$T(s,\Phi')w=|\mathrm{det}h|^{-(s+\frac{1}{2})}\pi(h^{-1})\,T(s,\Phi)\,w$$

Since π is irreducible we need only show that there is at least one non-zero vector in the set under consideration. Moreover there is an r such that $\alpha_F^r\otimes\pi$ is unitary and we may as well suppose that π itself is unitary. Let (u,v) be a positive invariant form on V.

Choose $v=w$ and $\tilde v$ so that $\langle u,\tilde v\rangle=(u,w)$ for all u. Let φ be the characteristic function of U_F. Then

$$\Phi(g)=\big(w,\pi(g)w\big)$$

if $|\mathrm{det}g|=1$ and is 0 otherwise. If

$$H=\{g\in G_F\,\big|\,|\mathrm{det}g|=1\}$$

then

$$T(s,\Phi)w=\int_H\big(w,\pi(g)w\big)\,\pi(g)w\,d^\times g$$

is independent of s and is non-zero because

$$\big(T(s,\Phi)w,w\big)=\int_H\big|\big(\pi(g)w,w\big)\big|^2\,d^\times g.$$

The fourth assertion follows immediately and the fifth will now be a consequence of the following lemma.

Lemma 13.1.8 *Suppose Φ belongs to $\mathcal{S}(A)$ and Ψ belongs to $\mathcal{S}_0(A)$. There is a vertical strip in which the integrals*

$$\iint \Phi(g)\,\Psi'(h)\,\langle\pi(g)v,\tilde\pi(h)\tilde v\rangle\,|\mathrm{det}g|^{s+\frac{1}{2}}\,|\mathrm{det}h|^{\frac{3}{2}-s}\,d^\times g\,d^\times h$$

and

$$\iint \Phi'(g)\,\Psi(h)\,\langle\pi^{-1}(g)v,\tilde\pi(h^{-1})\tilde v\rangle\,|\mathrm{det}g|^{\frac{3}{2}-s}\,|\mathrm{det}h|^{s+\frac{1}{2}}\,d^\times g\,d^\times h$$

exist and are equal.

A little juggling shows that there is no harm in supposing that the quasi-character η defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\,I$$

is a character. Fix v and $\tilde v$. Let C be a compact subset of G_F which contains the support of Ψ and Ψ'. The set

$$\{\tilde\pi(h)\tilde v \mid h \in C\}$$

is finite. Thus there is a compact set in G_F such that for any h in C the function

$$g \to \langle\pi(g)v,\tilde\pi(h)\tilde v\rangle$$

has its support in $Z_F\,C'$. Moreover these functions are uniformly bounded. The first integral is therefore absolutely convergent for $\mathrm{Re}\,s > -\frac{1}{2}$. The second is convergent for $\mathrm{Re}\,s < \frac{3}{2}$.

If $-\frac{1}{2} < \mathrm{Re}\,s < \frac{3}{2}$ the first integral is equal to

$$\int \Psi'(h)\,|\mathrm{det}h|^{\frac{3}{2}-s}\Big\{\int \Phi(g)\langle\pi(g)v,\tilde\pi(h)\tilde v\rangle\,|\mathrm{det}g|^{s+\frac{1}{2}}\,d^\times g\Big\}\,d^\times h.$$

Replacing g by hg we obtain

$$\int \Psi'(h)\,|\mathrm{det}h|^{2}\Big\{\int \Phi(gh)\langle\pi(g)v,\tilde v\rangle\,|\mathrm{det}g|^{s+\frac{1}{2}}\,d^\times g\Big\}\,d^\times h.$$

If we take the additive Haar measure to be $dh = |\mathrm{det}h|^2\,d^\times h$ this may be written as

$$\int \langle\pi(g)v,\tilde v\rangle\,|\mathrm{det}g|^{s+\frac{1}{2}}\Big\{\int \Phi(hg)\,\Psi'(h)\,dh\Big\}\,d^\times g.$$

The second integral is

$$\int \Psi(h)\,|\mathrm{det}h|^{s+\frac{1}{2}}\Big\{\int \Phi'(g)\langle\pi^{-1}(g)v,\tilde\pi^{-1}(h)\tilde v\rangle\,|\mathrm{det}g|^{\frac{3}{2}-s}\,d^\times g\Big\}\,d^\times h.$$

After a change of variables this becomes

$$\int \langle\pi^{-1}(g)v,\tilde v\rangle\,|\mathrm{det}g|^{\frac{3}{2}-s}\Big\{\int \Phi'(gh)\,\Psi(h)\,dh\Big\}d^\times g.$$

Replacing g by g^{-1} we obtain

$$\int \langle \pi(g)v, \tilde{v}\rangle \,|\mathrm{det} g|^{s+\frac{1}{2}} \Big\{ |\mathrm{det} g|^{-2} \int \Phi'(g^{-1}h)\,\Psi(h)\,dh \Big\}\, d^{\times} g.$$

Since

$$\int \Phi(hg)\,\Psi'(h)\,dh$$

is equal to

$$|\mathrm{det} g|^{-2} \int \Phi'(g^{-1}h)\,\Psi(h)\,dh$$

the lemma follows.

The theorem is now proved when π is absolutely cuspidal. Suppose that it is a constituent of $\tau = \rho(\mu_1, \mu_2)$. In this case the field may be archimedean. Although τ is not necessarily irreducible it is admissible and its matrix coefficients are defined. The contragredient representation $\widetilde{\tau}$ is $\rho(\mu_1^{-1}, \mu_2^{-1})$ and the space of τ is $\mathcal{B}(\mu_1, \mu_2)$ while that of $\widetilde{\tau}$ is $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$. If f belongs to $\mathcal{B}(\mu_1, \mu_2)$ and $\tilde{f}$ belongs to $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ then

$$\langle \tau(g)f, \tilde{f}\rangle = \int_K f(kg)\,\tilde{f}(k)\,dk$$

and

$$\langle f, \widetilde{\tau}(g)\tilde{f}\rangle = \int_K f(k)\,\tilde{f}(kg)\,dk$$

if K is the standard maximal compact subgroup of G_F.

If we set

$$L(s,\tau) = L(s,\mu_1)\,L(s,\mu_2)$$
$$L(s,\widetilde{\tau}) = L(s,\mu_1^{-1})\,L(s,\mu_2^{-1})$$

and

$$\varepsilon(s,\tau,\psi) = \varepsilon(s,\mu_1,\psi)\,\varepsilon(s,\mu_2,\psi)$$

the theorem may be formulated for the representation τ. We prove it first for τ and then for the irreducible constituents of τ.

We use a method of R. Godement. If Φ belongs to $\mathcal{S}(A)$ then for brevity the function $x \to \Phi(gxh)$ which also belongs to $\mathcal{S}(A)$ will be denoted by $h\Phi g$. Also let

$$\varphi_\Phi(a_1, a_2) = \int_F \Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) dx$$

where dx is the measure which is self dual with respect to ψ. φ_Φ belongs to $\mathcal{S}(F^2)$. The map $\Phi \to \varphi_\Phi$ of $\mathcal{S}(A)$ into $\mathcal{S}(F^2)$ is certainly continuous.

We are now going to define a kernel $K_\Phi(h, g, s)$ on $K \times K$. We set

$$K_\Phi(e,e,s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi).$$

Recall that the right-hand side is

$$\iint \varphi_\Phi(a_1,a_2)\,\mu_1(a_1)\,|a_1|^s\,\mu_2(a_2)\,|a_2|^s\,d^{\times}a_1\,d^{\times}a_2.$$

In general

$$K_\Phi(h,g,s) = K_{g\Phi h^{-1}}(e,e,s).$$

We also set

$$\widetilde{K}_\Phi(e,e,s) = Z(\mu_1^{-1}\alpha_F^s, \mu_2^{-1}\alpha_F^s, \varphi_\Phi)$$

and

$$\widetilde{K}_\Phi(h,g,s) = \widetilde{K}_{g\Phi h^{-1}}(e,e,s).$$

The kernels are defined for $\operatorname{Re} s$ sufficiently large and are continuous in h, g, and s and, for fixed h and g, holomorphic in s.

We now make some formal computations which will be justified by the result. The expression $Z(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f})$ is equal to

$$\int_{G_F} \Phi(g) \Big\{ \int_K f(kg)\, \tilde{f}(k)\, dk \Big\} |\mathrm{det} g|^{s+\frac{1}{2}}\, d^\times g$$

which is

$$\int_K \tilde{f}(k) \Big\{ \int_{G_F} \Phi(g)\, f(kg)\, |\mathrm{det} g|^{s+\frac{1}{2}}\, d^\times g \Big\}\, dk.$$

Changing variables in the inner integral we obtain

$$\int_K \tilde{f}(k) \Big\{ \int_{G_F} \Phi(k^{-1}g)\, f(g)\, |\mathrm{det} h|^{s+\frac{1}{2}}\, d^\times g \Big\}\, dk.$$

Using the Iwasawa decomposition to evaluate the integral over G_F we see that this is equal to

$$\int_{K\times K} K_\Phi(k_1,k_2,s)\, f(k_2)\, \tilde{f}(k_1)\, dk_1\, dk_2.$$

Since we could have put in absolute values and obtained a similar result all the integals are convergent and equal for $\operatorname{Re} s$ sufficiently large. A similar computation shows that

$$Z(\alpha_F^{s+\frac{1}{2}} \otimes \widetilde{\tau}, \Phi, f, \tilde{f})$$

is equal to

$$\int_{K\times K} \widetilde{K}_\Phi(k_1,k_2,s)\, f(k_1)\, \tilde{f}(k_2)\, dk_1\, dk_2$$

if $\operatorname{Re} s$ is large enough.

If ξ is an elementary idempotent such that $\tau(\xi)f = f$ and $\widetilde{\tau}(\xi)\tilde{f} = \tilde{f}$ then $Z(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f})$ is not changed if Φ is replaced by

$$\Phi_1(g) = \iint \Phi(k_1 g k_2^{-1})\, \xi(k_1)\, \xi(k_2)\, dk_1\, dk_2.$$

Thus, at least when proving the second and third assertions, we may suppose that Φ is K-finite on both sides and, in fact, transforms according to a fixed finite set of irreducible representations of K. Then, as s varies, the functions

$$K_\Phi(k_1,k_2,s)$$

stay in some fixed finite dimensional space U of continuous functions on $K \times K$. The map

$$F \to \iint F(k_1, k_2)\, f(k_2)\, \tilde{f}(k_1)\, dk_1\, dk_2$$

is a linear form on this space and we can find $g_1, \cdots, g_n$ and $h_1, \cdots, h_n$ in K such that it can be represented in the form

$$F \to \sum_{i=1}^{n} \lambda_i\, F(g_i, h_i).$$

Thus

$$Z(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f}) = \sum \lambda_i\, K_\Phi(g_i, h_i, s).$$

Thus to prove the second and third assertions we need only show that for each g and h in K the function

$$\frac{K_\Phi(g, h, s)}{L(s, \tau)}$$

is entire and $K_\Phi(g, h, s)$ itself is bounded at infinity in vertical strips. There is certainly no harm in supposing that $g = h = e$ so that

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$$

Thus the desired facts are consequences of the results obtained in paragraphs 3, 5, and 6 when proving the local functional equation for constituents of τ. Replacing τ by its contragredient representation we obtain the same results for $Z(\alpha_F^{s+\frac{1}{2}} \otimes \widetilde{\tau}, \Phi, f, \tilde{f})$.

To prove the functional equation we have to see what happens to the Fourier transform when we pass from the function Φ to to Φ_1. The answer is simple:

$$\Phi_1'(g) = \iint \Phi'(k_1 g k_2^{-1})\, \xi(k_1)\, \xi(k_2)\, dk_1\, dk_2.$$

Thus in proving the functional equation we may suppose that Φ is K-finite on both sides. We may also suppose that if $F(k_1, k_2)$ is in U so is $F'(k_1, k_2) = F(k_2, k_1)$. Then $Z(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi', f, \tilde{f}) = \sum \lambda_i\, \widetilde{K}_{\Phi'}(h_i, g_i, s)$. To prove the functional equation we have to show that

$$\frac{\widetilde{K}_{\Phi'}(h, g, 1-s)}{L(1-s, \tau)} = \varepsilon(s, \tau, \psi)\, \frac{K_\Phi(g, h, s)}{L(s, \tau)}$$

for any h and g in K. Since the Fourier transform of $g\Phi h^{-1}$ is $h\Phi' g$ it will be enough to do this for $h = g = e$. Then the equality reduces to

$$\frac{Z(\mu_1^{-1} \alpha_F^{1-s}, \mu_2^{-1} \alpha_F^{1-s}, \varphi_{\Phi'})}{L(1-s, \widetilde{\tau})} = \varepsilon(s, \tau, \psi)\, \frac{Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)}{L(s, \tau)}$$

and is a result of the facts proved in the first chapter and the next lemma.

Lemma 13.2.1 *The Fourier transform of the function φ_Φ is the function $\varphi_{\Phi'}$.*

The value of Φ' at

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is

$$\int \Phi\left(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \psi(\alpha x + \beta z + \gamma y + \delta t)\, dx\, dy\, dz\, dt$$

if dx, dy, dz, and dt are self-dual with respect to ψ. Thus $\varphi_{\Phi'}(\alpha, \delta)$ is equal to

$$\int \left\{ \int \Phi\left(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \psi(\alpha x + \delta t)\, \psi(\beta z)\, dx\, dy\, dz\, dt \right\} d\beta$$

Applying the Fourier inversion formula to the pair of variables β and z we see that this is equal to

$$\int \Phi\left(\begin{pmatrix} x & y \\ 0 & t \end{pmatrix}\right) \psi(\alpha x + \delta t)\, dx\, dy\, dt$$

which is the value of the Fourier transform of φ_Φ at (α, δ).

The theorem, with the exception of the fourth assertion, is now proved for the representation τ. We will now deduce it, with the exception of the fourth assertion, for the constituents of τ. We will return to the fourth assertion later.

If π is a constituent of τ either $\pi = \pi(\mu_1, \mu_2)$ or $\pi = \sigma(\mu_1, \mu_2)$. In the first case there is nothing left to prove. In the second only the third assertion remains in doubt. If F is the complex field even it is alright because we can always find another pair of quasi-characters μ_1' and μ_2' such that $\pi = \pi(\mu_1', \mu_2')$. We ignore this case and suppose that F is real or non-archimedean.

First take F to be non-archimedean. We may suppose that μ_1 and μ_2 are the form $\mu_1 = \chi\alpha_F^{\frac{1}{2}}$ and $\mu_2 = \chi\alpha_F^{-\frac{1}{2}}$. The one-dimensional representation $g \to \chi(\det g)$ is contained in $\widetilde{\tau} = \rho(\mu_1^{-1}, \mu_2^{-1})$ and acts on the function $g \to \chi(\det g)$. The matrix elements for π are the functions

$$g \to \langle \tau(g) f, \tilde{f} \rangle = \langle \pi(g) f, \tilde{f} \rangle$$

where $\tilde{f}$ belongs to $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ and

$$\int_K f(k)\, \chi(\det k)\, dk = 0.$$

For such an f there is an elementary idempotent ξ such that $\tau(\xi) f = f$ while

$$\int_K \xi(k)\, dk = 0$$

The value of $Z(\alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, f, \tilde{f})$ is not changed if we replace Φ by

$$\Phi_1(g) = \int_K \Phi(gh^{-1})\, \xi(h)\, dh.$$

Lemma 13.2.2 *If g_1 and g_2 belong to G_F then*

$$\iint \Phi_1 \left(g_1 \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} g_2 \right) \, dx \, dy = 0.$$

It will be enough to prove this when g_1 is the identity. Let

$$\varphi(x, y) = \Phi_1 \left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right).$$

If g_1 is the identity then, after a change of variables, the integral becomes

$$|\det g_2|^{-1} \iint \varphi(x, y) \, dx \, dy$$

so that we can also assume g_2 is the identity. Then the integral equals

$$\int_K \left\{ \iint \Phi \left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} k \right) \, dx \, dy \right\} \xi(k^{-1}) \, dk.$$

Changing variables as before we see that the inner integral does not depend on K. Since

$$\int_K \xi(k^{-1}) \, dk = 0$$

the lemma follows.

To establish the third assertion for the representation π all we need do is show that for any g and h in K the function

$$\frac{K_\Phi(g, h, s)}{L(s, \pi)}$$

is entire provided

$$\iint \Phi \left(g_1 \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} g_2 \right) \, dx \, dy = 0$$

for all g_1 and g_2 in G_F. As usual we need only consider the case that $g = h = e$. Since

$$\int \varphi_\Phi(x, 0) \, dx = 0$$

and

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$$

we need only refer to Corollary 3.7.

If F is the field of real numbers the proof is going to be basically the same but a little more complicated. We may assume that $\mu_1 \mu_2^{-1}(x) = |x|^{2p+1-m} (\operatorname{sgn} x)^m$, where p is a non-negative integer and m is 0 or 1, and that π acts on $\mathcal{B}_S(\mu_1, \mu_2)$. The restriction of π to $SO(2, \mathbb{R})$ contains only those representations κ_n for which $n \equiv 1 - m \pmod 2$ and $|n| \geq 2p + 1 - m$. Let ξ_n be the elementary idempotent corresponding to the representation κ_n. As before we may suppose that

$$\int_{SO(2,\mathbb{R})} \Phi(xk^{-1}) \, \xi_n(k) \, dk = 0 \tag{13.2.3}$$

if κ_n does not occur in the restriction of π to $SO(2, \mathbb{R})$.

Lemma 13.2.4 *If Φ satisfies (13.2.3), if g_1 and g_2 belong to G_F, and $\varphi = \varphi_{g_1\Phi g_2}$ then*

$$\int_{\mathbb{R}} x^i \, \frac{\partial^j}{\partial y^j} \varphi(x,0)\, dx = 0$$

if $i \geq 0$, $j \geq 0$ and $i+j = 2p-m$.

We may assume that $g_2 = e$. If $\varphi = \varphi_\Phi$ let

$$L(\Phi) = \int_{\mathbb{R}} x^i \, \frac{\partial^j}{\partial y^j} \varphi(x,0)\, dx$$

and let

$$F(g) = L(g\Phi).$$

We have to show that, under the hypothesis of the lemma, $F(g) = 0$ for all g. However F is defined for all Φ in $\mathcal{S}(A)$ and if Φ is replaced by $h\Phi$ the function F is replaced by $F(gh)$. Thus to establish the identity

$$F\left(\begin{pmatrix} a_i & z \\ 0 & a_2 \end{pmatrix} g\right) = \eta_1(a_2)\,\eta_2(a_2)\,F(g)$$

where $\eta_1(a_1) = a_1^{-i}\,|a_1|^{-1}$ and $\eta_2(a_2) = a_2^j\,|a_2|^{-1}$, we need only establish it for $g = e$.

Let

$$h = \begin{pmatrix} a_1 & z \\ 0 & a_2 \end{pmatrix}.$$

Then

$$h\Phi\left(\begin{pmatrix} x & u \\ 0 & y \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} a_1 y & xz + a_2 u \\ 0 & a_2 y \end{pmatrix}\right).$$

If $\varphi = \varphi_\Phi$ and $\varphi_1 = \varphi_{h\Phi}$ then $\varphi_1(x,y)$, which is given by

$$\int \Phi\left(\begin{pmatrix} a_1 x & xz + a_2 u \\ 0 & a_2 y \end{pmatrix}\right) du,$$

is equal to

$$|a_2|^{-1} \int \Phi\left(\begin{pmatrix} a_1 x & u \\ 0 & a_2 y \end{pmatrix}\right) du = |a_2|^{-1}\,\varphi(a_1 x, a_2 y).$$

Moreover $F(h)$ is equal to

$$\int x^i \, \frac{\partial^j \varphi_1}{\partial y^j}(x,0)\, dx$$

which equals

$$a_1^{-i}\,|a_1|^{-1}\,a_2^j\,|a_2|^{-1} \int x^i \, \frac{\partial^j \varphi}{\partial y^j}(x,0)\, dx$$

as required.

Finally if

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and $\varphi = \varphi_{g\Phi}$ then $F(g)$ is equal to

$$\int x^i \frac{\partial^j \varphi}{\partial y^j}(x,0)\,dv$$

and

$$\varphi(x,y) = \int \Phi\left(\begin{pmatrix} \alpha x + \gamma u & \beta x + \delta u \\ y\gamma & y\delta \end{pmatrix}\right) du.$$

Since we can interchange the orders of differentiation and integration,

$$\frac{\partial^j \varphi}{\partial y^j}(x,0) = \sum_{n=0}^{j} \lambda_n \gamma^n \delta^{j-n} \int \varphi_n(\alpha x + \gamma u, \beta x + \delta u)\,du$$

where

$$\varphi_n(x,y) = \frac{\partial^j \Phi}{\partial \gamma^n\,\partial \delta^{j-n}}\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right)$$

and the number λ_n are constants. Thus $F(g)$ is a linear combination of the functions

$$\gamma^n \delta^{j-n} \iint x^i\, \varphi_n(\alpha x + \gamma u, \beta x + \delta u)\,dx\,du.$$

If $\alpha \neq 0$ we may substitute $x - \frac{\gamma u}{2}$ for x to obtain

$$\gamma^n \delta^{j-n} \iint \frac{(x-\gamma u)^i}{2}\, \varphi_n\Big(\alpha x, \beta x + \frac{\Delta u}{2}\Big)\,dx\,du$$

where $\Delta = \det g$. Substituting $u - \frac{\alpha\beta}{\Delta}x$ for u we obtain

$$\gamma^n \delta^{j-n} \iint \Big(x + \frac{\beta\gamma}{\Delta}x - \frac{\gamma u}{2}\Big)^i\, \varphi_n\Big(\alpha x, \frac{\Delta u}{2}\Big)\,dx\,du.$$

After one more change of variables this becomes

$$\Delta^{-i}\,|\Delta|^{-1}\,\gamma^n \delta^{j-n} \iint (\delta x - \gamma u)^i\, \varphi_n(x,u)\,dx\,du.$$

In conclusion $F(g)$ is a function of the form

$$F\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \Delta^{-i}\,|\Delta|^{-1}\,P(\alpha,\beta,\gamma,\delta)$$

where P is a polynomial.

Thus the right translates of F by the elements of G_F span a finite-dimensional space. In particular it is $O(2,\mathbb{R})$ finite and if $\eta_1 = \mu_1'\alpha_F^{\frac{1}{2}}$ while $\eta_2 = \mu_2'\alpha_F^{-\frac{1}{2}}$ it lies in a finite-dimensional invariant subspace of $\mathcal{B}(\mu_1',\mu_2')$. Thus it lies in $\mathcal{B}_F(\mu_1',\mu_2')$. Since $\mu_1'\mu_2'^{-1} = \mu_1^{-1}\mu_2$ no representation of $SO(2,\mathbb{R})$ occurring in $\pi(\mu_1',\mu_2')$ can occur in $\pi = \sigma(\mu_1,\mu_2)$. If F is not zero then for at least one such representation κ_n

$$F_1(g) = \int_{SO(2,\mathbb{R})} f(gk^{-1})\,\xi_n(k)\,dk$$

is not identically 0. But F_1 is the result of replacing Φ by

$$\Phi_1(x) = \int_{SO(2,\mathbb{R})} \Phi(xk^{-1})\,\xi_n(k)\,dk$$

in the definition of F. In particular if Φ satisfies the conditions of the lemma both Φ_1 and F_1 are zero. Therefore F is also zero and the lemma is proved.

The third assertion can now be verified as in the non-archimedean case by appealing to Lemma 5.17. The fourth has still to be proved in general.

If F is the real field let $\mathcal{S}_1(A)$ be the space of functions of the form

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp\big(-\pi(a^2+b^2+c^2+d^2)\big)\,P(a,b,c,d)$$

where P is a polynomial. If F is the complex field $\mathcal{S}_1(A)$ will be the space of functions of the form

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp\big(-\pi(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})\big)\,P(a,\bar{a},b,\bar{b},c,\bar{c},d,\bar{d})$$

where P is again a polynomial. If F is non-archimedean $\mathcal{S}_1(A)$ will just be $\mathcal{S}(A)$. The space $\mathcal{S}_1(F^2)$ is defined in a similar manner.

