February 23, 1975

Dear Roger,

Yesterday I took up two of your letters, written to me some time ago, in which you describe some of your thoughts on the Weil representation. Reading them over and glancing through some of the relevant literature, I began to appreciate your view that these representations demand a systematic general treatment. To some extent simply as an excuse for writing to you, I thought I would communicate a couple of obvious questions which occurred to me. The first concerns your duality principle over $\mathbb{R}$, and you can probably answer it by now. The second refers to a clasical problem and concerns the global form of the duality. It may not be so easy to answer.

I notice in the examples over $\mathbb{R}$ available to me that your duality seems to be closely related to the functoriality provided by the associate group. Is this a general phenomenon? To give you a better idea what I mean let me describe the examples. G and H will be the paired groups with associate groups $G^\vee$ and $H^\vee$. With an appropriate labeling of the two groups as G and H there seems in each example to be a homomorphism $\psi : G^\vee \to H^\vee$ such that the pairing associates an element of $\prod_\varphi(G)$ to an element of $\prod_{\psi\circ\varphi}(H)$. I am using the notation of my preprint.* In particular $G^\vee$ is a semi-direct product $\overset{\circ}{G}{}^\vee \times W$ where W is the Weil group of $\mathbb{C}$ over $\mathbb{R}$. If $w \in W$ let $\overline{w}$ be its image in $\mathfrak{G} = \mathfrak{G}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$.

(i) $\quad G = SO(2)$ (compact form) $\qquad H = SL(2)$

$$G^\vee = \mathbb{C}^\times \times W \qquad\qquad H^\vee = SO(3,\mathbb{C}) \times W$$

$$\psi : z \times 1 \longrightarrow \begin{pmatrix} z & & \\ & z^{-1} & \\ & & 1 \end{pmatrix} \times 1$$

$$1 \times w \longrightarrow \omega(\overline{w}) \times 1$$

where

$$\omega(1) = I \qquad \omega(\sigma) = \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix}.$$

Here $\overset{\circ}{H}{}^\vee$ is taken with respect to the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* Item 16 or 39 of the bibliography

(ii) $\qquad G = SL(2,\mathbb{R}) \qquad H = SO(2n,\mathbb{R})$ (compact form), $n \geq 2$

H is an inner or an outer form according as n in even or odd.

$$G^\vee = SO(3,\mathbb{C}) \times W \qquad H^\vee = SO(2n,\mathbb{C}) \times W$$

$\overset{\circ}{H}{}^\vee$ is taken with respect to the form

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Denote the variables by $x_1, \ldots, x_n, y_1, \ldots, y_n$. When H is outer, σ acts on $\overset{\circ}{H}{}^\vee$ as $A \longrightarrow BAB^{-1}$ where B is the orthogonal matrix which interchanges x_n and y_n. I shall define a map $\psi_1 : G^\vee \longrightarrow SO(4,\mathbb{C}) \times W$ and a map $\psi_2 : G^\vee \longrightarrow SO(2n-4,\mathbb{C}) \times W$. Putting them together I will obtain a map $G^\vee \longrightarrow H^\vee$. $SO(4,\mathbb{C})$ is taken with respect to the variables $x_1, x_2, y_1, y_2; SO(2n-4,\mathbb{C})$ with respect to the remaining variables. ψ_2 is a composite of the projection $G^\vee \longrightarrow W$ and a map $W \longrightarrow SO(2n-4,\mathbb{C}) \times W$ $(W = \mathbb{C}^\times \times \mathfrak{G})$.

$$z \in \mathbb{C}^\times \longrightarrow \begin{pmatrix} \left(\frac{z}{\bar z}\right)^{n-1} & & & & & \\ & \ddots & & & & \\ & & \frac{z}{\bar z} & & & \\ & & & \left(\frac{\bar z}{z}\right)^{n-1} & & \\ & & & & \ddots & \\ & & & & & \frac{\bar z}{z} \end{pmatrix} \times \; z$$

$$\sigma \longrightarrow \begin{cases} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \times \sigma & n \text{ even} \\ \begin{pmatrix} 0 & & I & \\ & 1 & & 0 \\ I & & 0 & \\ & 0 & & 1 \end{pmatrix} \times \sigma & n \text{ odd} \end{cases}$$

I can define ψ_1 as a homomorphism $SO(3,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$ because I am dealing with a direct product. Since

$$SO(4,\mathbb{C}) \simeq SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$$

and

$$SO(3,\mathbb{C}) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$$

I can take it to be the diagonal map. Things will then be so arranged that

$$\begin{pmatrix} z & & \\ & z^{-1} & \\ & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} z & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & z^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

