Eisenstein Series*

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1. Preliminaries. In these lectures I want to discuss, with some indications of proofs, some of the elementary facts in the theory of Eisenstein series. Although the discussion can be carried out in more generality it is most convenient, in the context of this institute, to take for discrete group an arithmetically defined subgroup Γ of the group G of real points of a reductive group $G_{\mathbf{C}}$ defined over $\mathbf{Q}$ whose connected component $G^0_{\mathbf{Q}}$ has no rational character. It is also necessary to suppose that the centralizer of a maximal $\mathbf{Q}$ split torus of $G^0_{\mathbf{C}}$ meets every component of $G_{\mathbf{C}}$. The reduction theory of Borel applies, with trivial modifications, to G; it will be convenient to assume that Γ has a fundamental set with only one cusp. Fix a minimal parabolic subgroup $P^0_{\mathbf{C}}$ defined over $\mathbf{Q}$ and a maximal $\mathbf{Q}$-split torus $A^0_{\mathbf{C}}$ of $P^0_{\mathbf{C}}$ so that the standard parabolic $\mathbf{Q}$-subgroups are defined. A (standard) cuspidal (percuspidal) subgroup P is the normalizer in G of a (standard) parabolic (minimal parabolic) $\mathbf{Q}$-subgroup $P_{\mathbf{C}}$ of $G^0_{\mathbf{C}}$. To each standard cuspidal subgroup P is associated a subspace $\mathfrak{A}_{\mathbf{C}}$ of the Lie algebra $\mathfrak{a}^0_{\mathbf{C}}$ of $A^0_{\mathbf{C}}$; this subspace will be called the split component of P. By definition the rank of P is equal to its dimension. The set $\mathfrak{a}$ of real points on $\mathfrak{a}_{\mathbf{C}}$ will also be called the split component of P. P is a product AMN where A is the analytic subgroup of G with the Lie algebra $\mathfrak{a}$, N is the set of real points in the unipotent radical of $P_{\mathbf{C}}$, and M satisfies the same conditions as G. We identify M with $N\backslash MN$. Then $\Gamma \cap P \subseteq MN$ and $\Theta = \Gamma \cap N \backslash \Gamma \cap MN$ is an arithmetically defined subgroup of M. Assume that for each standard cuspidal subgroup P it also has a fundamental domain with only one cusp.

Suppose P and P' are two standard cuspidal subgroups with the split components $\mathfrak{a}$ and $\mathfrak{a}'$ respectively. If there is an element of Ω, the Weyl group (over $\mathbf{Q}$) of $\mathfrak{a}^0_{\mathbf{C}}$, taking $\mathfrak{a}_{\mathbf{C}}$ to $\mathfrak{a}'_{\mathbf{C}}$ we shall say that P and P' are associate; let $\Omega(\mathfrak{a}, \mathfrak{a}')$ be the set of distinct linear transformations from $\mathfrak{a}_{\mathbf{C}}$ to $\mathfrak{a}'_{\mathbf{C}}$ obtained by restricting such an element of Ω to $\mathfrak{A}_{\mathbf{C}}$. The relation of being associate is an equivalence relation. The normalizer of $\mathfrak{A}(\mathfrak{A}')$ in G leaves $M(M')$ invariant and consequently acts on the centre $Z(Z')$ of the universal enveloping algebra of the Lie algebra of $M(M')$ and on the set $\mathfrak{X}(\mathfrak{X}')$ of homomorphisms of $Z(Z')$ into $\mathbf{C}$. The orbits in $\mathfrak{X}(\mathfrak{X}')$ under this action are finite. If P and P' are associate, Z and Z' are isomorphic and there is a natural one-to-one corresondence between orbits in $\mathfrak{X}$ and $\mathfrak{X}'$. Every element of Z defines an unbounded operator on $L^2_0(\Theta\backslash M)$, the space of cusp forms on $\Theta\backslash M$. If $\xi \in \mathfrak{X}$ let

$$V(\xi) = \{\phi \in L^2_0(\Theta\backslash M) \mid X\phi = \xi(X)\phi \ \text{ for all } \ X \in Z\}$$