Lemma 13.2.5 *Suppose φ belongs to $\mathcal{S}_1(F^2)$. Then there is a Φ in $\mathcal{S}_1(A)$ such that*

$$K_\Phi(e,e,s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi)$$

and $f_1,\cdots,f_n$ in $\mathcal{B}(\mu_1,\mu_2)$ together with $\tilde{f}_1,\cdots,\tilde{f}_n$ in $\mathcal{B}(\mu_1^{-1},\mu_2^{-1})$ such that

$$\sum_{i=1}^n \int_{K\times K} K_\Phi(h,g,s)\,f_i(g)\,\tilde{f}_i(h)\,dg\,dh = K_\Phi(e,e,s).$$

Since there is a φ in $\mathcal{S}_1(F^2)$ such that

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi) = ae^{bs}\,L(s,\tau)$$

this lemma will imply the fourth assertion for the representation τ.

Given φ the existence of Φ such that $\varphi = \varphi_\Phi$ and therefore

$$K_\Phi(e,e,s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi)$$

is a triviality and we worry only about the existence of $f_1,\cdots,f_n$ and $\tilde{f}_1,\cdots,\tilde{f}_n$.

It is easily seen that if

$$\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$$

and

$$\begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix}$$

belong to K then

$$K_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} h, \begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix} g, s\right)$$

is equal to

$$\mu_1(a_1)\,\mu_2(a_2)\,\mu_1^{-1}(b_1)\,\mu_2^{-1}(b_2)\,K_\Phi(h,g).$$

Also

$$K_\Phi(hh_1, gg_1, s) = K_{g_1\Phi h_1^{-1}}(h,g,s).$$

Since Φ belongs to $\mathcal{S}_1(A)$ it is K-finite on the left and right. Thus there is a finite set S of irreducible representations of K such that if U_1 is the space of functions F on K which satisfy

$$F\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} h\right) = \mu_1(a_1)\,\mu_2(a_2)\,F(h)$$

for all

$$\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$$

in K and can be written as a linear combination of matrix elements of representations in S and U_2 is the space of functions F' on K which satisfy

$$F'\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} h\right) = \mu_1^{-1}(a_1)\,\mu_2^{-1}(a_2)\,F'(h)$$

and can be written as a linear combination of matrix elements of representations in S then, for every s, the function

$$(h,g) \to K_\Phi(h,g,s)$$

belongs to the finite dimensional space U spanned by functions of the form $(h,g) \to F(h)\,F'(g)$ with F in U_1 and F' in U_2.

Choose $F_1, \cdots, F_n$ and $F_1', \cdots, F_n'$ so that for every function F in U

$$F(e,e) = \sum_{i=1}^n \lambda_i \int_{K\times K} F(h,g)\,\bar{F}_i(h)\,\bar{F}_i'(g)\,dh\,dg.$$

Since $\bar{F}_i$ is the restriction to K of an element of $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ while $\bar{F}_i'$ is the restriction to K of an element of $\mathcal{B}(\mu_1, \mu_2)$ the lemma follows.

Unfortunately this lemma does not prove the fourth assertion in all cases. Moreover there is a supplementary condition to be verified.

Lemma 13.2.6 *Suppose F is non-archimedean and π is of the form $\pi = \pi(\mu_1, \mu_2)$ with μ_1 and μ_2 unramified. Suppose Φ is the characteristic function of $M(2, O_F)$ in $M(2, F)$. If v and $\tilde{v}$ are invariant under $K = GL(2, O_F)$ and if*

$$\int_K d^\times g = 1$$

then

$$Z(\alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, v, \tilde{v}) = L(s, \pi)\, \langle v, \tilde{v} \rangle.$$

Suppose f belongs to $\mathcal{B}(\mu_1, \mu_2)$ and is identically 1 on K while $\tilde{f}$ belongs to $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ and is identically 1 on K. Then

$$\langle f, \tilde{f} \rangle = \int_K f(k)\, \tilde{f}(k)\, dk = 1$$

and if $\tau = \rho(\mu_1, \mu_2)$ we are trying to show that

$$Z(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f}) = L(s, \tau).$$

The left side is equal to

$$\int_{K \times K} K_\Phi(h, g, s)\, f(h)\, \tilde{f}(g)\, dh\, dg.$$

Since Φ is invariant on both side under K this is equal to

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi)$$

if

$$\varphi(x, y) = \int \Phi\left(\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}\right) dz.$$

Since we have so normalized the Haar measure on G_F that

$$\int_{G_F} F(g)\, dg = \int_K \left\{ \int F\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) d^\times a_1\, d^\times a_2\, dx \right\} dk$$

where dk is the normalized measure on K, dx is the measure on F which assigns the measure 1 to O_F, and $d^\times a$ is the measure on $F^\times$ which assigns the measure 1 to U_F the function φ is the characteristic function of $O_F \times O_F$ and

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi) = L(s, \mu_1)\, L(s, \mu_2)$$

as required.

This lemma incidentally proves the fourth assertion for the one-dimensional representation $g \to \chi(\det g)$ if χ is unramified. If χ is ramified and π corresponds to χ then $\pi = \pi(\mu_1, \mu_2)$ if $\mu_1(a) = \chi(a)\, |a|^{\frac{1}{2}}$ and $\mu_2(a) = \chi(a)\, |a|^{-\frac{1}{2}}$. Thus $L(s, \pi) = 1$. If Φ is the restriction of the function χ^{-1} to K then

$$Z(\pi, \Phi, v, \tilde{v}) = \langle v, \tilde{v} \rangle \int_K d^\times g$$

and the fourth assertion is verified in this case.

Take μ_1 and μ_2 of this form with χ possibly unramified and suppose that $\pi = \sigma(\mu_1, \mu_2)$. Suppose first that χ is unramified. Let φ_0 be the characteristic function of O_F in F and let

$$\varphi_1(x) = \varphi_0(x) - |\varpi^{-1}|\,\varphi_0(\varpi^{-1}x).$$

It has O_F for support. Set

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \varphi_1(a)\,\varphi_0(b)\,\varphi_0(c)\,\varphi_0(d).$$

It has $M(2, O_F)$ for support and depends only on the residues of a, b, c, and d modulo $\mathfrak{p}_F$. If

$$K^1 = \{k \in K \mid k \equiv e (\mathrm{mod}\,\mathfrak{p})\}$$

then $K_\Phi(h, g, s)$ depends only on the cosets of h and g modulo K^1. Also

$$K_\Phi\left(e, w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, s\right) = 0$$

if x is in O_F. To see this we observe first that if

$$\Phi_1(g) = \Phi\left(gw\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

then $\varphi_{\Phi_1}(a_1, a_2)$ is equal to

$$\int_F \Phi\left(\begin{pmatrix} -y & a_1 - xy \\ -a_2 & -a_2 x \end{pmatrix}\right) dy$$

which equals

$$\varphi_0(a_2)\,\varphi_0(a_2 x) \int_{O_F} \varphi_1(y)\,\varphi_0(a_1 - xy)\,dy.$$

Since x is in O_F the function $\varphi_0(a_1 - xy)$ equals $\varphi_0(a_1)$ for y in O_F and this expression is 0 because

$$\int_{O_F} \varphi_1(y)\,dy = 0.$$

We choose f in $\mathcal{B}_S(\mu_1, \mu_2)$ so that $f(gk) = f(g)$ if k belongs to K_1, $f(e) = 1$, and

$$f(e) + \sum_{x \in O_F/\mathfrak{p}} f\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 0.$$

We choose $\tilde{f}$ in $\mathcal{B}(\mu_1, \mu_2)$ so that $\tilde{f}(gk) = \tilde{f}(g)$ if k belongs to K^1, $\tilde{f}(e) = 1$, and

$$\tilde{f}\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 0$$

if x belongs to O_F. Then

$$\int_{K \times K} K_\Phi(h, g, s)\,\tilde{f}(h)\,f(g)\,dh\,dg$$

is equal to

$$\int_K K_\Phi(e,g,s)\, f(g)\, dg = K_\Phi(e,e,s)$$

which equals

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi).$$

Moreover

$$\varphi_\Phi(a_1,a_2) = \varphi_1(a)\,\varphi_0(a_2)$$

so that, as we saw when proving Corollary 3.7, $L(s,\pi)$ is a constant times $Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi)$.

If χ is ramified $L(s,\pi)=1$. If Φ has support in K then $Z(\alpha_F^{s+1/2}\otimes\pi, \Phi, v, \tilde{v})$ is equal to

$$\int_K \Phi(k)\langle\pi(k)v, \tilde{v}\rangle\, dk$$

and we can certainly choose v, $\tilde{v}$ and Φ so that this is not 0.

We are not yet finished. We have yet to take care of the representations not covered by Lemma 13.2.5 when the field is archimedean. If F is the complex field we have only the finite dimensional representations to consider. There is a pair of characters μ_1 and μ_2 such that π is realized on the subspace $\mathcal{B}_f(\mu_1,\mu_2)$ of $\mathcal{B}(\mu_1,\mu_2)$. There will be positive integers p and q such that $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$. The representations $\sigma = \rho_{|q-p|}$ of $SU(2,\mathbb{C})$ which is of degree $|q-p|+1$ is contained in the restriction of π to $SU(2,\mathbb{C})$. In particular $\mathcal{B}_f(\mu_1,\mu_2)$ contains all functions f in $\mathcal{B}(\mu_1,\mu_2)$ whose restrictions to $SU(2,\mathbb{C})$ satisfy

$$f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k\right) = \mu_1(a_1)\,\mu_2(a_2)\,f(k)$$

and transform on the right according to σ.

We are going to use an argument like that used to prove Lemma 13.2.5. Suppose we can find a function Φ in $\mathcal{S}_1(A)$ such that

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi)$$

differs from $L(s,\pi)$ by an exponential factor and such that Φ transforms on the right under $SU(2,\mathbb{C})$ according to the representation σ. Then $K_\Phi(h,g,s)$ will satisfy the same conditions as in Lemma 13.2.5. Moreover the functions F' in the space we called U_2 can be supposed to transform on the right under $SU(2,\mathbb{C})$ according to σ so that the functions $\bar{F}'_i$ will correspond to functions f_i in $\mathcal{B}_f(\mu_1,\mu_2)$. Then

$$\int_{K\times K} K_\Phi(h,g,s)\,\tilde{f}_i(h)\,f_i(g)\,dh\,dg = Z(\alpha_F^{s+\frac{1}{2}}\otimes\tau, \Phi, f_i, \tilde{f}_i)$$

is equal to

$$Z(\alpha_F^{s+\frac{1}{2}}\otimes\pi, \Phi, v_i, \tilde{v}_i)$$

if $v_i = f_i$ and $\tilde{v}_i$ is the restriction of $\tilde{f}_i$, regarded as a linear functional, to $\mathcal{B}_f(\mu_1,\mu_2)$.

There are four possible ways of writing μ_1 and μ_2.

(i) $\mu_1(z) = z^{m_1}(z\bar{z})^{s_1}$, $\mu_2(z) = z^{m_2}(z\bar{z})^{s_2}$, $m_1 - m_2 = q - p$.
(ii) $\mu_1(z) = z^{m_1}(z\bar{z})^{s_1}$, $\mu_2(z) = \bar{z}^{m_2}(z\bar{z})^{s_2}$, $m_1 + m_2 = q - p$.
(iii) $\mu_1(z) = \bar{z}^{m_1}(z\bar{z})^{s_1}$, $\mu_2(z) = z^{m_2}(z\bar{z})^{s_2}$, $-m_1 - m_2 = q - p$.
(iv) $\mu_1(z) = \bar{z}^{m_1}(z\bar{z})^{s_1}$, $\mu_2(z) = \bar{z}^{m_2}(z\bar{z})^{s_2}$, $m_2 - m_1 = q - p$.

In all four cases m_1 and m_2 are to be non-negative integers. Φ is the product of $\exp(-\pi(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}))$ and a polynomial. We write down the polynomial in all four cases and leave the verifications to the reader.

(i.a) $m_1 \geq m_2 : \bar{a}^{m_1-m_2}(\bar{a}\bar{d}-\bar{b}\bar{c})^{m_2}$.
(i.b) $m_1 \leq m_2 : (\bar{a}\bar{d}-\bar{b}\bar{c})^{m_1}\bar{d}^{n_2-m_1}$.
(ii) $\bar{a}^{m_1}\,d^{m_2}$.
(iii) $a^{m_1}\,\bar{d}^{m_2}$.
(iv.a) $m_1 \geq m_2 : a^{m_1-m_2}(ad-bc)^{m_2}$.
(iv.b) $m_2 \geq m_1 : (ad-bc)^{m_1}\,d^{m_2-m_1}$.

For the real field the situation is similar. Suppose first that $\pi = \pi(\mu_1,\mu_2)$ is finite dimensional. If $\mu_1\mu_2(-1) = 1$ then π contains the trivial representation of $SO(2,\mathbb{R})$ and if $\mu_1\mu_2(-1) = 1$ it contains the representation

$$\kappa : \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \to e^{i\theta}.$$

We list the four possibilities for μ_1 and μ_2 and the polynomial P by which $\exp\big(-\pi(a^2+b^2+c^2+d^2)\big)$ is to be multiplied to obtain Φ.

(i) $\mu_1(-1) = \mu_2(-1) = 1$: $P(a,b,c,d) = 1$.
(ii) $\mu_1(-1) = \mu_2(-1) = 1$: $P(a,b,c,d) = ad - bc$.
(iii) $\mu_1(-1) = 1$, $\mu_2(-1) = -1$: $P(a,b,c,d) = c - id$.
(iv) $\mu_1(-1)$, $\mu_2(-1) = 1$: $P(a,b,c,d) = a - ib$.

Only the special representations remain to be considered. We may suppose that $\pi = \sigma(\mu_1,\mu_2)$ where μ_1 and μ_2 are of the form $\mu_1(x) = |x|^{r+\frac{q}{2}}$ and $\mu_2(x) = |x|^{r-\frac{q}{2}}(\operatorname{sgn} t)^m$ with $q = 2p+1-m$ and with p a non-negative integer. Moreover m is 0 or 1. $L(s,\pi)$ differs from

$$\Gamma\Big(\frac{s+r+\frac{q}{2}}{2}\Big)\,\Gamma\Big(\frac{s+r+\frac{q}{2}+1}{2}\Big)$$

by an exponential as does

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi)$$

if

$$\varphi(a_1,a_2) = e^{-\pi(a_1^2+a_2^2)}\,a_2^{q+1}.$$

Since the representation of κ_{q+1} occurs in the restriction of π to $SO(2,\mathbb{R})$ we may take

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp\big(-\pi(a^2+b^2+c^2+d^2)\big)\,(c+id)^{q+1}.$$

§**14. Automorphic forms and quaternion algebras.** Let F be a global field and let M' be a quaternion algebra over F. The multiplicative group G' of M' may be regarded as an algebraic group over F. In the ninth paragraph we have introduced the group $G'_{\mathbb{A}}$ and the Hecke algebra $\mathcal{H}'$. A continuous function φ on $G'_F \setminus G'_{\mathbb{A}}$ is said to be an automorphic form if for every elementary idempotent ξ in $\mathcal{H}'$ the space

$$\{\rho(\xi f)\,\varphi \,|\, f \in \mathcal{H}'\}$$

is finite dimensional.

If φ is an automorphic form it is $Z'_{\mathbb{A}}$ finite on the left if Z' is the centre of G'. Let $\mathcal{A}'$ be the space of automorphic forms on $G'_{\mathbb{A}}$ and if η is a quasi-character of $F^\times \setminus I$ let $\mathcal{A}'(\eta)$ be the space of φ in $\mathcal{A}'$ for which $\varphi'(ag) = \eta(a)\,\varphi'(g)$ for all a in $Z'_{\mathbb{A}}$ which, for convenience, we identify with I. The first assertion of the following lemma is easily proved by the methods of the eighth paragraph. The second is proved by the methods of the tenth. The proof is however a little simpler because $G'_F Z'_{\mathbb{A}} \setminus G'_{\mathbb{A}}$ is compact. Since, at least in the case of number fields, the proof ultimately rests on general facts from the theory of automorphic forms nothing is gained by going into details.

Lemma 14.1 (i) *If an irreducible admissible representation π of $\mathcal{H}'$ is a constituent of $\mathcal{A}'$ then for some η it is a constituent of $\mathcal{A}'(\eta)$.*

(ii) *The space $\mathcal{A}'(\eta)$ is the direct sum of subspaces irreducible and invariant under $\mathcal{H}'$. The representation of $\mathcal{H}'$ on each of these subspaces is admissible and no representation occurs more than a finite number of times in $\mathcal{A}'(\eta)$.*

Now we have to remind ourselves of some facts whose proofs are scattered throughout the previous paragraphs. Suppose $\pi = \otimes_v \pi_v$ is an irreducible admissible representation of $\mathcal{H}'$. For each v the representation π_v of $\mathcal{H}'_v$ is irreducible and admissible. Suppose ψ is a non-trivial additive character of $F \setminus A$ and ψ_v is its restriction to F_v. We have defined $L(s,\pi_v)$, $L(s,\tilde{\pi}_v)$, and $\varepsilon(s,\pi_v,\psi_v)$. If u_v is in the space of π_v and $\tilde{u}_v$ in the space of $\tilde{\pi}_v$ we have set

$$Z\big(\alpha_F^{s+\frac{1}{2}} \otimes \pi_v, \Phi, u_v, \tilde{u}_v\big)$$

equal to

$$\int_{G'_{F_v}} \Phi(g)\,\langle \pi_v(g)u_v, \tilde{u}_v\rangle\,|\nu(g)|^{s+\frac{1}{2}}\,d^\times g.$$

We know that

$$h_v\,\varepsilon(s,\pi_v,\psi_v)\left\{\frac{Z\big(\alpha_F^{s+\frac{1}{2}} \otimes \pi_v, \Phi, u_v, \tilde{u}_v\big)}{L(s,\pi_v)}\right\}$$

is entire and equals

$$\frac{Z\big(\alpha_F^{\frac{1}{2}-s} \otimes \tilde{\pi}_v, \Phi', u_v, \tilde{u}_v\big)}{L(1-s,\tilde{\pi}_v)}.$$

The factor h_v is 1 of G'_{F_v} is isomorphic to $GL(2,F_v)$ and is -1 otherwise. The case that G'_{F_v} is isomorphic to $GL(2,F_v)$ was treated in the previous paragraph. The other cases were treated in the fourth and fifth paragraphs.

Theorem 14.2 *Suppose π is a constituent of the space of automorphic forms on $G'_{\mathbb{A}}$. The infinite products*

$$\prod_v L(s,\pi_v)$$

and

$$\prod_v L(s,\tilde{\pi}_v)$$

are absolutely convergent for $\operatorname{Re} s$ *sufficiently large. The functions $L(s,\pi)$ and $L(s,\tilde{\pi})$ defined by them can be analytically continued to the whole complex plane as meromorphic functions. If F is a number field they will have only a finite number of poles and will be bounded at infinity in vertical strips of finite width. If*

$$\varepsilon(s,\pi) = \prod_v \varepsilon(s,\pi_v,\psi_v)$$

the functional equation

$$L(s,\pi) = \varepsilon(s,\pi)\, L(1-s,\tilde{\pi})$$

will be satisfied.

We may suppose that π acts on the subspace V of $\mathcal{A}'(\eta)$. Let φ be a non-zero function in V. For almost all v the algebra $M'_v = M' \otimes_F F_v$ is split and $G'_{F_v} = G'_v$ is isomorphic to $GL(2,F_v)$. Moreover for almost all such v, say for all v not in S, φ is an eigenfunction of the elements of $\mathcal{H}'_v = \mathcal{H}'_{F_v}$ which are invariant on both sides under translations by the elements of K'_v. Thus if f is such an element and $\varphi(g) \neq 0$ the corresponding eigenvalue $\lambda_v(f)$ is

$$\lambda_v(f) = \varphi(g)^{-1} \int_{G'_v} \varphi(gh)\, f(h)\, dh.$$

To prove the absolute convergence of the infinite products we have only to refer to Lemma 3.11 as in the proof of Theorem 11.1.

The representation $\tilde{\pi}$ contragredient to π can be defined. If $\pi = \otimes \pi_v$ acts on $V = \otimes_{u_v^0} V_v$ then $\tilde{\pi} = \otimes \tilde{\pi}_v$ acts on $\widetilde{V} = \otimes_{u_v^0} \widetilde{V}_v$ where $\tilde{u}_v^0$ is, for almost all v, fixed by K'_v and satisfies $\langle u_v^0, \tilde{u}_v^0 \rangle = 1$. The pairing between V and $\widetilde{V}$ is defined by

$$\langle \otimes u_v, \otimes \tilde{u}_v \rangle = \prod_v \langle u_v, \tilde{u}_v \rangle.$$

Almost all terms in the product are equal to 1. If u is in V and $\tilde{u}$ is in $\widetilde{V}$ the matrix element $\langle \pi(g)u, \tilde{u} \rangle$ can also be introduced. If f is in $\mathcal{H}'$

$$\langle \pi(f)u, \tilde{u} \rangle = \int_{G'_{\mathbb{A}}} f(g)\, \langle \pi(g)u, \tilde{u} \rangle\, d^{\times} g.$$

If $F(g)$ is a linear combination of such matrix elements and Φ belongs to the Schwartz space on $A'_{\mathbb{A}}$ we set*

$$Z\big(\alpha_F^{s+\frac{1}{2}}, \Phi, F\big) = \int_{G'_{\mathbb{A}}} \Phi(g)\, F(g)\, |\nu(g)|^{s+\frac{1}{2}}\, d^{\times} g.$$

* Unfortunately the symbol F plays two quite different roles on this page!

The function $\widetilde{F}(g) = F(g^{-1})$ is a linear combination of matrix coefficients for the representation $\tilde{\pi}$. We set

$$Z\big(\alpha_F^{s+\frac{1}{2}}, \Phi, \widetilde{F}\big) = \int_{G'_{\mathbb{A}}} \Phi(g)\, \widetilde{F}(g)\, |\nu(g)|^{s+\frac{1}{2}}\, d^\times g.$$

Before stating the next lemma we observe that if χ is a quasi-character of $F^\times \backslash I$ the one dimensional representation $g \to \chi\big(\nu(g)\big)$ is certainly a constituent of $\mathcal{A}'$.

Lemma 14.2.1 *If π is a constituent of $\mathcal{A}'$ the integrals defining $Z(\alpha_F^{s-\frac{1}{2}}, \Phi, F)$ and $Z(\alpha_F^{s-\frac{1}{2}}, \Phi, \widetilde{F})$ are absolutely convergent for $\operatorname{Re} s$ large enough. The two functions can be analytically continued to the whole complex plane as meromorphic functions with only a finite number of poles. If π is not of the form $g \to \chi\big(\nu(g)\big)$ they are entire. If F is a number field they are bounded at infinity in vertical strips of finite width. In all cases they satisfy the functional equation*

$$Z\big(\alpha_F^{s+\frac{1}{2}}, \Phi, F\big) = Z\big(\alpha_F^{\frac{3}{2}-s}, \Phi', \widetilde{F}\big)$$

if Φ' is the Fourier transform of Φ.

There is no harm in assuming that F is of the form

$$F(g) = \prod_v \langle \pi(g_v) u_v, \tilde{u}_v\rangle = \prod_v F_v(g_v)$$

and that Φ is of the form

$$\Phi(x) = \prod_v \Phi_v(x_v)$$

where, for almost all v, Φ_v is the characteristic function of $M(2, O_v)$. Recall that for almost all v we have fixed an isomorphism θ_v of M'_v with $M(2, F_v)$.