To tell the truth my discussion of this example is not based on any knowledge of the Weil representation but on what I infer about the Weil representation from the behaviour of theta series. Observe that for the pairing to have any sense for a given $\varphi \in \Phi(G)$, we must have $\psi \circ \varphi \in \Phi(H)$. If

$$\varphi : z \in \mathbb{C}^\times \longrightarrow \begin{pmatrix} \left(\frac{z}{\bar z}\right)^k & & \\ & \left(\frac{\bar z}{z}\right)^k & \\ & & 1 \end{pmatrix} \times z$$

$$\sigma \longrightarrow \begin{pmatrix} & 1 & \\ 1 & & \\ & & -1 \end{pmatrix} \times \sigma$$

this will be so only if $k \geq n$, as follows from the considerations of my preprint. This corresponds to what we know from theta series.

(iii) $G = U(n)$ $\qquad\qquad$ $H = U(n,n)$.

This is the Gross-Kunze situation.

$$G^\vee = GL(n,\mathbb{C}) \times W \qquad H^\vee = GL(2n,\mathbb{C}) \times W.$$

In both cases σ acts on $\overset{\circ}{G}{}^\vee$ or $\overset{\circ}{H}{}^\vee$ as

$$A \longrightarrow \begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{pmatrix}^{-1}.$$

Although the Gross-Kunze results are incomplete, the following choice of ψ seems to be compatible with them.

$$\psi : A \in \overset{\circ}{H}{}^{\vee} \longrightarrow \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

$$z \in \mathbb{C}^{\times} \subseteq W \quad \psi : 1 \times z \longrightarrow \begin{pmatrix} \left(\frac{z}{\bar{z}}\right)^{n/2} & & & & & & \\ & \ddots & & & & & \\ & & \left(\frac{z}{\bar{z}}\right)^{n/2} & & & & \\ & & & \left(\frac{\bar{z}}{z}\right)^{1/2} & & & \\ & & & & \left(\frac{\bar{z}}{z}\right)^{3/2} & & \\ & & & & & \ddots & \\ & & & & & & \left(\frac{\bar{z}}{z}\right)^{2n-1/2} \end{pmatrix} \times z$$

$$\psi : 1 \times \sigma \longrightarrow \begin{pmatrix} & & & & (-1)^{n/2} I \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \\ 1 & & & & \end{pmatrix} \times \sigma$$

One checks readily that this prescription does in fact yield a well-defined homomorphism.

(iv) $G = SO(2m)$ (compact form) $\qquad H = \mathrm{Sp}(2m)$.

This is the situation of Gelbart's paper, which I had described earlier.

$$G^{\vee} = SO(2m, \mathbb{C}) \times W \qquad H^{\vee} = SO(2m+1, \mathbb{C}) \times W$$

$SO(2m, \mathbb{C})$ is taken to be the orthogonal group of

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and $SO(2m+1, \mathbb{C})$ to be that of

$$\begin{pmatrix} 0 & I & \\ I & 0 & \\ & & 1 \end{pmatrix}.$$

We take the obvious imbedding $SO(2m,\mathbb{C}) \hookrightarrow SO(2m+1,\mathbb{C})$ and extend it to $G^\vee$ by sending $1 \times w$ to $1 \times w$ or to

$$\begin{pmatrix} I & & & & \\ & 0 & & 1 & \\ & & I & & \\ & 1 & & 0 & \\ & & & & -1 \end{pmatrix} \times w$$

according as $\bar{w} \in \{1,\sigma\}$ does or does not act trivially on the Dynkin diagram. Thus

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1^{-1} & & \\ & & & & \ddots & \\ & & & & & x_n^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 & & & & & & \\ & \ddots & & & & & \\ & & x_n & & & & \\ & & & x_1^{-1} & & & \\ & & & & \ddots & & \\ & & & & & x_n^{-1} & \\ & & & & & & 1 \end{pmatrix}$$

(v) $\qquad G = \mathrm{Sp}(2m) \qquad H = SO(2n), n = m+p > m$ (compact form)

This is the example mentioned at the end of Steve's paper; he has been kind enough to provide me with the information for this example and the next.

$$G^\vee = SO(2m+1,\mathbb{C}) \times W \qquad H^\vee = SO(2n,\mathbb{C}) \times W.$$