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and if Ξ is an orbit in $\mathfrak{X}$ let

$$V(\Xi) = \sum_{\xi \in \Xi} V(\xi).$$

$V(\Xi)$ is a closed subspace of $L_0^2(\Theta\backslash M)$ invariant under M and

$$L_0^2(\Theta\backslash M) = \sum_{\Xi} \oplus V(\Xi).$$

If Ξ' is the orbit in $\mathfrak{X}'$ corresponding to Ξ the space $V(\Xi')$ may be defined in a similar fashion. $V = V(\Xi)$ and $V' = (\Xi')$ are said to be associate. We shall call such a V a simple admissible subspace. The symbol W will denote the space of functions on a fixed maximal compact subgroup K of G spanned by the matrix elements of some irreducible representation of K.

2. Partial decomposition of $L^2(\Gamma\backslash G)$. If V is a simple admissible subspace of $L_0^2(\Theta\backslash M)$ let $\mathcal{E}(V, W)$ be the set of all continuous functions Φ on $NA(\Gamma \cap P)\backslash G$ such that $\Phi(mg)$ belongs to V for all g and $\Phi(gk^{-1})$ belongs to W for all g. $\mathcal{E}(V, W)$ is a finite dimensional Hilbert space with the inner product

$$(\Phi, \Psi) = \int_{\Theta\backslash M \times K} \Phi(mk)\bar{\Psi}(mk) dm\ dk.$$

Let $\mathcal{D}(V, W)$ be the space of all continuous functions on $N(\Gamma \cap P)\backslash G$ such that $\phi(mg)$ belongs to V and $\phi(gk^{-1})$ belongs to W for each g and such that the projection of the support of ϕ on $NM\backslash G$ is compact.

Lemma 1. *If* $\phi \in \mathcal{D}(V, W)$ *then*

$$\hat{\phi}(g) = \sum_{\Gamma \cap p\backslash\Gamma} \phi(\gamma g)$$

belongs to $L^2(\Gamma\backslash G)$.

The proof of this lemma requires the result in §6 of Godement's lecture on cusp forms.

Suppose $\{P\}$ is the set of all standard cuspidal subgroups associate to a given one and $\{V\} = \{V(P) \mid P \in \{P\}$ is a collection of associate simple admissible subspaces. Let $L(\{P\}, \{V\}, W)$ be the closed subspace of $L^2(\Gamma\backslash G)$ spanned by the functions $\hat{\phi}(\cdot)$ with ϕ in $\mathcal{D}(V(P), \{V\}, W)$ for some P in $\{P\}$.

Lemma 2. *$L^2(\Gamma\backslash G)$ is the orthogonal direct sum of the spaces $L(\{P\},\{V\},W)$ and for a fixed $\{P\}$ and $\{V\}$, $\sum_W \oplus L(\{P\},W)$ is invariant under G.*

This lemma is fairly easy consequence of Lemma 3 which will be stated below. To some extent it reduces the problem of decomposing $L^2(\Gamma\backslash G)$ to that of decomposing each of the spaces $L(\{P\},\{V\},W)$.

3. Eisenstein series. If P belongs to $\{P\}$ let $\mathfrak{a}_{\mathbf{C}}$ be the split component of P. Let Λ be the generic symbol for a linear function on $\mathfrak{a}_{\mathbf{C}}$. We can write any ϕ in $\mathcal{D}(V,W)$ as a Fourier integral

$$\phi(g)=\frac{1}{(2\pi)^q}\int_{\mathrm{Re}\Lambda=\Lambda_0}\exp(\Lambda(H(g))+\rho(H(g))\Phi(\Lambda,g)|d\Lambda|. \tag{1}$$

Here $\Phi(\cdot)$, which I call the Fourier transform of ϕ, is an entire function on the dual of $\mathfrak{a}_{\mathbf{C}}$ with values in $\mathcal{E}(V,W)$ and $\Phi(\Lambda,g)$ is the value of $\Phi(\Lambda)$ at g. The dimension of $\mathfrak{a}_{\mathbf{C}}$ is q; ρ is one-half the sum of the positive roots; and $a(g)=\exp H(g)$ if $g=na(g)mk, n\in N, a(g)\in A, m\in M, k\in K$. If $(\Lambda_0,\alpha)>(\rho,\alpha)$ for every positive root α then

$$\hat{\phi}(g)=\frac{1}{2\pi)^q}\int_{\mathrm{Re}\Lambda=\Lambda_0}\sum_{\Gamma\cap P\backslash\Gamma}\exp(\Lambda(H(\gamma g))+\rho(H(\gamma g)))\Phi(\Lambda,\gamma g)|d\Lambda|.$$