We know that each of the integrals

$$\int_{G'_v} \Phi_v(g_v)\, F_v(g_v)\, |\nu(g_v)|^{s+\frac{1}{2}}\, d^\times g_v$$

converges absolutely for $\operatorname{Re} s$ sufficiently large. Let S be a finite set of primes which contains all archimedean primes such that outside of S the vector u_v is u_v^0, the vector $\tilde{u}_v$ is $\tilde{u}_v^0$, Φ_v is the characteristic function of $M(2, O_v)$, and $\pi_v = \pi_v(\mu_v, \nu_v)$ where μ_v and ν_v are unramified. Let $\pi'_v = \pi_v(|\mu_v|, |\nu_v|)$. If v is not in S the integral

$$\int_{K'_v} \Phi(g_v)\, F_v(g_v)\, |\nu(g_v)|^{s+\frac{1}{2}}\, d^\times g_v = 1$$

and if $\sigma = \operatorname{Re} s$

$$\int_{G'_v} |\Phi_v(g_v)|\, |F_v(g_v)|\, |\nu(g_v)|^{\sigma+\frac{1}{2}}\, d^\times g_v$$

is, as we see if we regard π_v as acting on $\mathcal{B}(\mu_b, \nu_v)$, at most

$$\int_{G'_v} \Phi_v(g_v)\, \langle \pi'_v(g_v)\, f_v, \tilde{f}_v\rangle\, |\nu(g_v)|^{\sigma+\frac{1}{2}}\, d^\times g_v$$

if f_v and $\tilde{f}_v$ are the unique K'_v-invariant elements in $\mathcal{B}(|\mu_v|, |\nu_v|)$ and $\mathcal{B}(|\mu_v|^{-1}, |\nu_v|^{-1})$ which take the value 1 at the identity. We suppose that the total measure of K'_v is 1 so that $\langle f_v, \tilde{f}_v\rangle = 1$. According to Lemma 13.2.6 the integral is equal to $L(\sigma, \pi'_v)$. Since

$$\prod_{v\in S} L(\sigma, \pi'_v)$$

is absolutely convergent for σ sufficiently large the integral defining $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, F)$ is also and is equal to

$$\prod_v Z\big(\alpha_{F_v}^{s+\frac{1}{2}} \otimes \pi_v, \Phi_v, u_v, \tilde{u}_v\big)$$

and to

$$L(s,\pi)\prod_v \Xi(s, \Phi_v, u_v, \tilde{u}_v).$$

Notice that $\Xi(s, \Phi_v, u_v, \tilde{u}_v)$ is identically 1 for almost all v. $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, \widetilde{F})$ may be treated in a similar fashion. If we take π to be the trivial representation we see that

$$\int_{G'_A} \Phi(g)\,|\nu(g)|^{s+\frac{1}{2}}\, d^\times g$$

is absolutely convergent for $\operatorname{Re} s$ sufficiently large.

It will be enough to prove the remaining assertions of the lemma when η is a character. We may also assume that if η is of the form $\eta(a) = |a|^r$ then $r = 0$. We have identified V with a subspace of $\mathcal{A}'(\eta)$. We may take $\widetilde{V}$ to be $\{\tilde{\varphi} \mid \varphi \in V\}$. To see this observe that this space is invariant under $\mathcal{H}'$ and that

$$\langle \varphi_1, \varphi_2\rangle = \int_{G'_F Z'_{\mathbb{A}}\backslash G'_{\mathbb{A}}} \varphi_1(g)\,\varphi_2(g)\,dg$$

is a non-degenerate bilinear form. Here φ_1 belongs to V and $\tilde{\varphi}_2$ belongs to V. The remaining assertions need only be verified for functions of the form

$$F(g) = \int_{G'_F Z'_{\mathbb{A}}\backslash G'_{\mathbb{A}}} \varphi(hg)\,\tilde{\varphi}(h)\,dh$$

with φ in V and $\tilde{\varphi}$ in $\widetilde{V}$.

For such an F the function $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, F)$ is equal to

$$\int \Phi(g)\Big\{\int \varphi(hg)\,\tilde{\varphi}(h)\,dh\Big\}\,|\nu(g)|^{s+\frac{1}{2}}\, d^\times g.$$

Since φ and $\tilde{\varphi}$ are bounded this double integral converges absolutely for $\operatorname{Re} s$ sufficiently large. We first change variables by substituting $h^{-1}g$ for g. The integration with respect to g can then be carried out in three steps. We first sum over $G_{F'}$, then we integrate over $Z'_F \setminus Z'_{\mathbb{A}}$ which we identify with $F^\times \setminus I$, and finally we integrate over $G'_F Z'_{\mathbb{A}} \setminus G'_{\mathbb{A}}$. Thus if $K_\Phi(h_1, h_2, s)$ is

$$|\nu(h_1^{-1}|^{s+\frac{1}{2}}\,|\nu(h_2)|^{s+\frac{1}{2}} \int_{F^\times\backslash I} \sum_{G'_F} \Phi(h_1^{-1}\xi a h_2)\,\eta(a)\,|a|^{2s+1}\, d^\times a$$

the function $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, F)$ is equal to

$$\iint \varphi(h_2)\, \tilde{\varphi}(h_1)\, K_\Phi(h_1, h_2, s)\, dh_1\, dh_2.$$

The integrations with respect to h_1 and h_2 are taken over $G'_F Z'_{\mathbb{A}} \setminus G'_{\mathbb{A}}$. A similar result is of course valid for $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, \widetilde{F})$. If $\widetilde{K}_\Phi(h_1, h_2, s)$ is

$$|\nu(h_1^{-1})|^{s+\frac{1}{2}}\, |\nu(h_2)|^{s+\frac{1}{2}} \int_{F^\times \backslash I} \sum_{G'_F} \Phi(h_1^{-1} \xi a h_2)\, \eta^{-1}(a)\, |a|^{2s+1}\, d^\times a$$

then $Z(\alpha_F^{s+\frac{1}{2}}, \Phi, \widetilde{F})$ is equal to

$$\iint \varphi(h_2)\, \tilde{\varphi}(h_2)\, \widetilde{K}_\Phi(h_1, h_2, s)\, dh_1\, dh_2.$$

We first study

$$\theta(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi a)\, \eta(a)\, |a|_F^{2s+1}\, d^\times a$$

and

$$\widetilde{\theta}(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi a)\, \eta^{-1}(a)\, |a|_F^{2s+1}\, d^\times a.$$

The sums are taken over G'_F the set of non-zero elements of M'. Choose two non-negative continuous functions F_0 and F_1 on the positive real numbers so that $F_0(t) + F_1(t) = 1$, $F_1(t) = F_0(t^{-1})$, and so that F_0 vanishes near zero while F_1 vanishes near infinity. If

$$\theta_i(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi_a)\, \eta(a)\, |a|^{2s+1}\, F_i(|a|)\, d^\times a$$

we have

$$\theta(s, \Phi) = \theta_0(s, \Phi) + \theta_1(s, \Phi).$$

In the same way we may write

$$\widetilde{\theta}(s, \Phi) = \widetilde{\theta}_0(s, \Phi) + \widetilde{\theta}_1(s, \Phi)$$

$\theta_0(s, \Phi)$ and $\widetilde{\theta}_0(s, \Phi)$ are entire functions of s which are bounded in vertical strips.

Applying the Poisson formula we obtain

$$\Phi(0) + \sum_{\xi \neq 0} \Phi(\xi a) = |a|_F^{-4} \left\{ \Phi'(0) + \sum_{\xi \neq 0} \Phi'(\xi a^{-1}) \right\}.$$

Thus, for $\operatorname{Re} s$ sufficiently large, $\theta_1(s, \Phi)$ is equal to the sum of

$$\int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi'(\xi a^{-1})\, \eta(a)\, |a|^{2s-3}\, F_1(|a|)\, d^\times a,$$

which, after the substitution of a^{-1} for a, is seen to equal $\widetilde{\theta}_0(1-s,\Phi')$, and

$$\int_{F^\times\backslash I}\left\{\Phi'(0)\,|a|^{-4}-\Phi(0)\right\}\eta(a)\,|a|^{2s+1}\,F_1(|a|)\,d^\times a.$$

Thus if

$$\lambda(s)=\int_{F^\times\backslash I}|a|^s\,\eta(a)\,F_1(|a|)\,d^\times a$$

the function $\theta(s,\Phi)$ is equal to

$$\theta_0(s,\Phi)+\widetilde{\theta}_0(1-s,\Phi')+\Phi'(0)\,\lambda(2s-3)-\Phi(0)\,\lambda(2s+1).$$

A similar result is valid for $\widetilde{\theta}(s,\Phi)$. The function

$$\theta_0(s,\Phi)+\widetilde{\theta}_0(1-s,\Phi')$$

is entire and bounded in vertical strips and does not change when s and Φ are replaced by $1-s$ and Φ'.

If η is not of the form $\eta(a)=|a|^r$ the function $\lambda(s)$ vanishes identically. If η is trivial and I_0 is the group of idèles of norm 1

$$\lambda(s)=\int_{F^\times\backslash I}|a|^{2s+1}\,F_j(|a|)\,d^\times a.$$

It is shown in [10] that this function is meromorphic in the whole plane and satisfies $\lambda(s)+\lambda(-s)=0$. If F is a number field, its only pole is at $s=0$ and is simple. Moreover it is bounded at infinity in vertical strips of finite width. If F is a function field its poles are simple and lie at the zeros of $1-q^{-s}$. Here q is the number of elements in the field of constants.

Thus $\theta(s,\Phi)$ is meromorphic in the whole plane and is equal to $\widetilde{\theta}(1-s,\Phi')$. If $h\Phi g$ is the function $x\to\Phi(gxh)$ then

$$K_\Phi(h_1,h_2,s)=|\nu(h_1^{-1})|^{s+\frac12}\,|\nu(h_2)|^{s+\frac12}\,\theta(s,h_2\Phi h_1^{-1})$$

while

$$\widetilde{K}_\Phi(h_1,h_2,s)=|\nu(h_1^{-1})|^{s+\frac12}\,|\nu(h_2)|^{s+\frac12}\,\widetilde{\theta}(s,h_2\Phi h_1^{-1}).$$

Since the Fourier transform of $h_2\Phi h_1^{-1}$ is

$$|\nu(h_2)|^{-2}\,|\nu(h_1)|^2\,h_1\Phi' h_2^{-1}$$

we have

$$K_\Phi(h_1,h_2,s)=\widetilde{K}_{\Phi'}(h_2,h_1,s).$$

The functional equation of the lemma follows. So do the other assertions except the fact that the functions $Z(\alpha_F^{s+\frac12},\Phi,F)$ and $Z(\alpha_F^{s+\frac12},\Phi,\widetilde{F})$ are entire when η is trivial and π is not of the form $g\to\chi\big(\nu(g)\big)$. In this case the functions φ and $\tilde{\varphi}$ are orthogonal to the constant functions and the kernels $K_\Phi(h_1,h_2,s)$ and $\widetilde{K}_\Phi(h_1,h_2,s)$ may be replaced by

$$K'_{\Phi'}(h_1,h_2,s)=\widetilde{K}_{\Phi'}(h_1,h_2,s)+\Phi(0)\,\lambda(2s+1)-\Phi'(0)\,\lambda(2s-3)$$

and

$$\widetilde{K}'_{\Phi'}(h_1, h_2, s) = \widetilde{K}_{\Phi'}(h_1, h_2, s) + \Phi'(0)\,\lambda(2s+1) - \Phi(0)\,\lambda(2s-3).$$

The functional equation of the kernels is not destroyed but the poles disappear.

The theorem follows easily from the lemma. In fact suppose that the finite set of places S is so chosen that for v not in S

$$\Xi(s, \Phi_v^0, u_v^0, \tilde{u}_v^0) = 1$$

if Φ_v^0 is the characteristic function of $M(2, O_v)$. If v is in S choose Φ'_v, u_v^i, $\tilde{u}_v^i$, $1 \le i \le n_v$, so that

$$\sum_{i=1}^{n_v} \Xi(s, \Phi_v^i, u_v^i, \tilde{u}_v^i) = e^{b_v s}$$

where b_v is real. If α is a function from S to the integers and, for each v in S, $1 \le \alpha(v) \le n_v$, set

$$\Phi_\alpha(g) = \Big\{\prod_{v \in S} \Phi_v^{\alpha(v)}(g_v)\Big\}\Big\{\prod_{v \in S} \Phi_v^0(g_v)\Big\}$$

and set

$$F_\alpha(g) = \Big\{\prod_{v \in S} \langle \pi_v(g_v) u_v^{\alpha(v)}, \tilde{u}_v^{\alpha(v)}\rangle\Big\}\Big\{\prod_{v \notin S} \langle \pi_v(g_v) u_v^0, \tilde{u}_v^0\rangle\Big\}.$$

Then

$$\sum_\alpha Z\big(\alpha_F^{s+\frac{1}{2}}, \Phi_\alpha, F_\alpha\big) = c^{bs}\, L(s, \pi)$$

where b is real. The required analytic properties of $L(s, \pi)$ follow immediately.

To prove the functional equation choose for each v the function Φ_v and the vectors u_v and $\tilde{u}_v$ so that

$$\Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

is not identically 0. We may suppose that, for almost all v, $\Phi_v = \Phi_v^0$, $u_v = u_v^0$, and $\tilde{u}_v = \tilde{u}_v^0$. Let

$$\Phi(g) = \prod_v \Phi_v(g_v)$$

and let

$$F(g) = \prod_v \langle \pi_v(g_v) uv, \tilde{u}_v\rangle.$$

Then

$$Z\big(\alpha_F^{s+\frac{1}{2}}, \Phi, F\big) = L(s, \pi) \prod_v \Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

and

$$Z\big(\alpha_F^{\frac{3}{2}-s}, \Phi', \widetilde{F}\big) = L(1-s, \tilde{\pi}) \prod_v \widetilde{\Xi}(1-s, \Phi'_v, u_v, \tilde{u}_v).$$

Since

$$\widetilde{\Xi}(1-s, \Phi'_v, u_v, \tilde{u}_v) = h_v \varepsilon(s, \pi_v, \psi_v)\, \Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

the functional equation of the lemma implies that

$$L(s, \pi) = \Big\{\prod_v h_v\Big\}\, \varepsilon(s, \pi)\, L(1-s, \tilde{\pi}).$$

Since, by a well-known theorem, the algebra M' is split at an even number of places the product $\prod_v h_v$ equals 1.

Corollary 14.3 *If π is a constituent of $\mathcal{A}'$ which is not of the form $g \to \chi(\nu(g))$ then for any quasi-character ω of $F^\times$ the functions $L(s, \omega \otimes \pi)$ and $L(s, \omega^{-1} \otimes \tilde{\pi})$ are entire and bounded in vertical strips of finite width. Moreover they satisfy the functional equation*

$$L(s, \omega \otimes \pi) = \varepsilon(s, \omega \otimes \pi)\, L(1 - s, \omega^{-1} \otimes \tilde{\pi}).$$

We have only to observe that if π is a constituent of $\mathcal{A}'$ then $\omega \otimes \pi$ is also.

Now we change the notation slightly and let $\pi' = \otimes \pi'_v$ be an irreducible admissible representation of $\mathcal{H}'$. We want to associate to it a representation $\pi = \otimes \pi_v$ of $\mathcal{H}$ the Hecke algebra of $GL(2, \mathbb{A})$. If M'_v is split then π_v is just the representation corresponding to π'_v by means of the isomorphism θ_v of G_{F_v} and G'_{F_v}. If M'_v is not split π_v is the representation $\pi_v(\pi'_v)$ introduced in the fourth and fifth paragraphs. In both cases π_v is defined unambiguously by the following relations

$$\begin{aligned} L(s, \omega_v \otimes \pi_v) &= L(s, \omega_v \otimes \pi'_v) \\ L(s, \omega_v \otimes \tilde{\pi}_v) &= L(s, \omega_v \otimes \tilde{\pi}'_v) \\ \varepsilon(s, \omega_v \otimes \pi_v, \psi_v) &= \varepsilon(s, \omega_v \otimes \pi'_v, \psi_v) \end{aligned}$$

which holds for all quasi-characters ω_v of $F_v^\times$.

Applying the previous corollary and Theorem 11.3 we obtain the following theorem.

Theorem 14.4 *If π' is a constituent of $\mathcal{A}'$ and π'_v is infinite dimensional at any place where M' splits then π is a constituent of $\mathcal{A}_0$.*

Some comments on the assumptions are necessary. If π' is a constituent of $\mathcal{A}'$ we can always find a quasi-character of ω of $F^\times \setminus I$ such that $\omega \otimes \pi'$ is unitary. If $\pi' = \otimes \pi'_v$ the same is true of the representations π'_v. In particular if M' splits at v the representation π'_v will not be finite dimensional unless it is one dimensional. Various density theorems probably prevent this from happening unless π' is of the form $g \to \chi(\nu(g))$. If π' is of this form then all but a finite number of the representations π_v are one dimensional. But if M' does not split at v the representation π_v is infinite dimensional. Thus π cannot act on a subspace of $\mathcal{A}$. However it can still be a constituent of $\mathcal{A}$. This is in fact extremely likely. Since the proof we have in mind involves the theory of Eisenstein series we prefer to leave the question unsettled for now.

§**15. Some orthogonality relations.** It is of some importance to characterize the range of the map $\pi' \to \pi$ from the constituents of $\mathcal{A}'$ to those of $\mathcal{A}$ discussed in the last chapter. In this paragraph we take up the corresponding local question. Suppose F is a local field and M' is the quaternion algebra over F. G'_F is the group of invertible elements of M'. We know how to associate to every irreducible admissible representation π' of $\mathcal{H}'_F$ an irreducible admissible representation $\pi = \pi(\pi')$ of $\mathcal{H}_F$ the Hecke algebra of $GL(2, F)$.

Theorem 15.1 *Suppose F is non-archimedean. Then the map $\pi' \to \pi$ is injective and its image is the collection of special representations together with the absolutely cuspidal representations.*

The proof requires some preparation. We need not distinguish between representations of G'_F and $\mathcal{H}'_F$ or between representations of G_F and $\mathcal{H}_F$. An irreducible admissible representation π of G_F is said to be square-integrable if for any two vectors u_1 and u_2 in the space of π and any two vectors $\tilde{u}_1$ and $\tilde{u}_2$ in the space of $\tilde{\pi}$ the integral

$$\int_{Z_F \backslash G_F} \langle \pi(g)u_1, \tilde{u}_1 \rangle \, \langle u_2, \tilde{\pi}(g)\tilde{u}_2 \rangle \, dg$$

is absolutely convergent. Since $\tilde{\pi}$ is equivalent to $\eta^{-1} \otimes \pi$ if

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

this is equivalent to demanding that

$$\int_{Z_F \backslash G_F} |\langle \pi(g)u_1, \tilde{u}_1 \rangle|^2 \, |\eta^{-1}(\det g)| \, dg$$

be finite for every u_1 and u_2.

If π is square-integrable and ω is a quasi-character of $F^\times$ then $\omega \otimes \pi$ is square integrable. We can always choose ω so that $\omega^2\eta$ is a character. If η is a character choose u_0 different from 0 in the space V of π. Then

$$(u_1, u_2) = \int_{Z_F \backslash G_F} \langle \pi(g)u_1, u_0 \rangle \, \overline{\langle \pi(g)u_2, u_0 \rangle} \, dg$$

is a positive-definite form on the space V of π so that π is unitary and square-integrable in the usual sense.

The Schur orthogonality relations when written in the form

$$\int_{Z_F \backslash G_F} \langle \pi(g)u_1, \tilde{u}_1 \rangle \, \langle u_2, \tilde{\pi}(g)\tilde{u}_2 \rangle \, dg = \frac{1}{d(\pi)} \, \langle u_2, \tilde{u}_1 \rangle \, \langle u_1, \tilde{u}_2 \rangle$$

are valid not only for representations which are square-integrable in the usual sense but also for represntations which are square-integrable in our sense. The formal degree $d(\pi)$ depends on the choice of Haar measure. Notice that $d(\omega \otimes \pi) = d(\pi)$.

The absolutely cuspidal representations are certainly square-integrable because their matrix elements are compactly support modulo Z_F.

Lemma 15.2 *The special representations are square-integrable.*

Suppose $\sigma = \sigma(\alpha_F^{\frac{1}{2}}, \alpha_F^{-\frac{1}{2}})$. Since

$$\chi \otimes \sigma = \sigma\big(\chi\alpha_F^{\frac{1}{2}}, \chi\alpha_F^{-\frac{1}{2}}\big)$$

it is enough to show that σ is square-integrable. If φ belongs to $\mathcal{B}_s(\alpha_F^{\frac{1}{2}}, \alpha_F^{-\frac{1}{2}})$ and $\tilde{\varphi}$ belongs to $\mathcal{B}(\alpha_F^{-\frac{1}{2}}, \alpha_F^{\frac{1}{2}})$ then

$$f(g) = \langle \varphi, \rho(g^{-1})\tilde{\varphi} \rangle$$

is the most general matrix coefficient of σ. $\mathcal{B}(\alpha_F^{-\frac{1}{2}}, \alpha_F^{\frac{1}{2}})$ is the space of locally constant functions on $N_F A_F \setminus G_F$ and $\mathcal{B}_s(\alpha_F^{\frac{1}{2}}, \alpha_F^{-\frac{1}{2}})$ is the space of locally constant functions φ on G_F that satisfy

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| \varphi(g)$$

and

$$\int \varphi\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx = 0.$$

Since

$$G_F = \bigcup_{n \geq 0} Z_F\, K \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} K$$

we can choose the Haar measure on $Z_F \setminus G_F$ so that

$$\int_{Z_F \setminus G_F} |f(g)|^2 \, dg$$

is equal to

$$\sum_{n \geq 0} c(n) \int \left| f\left(k_1 \begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix} k_2\right)\right|^2 dk_1\, dk_2$$

where $c(0) = 1$ and

$$c(n) = q^n\Big(1 + \frac{1}{q}\Big)$$

if $n > 0$. Here $q = |\varpi|^{-1}$. Since f is K-finite on both sides and its translates are also matrix coefficients we need only show that

$$\sum_{n=0}^{\infty} \left| f\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)\right|^2 q^n$$

is finite. It will be more than enough to show that

$$\Phi(a) = f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|a|^{-1})$$

as $a \to \infty$.

We recall that

$$\Phi(a)=\int_F \varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\tilde{\varphi}\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)dx.$$

The function

$$\varphi_1(x)=\varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

is integrable and the function

$$\varphi_2(x)=\tilde{\varphi}\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

is bounded and locally constant. Moreover

$$\Phi(a)=\int_F \varphi_1(x)\,\varphi_2(a^{-1}x)\,dx.$$

Suppose $\varphi_2(x)=\varphi_2(0)$ for $|x|\le M$. If $|a|\ge 1$

$$\Phi(a)=\varphi_2(0)\int_{\{x\,\big|\,|x|\le|a|M\}}\varphi_1(x)\,dx+\int_{\{x\,\big|\,|x|>|a|M\}}\varphi_1(x)\,\varphi_2(a^{-1}x)\,dx.$$

Since

$$\int_F \varphi_1(x)\,dx=0$$

$\Phi(a)$ is equal to

$$\int_{\{x\,\big|\,|x|>|a|M\}}\big(\varphi_2(a^{-1}x)-\varphi_2(0)\big)\,\varphi_1(x)\,dx.$$

The function φ_2 is bounded so we need only check that

$$\int_{\{x\,\big|\,|x|>|a>\}}|\varphi_1(x)|\,dx=O(|a|^{-1})$$

as $|a|\to\infty$. The absolute value of the function φ is certainly bounded by some multiple of the function φ' in $\mathcal{B}(\alpha_F^{\frac{1}{2}},\alpha_F^{-\frac{1}{2}})$ defined by

$$\varphi'\left(\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix}k\right)=\left|\frac{a_1}{a_2}\right|$$

if k is in $GL(2,O_F)$. Since

$$w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & -x\end{pmatrix}=\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}k$$

with y in F and k in $GL(2,O_F)$ if $|x|>1$

$$\int_{\{x\,\big|\,|x|>|\varpi|^{-n}\}}|\varphi_1(x)|\,dx=O\Big(\sum_{k=n+1}^{\infty}|\varpi|^k\Big)=O(|\varpi|^{-n}).$$

Since we need to compare orthogonality relations on the two groups $G_F = GL(2,F)$ and G'_F we have to normalize their Haar measure simultaneously. There are two ways of doing this. We first describe the simplest. Choose a non-trivial additive character ψ of F. Then $\psi_M(x) = \psi(\operatorname{tr} x)$ and $\psi_{M'}(x) = \psi\big(\tau(x)\big)$ are non-trivial additive characters of $M = M(2,F)$ and M'. Let dx and dx' be the Haar measures on M and M' self-dual with respect to ψ_M and $\psi_{M'}$. Then

$$d^\times x = |x|_M^{-1}\, dx = |\det x|_F^{-2}\, dx$$

and

$$d^\times x' = |x'|_{M'}^{-1}\, dx' = |\nu(x')|_F^{-2}\, dx'$$

are Haar measure on G_F and G'_F.

The second method takes longer to describe but is more generally applicable and for this reason well worth mentioning. Suppose G and G' are two linear groups defined over F and suppose there is an isomorphism φ of G' with G defined over the finite Galois extension K. Suppose the differential form ω on G is defined over F. In general the form $\omega' = \varphi_*\omega$ on G' is not defined over F. Suppose however that ω is left and right invariant and under an arbitrary isomorphism it is either fixed or changes sign. Suppose moreover that for every σ in $\mathfrak{G}(K/F)$ the automorphism $\sigma(\varphi)\varphi^{-1}$ of G is inner. Then

$$\sigma(\omega') = \sigma(\varphi)_*\,\sigma\omega = \sigma(\varphi)_*\,\omega = \varphi_*\big(\sigma(\varphi)\,\varphi^{-1}\big)_*\omega = \varphi_*\omega = \omega'$$

and ω' is also defined over F. If ξ is another such isomorphism of G' with G then

$$\xi_*(\omega) = \varphi_*(\xi\varphi^{-1})_*\omega = \pm\varphi_*\omega = \pm\omega'$$

and the measures associated to $\varphi_*\omega$ and $\xi_*\omega$ are the same. Thus a Tamagawa measure on G_F determines one on G'_F.