If $SO(2n,\mathbb{C})$ acts on variables $x_1, \dots x_n, y_1, \dots, y_n$ we imbed $SO(2m+1,\mathbb{C}) \longrightarrow SO(2n,\mathbb{C})$ by taking $SO(2n,\mathbb{C})$ to be the orthogonal group of the form

$$\left(\begin{array}{ccc|ccc} & & & I & & \\ & 1 & & & 0 & \\ & & & & & I \\ \hline I & & & & & \\ & 0 & & & -1 & \\ & & I & & & \end{array}\right)$$

and letting $SO(2m+1, \mathbb{C})$ act on the variables $x_1, \ldots, x_m, y_1, \ldots, y_m, x_{m+1}$. We extend to ψ as follows. If $z \in \mathbb{C}^X \subseteq W$ we send z to

$$\left(\begin{array}{cccccc|cccccc}
1 &&&&&&&&&&& \\
& \ddots &&&&&&&&&& \\
&& 1 &&&&&&&&& \\
&&& \left(\frac{z}{\bar{z}}\right)^{p-1} &&&&&&&& \\
&&&& \left(\frac{z}{\bar{z}}\right)^{p-2} &&&&&&& \\
&&&&& \ddots &&&&&& \\
&&&&&& \frac{z}{\bar{z}} &&&&& \\
\hline
&&&&&&& 1 &&&& \\
&&&&&&&& 1 &&& \\
&&&&&&&&& \ddots && \\
&&&&&&&&&& 1 & \\
&&&&&&&&&&& \left(\frac{\bar{z}}{z}\right)^{p-1} \\
&&&&&&&&&&& \ddots \\
&&&&&&&&&&& \frac{\bar{z}}{z}
\end{array}\right)$$

$$\sigma \to \left(\begin{array}{ccc|ccc}
I &&&&& \\
& 1 &&&& \\
&& 0 &&& I \\
\hline
&&& I && \\
& \mathrm{I} &&& \pm 1 & \\
&&&&& 0
\end{array}\right) \times \sigma \text{ if } n \text{ even.}$$

The sign is so chosen that the determinant is 1.

$$\sigma \to \left(\begin{array}{cccc|cccc}
I &&&&&&& \\
& 1 &&&&&& \\
&& 0 &&& I && \\
&&& 1 &&& 0 & \\
\hline
&&&& I &&& \\
&&&&& \pm 1 && \\
&& I &&&& 0 & \\
&&& 0 &&&& 1
\end{array}\right) \times \sigma \text{ if } n \text{ is odd.}$$

(vi) $G = SO(2m-2)$ $H = \mathrm{Sp}(2m)$

According to Steve the representation of G with highest weight $\omega_1, \ldots, \omega_{m-1}$ gives rise to the representation of H which is in some sense a limit of holomorphic discrete series and is associated to the representation of $U(n)$ with highest weight

$*$ $$\omega_1 + m + 1, \ldots \omega_{m-1} + m + 1, m.$$

However I received the information over the telephone and I fear I may have garbled it. On formal grounds I would prefer

$$** \qquad \omega_1 + m - 1, \ldots, \omega_{m-1} + m - 1, m$$

to *. *If ** is correct* ψ should be obtained in the following manner. Define $\overset{\circ}{G}{}^{\vee}$ with respect to the form

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

in $2m-2$ variables and $\overset{\circ}{H}{}^{\vee}$ with respect to

$$\begin{pmatrix} 0 & I & \\ I & 0 & \\ & & 1 \end{pmatrix}$$

with variables $x_1, \ldots, x_m, y_1, \ldots, y_m, z$. We define $\psi : \overset{\circ}{H}{}^{\vee} \longrightarrow \overset{\circ}{G}{}^{\vee}$ by letting $\overset{\circ}{H}{}^{\vee}$ act on the variables $x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}$. We extend it to $\mathbb{C}^\times \subseteq W$ by sending $1 \times z \longrightarrow 1 \times z$ and $1 \times w$ with $\bar{w} = \sigma$ to $1 \times w$ if m is odd and to

$$\left(\begin{array}{ccc|ccc|c} 1 & & & & & & \\ & 0 & & & 1 & & \\ & & 1 & & & & \\ \hline & & & 1 & & & \\ & 1 & & & 0 & & \\ & & & & & 1 & \\ \hline & & & & & & -1 \end{array}\right)$$

if m is even.

[Added 1978: The correspondence at the finite places should function like that at the infinite. Why does no-one verify this at the level of Hecke algebras?] *

Before going on to the second question let me explain the difficulties I have with your answer to the question I posed last year. First of all I think I have a rough understanding of the philosophy of your letter, but the occasional specific statement baffles me. As an illustration, in the last example in which you describe the absolutely cuspidal representations associated to characters of O (the orthogonal group of an isotropic form in 4 variables) which are not trivial in SO you seem to say it is the character

* Added 1998 - See the paper of S. R Rallis, *Langlands' functoriality and the Weil representation*, Amer. Jour. Math., vol 104 (1982)

on O which matters and not merely its restriction to SO. Since there is an element of O fixing each vector in the corresponding Y_2, I should have thought only one of the extensions was relevant.