To study the map $\phi\to\hat{\phi}$ we shall, for an arbitrary Φ in $\mathcal{E}(V,W)$, study the series

$$\sum_{\Gamma\cap P\backslash\Gamma}\exp(\Lambda(H(\gamma g))+\rho(H(\gamma g)))\Phi(\gamma g).$$

This series is of interest for all functions Φ on $NA(\Gamma\cap P)\backslash G$ such that, for each $g, \Phi(mg)$ is an automorphic form, in the sense of Harish-Chandra, on $\Theta\backslash M$ which is square integrable on $\Theta\backslash M$ and $\Phi(gk^{-1})$ belongs to some space W. It is called an Eisenstein series. Denote its sum by $E(g,\Phi,\Lambda)$. For each g and Φ this function is defined and holomorphic in the domain $\{\Lambda \mid \mathrm{Re}(\Lambda,\alpha)>(\rho,\alpha)$ for all $\alpha>0\}$. One of the basic facts in the theory of Eisenstein series is that it can be continued to all of the dual space of $\mathfrak{a}_{\mathbf{C}}$ as a meromorphic function. This has first to be done when Φ belongs to one of the spaces $\mathcal{E}(V,W)$ and for the moment we concentrate on that.

Lemma 3. *If P' is another standard cuspidal subgroup of rank g then*

$$\int_{\Gamma\cap N'\backslash N'}E(ng,\Phi,\Lambda)dn=0 \tag{a}$$

if P and P' are not associate. However, if P and P' are associate

$$\int_{\Gamma\cap N'\backslash N'}E(ng,\Phi,\Lambda)dn=\sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}')}\exp(s\Lambda(H'(g))+\rho(H'(g)))(M(s,\Lambda)\Phi)(g) \tag{b}$$

where $M(s,\Lambda)$ is a linear transformation from $\mathcal{E}(V,W)$ to $\mathcal{E}(V',W)$ analytic as a function of Λ in $\{\Lambda \mid \mathrm{Re}(\Lambda,\alpha) > (\rho,\alpha)$ for $\alpha > 0\}$. Here V' is associate to V.

In order to gain some understanding of this lemma we consider the case that P is the standard cuspidal subgroup, $P' = P$, and Φ is a constant function. The sum on the right of (b) is then a sum over the Weyl group. The left side equals

$$\int_{\Gamma\cap N\backslash N} \sum_{\Gamma\cap P\backslash L} \exp(\Lambda(H(\gamma ng)) + \rho(H(\gamma ng)))\Phi(\gamma ng)dn$$

or

$$\sum_{\Gamma\cap P\backslash\Gamma/\Gamma\cap N} \mu(\Gamma\cap N\cap\gamma^{-1}P\gamma\backslash N\cap\gamma^{-1}P\gamma)\int_{N\cap\gamma^{-1}P\gamma\backslash N} \exp(\Lambda(H(\gamma ng)) + \rho(H(\gamma ng)))\Phi(\gamma ng)dn.$$

We consider the integrals in this sum individually. Using the Bruhat decomposition to write γ as $pn_W u$, we see that the integral equals

$$\exp(\Lambda(H(p)) + \rho(H(p)))\left\{\int_{N\cap n_W^{-1}Pn_W\backslash N} \exp(\Lambda(H(n_W ng)) + \rho(H(n_W ng))dn\right\}\Phi(g).$$

The expression in brackets equals

$$\exp(\Lambda(Adn_W(H(g) + \rho(H(g))))\int_{N\cap n_W^{-1}pn_W\backslash N} \exp(\Lambda(H(n_W n) + \rho(H(n_W n)))dn$$

and we are done. Observe that if, as we suppose, the measure of $\Gamma\cap N\backslash N$ is one then $M(1,\Lambda) = I$.