We apply this method to the simple case under consideration. If

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a typical element of M then

$$\mu = da \wedge db \wedge dc \wedge dd$$

is a differential form invariant under translations and the associated measure is self-dual with respect to ψ_M. If $\omega = (\det x)^{-2}\mu$ then ω is an invariant form on G and the associated measure is $d^\times x$.

If K is any separable quadratic extension of F we may imbed K in both M and M'. Let σ be the non-trivial element of $\mathfrak{G}(K/F)$. There is a u in M and a u' in M' such that $M = K + Ku$ and $M' = K + Ku'$ while $uxu^{-1} = x^\sigma$ and $u'xu'^{-1} = x^\sigma$ for all x in K. u^2 is a square in $F^\times$ and $u'^2 = \gamma$ is an element of $F^\times$ which is not the norm of any element of K. We may suppose that $u^2 = 1$. If we let K act to the right the algebra $L = K \otimes_F K$ is an algebra over K. σ acts on L through its action on the first factor. There is an isomorphism $L \to K \oplus K$ which transforms σ into the involution $(x,y) \to (y,x)$. In particular every element of $K \otimes 1$ is of the form $\delta\delta^\sigma$ with δ in L. Choose δ so that $\gamma = \delta\delta^\sigma$. If

$$M'_K = M' \otimes_F K = L \otimes Lu'$$

and

$$M_K = M \otimes_F K = L \otimes Lu,$$

let φ be the linear map from M'_K to M_K which sends $x + yu'$ to $x + y\delta u$. φ is easily seen to be an isomorphism of M'_K and M_K as algebras over K. $\sigma(\varphi)\varphi^{-1}$ takes $x + yu$ to

$$x + y\delta^\sigma\delta^{-1}u = \delta^{-1}(x + yu)\delta$$

and is therefore inner. Thus φ determines an isomorphism of G' the multiplicative group of M' with G the multiplicative group of M. φ is defined over K and $\sigma(\varphi)\varphi^{-1}$ is inner. Let $|\omega'|$ be the Haar measure on G'_F associated to the Haar measure $|\omega| = d^\times x$ of G_F. We want to show that $|\omega'|$ is just $d^\times x'$.

Let θ be an invariant form on K. The obvious projections of $M = K \oplus Ku$ on K define differential forms θ_1 and θ_2 on M. Let $\theta_1 \wedge \theta_2 = c\mu$. In the same way the projections of $M' = K \oplus Ku'$ on K define differential forms θ'_1 and θ'_2 on M'. If we extend the scalars from F to K we can consider the map $x \to x\delta$ of L into itself. θ is a form on L and its inverse image is $N(\delta)\theta = \gamma\theta$. Thus

$$\varphi_*(\theta_1 \wedge \theta_2) = \gamma\theta'_1 \wedge \theta'_2.$$

Thus if $\mu' = \varphi_*(\mu)$

$$c\mu' = \gamma\theta'_1 \wedge \theta'_2.$$

Suppose $c_1|\theta|$ is self-dual with respect to the character $\psi_K(x) = \psi\big(\tau(x)\big)$ on K. Then

$$\int \Big\{ \int \Phi(a,b)\, \psi_K(ax + by^\sigma)\, |\theta(a)|\, |\theta(b)| \Big\}\, |\theta(x)|\, |\theta(y)| = c_1^{-4}\Phi(0,0)$$

and

$$|\gamma|_F^2 \int \Big\{ \int \Phi(a,b)\, \psi_K(ax + by^\sigma\gamma)\, |\theta(a)|\, |\theta(b)| \Big\}\, |\theta(x)|\, |\theta(y)| = c_1^{-4}\Phi(0,0).$$

If $x + yu$ belongs to M with x and y in K then, since $\tau(u) = 0$,

$$\tau(x + yu) = \tau(x) = \mathrm{Tr}_{K/F}(x).$$

In the same way

$$\tau(x + yu') = \mathrm{Tr}_{K/F}(x).$$

Thus

$$\psi_M\big((x + yu)(a + bu)\big) = \psi_K(xa + yb^\sigma)$$
$$\psi_{M'}\big((x + yu')(a + bu')\big) = \psi_K(xa + yb^\sigma\gamma).$$

Thus $c_1^2|\theta_1 \wedge \theta_2|$ is self-dual with respect to ψ_M and $c_1^2|\gamma|_F\, |\theta'_1 \wedge \theta'_2|$ is self-dual with respect to $\psi_{M'}$. Since $c_1^2 = |c|_F$ the measure $|\mu'|$ is self-dual with respect to $\psi_{M'}$. Finally $\omega' = \nu(x')^{-2}\, dx'$ so that $|\omega'|$ is just $d^\times x'$. Thus the two normalizations lead to the same result.

If b is in M or M' the eigenvalues of b are the roots α_1 and α_2 of the equation

$$X^2 - \tau(b)\, X + \nu(b) = 0.$$

If b is in G_F or G'_F it is said to be regular of α_1 and α_2 are distinct; otherwise it is singular. We set

$$\delta(b) = \left| \frac{(\alpha_1 - \alpha_2)^2}{\alpha_1\alpha_2} \right|_F .$$

The set of singular elements is of measure 0. If b is regular the subalgebra of M or M' generated by b is a separable quadratic extension E of F and the multiplicative group of E is a Cartan subgroup of G_F or G'_F. To obtain a set of representatives for the conjugacy classes of Cartan subgroups of G_F or G'_F we choose once and for all a set S' of representatives for the classes of separable quadratic extensions of F. We also choose for each E in S' an imbedding of E in M and in M'. The multiplicative group of E may be regarded as a Cartan subgroup B_F of either G_F or G'_F. S' will also stand for the collection of Cartan subgroups obtained in this way. It is a complete set of representatives for the conjugacy classes of Cartan subgroups of G'_F. If S is the result of adjoining to S' the group A_F of diagonal matrices then S is a complete set of representatives for the conjugacy classes of Cartan subgroups of G_F. If B_F is in S' we choose the Tamagawa measure μ_B on B_F as in the seventh paragraph. The analogue for G'_F of the formula (1.2.2) is

$$\int_{Z'_F\backslash G'_F} f(g)\,\omega'_0(g) = \sum_{S'}\frac{1}{2}\int_{Z_F\backslash B_F}\delta(b)\Big\{\int_{B_F\backslash G'_F} f(g^{-1}bg)\,\omega'_B(g)\Big\}\,\mu_B^0(b).$$

Let $\widehat{B}_F$ be the set of regular elements in B_F and let

$$C = \bigcup_{S'} Z_F \setminus \widehat{B}_F.$$

We may regard C as the discrete union of the spaces $Z_F \setminus \widehat{B}_F$. We introduce on C the measure $\mu(c)$ defined by

$$\int_C f(c)\,\mu(c) = \frac{1}{2}\sum_{S'}\frac{1}{\text{measure}(Z_F\setminus B_F)}\int_{Z_F\backslash\widehat{B}_F} f(b)\,\delta(b)\,\mu_B^0(b).$$

Lemma 15.3 *Let η be a quasi-character of $F^\times$ and let $\Omega'(\eta)$ be the set of equivalence classes of irreducible representations π of G'_F such that $\pi(a)=\eta(a)$ for a in Z'_F, which we identify with $F^\times$. If π_1 and π_2 belong to $\Omega'(\eta)$ and*

$$f(g) = \chi_{\pi_1}(g)\,\chi_{\tilde{\pi}_2}(g)$$

where $\chi_\pi(g) = \operatorname{Tr}\pi(g)$ then

$$\int_C f(c)\,\mu(c) = 0$$

if π_1 and π_2 are not equivalent and

$$\int_C f(c)\,\mu(c) = 1$$

if they are.

Since $Z'_F \setminus G'_F$ is compact we may apply the Schur orthogonality relations for characters to see that

$$\frac{1}{\text{measure}\, Z'_F\setminus G'_F}\int_{Z'_F\backslash G'_F} f(g)\,\omega'_0(g)$$

is 0 if π_1 and π_2 are not equivalent and is 1 if they are. According to the integration formula remarked above this expression is equal to

$$\frac{1}{\text{measure}\, Z'_F\setminus G'_F}\sum_{S'}\frac{1}{2}\int_{Z_F\backslash B_F} f(b)\,\delta(b)\,(\text{measure}\, B_F\setminus G'_F)\,\mu_b^0(b).$$

Since

$$\text{measure}\, Z'_F\setminus G'_F = (\text{measure}\, Z_F\setminus B_F)(\text{measure}\, B_F\setminus G'_F)$$

the lemma follows. Observe that Z_F and Z'_F tend to be confounded.

There is form of this lemma which is valid for G_F.

Lemma 15.4 *Let η be a quasi-character of $F^\times$. Let $\Omega_0(\eta)$ be the set of equivalence classes of irreducible admissible representations π of G_F which are either special or absolutely cuspidal and satisfy $\pi(a) = \eta(a)$ for all a in Z_F. Suppose π_1 and π_2 belong to $\Omega_0(\eta)$. Let $f = f_{\pi_1,\pi_2}$ be the function*

$$f(b) = \chi_{\pi_1}(b)\,\chi_{\tilde{\pi}_2}(b)$$

on C. Then f is integrable and

$$\int_C f(c)\,\mu(c)$$

is 1 if π_1 and π_2 are equivalent and 0 otherwise.

It is enough to prove the lemma when η is a character. Then $\chi_{\tilde{\pi}}$ is the complex conjugate of χ_π and $f_{\pi,\pi}$ is positive. If the functions $f_{\pi,\pi}$ are integrable then by the Schwarz inequality all the functions f_{π_1,π_2} are integrable.

Let $\Omega(\eta)$ be the set of irreducible admissible representations π of G_F such that $\pi(a) = \eta(a)$ for a in Z_F. If φ is a locally constant function on G_F such that

$$\varphi(ag) = \eta^{-1}(a)\,\varphi(g)$$

for a in Z_F and such that the projection of the support of φ on $Z_F \setminus G_F$ is compact then we define $\pi(\varphi)$, if π is in $\Omega(\eta)$, by

$$\pi(\varphi) = \int_{Z_F\setminus G_F} \varphi(g)\,\pi(g)\,\omega^0(g).$$

It is easily seen that $\pi(\varphi)$ is an operator of finite rank and that the trace of $\pi(\varphi)$ is given by the convergent integral

$$\int_{Z_F\setminus G_F} \varphi(g)\,\chi_\pi(g)\,\omega^0(g).$$

In fact this follows from the observation that there is a φ_1 in $\mathcal{H}_F$ such that

$$\varphi_1(g) = \int_{Z_F} \varphi_1(ag)\,\eta(a)\,\mu_Z(a)$$

and the results of the seventh paragraph.

Suppose π_1 is absolutely cuspidal and unitary and acts on the space V_1. Suppose also that $\pi_1(a) = \eta(a)$ for a in Z_F. Choose a unit vector u_1 and V_1 and set

$$\varphi(g) = d(\pi_1)\,\big(u_1, \pi_1(g)u_1\big).$$

Since π_1 is integrable it follows from the Schur orthogonality relations that $\pi_2(\varphi) = 0$ if π_2 in $\Omega(\eta)$ is not equivalent to π_1 but that $\pi_2(\varphi)$ is the orthogonal projection on $\mathbb{C}u_1$ if $\pi_2 = \pi_1$. In the first case $\operatorname{Tr}\pi_1(\varphi) = 0$ and in the second $\operatorname{Tr}\pi_2(\varphi) = 1$.

On the other hand

$$\operatorname{Tr}\pi_2(\varphi) = \int_{Z_F\setminus G_F} \chi_{\pi_2}(g)\,\varphi(g)\,\omega^0(g).$$

We apply formula (7.2.2) to the right side to obtain

$$\sum_S \frac{1}{2}\int_{Z_F\setminus B_F} \chi_{\pi_2}(b)\,\delta(b)\Big\{\int_{B_F\setminus G_F} \varphi(g^{-1}bg)\,\omega_B(g)\Big\}\,\mu_B^0(g).$$

If B_F belongs to S' the inner integral is equal to

$$\frac{1}{\operatorname{measure} Z_F \setminus B_F} d(\pi_1) \int_{Z_F \setminus G_F} \big(u_1, \pi_1(g^{-1}bg)u_1\big)\, \omega_B(g)$$

which by Proposition 7.5 is equal to

$$\frac{1}{\operatorname{measure} Z_F \setminus B_F} \chi_{\tilde{\pi}_1}(b).$$

If B_F is A_F the group of diagonal matrices the inner integral is, apart from a constant relating Haar measures, the product of $d(\pi_1)$ and the integral over $GL(2, O_F)$ of

$$\int_F \left(\pi_1\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} b \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \pi_1(k)u_1, \pi_1(k)u_1\right) dx.$$

If

$$b = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

this is

$$\left|1 - \frac{\alpha_2}{\alpha_1}\right|^{-1} \int_F \left(\pi_1(b)\, \pi_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \pi_1(k)u_1, \pi_1(k)u_1\right) dx$$

which we know is 0. Collecting these facts together we see that $f = f_{\pi_2,\pi_1}$ is integrable on C if π_1 is absolutely cuspidal and that its integral has the required value.

To complete the proof all we need do is show that if $\pi = \sigma(\chi\alpha_F^{\frac{1}{2}}, \chi\alpha_F^{-\frac{1}{2}})$ is a special representation then $f = f_{\pi,\pi}$ is integrable on C and

$$\int_C f(c)\, \mu(c) = 1.$$

If π' is the one-dimensional representation $g \to \chi\big(\nu(g)\big)$ of G'_F then $\pi = \pi(\pi')$. To prove the existence of χ_π we had to show in effect that if B_F was in S' and b was in $\widehat{B}_F$ then

$$\chi_\pi(b) = -\chi_{\pi'}(b).$$

Thus $f_{\pi,\pi} = f_{\pi',\pi'}$ and the assertion in this case follows from the previous lemma.

The relation just used does not seem to be accidental.

Proposition 15.5 *Suppose π' is an irreducible admissible representation of G'_F and $\pi = \pi(\pi')$ the corresponding representation of G_F. If B_F is in S' and b is in $\widehat{B}_F$*

$$\chi_{\pi'}(b) = -\chi_\pi(b).$$

We may suppose that π' is not one-dimensional and that π is absolutely cuspidal. We may also suppose that they are both unitary. We take π in Kirillov form with respect to some additive character ψ. If φ is in $\mathcal{S}(F^\times)$ the function

$$\varphi' = \pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \varphi$$

is also.

Since the measure μ and μ' are self-dual with respect to the characters ψ_M and $\psi_{M'}$ Lemma 13.1.1 and Proposition 4.5 show us that for any λ in $F^\times$

$$\int_{G_F} \varphi(\det g)\big(\pi(g^{-1})u, u\big)\,|\det g|\,\psi_M(\lambda g)\,\omega(g) \tag{15.5.1}$$

is equal to

$$\varphi'(\lambda^2)\,\eta^{-1}(\lambda)\,|\lambda|^{-2}$$

and that

$$\int_{G'_F} \varphi(\det g)\big(\pi'(g^{-1})u', u'\big)\,|\det g|\,\psi_{M'}(\lambda g)\,\omega'(g) \tag{15.5.2}$$

is equal to

$$-\varphi'(\lambda^2)\,\eta^{-1}(\lambda)\,|\lambda|^{-2}.$$

Here u is a unit vector in the space of π and u' a unit vector in the space π'. In any case (15.5.1) is just the negative of (15.5.2).

If we use formula (7.2.1) to express the integral (15.5.1) as a sum over S we obtain

$$\frac{1}{2}\sum_{S'} \frac{1}{\text{measure}\, Z_F \setminus B_F} \int_{B_F} \varphi(\det b)\,|\det b|\,\frac{\chi_\pi(b^{-1})}{d(\pi)}\,\delta(b)\,\psi_M(\lambda b)\,\mu_B(b).$$

The contribution from A_F vanishes as in the previous lemma. The other integrals have been simplified by means of Proposition 7.5. There is of course an obvious analogue for the group G'_F of the formula (7.2.1). If we apply it we see that (15.5.2) is equal to

$$\frac{1}{2}\sum_{S'} \frac{1}{\text{measure}\, Z_F \setminus B_F} \int_{B_F} \varphi(\det b)\,|\nu(b)|\,\frac{\chi_{\pi'}(b^{-1})}{d(\pi')}\,\delta(b)\,\psi_{M'}(\lambda b)\,\mu_B(b)$$

if $\nu(b)$ is the reduced norm. Of course on B_F the functions $\nu(b)$ and $\det b$ are the same. Choose B_F^0 in S' and b_0 in $\widehat{B}_F^0$. We shall show that

$$\frac{\chi_{\pi'}(b_0^{-1})}{d(\pi')} = \frac{-\chi_\pi(b_0^{-1})}{d(\pi)}.$$

The orthogonality relations of the previous two lemmas will show that $d(\pi) = d(\pi')$ and we will conclude that

$$\chi_{\pi'}(b_0^{-1}) = -\chi_\pi(b_0^{-1}).$$

The norm and the trace of b_0 are the same whether it is regarded as an element of M or of M'. In fact if $\widehat{B}_F^0$ is the multiplicative group of E in S' the norm and the trace are in both cases the norm and the trace of b_0 as an element of E. Since b_0 and its conjugate in E are conjugate in G_F and G'_F we can choose an open set U in $E^\times$ containing both b_0 and its conjugate so that

$$|\nu(b)|\,\chi_{\pi'}(b^{-1})\,\delta(b) = |\nu(b_0)|\,\chi_{\pi'}(b_0^{-1})\,\delta(b_0)$$

if b is in U. Lemma 7.4.2 shows that χ_π is locally constant in $\widehat{B}_F^0$. Thus we can also suppose that

$$|\det b|\,\chi_\pi(b^{-1})\,\delta(b) = |\det b_0|\,\chi_\pi(b_0^{-1})\,\delta(b_0)$$

if b is in U. Suppose α_0 and β_0 are the trace and norm of b_0. We can choose a positive integer m so that if $\alpha-\alpha_0$ and $\beta-\beta_0$ belong to $\mathfrak{p}_F^m$ the roots of

$$X^2-\alpha X+b$$

belong to E and in fact lie in U.

Let $\xi(\lambda)$ be the expression (15.5.1) regarded as a function of λ. Keeping in mind the fact that

$$\psi_M(\lambda b)=\psi_{M'}(\lambda b)=\psi(\lambda\operatorname{tr}b),$$

we compute

$$\frac{1}{\text{measure}\,\mathfrak{p}_F^{-m-n}}\int_{\mathfrak{p}_F^{-m-n}}\xi(\lambda)\,\psi(-\lambda\alpha_0)\,d\lambda \tag{15.5.3}$$

where $\mathfrak{p}_F^{-n}$ is the largest ideal on which ψ is trivial. Since

$$\frac{1}{\text{measure}\,\mathfrak{p}_F^{-m-n}}\int_{\mathfrak{p}_F^{-m-n}}\psi\big(\lambda(\operatorname{tr}b-\alpha_0)\big)\,d\lambda$$

is 0, unless $\operatorname{tr}b-\alpha_0$ belongs to $\mathfrak{p}_F^m$ when it is 1, the integral (15.5.3) is equal to

$$\frac{1}{2}\sum_{S'}\frac{1}{\text{measure}\,Z_F\setminus B_F}\int_{V(B_F)}\varphi(\det b)\,|\det b|\,\frac{\chi_\pi(b^{-1})}{d(\pi)}\,\delta(b)\,\mu_B(b)$$

if

$$V(B_F)=\{b\in B_F\,\big|\,\operatorname{tr}b-\alpha_0\in\mathfrak{p}_F^m\}.$$

If we take φ to be the characteristic function of

$$\{\beta\in F\,\big|\,\beta-\beta_0\in\mathfrak{p}_F^m\}$$

the summation disappears and we are left with

$$\frac{1}{2}\cdot\frac{1}{\text{measure}\,Z_F\setminus B_F}\,|\det b_0|\,\frac{\chi_\pi(b_0^{-1})}{d(\pi)}\,\delta(b_0)\int_{V(B_F^0)}\varphi(\det b)\,\mu_B(b).$$

If we replace $\xi(\lambda)$ by the expression (15.5.2) the final result will be

$$\frac{1}{2}\cdot\frac{1}{\text{measure}\,Z_F\setminus B_F}\,|\nu(b_0)|\,\frac{\chi_{\pi'}(b_0^{-1})}{d(\pi)}\,\delta(b_0)\int_{V(B_F^0)}\varphi(\det b)\,\mu_B(b).$$

Since these differ only in sign the proposition follows.

We are now in a position to prove Theorem 15.1. The orthogonality relations and the previous lemma show that the map $\pi'\to\pi$ is injective because the map takes $\Omega'(\eta)$ into $\Omega_0(\eta)$. It is enough to verify that V is surjective when η is unitary. Let $\mathcal{L}^2(\eta)$ be the space of all measurable functions f on

$$\cup_{S'}\widehat{B}_F$$

such that $f(ab) = \eta(a)\, f(b)$ if a is in Z_F and

$$\int_C |f(c)|^2\, \mu(c)$$

is finite. By the Peter–Weyl theorem the set of functions $\chi_{\pi'}$, $\pi' \in \Omega'(\eta)$, form an orthonormal basis of $\mathcal{L}^2(\eta)$. The family χ_π, $\pi \in \Omega_0(\eta)$, is an orthonormal family in $\mathcal{L}^2(\eta)$. By the previous proposition the image of $\Omega'(\eta)$ in $\Omega_0(\eta)$ is actually an orthonormal basis and must therefore be the whole family.

We observe that it would be surprising if the relation $d(\pi) = d(\pi')$ were not also true when π' is one-dimensional. The facts just discussed are also valid when F is the field of real numbers. They follow immediately from the classification and the remarks at the end of the seventh paragraph.

We conclude this paragraph with some miscellaneous facts which will be used elsewhere. F is again a non-archimedean field. Let $K = GL(2, O_F)$ and let K_0 be the set of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in K for which $c \equiv 0 (\operatorname{mod} \mathfrak{p}_F)$. Suppose π is an irreducible admissible representation of G_F in the space V. We are interested in the existence of a non-zero vector v in V such that

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = \omega_1(a)\, \omega_2(d)\, v$$

for all matrices in K_0 while

$$\pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right) v = \omega_0\, v$$

ω_0 is a constant and ω_1 and ω_2 two characters of U_F. ϖ is a generator of $\mathfrak{p}_F$. Since

$$\begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} d & \varpi^{-1} c \\ \varpi b & a \end{pmatrix}$$

such a vector can exist only if $\omega_1 = \omega_2 = \omega$.

Lemma 15.6 *Suppose ω and ω_0 are given. Let π be $\rho(\mu_1, \mu_2)$ which may not be irreducible. There is a non-zero vector φ in $\mathcal{B}(\mu_1, \mu_2)$ satisfying the above conditions if and only if the restrictions of μ_1 and μ_2 to U_F, the group of units of O_F, are equal to ω and*

$$\omega_0^2 = \mu(-\varpi)\, \mu_2(-\varpi)$$

Moreover φ if it exists is unique apart from a scalar factor.

It is easily seen that K is the disjoint union of K_0 and

$$K_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K_0 = K_0 w K_0$$

Let φ_1 be the function which is 0 on $K_0 w K_0$ and on K_0 is given by

$$\varphi_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \omega(ad).$$

Let φ_2 be the function which is 0 on K_0 and takes the value $\omega(a'd'ad)$ at

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If φ in $\mathcal{B}(\mu_1, \mu_2)$ satisfies

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\varphi = \omega(ad)\varphi$$

for all matrices in K_0 then the restrictions of φ to K must be a linear combination of φ_1 and φ_2. This already implies that ω is the restriction of μ_1 and μ_2 to U_F. Suppose $\varphi = a\varphi_1 + b\varphi_2$. Since

$$\pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right)\varphi_1 = |\varpi|^{\frac{1}{2}}\,\mu_1(\varpi)\,\mu_2(-1)\,\varphi_2$$

and

$$\pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right)\varphi_2 = |\varpi|^{-\frac{1}{2}}\,\mu_2(-\varpi)\,\varphi_1$$

while $\mu_1(-1) = \mu_2(-1) = \omega(-1)$, we have

$$\omega_0 b = |\varpi|^{\frac{1}{2}}\,\mu_1(-\varpi)a$$

and

$$\omega_0 a = |\varpi|^{-\frac{1}{2}}\,\mu_2(-\varpi)b$$

Apart from scalar factors there is at most one solution of this equation. There is one non-trivial solution if and only if $\omega_0^2 = \mu_1(-\varpi)\,\mu_2(-\varpi)$.