Being that as it may, let's suppose that over a non-archimedean field the functoriality still works as in Example (iv) with $m = 2$. Since $m = 2$ I can regard $\overset{\circ}{H}{}^{\vee}$ as the adjoint group of $\mathrm{Sp}(4, \mathbb{C})$. This I prefer to do.

$\overset{\circ}{G}{}^{\vee}$ I represent as

$$SL(2, \mathbb{C}) \times SL(2, \mathbb{C})/\mathbb{Z}_2.$$

Since

$$G \subseteq D^{\times} \times D^{\times}/F = G_1 \ (F \text{ diagonally embedded}),$$

I can regard representations of G as components of representations of $D^X \times D^X = G_2$. Also

$$\overset{\circ}{G}{}^{\vee}_2 = GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \qquad \overset{\circ}{G}{}^{\vee} = \{(x, y) \in \overset{\circ}{G}{}^{\vee}_2 \mid \det x \det y = 1\}$$
$$\overset{\circ}{G}{}^{\vee} = \overset{\circ}{G}{}^{\vee}_1/\{(\lambda, \lambda^{-1}) \mid \lambda \in \mathbb{C}^{\times}\}.$$

Since we have

$$\begin{array}{ccc} G_2^{\vee} & \leftarrow & G_1^{\vee} \\ & & \downarrow \\ & & G^{\vee} \end{array}$$

(Note $\Phi(G_1^{\vee}) \to \Phi(G^{\vee})$ is surjective)

and

$$\begin{array}{ccc} \prod^{\sim}(G_2) & \to & \prod^{\sim}(G_1) \\ & & \updownarrow \\ & & \prod^{\sim}(G) \end{array}$$

and we know the role the associate group plays for G_2 we know the role it plays for $G = SO$.

A character of SO trivial on $D' \times D'/F$ (as in your letter, D' consists of the elements of norm 1) corresponds to two special homomorphisms

$$(\varphi_1, \varphi_2) \to GL(2) \times GL(2) = \overset{\circ}{G}{}^{\vee}_2.$$

I think you know what I mean. Some further comments are provided in the enclosed letter. Here

$$\varphi_1 = \begin{pmatrix} \mu|\cdot|^{-1/2} & * \\ 0 & \mu|\cdot|^{1/2} \end{pmatrix} \qquad |\cdot| = \text{absolute value}$$
$$\varphi_2 = \begin{pmatrix} \lambda|\cdot|^{1/2} & * \\ & \lambda|\cdot|^{1/2} \end{pmatrix}$$

λ and μ are two characters with $(\lambda\mu)^2 = 1$. Composing with ψ to obtain a map to

$$\overset{\circ}{H}{}^{\vee} = \mathrm{Sp}(4,\mathbb{C})\backslash\mathbb{Z}_2 = Gp(4,\mathbb{C})/\mathbb{C}^{\times}$$

we find

$$\begin{pmatrix} \mu \quad |\cdot|^{-1/2} & 0 & * & 0 \\ 0 & \lambda^{-1}|\cdot|^{-1/2} & 0 & * \\ 0 & 0 & \mu|\cdot|^{1/2} & 0 \\ 0 & 0 & 0 & \lambda^{-1}|\cdot|^{1/2} \end{pmatrix}$$

Only $\lambda\mu$ is relevant. If I knew that for one specific choice of $\lambda\mu$ the corresponding supercuspidal representation of $H = \mathrm{Sp}(4, F)$ was that induced by the anomalous Srinivasan representation I would be a happy man. This you do not assure me and I have not yet made any calculations, for I am reluctant to get into the ring with the Weil representation; so my happiness must be postponed.

However you have assured me that if one starts from the trivial representation of $SO(2)$ for an anisotropic form associated to the unramified quadratic extension then one ends up with exactly this representation of $\mathrm{Sp}(4, F)$. We should be able to appeal to Example (vi) to decide the correct choice of ψ. However this example provides us with no unambiguous answer. To clarify I first observe that the enclosed letter allows me to pass back and forth rather freely between the Weil form and the Galois form of the associate group. Thus some of what I write below must be taken as a symbolic representation of objects introduced more explicitly in that letter.