Lemma 15.7 *Suppose $\pi = \sigma(\mu_1, \mu_2)$ is the special representation corresponding to the quasi-chacters $\mu_1 = \chi\alpha_F^{-\frac{1}{2}}$ and $\mu_2 = \chi\alpha_F^{\frac{1}{2}}$. There is a non-zero vector v in the space of π such that*

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)v = \omega(ad)v$$

for all matrices in K_0 while

$$\pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right)v = \omega_0 v$$

if and only if ω is the restriction of χ to U_F and $\omega_0 = -\chi(-\varpi)$. If v exists it is unique apart from a scalar.

We first let π act on $\mathcal{B}_s(\mu_2, \mu_2)$ a subspace of $\mathcal{B}(\mu_2, \mu_1)$. The condition on ω follows from the previous lemma which also shows that ω_0 must be $\pm\chi(-\varpi)$. If we take the plus sign we see that v must correspond to the function whose restriction to K is constant. Since this function does not lie in $\mathcal{B}_s(\mu_2, \mu_1)$ only the minus sign is possible. To see the existence we let π act on

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_f(\mu_1, \mu_2)$$

In $\mathcal{B}(\mu_1, \mu_2)$ there are two functions satisfying the conditions of the lemma. One with $\omega_0 = -\chi(-\varpi)$ and one with $\omega_0 = \chi(-\varpi)$. One of the two, and we know which, must have a non-zero projection on $\mathcal{B}_s(\mu_1, \mu_2)$.

The above lemmas together with the next one sometimes allow us to decide whether or not a given representation is special.

Lemma 15.8 *If the absolutely cuspidal representation π acts on V there is no non-zero vector v in V such that*

$$\pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right) v = \omega_0 v$$

and

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = \omega(ad)v$$

for all matrices in K_0.

We may suppose that π is the Kirillov form with respect to an additive character ψ such that O_F is the largest ideal on which ψ is trivial. Then v is a function φ in $\mathcal{S}(F^\times)$. If a is in U_F and b is in $F^\times$ we must have $\varphi(ab) = \omega(a)\,\varphi(b)$. Moreover if b is in $F^\times$ and x is in O_F then $\varphi(b) = \psi(xb)\,\varphi(b)$. Thus $\varphi(b) = 0$ if b is not in O_F. Consequently $\hat{\varphi}(\nu, t)$ is 0 if $\nu \neq \omega^{-1}$ but $\hat{\varphi}(\omega^{-1}, t)$ is a polynomial of the form

$$a_m t^m + \cdots + a_n t^n$$

with $a_m a_n \neq 0$. If $\varphi_1(b) = \varphi(-\varpi b)$ then

$$\hat{\varphi}_1(\omega^{-1}, t) = \frac{\omega(-1)}{t}\,\hat{\varphi}(\omega^{-1}, t).$$

Let

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)\, I$$

and let ν_0 be the restriction of η to U_F while $z_0 = \eta(\varpi)$. The character ν_0 will have to be equal to ω^2. The relation

$$\omega_0\varphi = \pi\left(\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}\right)\varphi = \pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_2$$

implies that

$$\omega_0\hat{\varphi}(\omega^{-1}, t) = C(\omega^{-1}, t)\,\omega(-1)\,z_0 t\,\hat{\varphi}(\omega^{-1}, z_0^{-1}t^{-1}).$$

By Proposition 2.23, $C(\omega^{-1}, t)$ is of the form $ct^{-\ell}$ with $\ell \geq 2$. Thus the right side has a pole at 0 not shared by the left. This is a contradiction.

§**16. An application of the Selberg trace formula.** In the fourteenth paragraph we saw that if $\pi' = \otimes_v \pi'_v$ is a constituent of $\mathcal{A}'$ and π' is not of the form $g \to \chi(\nu(g))$ where χ is a quasi-character of $F^\times \setminus I$ then $\pi = \otimes_v \pi_v$, with $\pi_v = \pi(\pi'_v)$, is a constituent of $\mathcal{A}_0$. Let S be the set of places at which the quaternion algebra M' does not split. Given the results of the previous paragraph it is tempting to conjecture that the following theorem is valid.

Theorem 16.1 *Suppose $\pi = \otimes \pi_v$ is a constituent of $\mathcal{A}_0$. If for every v in S the representation π_v is special or absolutely cuspidal then for every v there is a representation π'_v such that $\pi_v = \pi(\pi'_v)$ and $\pi' = \otimes \pi'_v$ is a constituent of $\mathcal{A}'$.*

The existence of π'_v has been shown. What is not clear is that π' is a constituent of $\mathcal{A}'$. It seems to be possible to prove this by means of the Selberg trace formula. Unfortunately a large number of analytical facts need to be verified. We have not yet verified them. However the theorem and its proof seem very beautiful to us; so we decided to include a sketch of the proof with a promise to work out the analytical details and publish them later. We must stress that the sketch is merely a formal argument so that the theorem must remain, for the moment, conjectural.

We first review some general facts about traces and group representations. Suppose G is a locally compact unimodular group and Z is a closed subgroup of the centre of G. Let η be a character of Z. We introduce the space $L^1(\eta)$ of all measurable functions f on G which satisfy $f(ag) = \eta^{-1}(a)\, f(g)$ for all a in Z and whose absolute values are integrable on $Z \setminus G$. If f_1 and f_2 belong to $L^1(\eta)$ so does their product $f_1 * f_2$ which is defined by

$$f_1 * f_2(g) = \int_{Z\setminus G} f_1(gh^{-1})\, f_2(h)\, dh$$

If f belongs to $L^1(\eta)$ let f^* be the function $f^*(g) = \bar{f}(g^{-1})$. It also belongs to $L^1(\eta)$. A subalgebra B of $L^1(\eta)$ will be called ample if it is dense and closed under the operation $f \to f^*$.

Let π be a unitary representation of G on the Hilbert space H such that $\pi(a) = \eta(a)\, I$ for all a in Z. We do not suppose that π is irreducible. If f belongs to $L^1(\eta)$ we set

$$\pi(f) \int_{Z\setminus G} f(g)\, \pi(g)\, dg$$

If $\pi(f)$ is compact for all f in some ample subalgebra B then π decomposes into the direct sum of irreducible representations no one of which occurs more than a finite number of times.

Lemma 16.1.1 *Suppose π_1 and π_2 are two unitary representations of G such that $\pi_1(a) = \eta(a)\, I$ and $\pi_2(a) = \eta(a)\, I$ for all a in Z. Suppose there is an ample subalgebra B of $L^1(\eta)$ such that $\pi_1(f)$ and $\pi_2(f)$ are of Hilbert–Schmidt class for all f in B.*

(i) *If for every f in B*

$$\operatorname{trace} \pi_1(f)\, \pi_1(f^*) \geq \operatorname{trace} \pi_2(f)\, \pi_2(f^*)$$

then π_2 is equivalent to a subrepresentation of π_1.

(ii) *If for every f in B*

$$\operatorname{trace} \pi_1(f)\, \pi_1(f^*) = \operatorname{trace} \pi_2(f)\, \pi_2(f^*)$$

then π_2 is equivalent to π_1.

Let π_1 act on H_1 and let π_2 act on H_2. A simple application of Zorn's lemma shows that we can choose a pair of closed invariant subspaces M_1 and M_2, of H_1 and H_2 respectively, such that the restrictions of π_1 to M_1 and π_2 to M_2 are equivalent and such that the pair M_1, M_2 is maximal with respect to this property. Replacing H_1 and H_2 by the orthogonal complements of M_1 and M_2 we may suppose that $M_1 = 0$ and that $M_2 = 0$. To prove the first assertion of the lemma we have to show that with this assumption $H_2 = 0$. If the second condition is satisfied we can reverse the roles of π_1 and π_2 to see that H_1 is also 0.

Before beginning the proof we make a simple remark. Suppose σ is an irreducible unitary representation of G on L and σ_α, $\alpha \in A$, is an irreducible unitary representation of G on L_α. Suppose that $\sigma(a) = \eta(a)\,I$ for all a in Z and $\sigma_\alpha(a) = \eta(a)\,I$ for all a in Z and all α in A. Suppose that σ is equivalent to none of the σ_α and that a non-zero vector x in L and vectors x_α in L_α are given. Finally suppose that

$$\sum_\alpha \|\sigma_\alpha(f)\,x_\alpha\|^2$$

is finite for every f in B. Then if ε is any positive number there is an f in B such that

$$\sum_\alpha \|\sigma_\alpha(f)x_\alpha\|^2 < \varepsilon\|\sigma(f)x\|^2.$$

Suppose the contrary and let L' be the closure in $\oplus_\alpha L_\alpha$ of

$$\{\otimes\sigma_\alpha(f)\,x_\alpha \bigm| f \in B\}$$

L' is invariant under G and the map

$$\oplus\sigma_\alpha(f)x_\alpha \to \sigma(f)x$$

may be extended to a continuous G-invariant map A' of L' into L. If A' were 0 then $\sigma(f)x = 0$ for all f in B which is impossible. Let A be the linear transformation from $\oplus L_\alpha$ to L which is A' on L' and 0 on its orthogonal complement. A commutes with G and is not 0. Let A_α be the restriction of A to L_α. A_α is a G-invariant map of L_α into L and is therefore 0. Thus A is 0. This is a contradiction.

Suppose H_2 is not 0. There is an h in B such that $\pi_1(h) \neq 0$. If $f = h * h^*$ then $\pi_2(f)$ is positive semi-definite and of trace class. It has a positive eigenvalue and with no loss of generality we may suppose that its largest eigenvalue is 1. Let $\pi_2 = \otimes\pi_2^\beta$, where π_2^β acts on H_2^β, be a decomposition of π_1 into irreducible representations. There is a β_0 and a unit vector x in $H_2^{\beta_0}$ such that $\pi_2(f)x = x$. Let $\pi_1 = \oplus\pi_1^\alpha$, where π_1^α acts on H_1^α, be a decomposition of π_1 into irreducible representations. Choose an orthogonal basis $\{x^{\alpha,\gamma} \bigm| \gamma \in \Gamma_\alpha\}$ of H_1^α consisting of eigenvectors of $\pi_1(f)$. Since

$$\operatorname{trace}\pi_1(f) \geq \operatorname{trace}\pi_2(f)$$

the largest eigenvalue of $\pi_1(f)$ is positive. Let it be λ.

If f_1 belongs to B,

$$\sum_\alpha\sum_\gamma \|\pi_1^\alpha(f_1)\,x^{\alpha,\gamma}\|^2$$

is the Hilbert–Schmidt norm of $\pi_1(f_1)$ and is therefore finite. By assumption $\pi_2^{\beta_0}$ is not equivalent to any of the representations π_1^α so that we can apply our earlier remark to the vector x and the family of representations $\pi_1^{\alpha,\gamma} = \pi_1^\alpha$ together with the family of vectors $x^{\alpha,\gamma}$ to infer the existence of an f_1 in B such that

$$\sum_\alpha\sum_\gamma \|\pi_1(f_1)\,x^{\alpha,\gamma}\|^2 < \frac{1}{2\lambda}\,\|\pi_2(f_1)\,x\|^2.$$

Then

$$\operatorname{trace}\pi_1(f_1 f)\,\pi_1^*(f_1 f) = \operatorname{trace}\pi_1^*(f_1 f)\,\pi_1(f_1 f)$$

is equal to

$$\sum_{\alpha,\gamma} \|\pi_1(f_1)\,\pi_1(f)\,x^{\alpha,\gamma}\|^2 \le \lambda \sum_{\alpha,\gamma} \|\pi_1(f_1)\,x^{\alpha,\gamma}\|^2.$$

The right side is less than

$$\frac{1}{2}\|\pi_2(f_1)\,x\|^2 = \frac{1}{2}\|\pi_2(f_1 f)\,x\|^2$$

which is at most

$$\frac{1}{2}\operatorname{trace}\pi_2(f_1 f)\,\pi_2^*(f_1 f).$$

This is a contradiction.

The next lemma is a consequence of the results of [35].

Lemma 16.1.2 *Suppose η is trivial so that $L^1(\eta) = L^1(Z \setminus G)$. Suppose that B is an ample subalgebra of $L^1(\eta)$ which is contained in $L^2(Z \setminus G)$. If there is a positive constant γ and a unitary representation π of $Z \setminus G$ such that $\pi(f)$ is of Hilbert–Schmidt class for all f in B and*

$$\operatorname{trace}\pi(f)\,\pi(f^*) = \gamma \int_{Z\setminus G} |f(g)|^2\,dg$$

then $Z \setminus G$ is compact.

In proving the theorem it is better to deal with representations in the adèle groups than to deal with representations of the global Hecke algebras. We have to assume that the reader is sufficiently well acquainted with the theory of group representations to pass back and forth unaided between the two viewpoints.

If F is a global field, $\mathbb{A}$ is the adèle ring of F, $G = GL(2)$, and η is a character of the idèle class group $F^\times \setminus I$ the space $A(\eta)$ of all measurable functions φ on $G_F \setminus G_{\mathbb{A}}$ that satisfy

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\,\varphi(g)$$

for all a in I and whose absolute values are square-integrable on $G_F\,Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$ is a Hilbert space. If φ belongs to this space

$$\int_{N_F\setminus N_{\mathbb{A}}} \varphi(ng)\,dn$$

is defined for almost all g. If it is 0 for almost all g the function φ is said to be a cusp form. The space $A_0(\eta)$ of all such cusp forms is closed and invariant under $G_{\mathbb{A}}$. It is in fact the closure of $\mathcal{A}_0(\eta)$. It decomposes in the same way but now into a direct sum of closed orthogonal subspaces V on which $G_{\mathbb{A}}$ acts according to an irreducible representation $\pi = \otimes\pi_v$. Thus V is now isomorphic to a tensor product of Hilbert spaces. Of course the same representations occur now as occurred before. Similar remarks apply to the multiplicative group G' of quaternion algebra M' over F.

It will be enough to prove the theorem when π is a constituent of some $\mathcal{A}_0(\eta)$ or $A_0(\eta)$ and η is a character because we can always take the tensor product of π with a suitable quasi-character. Suppose

η is given. Let S be the set of places at which M' does not split. Suppose that for each v in S we are given an irreducible unitary representation σ'_v of $G'_{F_v} = G'_v$ such that

$$\sigma'_v(a) = \eta_v(a)\, I$$

for all a in F^*_v which we identify with $Z'_v = Z'_{F_v}$. Let $\sigma_v = \pi(\sigma'_v)$ be the representation of G_v corresponding to σ'_v. We may take σ_v unitary. Let σ_v act on U_v and let σ'_v act on U'_v. Fix a unit vector u'_v in U_v and a unit vector u_v in U_v which is K_v-finite. u'_v is automatically K'_v-finite.

Write $A_0(\eta)$ as the direct sum, in the Hilbert space sense, of mutually orthogonal invariant irreducible subspaces $V^1, V^2, \cdots$. Let the factorization of the representation π^i on V^i be $\otimes\pi^i_v$. Let π^i_v act on V^i_v. For simplicity of notation we identify V^i with $\otimes V^i_v$. We also suppose that if v is in S and π^i_v is equivalent to σ_v then $U_v = V^i_v$ and $\pi^i_v = \sigma_v$. Let X be the set of all i such that $\pi^i_v = \sigma_v$ for all v in S and if i belongs to X let

$$M^i = \{\otimes_{v\in S} u_v\} \otimes \{\otimes_{v \in S} V^i_v\}.$$

M^i is invariant and irreducible under the action of

$$\widehat{G}_S = \{g = (g_v) \,\big|\, g_v = 1 \text{ for all } v \text{ in } S\}.$$

Let

$$M = \oplus_{i\in X} M^i.$$

M is a Hilbert space and $\widehat{G}_S$ acts on M. If at least one of the representations σ'_v, $v \in S$, is not one-dimensional set $N = M$. If they are all one-dimensional let N be the subspace of $A_0(\eta)$ spanned, in the Hilbert space sense, by M and the functions $g \to \chi(\det g)$ where χ is a character of $F^\times \setminus I$ such that $\chi^2 = \eta$ and $\sigma'_v(g) = \chi_v\big(\nu(g)\big)$ for all g in G'_v if v is in S. If v is non-archimedean this last condition determines χ_v uniquely. If v is real it only determines it on the positive numbers.

Let $A'(\eta)$ be the space of all measurable functions φ on $G'_F \setminus G'_{\mathbb{A}}$ that satisfy $\varphi(ag) = \eta(a)\,\varphi(g)$ for all a in I and whose absolute values are square integrable on $G'_F\, Z'_{\mathbb{A}} \setminus G'_{\mathbb{A}}$. Replacing σ_v by σ'_v and u_v by u'_v we define N' in the same way as we defined M. If at least one of the representations σ'_v, $v \in S$, is not one-dimensional we set $M' = N'$. However if they are all one-dimensional and χ is a character of $F^\times \setminus I$ such that $\chi^2 = \eta$ and $\sigma'_v(g) = \chi_v\big(\nu(g)\big)$ for all G in G'_V if v is in S then the function $g \to \chi\big(\nu(g)\big)$ belongs to N'. We let M' be the orthogonal complement in $N'$of the set of such functions. The group $\widehat{G}'_S$ acts on M' and N'. However by means of the local isomorphisms θ_v we can define an isomorphism of $\widehat{G}_S$ and $\widehat{G}'_S$. Thus $\widehat{G}_S$ acts on M and M'. To prove the theorem we need only show that the representations on these two spaces are equivalent. To do this we combine Lemma 16.1.1 with the Selberg trace formula.

To apply Lemma 16.1.1 we have to introduce an algebra B. B will be the linear span of B_0, the set of functions f on $\widehat{G}_S$ of the form

$$f(g) = \prod_{v \notin S} f_v(g_v)$$

where the functions f_v satisfy the following conditions.

(i) If a_v belongs to $F^\times_v$ then

$$f_v(a_v g_v) = \eta_v^{-1}(a_v)\, f_v(g_v).$$

(ii) f_v is K_v-finite on both sides and the projection of the support of f_v on $Z_v \setminus G_v$ is compact.
(iii) If v is archimedean f_v is infinitely differentiable.
(iv) If v is non-archimedean f_v is locally constant.

(v) For almost all non-archimedean v the functions f_v is 0 outside of Z_vK_v but on Z_vK_v is given by

$$f_v(g) = \omega_v^{-1}(\det g)$$

where ω_v is unramified and satisfies $\omega_v^2 = \eta_v$.

We introduce B' in the same way. We may identify B and B' and to verify the conditions of the lemma we need only show that if $f = f_1 * f_2$ with f_1 and f_2 in B_0 then

$$\operatorname{trace}\sigma(f) = \operatorname{trace}\sigma'(f)$$

if σ is the representation on M and σ' that on M'. Let τ be the representation on N and τ' that on N'. Since

$$\operatorname{trace}\tau(f) = \operatorname{trace}\sigma(f) + \sum \int_{\widehat{Z}_S\backslash\widehat{G}_S} \chi(g)\, f(g)\, dg$$

and

$$\operatorname{trace}\tau'(f) = \operatorname{trace}\sigma'(f) + \sum \int_{\widehat{Z}_S\backslash\widehat{G}_S} \chi(g)\, f(g)\, dg$$

we need only show that

$$\operatorname{trace}\tau(f) = \operatorname{trace}\tau'(f).$$

Before beginning the proof we had better describe the relation between the Haar measures on the groups $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$ and $Z'_{\mathbb{A}} \backslash G'_{\mathbb{A}}$. Choose a non-trivial character ψ of $F \backslash \mathbb{A}$. If ω_0 is any invariant form of maximal degree on $Z \backslash G$ defined over F and therefore over each F_v we can associate to ω_0 and ψ_v a Haar measure $\omega_0(v)$ on $Z_v \backslash G_v$. Then $\prod_{v\notin S} \omega_0(v)$ determines a Haar measure ω_0 on $\widehat{Z}_S \backslash \widehat{G}_S$ and $\prod_v \omega_0(v)$ determines a Haar measure ω_0 on $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. The measure on $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$ is independent of ψ and is called the Tamagawa measure. As in the previous paragraph we can associate to $\omega_0(v)$ a measure $\omega'_0(v)$ on $Z'_v \backslash G'_v$ and therefore to ω_0 a measure ω'_0 on $\widehat{Z}'_S \backslash \widehat{G}'_S$ or $Z'_{\mathbb{A}} \backslash G'_{\mathbb{A}}$.

We first take $f = f_1 * f_2$ in B' and find a formula for $\operatorname{trace}\tau'(f)$. Let $d(\sigma'_v)$ be the formal degree of σ'_v with respect to the measure $\omega'_0(v)$ and let ξ'_v be the function

$$\xi'_v(g) = d(\sigma'_v)\, \overline{(\sigma'_v(g)u'_v, u'_v)}$$

on G'_v. Let $\Phi' = \Phi'_f$ be the function

$$\Phi'(g) = \Big\{ \prod_{v\in S} \xi'_v(g_v) \Big\}\, f(\hat{g}_S)$$

on $G'_{\mathbb{A}}$. Here $\hat{g}_S$ is the projection of g on $\widehat{G}'_S$. If ρ' is the representation of $G'_{\mathbb{A}}$ on $A'(\eta)$ the restriction of $\rho'(\Phi')$ to N' is $\tau'(f)$ and $\rho'(\Phi)$ annihilates the orthogonal complement of N'. Thus

$$\operatorname{trace}\rho'(\Phi') = \operatorname{trace}\tau'(f).$$

If φ is in $A'(\eta)$ then $\rho'(\Phi')\,\varphi(g)$ is equal to

$$\int_{Z'_{\mathbb{A}}\backslash G'_{\mathbb{A}}} \varphi(gh)\, \Phi'(h)\, \omega'_0(h) = \int_{Z'_A\backslash G'_A} \varphi(h)\, \Phi'(g^{-1}h)\, \omega'_0(h).$$

The integration on the right can be performed by first summing over $Z'_F \setminus G'_F$ and then integrating over $Z_{\mathbb{A}} G'_F \setminus G'_{\mathbb{A}}$. If

$$\Phi'(g,h) = \sum_{Z'_F \setminus G'_F} \Phi'(g^{-1}\gamma h)$$

the result is

$$\int_{Z'_{\mathbb{A}} G'_F \setminus G'_{\mathbb{A}}} \varphi(h)\, \Phi'(g,h)\, \omega'_0(h).$$

Thus the trace of $\rho'(\Phi)$ is equal to

$$\int_{Z'_A G'_F \setminus G'_A} \Phi'(g,g)\, dg.$$

If we write out the integrand and perform the usual manipulations (cf [29]) we see that this integral is

$$\sum_{\{\gamma\}} \text{measure}\left(Z'_{\mathbb{A}}\, G'_F(\gamma) \setminus G'_{\mathbb{A}}(\gamma)\right) \int_{G'_A(\gamma)\setminus G'_A} \Phi'(g^{-1}\gamma g). \tag{16.1.3}$$

The sum is over a set of representatives of the conjugacy classes in G'_F. $G'_{\mathbb{A}}(\gamma)$ is the centralizer of γ in $G'_{\mathbb{A}}$ and $G'_F(\gamma)$ is its centralizer in G'_F.

Let Q' be a set of representatives for the equivalence classes of quadratic extensions E of F such that $E \otimes_F F_v$ is a field for all v in S. For each E in Q' fix an imbedding of E in the quaternion algebra M'. Let $B_F = B_F(E)$ be the multiplicative group of E, considered as a subalgebra of M', or what is the same the centralizer of E in G'_F. Let $B_{\mathbb{A}} = B_{\mathbb{A}}(E)$ be the centralizer of E in $G'_{\mathbb{A}}$. Let Q'_1 be the separable extensions in Q' and Q'_2 the inseparable ones if they exist. Then (16.1.3) is the sum of

$$\text{measure}(Z'_{\mathbb{A}}\, G'_F \setminus G'_{\mathbb{A}})\, \Phi'(e), \tag{16.1.4}$$

if e is the identity,

$$\frac{1}{2} \sum_{Q'_1} \sum_{\substack{\gamma \in Z'_F \setminus B_F \\ \gamma \notin Z'_F}} \text{measure}(Z'_{\mathbb{A}}\, B_F \setminus B_{\mathbb{A}}) \int_{B_{\mathbb{A}} \setminus G_{\mathbb{A}}} \Phi'(g^{-1}\gamma g)\, \omega_B(g) \tag{16.1.5}$$

and

$$\sum_{Q'_2} \sum_{\substack{\gamma \in Z'_F \setminus B_F \\ \gamma \notin Z_F}} \text{measure}(Z'_{\mathbb{A}}\, B_F \setminus B_{\mathbb{A}}) \int_{B_{\mathbb{A}} \setminus G_{\mathbb{A}}} \Phi'(g^{-1}\gamma g)\, \omega_B(g). \tag{16.1.6}$$

The last sum is deceptive because Q'_2 has at most one element. ω_B is the quotient of the measure on $Z'_{\mathbb{A}} \setminus G'_{\mathbb{A}}$ by that on $Z'_{\mathbb{A}} \setminus B_{\mathbb{A}}$. The choice of the measure on $Z'_{\mathbb{A}} \setminus B_{\mathbb{A}}$ is not too important. We do suppose that it is a product measure.*

The expression (16.1.4) is equal to

$$\text{measure}(Z'_{\mathbb{A}}\, G'_F \setminus G'_{\mathbb{A}}) \Big\{ \prod_{v \in S} d(\sigma'_v) \Big\}\, f(e).$$

* In (16.1.5) the factor $\frac{1}{2}$ is not quite correct. If we want to leave it in, both γ and its conjugate must be counted, even if they differ only by an element of F.

The integrals of (16.1.5) and (16.1.6) are equal to the product

$$\prod_{v\in S}\frac{\chi_{\sigma'_v}(\gamma^{-1})}{\text{measure}(Z'_v\setminus B_v)}$$

and

$$\int_{\widehat{B}_S\setminus\widehat{G}'_S} f(g^{-1}\gamma g)\,\omega_B.$$

Now regard $f=f_1*f_2$ as an element of B. We can still introduce for each v in S the function

$$\xi_v(g)=d(\sigma_v)\overline{\big(\sigma_v(g)u_vu_v\big)}$$

on G_v. $d(\sigma_v)$ is the formal degree of σ_v with respect to the measure $\omega_0(v)$. If σ'_v is not one-dimensional ξ_v is integrable and we can use it to define a function Φ to which we can hope to apply the trace formula. When σ'_v is one-dimensional the function ξ_v is not even integrable so it is of no use to us. However in this case we can find an integrable function ζ_v with the following properties.

(i) For all a in F_v

$$\zeta_v\left(\begin{pmatrix}a&0\\0&a\end{pmatrix}g\right)=\eta_v^{-1}(a)\,\zeta_v(g).$$

(ii) For a suitable choice of u_v the operator $\sigma_v(\zeta_v)$ is the orthogonal projection on the space $\mathbb{C}u_v$.

(iii) If χ_v is a character of $F_v^\times$ such that $\chi_v^2=\eta_v$ then

$$\int_{Z_v\setminus G_v}\chi_v(\det g)\,\zeta_v(g)\,\omega_0(v)$$

is -1 if $\sigma'_v(h)=\chi_v\big(\nu(h)\big)$ for all h in G'_v and is 0 otherwise.

(iv) If π_v is a unitary infinite-dimensional irreducible admissible representation of G_v which is not equivalent to σ_v but satisfies

$$\pi_v\left(\begin{pmatrix}a&0\\0&a\end{pmatrix}\right)=\eta_v(a)\,I$$

for all a in $F_v^\times$ then

$$\operatorname{trace}\pi_v(\zeta_v)=0.$$

If v is real we cannot describe ζ_v without a great deal more explanation than is desirable at present. However after a few preliminary remarks we will be able to describe it when v is non-archimedean.

Suppose $\sigma'_v(g)=\chi_v\big(\nu(g)\big)$ for g in G'_v and π_v is a representation of G_v such that

$$\pi_v\left(\begin{pmatrix}a&0\\0&a\end{pmatrix}\right)=\eta_v(a)\,I$$

for all a in $F_v^\times$. Applying Lemma 3.9 to $\chi_v^{-1}\otimes\pi_v$ we see that the restriction of π_v to K_v contains the representation $k\to\chi_v(\det k)$ if and only if $\pi_v=\pi(\mu_v,\nu_v)$, $\mu_v\nu_v=\eta_v$, and the restrictions of μ_v and ν_v to U_v, the group of units of F_v, are both equal to the restriction of χ_v. Let ζ'_v be the function on G_v which is 0 outside of Z_vK_v but on K_v is equal to

$$\frac{1}{\text{measure}(Z_v\setminus Z_v\,K_v)}\,\chi_v^{-1}(\det g).$$

Let H_v be the group generated by Z_v, the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in K_v for which $c \equiv 0 (\mathrm{mod}\, \mathfrak{p}_v)$, and

$$\begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}.$$

Let ω_v be the character $\omega_v(a) = (-1)^n \chi_v(a)$ if $|a| = |\varpi_v|^n$. According to the concluding lemmas of the previous paragraph there is a non-zero vector u in the space of π_v such that

$$\pi_v(g)\, u = \omega_v(\det g)\, u$$

for all g in H_v if and only if π_v is equivalent to σ_v, $\pi_v = \pi(\mu_v, \nu_v)$ is infinite-dimensional, $\mu_v \nu_v = \eta_v$, and the restrictions of μ_v and ν_v to U_v are equal to the restriction of χ_v, or π_v is the one-dimensional representation

$$g \to \omega_v(\det g).$$

Let ζ_v'' be the function which is 0 outside of H_v and equal to

$$\frac{1}{\text{measure}\, Z_v \setminus H_v}\, \omega_v^{-1}(\det g)$$

on H_v. We may take

$$\zeta_v = \zeta_v'' - \zeta_v'.$$

There are some consequences of the four conditions on ζ_v which we shall need. If μ_v and ν_v are two characters of $F_v^\times$ such that $\mu_v \nu_v = \eta_v$, the trace of $\rho(\zeta_v, \mu_v, \nu_v)$ is a multiple of

$$\int_{Z_v \setminus A_v} \mu_v(\alpha)\, \nu_v(\beta) \left|\frac{\alpha}{\beta}\right|^{\frac{1}{2}} \left\{ \int_{N_v} \int_{K_v} \zeta_v(k^{-1} a n k)\, dn\, dk \right\} da$$

if

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Since this is 0 for all possible choice of μ_v and ν_v

$$\int_{N_v} \int_{K_v} \zeta_v(k^{-1} a n k)\, dk\, dn = 0$$

for all a. We also observe that if σ_v' is not one-dimensional then

$$\int_{N_v} \int_{K_v} \xi_v(k^{-1} a n k)\, dk\, dn = 0$$

for all a.

If π_v is special or absolutely cuspidal trace $\pi_v(\zeta_v)$ is therefore equal to

$$\frac{1}{2}\sum_{S'}\int_{Z_v\backslash B_v}\Big\{\int_{B_v\backslash G_v}\zeta_v(g^{-1}bg)\,\omega_b(v)\Big\}\,\chi_{\pi_v}(b)\,\delta(b)\mu_B^0(b).$$

Since $\mathrm{trace}\,\pi_v(\zeta_v)$ is 1 if π_v is equivalent to σ_v and 0 otherwise the orthogonality relations imply that

$$\int_{B_v\backslash G_v}\zeta_v(g^{-1}bg)\,\omega_B(v)=\frac{-1}{\mathrm{measure}\,Z_v\setminus B_v}\,\chi_{\sigma_v}(b^{-1})$$

for all regular b and therefore, by continuity, for all b whose eigenvalues do not lie in F_v. It probably also follows from the Plancherel theorem that $\zeta_v(e)=d(\sigma_v)$. We do not need this but we shall eventually need to know that $\zeta_v(e)=d(\sigma_v')$. For the moment we content ourselves with observing that if ω_v is a character of $F_v^\times$ and σ_v' is replaced by $\omega_v\otimes\sigma_v'$ the formal degree does not change and ζ_v is replaced by the function $g\to\omega_v^{-1}(\mathrm{det}g)\,\zeta_v(g)$ so that $\zeta_v(e)$ does not change. Thus the relation $\zeta_v(e)=d(\sigma_v')$ need only be proved when σ_v' is trivial.

Let S_1 be the subset of v in S for which σ_v' is one-dimensional and let S_2 be the complement of S_1 in S. Given $f=f_1*f_2$ in B we set

$$\Phi(g)=\Big\{\prod_{v\in S_1}\zeta_v(g_v)\Big\}\Big\{\prod_{v\in S_2}\xi_v(g_v)\Big\}\,f(\hat g_S).$$

Let ρ_0^+ be the representation of $G_{\mathbb{A}}$ on $A_0^+(\eta)$ the sum, in the Hilbert space sense, of $A_0(\eta)$ and the functions $\chi:g\to\chi(\mathrm{det}g)$ where χ is a character of $F^\times\setminus I$ such that $\chi^2=\eta$ and let ρ be the representation on $A(\eta)$. If at least one of the representations ρ_v' is not one-dimensional $\rho_0^+(\Phi)$ annihilates the orthogonal complement of $A_0(\eta)$. If they are all one-dimensional we apply the third condition on the functions ζ_v together with the fact that the number of places in S is even to see that $\rho_0^+(\Phi)\,\chi=0$ unless $\sigma_v'(h)=\chi_v\big(\nu(h)\big)$ for all h in G_v' and all v in S but that if this is so

$$\rho_0^+(\Phi)\,\chi=\tau(f)\,\chi.$$

Recall that $A_0(\eta)$ is the direct sum of spaces V^i on which $G_{\mathbb{A}}$ acts according to representations $\pi^i=\otimes\pi_v^i$. If at least one of the representations σ_v' is not one-dimensional $\rho_0^+(\Phi)$ is equal to $\sigma(f)$ on M and annihilates the orthogonal complement of M in $A_0(\eta)$. Suppose they are all one-dimensional. If i belongs to X the restrictions of $\rho_0^+(\Phi)$ and $\sigma(f)$ or $\tau(f)$ to M^i are equal and $\rho_0^+(\Phi)$ annihilates the orthogonal complement of M^i in V^i. If i is not in X the trace of the restriction of $\rho_0^+(\Phi)$ to V^i is

$$\Big\{\prod_{v\in S}\mathrm{trace}\,\pi_v^i(\zeta_v)\Big\}\{\mathrm{trace}\,\hat\pi_S^i(f)\}$$

in $\hat\pi_S^i=\otimes_{v\notin S}\pi_v^i$. Since π_v^i, $v\in S$, are all infinite-dimensional and for at least one such v the representation π_v^i is not equivalent to σ_v,

$$\prod_{v\in S}\mathrm{trace}\,\pi_v^i(\zeta_v)=0.$$

We conclude that

$$\mathrm{trace}\,\rho_0^+(\Phi)=\mathrm{trace}\,\tau(f).$$

To show that

$$\operatorname{trace}\tau(f) = \operatorname{trace}\tau'(f)$$

we have to apply the trace formula to find a suitable expression for $\operatorname{trace}\rho_0^+(\Phi)$. In order to describe the formula we need to state some results in the theory of Eisenstein series.

Consider the collection of pairs of character μ, ν of $F^\times \setminus I$ such that $\mu\nu = \eta$. To such pairs, μ, ν and μ', ν' are said to be equivalent if there is a complex number r such that $\mu' = \mu\alpha_F^r$ and $\nu' = \nu\alpha_F^{-r}$. If a belongs to I then $\alpha_F^r(a) = |a|^r$. Let P be a set of representatives for these equivalence classes.

Suppose (μ, ν) belongs to P. If s is a complex number the space $\mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ of functions on $N_{\mathbb{A}} \setminus G_{\mathbb{A}}$ is defined as in the tenth paragraph. Since the functions in this space are determined by their restrictions to K we may think of it as a space of functions on K in which case it is independent of s. Thus we have isomorphisms

$$T_s : \mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}}) \to \mathcal{B}(\mu, \nu).$$

The theory of Eisenstein series provides us with a function $(\varphi, s) \to E(\varphi, s)$ from $\mathcal{B}(\mu, \nu) \times \mathbb{C}$ to $\mathcal{A}(\eta)$. $E(g, \varphi, s)$ is the value of $E(\varphi, s)$ at g. For a given φ the function $E(g, \varphi, s)$ is continuous in g and meromorphic in s. Moreover there is a discrete set of points in $\mathbb{C}$ such that outside of this set it is holomorphic in s for all g and φ. If s is not in this set the map $\varphi \to E(T_s\varphi, s)$ of $\mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ into $\mathcal{A}(\eta)$ commutes with the action of $\mathcal{H}$.

If the total measure of $N_F \setminus N_{\mathbb{A}}$ is taken to be 1 the integral

$$\int_{N_F \setminus N_{\mathbb{A}}} E(ng, T_s\varphi, s)\, dn$$

is equal to

$$\varphi(g) + \big(M(s)\varphi\big)(g).$$

$M(s)$ is a linear transformation from $\mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ to $\mathcal{B}(\nu\alpha_F^{-\frac{s}{2}}, \mu\alpha_F^{\frac{s}{2}})$ which commutes with the action of $\mathcal{H}$. It is mermorphic in the sense that

$$\langle M(s)T_s^{-1}\varphi_1, T_s^{-1}\varphi_2\rangle$$

is meromorphic if φ_1 belongs to $\mathcal{B}(\mu, \nu)$ and φ_2 belongs to $\mathcal{B}(\nu^{-1}, \mu^{-1})$. The quotient of $M(s)$ by

$$\frac{L(1-s, \nu\mu^{-1})}{L(1+s, \mu\nu^{-1})} = \varepsilon(1-s, \nu\mu^{-1})\, \frac{L(s, \mu\nu^{-1})}{L(1+s, \mu\nu^{-1})}$$

is holomorphic for $\operatorname{Re} s \geq 0$. Since the analytic behavior of $E(g, \varphi, s)$ is controlled by that of $M(s)$ it should be possible, as we observed before, to use the Eisenstein series to show that a constituent of $\mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ is also a constituent of $\mathcal{A}(\eta)$.

To indicate the dependence of $M(s)$ on μ and ν we write $M(\mu, \nu, s)$. Then

$$M(\mu, \nu, s)M(\nu, \mu, -s) = I.$$

If s is purely imaginary we can introduce the inner product

$$(\varphi_1, \varphi_2) = \int_K \varphi_1(k)\, \bar{\varphi}_s(k)\, dk$$

on $\mathcal{B}(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$. Let $B(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ be its completion with respect to this inner product. $B(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ may be thought of as a function space on $G_{\mathbb{A}}$ on which $G_{\mathbb{A}}$ acts by right translations. The representation of $G_{\mathbb{A}}$ on $B(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ is unitary. Let g correspond to the operator $\rho(g, \mu, \nu, s)$ and if f is in $L^1(\eta)$ let

$$\rho(f, \mu, \nu, s) = \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} f(g)\,\rho(g, \mu, \nu, s)\,\omega_0(g)$$

T_s extends to an isometry, from $B(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ to $B(\mu, \nu)$ and $M(\mu, \nu, s)$ extends to an isometry from $B(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}})$ to $B(\nu\alpha_F^{-\frac{s}{2}}, \mu\alpha_F^{\frac{s}{2}})$. In particular

$$M^*(\mu, \nu, s) = M(\nu, \mu, -s).$$

Suppose (μ, ν) is in P and, for some r, $\nu = \mu\alpha_F^r$ and $\mu = \nu\alpha_F^{-r}$. Replacing μ by $\mu\alpha_F^{\frac{r}{2}}$ and ν by $\nu\alpha_F^{-\frac{r}{2}}$ if necessary we may suppose that $\mu = \nu$. We may also suppose that if (μ, ν) is in P and is not equivalent to (ν, μ) then (ν, μ) is also in P. Let L be the Hilbert space sum

$$\oplus_P B(\mu, \nu)$$

and let $\mathcal{L}$ be the algebraic sum

$$\oplus_P \mathcal{B}(\mu, \nu).$$

If we define $L(s)$ to be

$$\oplus_p B\big(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}}\big)$$

and $\mathcal{L}(s)$ to be

$$\oplus_P \mathcal{B}\big(\mu\alpha_F^{\frac{s}{2}}, \nu\alpha_F^{-\frac{s}{2}}\big)$$

we can introduce the map

$$T_s : L(s) \to L.$$

The representation $g \to \rho(g, s)$ is the representation

$$g \to \oplus\rho(g, \mu, \nu, s)$$

on $L(s)$. $M(s)$ will be the operator on $L(s)$ which takes $\oplus\varphi(\mu, \nu)$ to $\oplus\varphi_1(\mu, \nu)$ with

$$\varphi_1(\nu, \mu) = M(\mu, \nu, s)\,\varphi(\mu, \nu).$$

It is unitary.

If F has characteristic O let H be the space of all square integrable functions φ from the imaginary axis to L such that

$$T_{-s}^{-1}\varphi(-s) = M(s)\,T_s^{-1}\varphi(s)$$

with the norm

$$\frac{c}{\pi}\int_{-i\infty}^{i\infty} \|\varphi(s)\|^2\,d|s|$$

c is a positive constant relating various Haar measures. It will be defined more precisely later. If F is a function field with field of constants $\mathbb{F}_q$ the functions in H are to be periodic of period $\frac{\log q}{2\pi} i$ and the norm is to be

$$\frac{c \log u}{\pi} \int_0^{\frac{2\pi}{\log q}} \|\varphi(s)\|^2 \, d|s|.$$

On the whole we shall proceed as though F had characteristic O merely remarking from time to time the changes to be made when the characteristic is positive.

If $\varphi = \oplus \varphi(\mu, \nu)$ is in $\mathfrak{L}$ we set

$$E(g, \varphi, s) = \sum E\big(g, \varphi(\mu, \nu), s\big).$$

If φ in H takes values in $\mathfrak{L}$

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-iT}^{iT} E\big(g, \varphi(s), s\big) \, d|s| = \tilde{\varphi}(g)$$

exists in $A(\eta)$. The map $\varphi \to \tilde{\varphi}$ extends to an isometry of H with a subspace $A_1(\eta)$ of $A(\eta)$. If g is in $G_{\mathbb{A}}$ and φ' is defined by

$$\varphi'(s) = T_s \, \rho(g, s) \, T_s^{-1} \varphi(s)$$

then $\tilde{\varphi}'$ is $\rho(g)\tilde{\varphi}$.

The orthogonal complement of $A_1(\eta)$ is $A_0^+(\eta)$. Thus if E is the orthogonal projection of $A(\eta)$ on $A_1(\eta)$ the trace of $\rho_0^+(\Phi)$ is the trace of $\rho(\Phi) - E\rho(\Phi)$ which, according the the Selberg trace formula, is the sum of the following expressions which we first write out and then explain.

(i)

$$\text{measure}(Z_{\mathbb{A}} \, G_F \setminus G_{\mathbb{A}}) \, \Phi(e).$$

(ii)

$$\frac{1}{2} \sum_{Q_1} \sum_{\substack{\gamma \in Z_F \setminus B_F \\ \gamma \notin Z_F}} \text{measure}(Z_{\mathbb{A}} \, B_F \setminus B_{\mathbb{A}}) \int_{B_{\mathbb{A}} \setminus G_{\mathbb{A}}} \Phi(g^{-1} \gamma g) \, \omega_B(g).$$

(iii)

$$\sum_{Q_2} \sum_{\substack{\gamma \in Z_F \setminus B_F \\ \gamma \notin Z_F}} \text{measure}(Z_{\mathbb{A}} \, B_F \setminus B_{\mathbb{A}}) \int_{B_{\mathbb{A}} \setminus G_{\mathbb{A}}} \Phi(g^{-1} \gamma g) \, \omega_b(g).$$

(iv)

$$-c \sum_{\substack{\gamma \in Z_F \setminus A_F \\ \gamma \notin Z_F}} \sum_v \Big\{ \prod_{w \neq v} \omega(\gamma, f_w) \Big\} \, \omega(\gamma, f_v).$$

(v)

$$c \Big[\lambda_0 \prod_v \theta(0, f_v) + \lambda_{-1} \Big\{ \sum_v \theta'(0, f_v) \prod_{w \neq v} \theta(0, f_w) \Big\} \Big].$$

(vi) If F is a number field

$$-\frac{1}{4} \, \text{trace} \, M(0) \, \rho(\Phi, 0),$$

but

$$-\frac{\log q}{4} \left\{ \text{trace} \, M(0) \, \rho(\Phi, 0) + \text{trace} \, M\Big(\frac{\pi}{\log q}\Big) \, \rho\Big(\Phi, \frac{\pi}{\log q}\Big) \right\}$$

if F is a function field.

(vii) If F is a number field

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \operatorname{trace} m^{-1}(s)\, m'(s)\, \rho(\Phi, s)\, d|s|,$$

but

$$\frac{\log q}{4\pi}\int_{0}^{\frac{2\pi}{\log q}} \operatorname{trace} m^{-1}(s)\, m'(s)\, \rho(\Phi, s)\, d|s|$$

if F is a function field.

(viii) The sum over (μ, ν) and v of

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \left\{ \operatorname{tr} R^{-1}(\mu_v, \nu_v, s)\, R'(\mu_v, \nu_v, s)\, \rho(f, \mu_v, \nu_v, s)\right\} \left\{ \prod_{w \neq v} \operatorname{tr} \rho(f_w, \mu_w, \nu_w, s)\right\} d|s|$$

if F is a number field and of

$$\frac{\log q}{4\pi}\int_{0}^{\frac{2\pi}{\log q}} \left\{ \operatorname{tr} R^{-1}(\mu_v, \nu_v, s)\, R'(\mu_v, \nu_v, s)\, \rho(f, \mu_v, \nu_v, s)\right\} \left\{ \prod_{w \neq v} \operatorname{tr} \rho(f_w, \mu_w, \nu_w, s)\right\} d|s|$$

if F is a function field.

The function Φ is of the form

$$\Phi(g) = \prod_v f_v(g_v).$$

Let Q be a set of representatives for the equivalence clases of quadratic extensions of F. For each E in Q fix an imbedding of E in the matrix algebra $M = M(2, F)$. Let $B_F = B_F(E)$ be the multiplicative group of E, considered as a subalgebra of M. B_F is the centralizer of E in G_F. Let $B_{\mathbb{A}} = B_{\mathbb{A}}(E)$ be the centralizer of E in $G_{\mathbb{A}}$. Q_1 is the collection of separable extensions in Q and Q_2 is the collection of inseparable extensions. A_F is the group of diagonal matrices in G_F.