Global compatibility leaves no choice for the restriction of ψ to $\overset{\circ}{G}{}^{\vee} = \mathbb{C}^{\times}$. It must be the same as at infinity and takes

$$\lambda \longrightarrow \begin{pmatrix} \lambda^{1/2} & & & \\ & \lambda^{1/2} & & \\ & & \lambda^{-1/2} & \\ & & & \lambda^{-1/2} \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{C})/\mathbb{Z}_2.$$

Let $\{1, \sigma\}$ be the Galois group of the quadratic field F' associated to G. If $w \in W$ let $\bar{w}$ be its image in this Galois group. If $\bar{w} = 1$ then $\psi(w)$ must be of the form

$$\begin{pmatrix} A & 0 \\ 0 & {}^t - A^{-1} \quad 0 \end{pmatrix} \times w$$

and if $\bar{w} = \sigma$ of the form

$$\begin{pmatrix} 0 & A \\ -{}^tA^{-1} & 0 \end{pmatrix} \times w.$$

One possibility is to choose w_0 with $\bar{w}_0 = \sigma$ and to set

$$A = \psi_1(w) = \begin{pmatrix} |\cdot|^{1/2} & * \\ 0 & |\cdot|^{1/2} \end{pmatrix}$$

the special representation of $W_{F'}$ if $\bar{w} = 1$. Let

$$\psi_1(w_0^{-1} w w_0) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \psi_1(w) \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then map

$$w_0 \longrightarrow \begin{pmatrix} & & & \alpha^{-1} \\ & & -1 & \\ & -\alpha & & \\ 1 & & & \end{pmatrix} \times w_0.$$

Conjugating by

$$\begin{pmatrix} \alpha^{1/2} & & 1 & 0 & 0 \\ \alpha^{1/2} & & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2\alpha^{1/2}} & \multicolumn{2}{c}{\begin{pmatrix} 1 & \alpha^{1/2} \\ 1 & \alpha^{-1/2} \end{pmatrix}} \\ 0 & 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

we obtain

$$w \longrightarrow \begin{pmatrix} |\cdot|^{-1/2} & 0 & * & 0 \\ 0 & \mu|\cdot|^{1/2} & 0 & * \\ 0 & 0 & |\cdot|^{1/2} & 0 \\ 0 & 0 & 0 & \mu|\cdot|^{1/2} \end{pmatrix} \times w$$

where μ is the quadratic character defined by F'. All this means is that with an appropriate choice of ψ we can achieve consistency with the information obtained from quadratic forms in four variables. However there is yet no real justification for this choice of ψ.

There is one other place to look but there we find even less information. When I discussed the question with Bill Casselman, he referred me to the Comptes Rendus notes of Soto Andrade. There is something there; but not enough to allow one to come to grips with the question.

Let F' be the unramified quadratic extension of F, $G = SL(2, F')/\mathbb{Z}_2$, an orthogonal group in four variables, H is as before. Now of course G is non-compact, which I understand entails all sorts of complications. Nonetheless the results of Soto Andrade do suggest that the special representation of G corresponds to the representation of H induced from the anomalous Srinivasan representation.

There is only one reasonable homomorphism of

$$G^\vee = (SL(2, \mathbb{C}) \times SL(2, \mathbb{C})) \times W$$

to $H^\vee$,

$$\begin{pmatrix} a & b \\ a & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longrightarrow \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix} \text{(both sides modulo } \mathbb{Z}_2\text{)}$$

If $\bar{w} = 1$ then $I \times w \longrightarrow I \times w$ but if $\bar{w} = \sigma$ then

$$I \times w \longrightarrow \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \times \sigma.$$

The map $\varphi : w \longrightarrow G^\vee$ corresponding to the special representation is of course a special representation

$$\varphi : w \longrightarrow (\varphi_1(w), \varphi_1(w)) \times w \qquad \bar{w} = 1$$
$$w_0 \longrightarrow \left(\begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \times \begin{pmatrix} \alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2} \end{pmatrix} \right) \times w_0.$$

This leads to the same $\psi \circ \varphi$ as before, so all is consistent. The only difficulty is that there are not enough hard facts available.

Now that I've poured out my heart to you on this matter, let me come to the second question. It is more of a problem than a question. I would like to see someone carry out the analysis of Example (ii) over a non-archimedean field and then turn to the global situation, apply the local information, and obtain a definitive theorem about which modular forms (on the upper half-plane) are representable by theta series in $2n$ variables. $n = 1, 2$ are of course o.k. This problem may be beyond us at present, but it is important.

Whenever you are sufficiently far along with your work in the Weil representation that you would like to give a talk on it here, let me know. Since I have a tiny bit of money for such purposes, I can invite you down.

All the best,

Bob