Choose on $N_{\mathbb{A}}$ that Haar measure which makes the measure of $N_F \setminus N_{\mathbb{A}}$ equal to 1. Choose on K the normalized Haar measure. On the compact group H obtained by taking the quotient of

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A_{\mathbb{A}} \;\middle|\; |\alpha| = |\beta| \right\}$$

by $Z_{\mathbb{A}} A_F$ choose the normalized Haar measure. H is the kernel of the map

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to \log \left|\frac{\alpha}{\beta}\right|^{\frac{1}{2}}$$

of $A_F\, Z_{\mathbb{A}} \setminus A_{\mathbb{A}}$ onto $\mathbb{R}$ or $\log q\,\mathbb{Z}$. On $\mathbb{R}$ one has the standard measure dx and on $\log q\,\mathbb{Z}$ one has the standard measure which assigns the measure 1 to each point. The measures on H and on $H \setminus (A_F\, Z_{\mathbb{A}} \setminus A_{\mathbb{A}})$ together with the measure on $Z_{\mathbb{A}} \setminus A_F\, Z_A$ which assigns the measure 1 to each point serve to define a measure da on $Z_{\mathbb{A}} \setminus A_{\mathbb{A}}$. The constant c is defined by demanding that

$$\int_{Z_{\mathbb{A}} \setminus G_{\mathbb{A}}} f(g)\, \omega_0(g)$$

be equal to

$$c \int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}} \int_{N_{\mathbb{A}}} \int_K f(ank)\, da\, dn\, dk$$

if f is an integrable function on $Z_{\mathbb{A}} \backslash G_{\mathbb{A}}$. We may suppose that the measures on $Z_{\mathbb{A}} \backslash A_{\mathbb{A}}$, $N_{\mathbb{A}}$, and K are given as product measures and in particular that

$$\int_{K_v} dk_v = 1$$

and

$$\int_{N_v} \chi(n_v)\, dn_v = 1$$

for almost all v if χ is the charcteristic function of

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in O_{F_v} \right\}.$$

The factors $\omega(\gamma, f_v)$ and $\omega_1(\gamma, f_v)$ appearing in the fourth expression are defined by

$$\omega(\gamma, f_v) = \int_{N_v} \int_{K_v} f_v(k_v^{-1} n_v^{-1} \gamma n_v k_v)\, dn_v\, dk_v$$

and

$$\omega_1(\gamma, f_v) = \int_{N_v} \int_{K_v} f_v(k_v^{-1} n_v^{-1} \gamma n_v k_v)\, \log \lambda(n_v)\, dn_v\, dk_v.$$

If

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix} n'k'$$

then

$$\lambda(n) = \left| \frac{\alpha'}{\beta'} \right|.$$

Set $\theta(s, f_v)$ equal to

$$\frac{1}{L(1+s, 1_v)} \int_{Z_v \backslash A_v} \int_{K_v} f_v(k_v^{-1} a_v^{-1} n_0 a_v k_v) \left| \frac{\alpha_v}{\beta_v} \right|^{-1-s} da_v\, dk_v$$

where

$$a_v \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}$$

and

$$n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

1_v is the trivial character of $F_v^\times$. Then $\theta(s, f_v)$ is analytic at least for $\operatorname{Re} s > -1$. It derivative at 0 is $\theta'(0, f_v)$. If

$$L(1+s, 1_F) = \prod_v L(1+s, 1_v)$$

the Laurent expansion of $L(1+s,1_F)$ about $s=0$ is

$$\frac{\lambda_{-1}}{s}+\lambda_0+\cdots.$$

The operator $m(s)$ is the operator on $L(s)$ which for each (μ,ν) multiplies every element of $B(\mu\alpha_F^{\frac{s}{2}},\nu\alpha_F^{-\frac{s}{2}})$ by

$$\frac{L(1-s,\nu\mu^{-1})}{L(1+s,\mu\nu^{-1})}.$$

We may represent $B(\mu\alpha_{F_v}^{\frac{s}{2}},\nu\alpha_F^{-\frac{s}{2}})$ as

$$\otimes_v B(\mu_v\alpha_{F_v}^{\frac{s}{2}},\nu_v\alpha_{F_v}^{-\frac{s}{2}})$$

when s is purely imaginary. If $\mathrm{Re}\, s>0$ let $R(\mu_v,\nu_v,s)$ be the operator from $\mathcal{B}(\mu_v\alpha_{F_v}^{\frac{s}{2}},\nu_v\alpha_{F_v}^{-\frac{s}{2}})$ to $\mathcal{B}(\nu_v\alpha_{F_v}^{-\frac{s}{2}}\mu_v\alpha_{F_v}^{\frac{s}{2}})$ defined by setting

$$R(\mu_v,\nu_v,s)\,\varphi(g)$$

equal to

$$\varepsilon(1-s,\mu_v^{-1}\nu_v,\psi_v)\,\frac{L(1+s,\mu_v\nu_v^{-1})}{L(s,\mu_v\nu_v^{-1})}\int_{N_v}\varphi\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}ng\right)dn.$$

These operators can be defined for s purely imaginary by analytic continuation. They are then scalar multiples of unitary operators and for a given μ,ν are in fact unitary for almost all v. Thus $R(\mu_v,\nu_v,s)$ can be defined as an operator $B(\mu_v\alpha_{F_v}^{\frac{s}{2}},\nu_v\alpha_{F_v}^{-\frac{s}{2}})$ when s is purely imaginary and

$$M(s)=\sum_{(\mu,\nu)}\left\{\,\otimes_v\, R(\mu_v,\nu_v,s)\right\}\frac{L(1-s,\nu\mu^{-1})}{L(1+s,\mu\nu^{-1})}.$$

Set

$$N(s)=T_S M(s)\,T_S^{-1}$$

and if $N'(s)$ is the deriviative of $N(s)$ set

$$M'(s)=T_s^{-1}\,N'(s)\,T_s$$

Define $R'(\mu_v,\nu_v,s)$ in a similar fashion. Then

$$\mathrm{trace}\,M^{-1}(s)\,M'(s)\,\rho(\Phi,s)$$

is the sum of

$$\mathrm{trace}\,m^{-1}(s)\,m'(s)\,\rho(\Phi,s)$$

and

$$\sum_{(\mu,\nu)}\sum_{v}\left\{\mathrm{tr}\,R^{-1}(\mu_v,\nu_v,s)\,R'(\mu_v,\nu_v,s)\,\rho(f_v,\mu_v,\nu_v,s)\right\}\Big\{\prod_{w\neq v}\mathrm{tr}\,\rho(f_w,\mu_w,\nu_w,s)\Big\},$$

where $\rho(f_v,\mu_v,\nu_v,s)$ is the restriction of $\rho(f_v)$ to $B(\mu_v\alpha_{F_v}^{\frac{s}{2}},\nu_v\alpha_{F_v}^{-\frac{s}{2}})$.

If $E(\mu,\nu,s)$ is the projection of $L(s)$ on $B(\mu\alpha_F^{\frac{s}{2}},\nu\alpha_F^{-\frac{s}{2}})$ we can write

$$m(s)=\sum a(\mu,\nu,s)\,E(\mu,\nu,s)$$

where the $a(\mu,\nu,s)$ are scalars. Thus

$$\operatorname{trace} m^{-1}(s)\,m'(s)\,\rho(\Phi,s)$$

is equal to

$$\sum \frac{a'(\mu,\nu,s)}{a(\mu,\nu,s)}\Big\{\prod_v \operatorname{trace}\rho(f_v,\mu_v,\nu_v,s)\Big\}.$$

We can also write

$$M(0)=\sum a(\mu,\nu)\,E(\mu,\nu,0)$$

so that

$$\operatorname{trace} M(0)\,\rho(\Phi,0)$$

is equal to

$$\sum a(\mu,\nu)\Big\{\prod_v \operatorname{trace}\rho(f_v,\mu_v,\nu_v,0)\Big\}.$$

If F is a function field

$$M\Big(\frac{\pi}{\log q}\Big)=\sum B(\mu,\nu)\,E\big(\mu,\nu,\frac{\pi}{\log q}\big).$$

If

$$\int_{K_v}\int_{N_v} f_v(k^{-1}ank)\,dn\,dk=0 \tag{16.1.7}$$

for all a in $A_v=A_{F_v}$ then $\omega(\gamma,f_v)=0$ for all γ, $\theta(0,f_v)=0$, and

$$\operatorname{trace}\rho(f_v,\mu_v,\nu_v,s)=0$$

for all μ_v, ν_v, and s. In particular if (16.1.7) is satisifed for at least two v the expressions (iv) to (viii) vanish and the trace formula simplifies considerably.

We now apply this formula to the function

$$\Phi(g)=\Big\{\prod_{v\in S_1}\zeta_v(g_v)\Big\}\Big\{\prod_{v\in S_2}\xi_v(g_v)\Big\}\,f(\hat g_S)$$

where $f=f_1*f_2$ with f_1 and f_2 in B is of the form

$$f(\hat g_S)=\prod_{v\notin S} f_v(g_v).$$

Since S has at least two elements and the functions ζ_v and ξ_v satisfy (16.1.7) only the expressions (i) to (iii) do not vanish identically. The expression (i) is now equal to

$$\Big\{\prod_{v\in S_1}\zeta_v(e)\Big\}\Big\{\prod_{v\in S_2} d(\sigma_v)\Big\}\,f(e)$$

We recall that $d(\sigma_v) = d(\sigma'_v)$ if v is in S_2.

We may suppose that Q_2 is equal to Q'_2 and that Q'_1 is a subset of Q_1. If E is in Q_1 or Q_2 and γ is in $B_F = B_F(E)$ but not in Z_F

$$\int_{B_{\mathbb{A}}\backslash G_{\mathbb{A}}} \Phi(g^{-1}\gamma g)\,\omega_B(g)$$

is equal to the product of

$$\Big\{\prod_{v\in S_1}\int_{B_v\backslash G_v}\zeta_v(g_v^{-1}\gamma g_v)\,\omega_B(v)\Big\}\Big\{\prod_{v\in S_2}\int_{B_v\backslash G_v}\xi_v(g_v^{-1}\gamma g_v)\,\omega_B(v)\Big\}$$

and

$$\int_{\widehat{B}_S\backslash\widehat{G}_S} f(g^{-1}\gamma g)\,\omega_B.$$

If v is in S and $E\otimes_F F_v$ is not a field so that B_v is conjugate to A_v the corresponding factor in the first of these two expressions vanishes. Thus the sum in (ii) need only be taken over Q'_1. If E is in Q'_1 or Q_2 the first of these two expressions is equal to

$$\prod_{v\in S}\frac{\chi_{\sigma_v}(\gamma^{-1})}{\text{measure}\,Z_v\setminus B_v}.$$

Thus, in the special case under consideration, (ii) is equal to (16.1.5) and (iii) is equal to (16.1.6) so that

$$\text{trace}\,\tau(f) - \Big\{\prod_{v\in S_1}\zeta_v(e)\Big\}\Big\{\prod_{v\in S_2}d(\sigma)\Big\}\,\text{measure}(Z_{\mathbb{A}}\,G_F\setminus G_{\mathbb{A}})\,f(e)$$

is equal to

$$\text{trace}\,\tau'(f) - \Big\{\prod_{v\in S}d(\sigma'_v)\Big\}\,\text{measure}(Z'_{\mathbb{A}}\,G'_F\setminus G'_{\mathbb{A}})\,f(e).$$

We may take η to be trivial and apply Lemmas 16.1.1 and 16.1.2 to see that, in this case,

$$\text{trace}\,\tau(f) = \text{trace}\,\tau'(f)$$

and

$$\Big\{\prod_{v\in S}d(\sigma'_v)\Big\}\,\text{measure}(Z'_{\mathbb{A}}\,G'_F\setminus G'_{\mathbb{A}})$$

is equal to

$$\Big\{\prod_{v\in S_1}\zeta_v(e)\Big\}\Big\{\prod_{v\in S_2}d(\sigma_v)\Big\}\,\text{measure}(Z_{\mathbb{A}}\,G_F\setminus G_{\mathbb{A}}).$$

Still taking η trivial we choose the σ'_v so that none of them are one-dimensional and conclude that

$$\text{measure}(Z'_{\mathbb{A}}\,G'_F\setminus G'_{\mathbb{A}}) = \text{measure}(Z_{\mathbb{A}}\,G_F\setminus G_{\mathbb{A}}). \tag{16.1.8}$$

Then we take exactly one of them to one-dimensional and conclude that $\zeta_v(e) = d(\sigma'_v)$. Thus $\zeta_v(e) = d(\sigma'_v)$ and

$$\text{trace}\,\tau(f) = \text{trace}\,\tau'(f)$$

in general.

The relation (16.1.8) is well-known. One can hope however that the proof of it just given can eventually be used to show that the Tamagawa numbers of two groups which differ only by an inner twisting are the same or at least differ only by an explicitly given factor. Since the method of [33] can probably be used to evaluate the Tamagawa numbers of quasi-split groups the problem of evaluating the Tamagawa numbers of reductive groups would then be solved. However a great deal of work on the representation theory of groups over local fields remains to be done before this suggestion can be carried out.

To complete our formal argument we need to sketch a proof of the trace formula itself. One must use a bootstrap method. The first step, which is all we shall discuss, is to prove it for some simple class of functions Φ. We take Φ of the form $\Phi = f' * f''$ with

$$f'(g) = \prod_v f'_v(g_v)$$

and

$$f''(g) = \prod_v f''_v(g_v)$$

where f'_v and f''_v satisfy the five conditions on page 325. The function f_v is $f'_v * f''_v$.

Suppose φ is a K-finite compactly supported function in $A(\eta)$. For each purely imaginary s define $\tilde{\varphi}(s)$ in $\mathcal{L}$ by demanding that

$$\frac{1}{2c}\int_{G_F\, Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,\bar{E}(g,\varphi',s)\,\omega_0(g) = \big(\tilde{\varphi}(s),\varphi'\big)$$

be valid for all φ' in $\mathcal{L}$. The map $\varphi \to \tilde{\varphi}(s)$ extends to a continuous map of $A(\eta)$ onto H. $\tilde{\varphi}(s)$ is the function in H corresponding to $E\varphi$ in $A_1(\eta)$.

For each (μ,ν) in P choose an orthonormal basis $\{\varphi_i(\mu,\nu)\}$ of $\mathcal{B}(\mu,\nu)$. We may suppose that any elementary idemotent in $\mathcal{H}$ annihilates all but finitely many elements of this basis. If

$$\tilde{\varphi}(s) = \sum_{(\mu,\nu)}\sum_i a_i(\mu,\nu,s)\,\varphi_i(\mu,\nu)$$

then

$$a_i(\mu,\nu,s) = \frac{1}{2c}\int_{G_F\, Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,\bar{E}\big(g,\varphi_i(\mu,\nu),s\big)\,\omega_0(g).$$

Let

$$\rho(\Phi,s)\,T_S^{-1}\,\varphi_i(\mu,\nu) = \sum_j \rho_{ji}(\Phi,\mu,\nu,s)\,T_S^{-1}\,\varphi_j(\mu,\nu).$$

For all but finitely many μ, ν, i and j the functions $\rho_{ji}(\Phi,\mu,\nu,s)$ vanish identically. $E\rho(\Phi)\varphi$ is equal to

$$\lim_{T\to\infty}\sum_{\mu,\nu}\sum_{i,j}\int_{-iT}^{iT}\rho_{ij}(\Phi,\mu,\nu,s)\,a_j(\mu,\nu,s)\,E\big(g,\varphi_i(\mu,\nu),s\big)\,d|s|.$$

A typical one of these integrals is equal to the integral over $G_F\, Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$ of the product of $\varphi(g)$ and

$$\int_{-iT}^{iT}\rho_{ij}(\Phi,\mu,\nu,s)\,E\big(g,\varphi_i(\mu,\nu),s\big)\,\bar{E}\big(h,\varphi_j(\mu,\nu),s\big)\,d|s|.$$

Thus the kernel of $E\,\rho(\Phi)$ is the sum over (μ,ν) and i,j of

$$\frac{1}{4\pi c}\int_{-i\infty}^{i\infty}\rho_{ij}(\Phi,\mu,\nu,s)\,E\big(g,\varphi_i(\mu,\nu),s\big)\,\bar{E}\big(h,\varphi_j(\mu,\nu),s\big)\,d|s|.$$

The kernel of $\rho(\Phi)$ is

$$\Phi(g,h)=\sum_{Z_F\backslash G_F}\Phi(g^{-1}\gamma h).$$

To compute the trace of $\rho(\Phi)-E\,\rho(\Phi)$ we integrate the difference of these two kernels over the diagonal.

The function $\Phi(g,g)$ may be written as the sum of

$$\sum_{\delta\in P_F\backslash G_F}\ \sum_{\substack{\gamma\in N_F\\ \gamma\neq e}}\Phi(g^{-1}\delta^{-1}\gamma\,\delta g), \tag{16.2.1}$$

where P_F is the group of super-triangular matrices in G_F,

$$\frac{1}{2}\sum_{\substack{\gamma\in Z_F\backslash A_F\\ \gamma\notin Z_F}}\ \sum_{\delta\in A_F\backslash G_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g), \tag{16.2.2}$$

where A_F is the group of diagonal matrices in G_F,

$$\frac{1}{2}\sum_{Q_1}\ \sum_{\substack{\gamma\subset Z_F\backslash B_F\\ \gamma\notin Z_F}}\ \sum_{\delta\in B_F\backslash G_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g) \tag{16.2.3}$$

and

$$\sum_{Q_2}\ \sum_{\substack{\gamma\in Z_F\backslash B_F\\ \gamma\notin Z_F}}\ \sum_{\delta\in B_F\backslash G_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g) \tag{16.2.4}$$

together with

$$\Phi(e). \tag{16.2.5}$$

The constant $\Phi(e)$ can be integrated over $G_F\,Z_{\mathbb{A}}\setminus G_A$ immediately to give the first term of the trace formula. The standard manipulations convert (16.2.3) and (16.2.4) into the second and third terms of the trace formula.

The expressions (16.2.1) and (16.2.2) have to be treated in a more subtle fashion. We can choose a constant $e_1>0$ so that if

$$g=\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}k$$

with x in $\mathbb{A}$, α and β in I such that $|\frac{\alpha}{\beta}|\geq c_1$, and k in K and if

$$\gamma g=\begin{pmatrix}1&x'\\0&1\end{pmatrix}\begin{pmatrix}\alpha'&0\\0&\beta'\end{pmatrix}k'$$

with γ in G_F, x' in $\mathbb{A}$, α' and β' in I such that $|\frac{\alpha'}{\beta'}|\geq c_1$, and k' in K then γ belongs to P_F. Let χ be the characteristic function of

$$\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}k\ \Big|\ \Big|\frac{\alpha}{\beta}\Big|\geq c_1\right\}.$$

The expression (16.2.2) is the sum of

$$\frac{1}{2}\sum_{\delta\in P_F\backslash G_F}\ \sum_{\substack{\gamma\in Z_F\backslash P_F\\ \gamma\notin Z_F N_F}}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\big(\chi(\delta g)+\chi(\varepsilon(\gamma)\delta g)\big)$$

and

$$\frac{1}{2}\sum_{\delta\in P_F\backslash G_F}\ \sum_{\substack{\gamma\in Z_F\backslash P_F\\ \gamma\notin Z_F N_F}}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\big(1-\chi(\delta g)-\chi(\varepsilon(\gamma)\delta g)\big).$$

Here $\varepsilon(\gamma)$ is any element of G_F not in P_F such that

$$\varepsilon(\gamma)\,\gamma\,\varepsilon^{-1}(\gamma)\in P_F.$$

There is always at least one such $\varepsilon(\gamma)$. The integral of the second sum over $G_F\,Z_{\mathbb{A}}\setminus G_{\mathbb{A}}$ converges. It is equal to

$$\frac{1}{2}\int_{Z_{\mathbb{A}}P_F\backslash G_{\mathbb{A}}}\ \sum_{\substack{\gamma\subset Z_F\backslash P_F\\ \gamma\notin Z_F N_F}}\Phi(g^{-1}\gamma g)\big(1-\chi(g)-\chi(\varepsilon(\gamma)g)\big)\,\omega_0(g).$$

Every γ occurring in the sum can be written as $\delta^{-1}\gamma_0\delta$ with γ_0 in A_F and δ in P_F. Then

$$\big(\delta^{-1}\varepsilon(\gamma_0)\delta\big)\big(\delta^{-1}\gamma_0\delta\big)\big(\delta^{-1}\varepsilon(\gamma_0)\delta\big)^{-1}=\delta^{-1}\big(\varepsilon(\gamma_0)\gamma_0\varepsilon^{-1}(\gamma_0)\big)\delta$$

so that we can take $\varepsilon(\gamma)=\delta^{-1}\varepsilon(\gamma_0)\delta$. We take

$$\varepsilon(\gamma_0)=w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Since $\chi(\delta g)=\chi(g)$ and

$$\chi(\delta^{-1}w\delta g)=\chi(w\delta g)$$

the integrand is

$$\sum_{\substack{\gamma\in Z_F\backslash A_F\\ \gamma\notin Z_F}}\ \sum_{\delta\in A_F\backslash P_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\big(1-\chi(\delta g)-\chi(w\delta g)\big).$$

The integral itself is equal to

$$\frac{1}{2}\sum_{\substack{\gamma\in Z_F\backslash A_F\\ \gamma\notin Z_F}}\int_{Z_{\mathbb{A}}A_F\backslash G_{\mathbb{A}}}\Phi(g^{-1}\gamma g)\big(1-\chi(g)-\chi(wg)\big)\,\omega_0(g).$$

All but a finite number of the integrals in this sum are 0.

It is convenient to write each of them in another form. If

$$g=\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}nk$$

then $\chi(g)$ is 1 if $|\frac{\alpha}{\beta}| \geq c_1$ and is 0 if $|\frac{\alpha}{\beta}| < c_1$. If

$$wn = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix} n'k'$$

and $\lambda(n)$ is $|\frac{\alpha'}{\beta'}|$ then $\chi(wg)$ is 1 if $|\frac{\alpha}{\beta}| \leq \frac{\lambda(n)}{c_1}$ and is 0 if $|\frac{\alpha}{\beta}| > \frac{\lambda(n)}{c_1}$. It is easily seen that $\lambda(n) \leq 1$. Thus if $c_1 > 1$, as we may suppose, one of $\chi(g)$ and $\chi(wg)$ is always 0. The integral

$$\int_{Z_{\mathbb{A}} A_F \backslash G_{\mathbb{A}}} \Phi(g^{-1}\gamma g)\big(1 - \chi(g) - \chi(wg)\big)\,\omega_0(g)$$

is equal to

$$c \int_{N_{\mathbb{A}}} \int_K \Phi(k^{-1}n^{-1}\gamma nk)\big(2\log c_1 - \log\lambda(n)\big)\,dn\,dk$$

which we write as the sum of

$$2c\log c_1 \int_{N_{\mathbb{A}}} \int_K \Phi(k^{-1}n^{-1}\gamma nk)\,dn\,dk \tag{16.2.6}$$

and

$$-\sum_v c \int_{N_{\mathbb{A}}} \int_K \Phi(k^{-1}n^{-1}\gamma nk)\ \log\lambda(n_v)\,dn\,dk.$$

if we express each of the integrals in the second expression as a product of local integrals we obtain the fourth term of the trace formula. All but a finite number of the integrals are 0 so that the sum is really finite. We will return to (16.2.6) later. If F is a function field over $\mathbb{F}_q$ it is best to take c_1 to be a power of q^n of q. Then $2\log c_1$ is replaced by $2n-1$.

The expression (16.2.1) is the sum of

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi(\delta g)$$

and

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1}\delta^{-1}\gamma\delta g)\big(1 - \chi(\delta g)\big).$$

The integral of the second expression over $G_F\,Z_{\mathbb{A}} \setminus G_{\mathbb{A}}$ converges. It is equal to

$$\int_{P_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1}\gamma g)\big(1 - \chi(g)\big)\,\omega_0(g).$$

If

$$n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

the integrand is equal to

$$\sum_{N_F Z_F \backslash P_F} \Phi(g^{-1}\delta^{-1}n_0\delta g)\big(1 - \chi(\delta g)\big)$$

so that the integral itself is equal to

$$\int_{N_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Phi(g^{-1} n_0 g)\big(1-\chi(g)\big)\,\omega_0(g)$$

which is

$$c\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}a^{-1}n_0 ak)\big(1-\chi(a)\big)\left|\frac{\alpha}{\beta}\right|^{-1} da\,dk$$

if

$$a=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}.$$

The integrand vanishes outside of a compact set. Thus the integral is the limit as s approaches 0 from above of

$$c\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}a^{-1}n_0 ak)\big(1-\chi(a)\big)\left|\frac{\alpha}{\beta}\right|^{-1-s} da\,dk,$$

which is the difference of

$$c\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}a^{-1}n_0 ak)\left|\frac{\alpha}{\beta}\right|^{-1-s} da\,dk$$

and

$$c\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}a^{-1}n_0 ak)\left|\frac{\alpha}{\beta}\right|^{-1-s}\chi(a)\,da\,dk.$$

The first of these two expressions is equal to

$$c\Big\{\prod_v\int_{Z_v\backslash A_v}\int_{K_v} f_v(k_v^{-1}a_v^{-1}n_0 a_v k_v)\left|\frac{\alpha_v}{\beta_v}\right|^{-1-s} da_v\,dk_v\Big\}$$

which is

$$c\,L(1+s,1_F)\Big\{\prod_v\theta(s,f_v)\Big\}. \tag{16.2.7}$$

Observe that if v is non-archimedean and f_v is 0 outside of $Z_v\,K_v$ and is 1 on the elements of $Z_v\,K_v$ of determinant 1 then

$$\int_{Z_v\backslash A_v}\int_{K_v} f_v(k_v^{-1}a_v^{-1}n_0 a_v k_v)\left|\frac{\alpha_v}{\beta_v}\right|^{-1-s} da_v\,dk_v$$

is the product of the measure of

$$\left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\in Z_v\setminus A_v \;\Big|\; |\alpha|=|\beta|\right\}$$

and

$$\sum_{n=0}^{\infty}|\varpi_v^n|^{1+s}=L(1+s,1_v),$$

so that

$$\prod_v\theta(s,f_v)=\theta(s,\Phi)$$

is analytic for $\operatorname{Re} s > -1$ and its derivative at 0 is

$$\sum_v \theta'(s, f_v)\Big\{\prod_{w\neq v} \theta(s, f_w)\Big\}.$$

The function (16.2.7) has a simple pole at $s = 0$. The constant term in its Laurent expansion is

$$c\Big[\lambda_0 \prod_v \theta(0, f_v) + \lambda_{-1}\Big\{\sum_v \theta'(0, f_v)\prod_{w\neq v} \theta(0, f_w)\Big\}\Big]$$

which is the fifth term of the trace formula.

The expression

$$c\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}a^{-1}n_0ak)\Big|\frac{\alpha}{\beta}\Big|^{-1-s}\chi(a)\,da\,dk$$

is equal to

$$c\int_{Z_{\mathbb{A}}A_F\backslash A_{\mathbb{A}}}\int_K \sum_{\substack{\gamma\in N_F\\ \gamma\neq e}} \Phi(k^{-1}a^{-1}\gamma ak)\Big|\frac{\alpha}{\beta}\Big|^{-1-s}\chi(a)\,da\,dk.$$

Choose a non-trivial character ψ of $F\setminus\mathbb{A}$ and let

$$\Psi(y, g) = \int_{\mathbb{A}} \Phi\left(g^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(xy)\,dx.$$

Then

$$\Psi(y, ag) = \Big|\frac{\beta}{\alpha}\Big|^{-1}\Psi\Big(\frac{\alpha}{\beta}y, g\Big).$$

Moreover by the Poisson summation formula

$$\sum_{\substack{\gamma\in N_F\\ \gamma\neq e}} \Phi(k^{-1}a^{-1}\gamma ak)$$

is equal to

$$\sum_{y\neq 0}\Big|\frac{\beta}{\alpha}\Big|^{-1}\Psi\Big(\frac{\alpha}{\beta}y, k\Big) + \Big|\frac{\beta}{\alpha}\Big|^{-1}\Psi(0, k) - \Phi(e).$$

The integral

$$c\int_{Z_{\mathbb{A}}A_F\backslash A_{\mathbb{A}}}\int_K \Big|\frac{\alpha}{\beta}\Big|^{-s}\chi(a)\Big\{\sum_{y\neq 0}\Psi\Big(\frac{\alpha}{\beta}y, k\Big)\Big\}\,da\,dk$$

is a holomorphic function of s and its value at $s = 0$ approaches 0 as c_1 approaches ∞. Since we shall eventually let c_1 approach ∞ it contributes nothing to the trace formula. If F is a number field

$$c\int_{Z_{\mathbb{A}}A_F\backslash A_{\mathbb{A}}}\int_K \Phi(e)\Big|\frac{\alpha}{\beta}\Big|^{-1-s}\chi(a)\,da\,dk$$

is a multiple of

$$\frac{1}{1+s}\cdot\frac{1}{c_1^{1+s}}$$

which is defined at $s=0$. Its value there approaches 0 as c_1 approaches ∞. Finally

$$c\int_{Z_{\mathbb{A}}A_F\backslash A_{\mathbb{A}}}\int_K \Psi(0,k)\left|\frac{\alpha}{\beta}\right|^{-s}\chi(a)\,da\,dk$$

is equal to

$$\frac{c}{s\,c_1^s}\int_K \Psi(0,k)\,dk.$$

The pole of this function at $s=0$ must cancel that of (16.2.7). Consequently

$$\int_K \Psi(0,k)\,dk=\lambda_{-1}\,\theta(0,\Phi).$$

The constant term in its Laurent expansion about 0 is

$$-c\log c_1\int_K \Psi(0,k)\,dk.$$

Not this expression but its negative

$$c\log c_1\int_K \Psi(0,k)\,dk \tag{16.2.8}$$

enters into the integral of the kernel of $\rho(\Phi)-E\,\rho(\Phi)$ over the diagonal. If F is a function field $\frac{c}{s\,c_1^s}$ is to be replaced by

$$\frac{cq^{-ns}}{1-q^{-s}}$$

and $\log c_1$ by $n-\frac{1}{2}$.

The Poisson summation formula can be used to simplify the remaining part of (16.2.1). We recall that it is

$$\sum_{\delta\in P_F\backslash G_F}\ \sum_{\substack{\gamma\in N_F\\ \gamma\neq e}}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi(\delta g).$$

We subtract from this

$$\sum_{\delta\in P_F\backslash G_F}\Psi(0,\delta g)\,\chi(\delta g)$$

to obtain the difference betwen

$$\sum_{\delta\in P_F\backslash G_F}\sum_{y\neq 0}\Psi(y,\delta g)\,\chi(\delta g)$$

and

$$\sum_{\delta\in P_F\backslash G_F}\Phi(e)\,\chi(\delta g).$$

The integrals of both these functions over $Z_{\mathbb{A}}\,G_F\setminus G_{\mathbb{A}}$ converge and approach 0 as c_1 approaches ∞. They may be ignored.

The remaining part of (16.2.2) is the sum of

$$\frac{1}{2}\sum_{\delta\in P_F\backslash G_F}\ \sum_{\substack{\gamma\in Z_F\backslash P_F\\ \gamma\notin Z_F N_F}}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi(\delta g)$$

and
$$\frac{1}{2}\sum_{\delta\in P_F\backslash G_F}\sum_{\substack{\gamma\in Z_F\backslash P_F\\ \gamma\notin Z_F N_F}}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi\big(\varepsilon(\gamma)\delta g\big).$$

These two sums may be written as
$$\frac{1}{2}\sum_{\substack{\gamma\in Z_F\backslash A_F\\ \gamma\notin Z_F}}\sum_{A_F\backslash G_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi(\delta g)$$

and
$$\frac{1}{2}\sum_{\substack{\gamma\in Z_F\backslash A_F\\ \gamma\notin Z_F}}\sum_{A_F\backslash G_F}\Phi(g^{-1}\delta^{-1}\gamma\delta g)\,\chi(w\delta g).$$

Replacing δ by $w^{-1}\delta$ in the second sum we see that the two expressions are equal. Their sum is equal to twice the first which we write as
$$\sum_{\substack{\gamma_1\in Z_F\backslash A_F\\ \gamma_1\notin Z_F}}\sum_{\delta\in P_F\backslash G_F}\sum_{\gamma_2\in N_F}\Phi(g^{-1}\delta^{-1}\gamma_1\gamma_2\delta g)\,\chi(\delta g).$$

For a given Φ all but finitely many of the sums
$$\sum_{\delta\in P_F\backslash G_F}\sum_{\gamma_2\in N_F}\Phi(g^{-1}\delta^{-1}\gamma_1\gamma_2\delta g)\,\chi(\delta g) \tag{16.2.9}$$

are zero. Set
$$\Psi(y,\gamma_1,g)=\int_{\mathbb{A}}\Phi\left(g^{-1}\gamma_1\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\,\psi(xy)\,dx.$$
The expression (16.2.9) is the sum of
$$\sum_{\delta\in P_F\backslash G_F}\sum_{y\neq 0}\Psi(y,\gamma_1,\delta g)\,\chi(\delta g)$$

and
$$\sum_{\delta\in P_F\backslash G_F}\Psi(0,\gamma_1,\delta g)\,\chi(\delta g).$$

The first of these two expressions is integrable on $G_F\,Z_{\mathbb{A}}\setminus G_{\mathbb{A}}$ and its integral approaches 0 as c_1 approaches ∞.

Since $\Psi(0,g)=\Psi(0,e,g)$ we have expressed $\Phi(g,g)$ as the sum of
$$\sum_{\delta\in P_F\backslash G_F}\sum_{\gamma\in Z_F\backslash A_F}\Psi(0,\gamma,\delta g)\,\chi(\delta g) \tag{16.2.10}$$

and a function which can be integrated over $G_F\,Z_{\mathbb{A}}\setminus G_{\mathbb{A}}$ to give the first five terms of the trace formula, the sum of (16.2.8) and one-half of the sum over γ in $Z_F\setminus A_F$ but not in Z_F of (16.2.6) which is
$$c\log c_1\sum_{\gamma\in Z_F\backslash A_F}\int_{N_{\mathbb{A}}}\int_K\Phi(k^{-1}\gamma nk)\,dn\,dk, \tag{16.2.11}$$

and an expression which goes to 0 as c_1 approaches ∞.

Now we discuss the kernel of $E\,\rho(\Phi)$ in the same way. Set $H(g;\mu,\nu,i,j,s)$ equal to

$$\rho_{ij}(\Phi,\mu,\nu,s)\,E\big(g,\varphi_i(\mu,\nu),s\big)\,\bar{E}\big(g,\varphi_j(\mu,\nu),s\big).$$

On the diagonal the kernel of $E\,\rho(\Phi)$ is equal to

$$\sum_{\mu,\nu}\sum_{i,j}\frac{1}{4\pi c}\int_{-i\infty}^{i\infty}H(g;\mu,\nu,i,j,s)\,d|s|$$

if F is a number field and to

$$\sum_{\mu,\nu}\sum_{i,j}\frac{\log q}{4\pi c}\int_{0}^{\frac{2\pi}{\log q}}H(g;\mu,\nu,i,j,s)\,d|s|$$

if F is a function field. We set $E_1(g,\varphi,s)$ equal to

$$\sum_{P_F\backslash G_F}\big\{T_s^{-1}\,\varphi(\delta g)+M(s)\,T_S^{-1}\,\varphi(\delta g)\big\}\,\chi(\delta g)$$

and let

$$E_2(g,\varphi,s)=E(g,\varphi,s)-E_1(g,\varphi,s).$$

If, for $m=1,2$, $n=1,2$, $H_{mn}(g;\mu,\nu,i,j,s)$ is

$$\rho_{ij}(\Phi,\mu,\nu,s)\,E_m\big(g,\varphi_i(\mu,\nu),s\big)\,\bar{E}_n\big(g,\varphi_j(\mu,\nu),s\big)$$

and $\Phi_{mn}(g)$ is, at least when F is a number field,

$$\sum_{\mu,\nu}\sum_{i,j}\frac{1}{4\pi c}\int_{-i\infty}^{i\infty}H_{m,n}(g;\mu,\nu,i,j,s)\,d|s|,$$

the kernel of $E\,\rho(\Phi)$ is

$$\sum_{m=1}^{n}\sum_{n=1}^{2}\Phi_{mn}(g)$$

on the diagonal.

If m or n is 2

$$\int_{G_F\,Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\Phi_{mn}(g)\,\omega_0(g)$$

is equal to

$$\frac{1}{4\pi c}\int_{-i\infty}^{i\infty}\sum_{\mu,\nu}\sum_{i,j}\Big\{\int_{G_F\,Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}H_{m,n}(g,\mu,\nu,i,j,s)\,\omega_0(g)\Big\}\,d|s|. \tag{16.2.12}$$

Take first $m=n=2$. If F is a number field a formula for the inner product

$$\int_{G_F\,Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}E_2(g,\varphi_1,s)\,\bar{E}_2(g,\varphi_2,s)\,\omega_0(g)$$

can be inferred from the formulae of [26] and [27]. The result is the sum of

$$c \lim_{t\to 0} \frac{1}{2t}\big\{c_1^{2t}(\varphi_1,\varphi_2) - c_1^{-2t}\big(N(t+s)\varphi_1, N(t+s)\varphi_2\big)\big\},$$

where $N(t+s) = T_{t+s}\, M(t+s)\, T_{t+s}^{-1}$, and

$$c \lim_{t\to 0} \frac{1}{2s}\big\{c_1^{2s}\big(\varphi_1, N(t+s)\varphi_2\big) - c_1^{-2s}\big(N(t+s)\varphi_1,\varphi_2\big)\big\}.$$

The second expression is equal to

$$\frac{c}{2s}\big\{c_1^{2s}\big(\varphi_2, N(s)\varphi_2\big) - c_1^{-2s}\big(N(s)\varphi_1,\varphi_2\big)\big\}.$$

The first is the sum of

$$2c\log c_1(\varphi_1,\varphi_2)$$

and

$$-\frac{c}{2}\big\{\big(N^{-1}(s)\,N'(s)\varphi_1,\varphi_2\big) + \big(\varphi_1, N^{-1}(s)\,N'(s)\,\varphi_2\big)\big\}.$$

If F is a function field over $\mathbb{F}_q$ and $c_1 = q^n$ the inner product is the sum of

$$c\log q\left\{\frac{1-q^s+q^{-s}}{1-q^{-2s}}\,\big(\varphi_1, N(s)\varphi_2\big)\,q^{2(n-1)s} + \frac{1-q^{-s}+q^s}{1-q^{2s}}\,\big(N(s)\varphi_1,\varphi_2\big)\,q^{-2(n-1)s}\right\}$$

and

$$(2n-1)c(\varphi_1,\varphi_2)$$

and

$$-\frac{c}{2}\big\{\big(N^{-1}(s)\,N'(s)\,\varphi_1,\varphi_2\big) + \big(\varphi_1, N^{-1}(s)\,N'(s)\,\varphi_2\big)\big\}.$$

Certainly

$$\sum_{\mu,\nu}\sum_{i,j}\rho_{ij}(\Phi,\mu,\nu,s)\,\big(\varphi_i(\mu,\nu),\varphi_j(\mu,\nu)\big) = \operatorname{trace}\rho(\Phi,s)$$

which equals

$$\sum_{\mu,\nu} c\int_{N_{\mathbb{A}}}\int_{Z_{\mathbb{A}}\backslash A_{\mathbb{A}}}\int_K \Phi(k^{-1}ank)\,\mu(\alpha)\,\nu(\beta)\Big|\frac{\alpha}{\beta}\Big|^{\frac{s+1}{2}}\,dn\,da\,dk$$

or

$$\sum_{\mu,\nu} c\int_{N_{\mathbb{A}}}\int_{Z_{\mathbb{A}}A_F\backslash A_{\mathbb{A}}}\int_K \sum_{\gamma\in Z_F\backslash A_F}\Phi(k^{-1}a\gamma nk)\,\mu(\alpha)\,\nu(\beta)\Big|\frac{\alpha}{\beta}\Big|^{\frac{s+1}{2}}.$$

Thus if H is the set of all

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

in $Z_{\mathbb{A}}A_F \setminus A_{\mathbb{A}}$ for which $|\alpha| = |\beta|$

$$\frac{1}{4\pi c}\int_{-i\infty}^{i\infty}\operatorname{trace}\rho(\Phi,s)\,d|s| \tag{16.2.13}$$

is equal to

$$\frac{1}{2}\sum_{\mu,\nu}\int_H\int_{N_{\mathbb{A}}}\int_K\sum_\gamma \Phi(k^{-1}a\gamma nk)\,\mu(\alpha)\,\nu(\beta)\,dn\,dk\,da$$

which is

$$\frac{1}{2}\int_{N_{\mathbb{A}}}\int_K \Phi(k^{-1}\gamma nk)\,dn\,dk.$$

When multiplied by $2c\log c_1$ the effect of this is to cancel the term (16.2.11). If F is a function field (16.2.13) is said to be replaced by

$$\frac{\log q}{4\pi c}\int_0^{\frac{2\pi}{\log q}}\operatorname{trace}\rho(\Phi,s)\,d|s|$$

but the conclusion is the same.

The expression

$$\sum_{\mu,\nu}\sum_{i,j}\rho_{ij}(\Phi,\mu,\nu,s)\,\big(\varphi_i(\mu,\nu),N(s)\varphi_j(\mu,\nu)\big)$$

is equal to

$$\operatorname{trace}M^{-1}(s)\,\rho(\Phi,s)$$

when s is purely imaginary and

$$\sum_{\mu,\nu}\sum_{i,j}\rho_{ij}(\Phi,\mu,\nu,s)\,\big(N(s)\varphi_i(\mu,\nu),\varphi_j(\mu,\nu)\big)$$

is equal to

$$\operatorname{trace}M(s)\,\rho(\Phi,s).$$

Since $M(0)=M^{-1}(0)$

$$\lim_{c_1\to\infty}\frac{1}{8\pi}\int_{-i\infty}^{i\infty}\frac{1}{2}\big\{c_1^{2s}\operatorname{trace}M^{-1}(s)\,\rho(\Phi,s)-c_1^{-2s}\operatorname{trace}M(s)\,\rho(\Phi,s)\big\}\,d|s|$$

is equal to

$$\frac{1}{4}\operatorname{trace}M(0)\,\rho(\Phi,0).$$

When multiplied by -1 this is the sixth term of the trace formula. For a function field it is to be replaced by

$$\frac{\log q}{4}\Big\{\operatorname{trace}M(0)\,\rho(\Phi,0)+\operatorname{trace}M\Big(\frac{\pi}{\log q}\Big)\,\rho\Big(\Phi,\frac{\pi}{\log q}\Big)\Big\}.$$

When s is purely imaginary

$$\big(N^{-1}(s)\,N'(s)\,\varphi_1,\varphi_2\big)=\big(\varphi_1,N^{-1}(s)\,N'(s)\,\varphi_2\big).$$

Moreover

$$\sum_{\mu,\nu}\sum_{i,j}\rho_{ij}(\Phi,\mu,\nu,s)\big(N^{-1}(s)\,N'(s)\,\varphi_i(\mu,\nu),\varphi_j(\mu,\nu)\big)$$

is equal to

$$\operatorname{trace} M^{-1}(s)\, M'(s)\, \rho(\Phi, s).$$

Thus

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty} \operatorname{trace} M^{-1}(s)\, M'(s)\, \rho(\Phi, s)\, d|s|$$

is to be added to the trace formula. It gives the seventh and eighth terms.

Next we consider (16.2.12) when $m = 2$ and $n = 1$. If $\varphi_2' = T_S^{-1}\varphi_2$ and $\varphi_2'' = M(s)\, T_s^{-1}\, \varphi_2$ the integral

$$\int_{G_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} E_2(g, \varphi_1, s)\, \bar{E}_1(g, \varphi_2, s)\, \omega_0(g) \tag{16.2.14}$$

is the sum of

$$\int_{P_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} E_2(g, \varphi_1, s)\, \bar{\varphi}_2'(g)\, \chi(g)\, \omega_0(g)$$

and

$$\int_{P_F Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} E_2(g, \varphi_1, s)\, \bar{\varphi}_2''(g)\, \chi(g)\, \omega_0(g).$$

Since φ_2', φ_2'' and χ are all functions on $Z_{\mathbb{A}} N_{\mathbb{A}} P_F \setminus G_{\mathbb{A}}$ while, as is known,

$$\chi(g) \int_{N_{\mathbb{A}}} E_2(ng, \varphi_1, s)\, dn = 0$$

when c_1 is sufficiently large, the integral (16.2.14) is 0. Thus (16.2.12) is 0 when $m = 2$ and $n = 1$ and also when $m = 1$ and $n = 2$.

Set

$$F(g, \varphi, s) = T_s^{-1}\, \varphi(g) + M(s)\, T_s^{-1}\, \varphi(g)$$

and set $H_0(g, \mu, \nu, i, j, s)$ equal to

$$\rho_{ij}(\Phi, \mu, \nu, s)\, F\big(g, \varphi_i(\mu, \nu), s\big)\, \bar{F}\big(g, \varphi_j(\mu, \nu), s\big)\, \chi(g).$$

If c_1 is so large that $\chi(\delta_1 g)\, \chi(\delta_2 g) = 0$ when δ_1 and δ_2 do not belong to the same coset of P_F the function $\Phi_{1,1}(g)$ is equal to

$$\sum_{\mu,\nu} \sum_{i,j} \sum_{P_F \backslash G_F} \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H_0(\delta g, \mu, \nu, i, j, s)\, d|s|.$$

If $\varphi_i'(g, \mu, \nu)$ is the value of $T_S^{-1}\, \varphi_i(\mu, \nu)$ at g then

$$\sum_{i,j} \rho_{ij}(\Phi, \mu, \nu, s)\, \varphi_i'(h, \mu, \nu)\, \bar{\varphi}_j'(g, \mu, \nu)$$

is the kernel of $\rho(\Phi, \mu, \nu, s)$ which is

$$c \int_{N_{\mathbb{A}}} \int_{Z_{\mathbb{A}} \backslash A_{\mathbb{A}}} \Phi(g^{-1} anh) \Big|\frac{\alpha}{\beta}\Big|^{\frac{s+1}{2}}\, \mu(\alpha)\, \nu(\beta)\, dn\, da.$$

If we set $h = g$, divide by $4\pi c$, integrate from $-i\infty$ to $i\infty$, and then sum over μ and ν we obtain

$$\frac{1}{2} \sum_{\gamma \in Z_F \backslash A_F} \Psi(0, \gamma, g).$$

If $\varphi_i''(g, \mu, \nu)$ is the value of $M(s)\, T_S^{-1}\, \varphi_i(\mu, \nu)$ at g

$$\sum_{i,j} \rho_{ij}(\Phi, \mu, \nu, s)\, \varphi_i''(h, \mu, \nu)\, \bar{\varphi}_j''(g, \mu, \nu)$$

is the kernel of

$$M(\mu, \nu, s)\, \rho(\Phi, \mu, \nu, s)\, M(\nu, \mu, -s) = \rho(\Phi, \nu, \mu, -s).$$

Thus $\Phi_{1,1}(g)$ is the sum of

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\gamma \in Z_F \backslash A_F} \Psi(0, \gamma, \delta g)\, \chi(\delta g) \tag{16.2.15}$$

and

$$\sum_{\mu,\nu} \sum_{i,j} \sum_{P_F \backslash G_F} \frac{\chi(\delta g)}{4\pi c} \int_{-i\infty}^{i\infty} \big\{ H_1(\delta g, \mu, \nu, i, j, s) + H_2(\delta g, \mu, \nu, i, j, s) \big\}\, d|s|$$

where $H_1(g, \mu, \nu, i, j, s)$ is

$$\rho_{ij}(\Phi, \mu, \nu, s)\, \varphi_i'(g, \mu, \nu)\, \bar{\varphi}_j''(g, \mu, \nu)$$

and $H_2(g, \mu, \nu, i, j, s)$ is

$$\rho_{ij}(\Phi, \mu, \nu, s)\, \varphi_i''(g, \mu, \nu)\, \bar{\varphi}_j'(g, \mu, \nu).$$

The expression (16.2.15) cancels (16.2.10). If $g = nak$ with

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$H_1(g, \mu, \nu, i, j, s)$ is equal to

$$\rho_{ij}(\Phi, \mu, \nu, s)\, \mu\Big(\frac{\alpha}{\beta}\Big) \left|\frac{\alpha}{\beta}\right|^{s+1} \varphi_i'(k)\, \bar{\varphi}_j''(k).$$

The functions $\rho_{ij}(\Phi, \mu, \nu, s)$ are infinitely differentiable on the imaginary axis. Thus

$$\frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H_1(g, \mu, \nu, i, j, s)\, d|s|$$

is, $O\big(|\frac{\alpha}{\beta}|^M\big)$ as $|\frac{\alpha}{\beta}| \to \infty$ for any real M. Thus if this expression is multiplied by $\chi(g)$ and averaged over $P_F \setminus G_F$ the result is integrable on $Z_{\mathbb{A}}\, G_F \setminus G_{\mathbb{A}}$ and its integral approaches 0 as c_1 approaches ∞. Thus it contributes nothing to the trace. Nor do the analogous integrals for $H_2(g, \mu, \nu, i, j, s)$.

References

for Chapter III

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They are also referred to, but rather obliquely, in:

25. Selberg, A., *Discontinuous groups and harmonic analysis*, Proc. Int. Cong. Math. (1962)

The theory of Eisenstein series is discussed in [11], [13], [14], and [25] as well as in:

26. Langlands, R.P., *On the functional equations satisifed by Eisenstein series*, Mimeographed notes

27. Langlands, R.P., *Eisenstein series*, in *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc. (1966)

28. Selberg, A., *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, Jour. Ind. Math. Soc., vol XX (1956)

[25] and [28] are of course the basic references for the Selberg trace formula. Some of its formal aspects are also described in:

29. Langlands, R.P., *Dimension of spaces of automorphic forms*, in *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc. (1966)

The theorem of §16 can still be stated and proved if M is replaced by a quaternion algebra which splits everywhere that M' does. The proof is in fact rather easier. However these apparently more general theorems are immediate consequences of the proof of the original theorem. Theorems very similar to that of §16 and its extensions have been proved by Shimizu. Our methods differ little from his.

30. Shimizu, H., *On discontinuous groups operating on the product of the upper half planes*, Ann. of Math., vol 77(1963)

31. Shimizu, H., *On traces of Hecke operators*, Jour. Fac. Sci. Univ. Tokyo, vol. 10 (1963)

32. Shimizu, H., *On zeta functions of quaternion algebras*, Ann. of Math., vol 81(1965)

We have also had occasion to refer to:

33. Langlands, R.P., *The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups*, in *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc. (1966)

Orthogonality relations for the characters of non-compact groups first appeared in:

34. Harish–Chandra, *Discrete series for semisimple Lie groups*, II, Acta Math., vol. 116 (1966)

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35. Rickert, Neil. W., *Convolution of L^2-functions*, Coll. Math, v. XIX (1968)