4. Some functional analysis. Combining Lemma 3 with the Fourier inversion formula we obtain a formula which is basic for everything to follow.

Corollary. *Suppose P and P' are associate standard cuspidal subgroups, V and V' are associate admissible subspaces, ϕ belongs to $\mathcal{D}(V,W)$, and ψ belongs to $\mathcal{D}(V',W)$ If the Haar measure on G is suitably chosen, then*

$$\int_{\Gamma\backslash G} \hat\phi(g)\bar{\hat\psi}(g)dg = \frac{1}{(2\pi)^q}\int_{\mathrm{Re}\Lambda=\Lambda_0} \sum_{s\in\Omega(\mathfrak{a},\mathfrak{a}')} (M(s,\Lambda)\Phi(\Lambda),\Psi(-s\bar\Lambda))|d\Lambda|. \tag{2}$$

Of course Λ_0 must be such that $(\Lambda_0,\alpha) > (\rho,\alpha)$ if α is a positive root of $\mathfrak{A}$. Simple approximation arguments now show that if $\phi(g)$ can be represented in the form (1) with a function $\Phi(\cdot)$, with values in $\mathcal{E}(V,W)$, which is defined and analytic in a tube over a ball of radius R with $R > (\rho,\rho)^{\frac{1}{2}}$ and behaves well at infinity then $\hat\phi(\cdot)$ is defined and square integrable and the formula (2) is valid. In particular $\Phi(\cdot)$

could be taken to lie in $\mathcal{H}(\mathcal{E}(V,W))$, the space of all functions analytic in some such tube which go to zero at infinity faster than the inverse of any polynomial.

Let $P^1,\dots,P^r$ be the elements of $\{P\}$, let $V^i = V(P^i)$ and set

$$\mathcal{H} = \sum_{i=1}^{r} \oplus \mathcal{H}(\mathcal{E}(V^i, W)).$$

Let $\Phi(\cdot) = (\Phi_1(\cdot),\dots \Phi_r(\cdot))$, where $\Phi_i(\cdot)$ is a function in $\mathcal{H}(\mathcal{E}(V^i,W))$, be the symbol for a generic element of $\mathcal{H}$. It is clear that we can define a linear map $\Phi(\cdot) \to \hat{\phi}(\cdot)$ of $\mathcal{H}$ into $L(\{P\},\{V\},W)$.

Suppose that, for $1 \leqq i \leqq r$, $f_i(\cdot)$ is a complex valued function defined, bounded, and analytic in the tube T^i_R over some ball of radius $R > (\rho,\rho)^{1|2}$ with center 0 in the dual of $\mathfrak{a}^i_{\mathbf{C}}$ and $f_i(s\Lambda) = r_i(\Lambda)$ if $s \in \Omega(\mathfrak{a}^i,\mathfrak{a}^j)$. Set

$$f\Phi(\cdot) = (f_1(\cdot)\Phi_1(\cdot),\dots,f_r(\cdot)\Phi_r(\cdot)).$$

The following lemma is quite useful.

Lemma 4. *If*

$$\max_{1\leqq i \leqq r} \sup_{\Lambda \in T^i_{\mathbf{R}}} |f_i(\Lambda)| = k$$

then there is a bounded operator $\lambda(f)$ on $L(\{P\},\{V\},W)$ of norm at most k such that if $\Psi(\cdot) = f\Phi(\cdot)$ then $\hat{\psi} = \lambda(f)\hat{\phi}$.

Suppose $\Phi(\cdot) = (\Phi_1(\cdot),\dots,\Phi_r(\cdot))$ and $\Psi(\cdot) = (\Psi_1(\cdot),\dots,\Psi_r(\cdot))$ are two arbitrary elements in $\mathcal{H}$. Then $(\hat{\phi},\hat{\psi})$ is equal to

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \frac{1}{(2\pi)^q} \int_{\mathrm{Re}\Lambda^i = \Lambda_\delta} \sum_{s\in\Omega(\mathfrak{a}^i,\mathfrak{a}^j)} (M(s,\Lambda^i)\Phi_i(\Lambda^i), \Psi_j(-s\bar{\Lambda}_i))|d\Lambda_i|.$$

Denote this expression by $(\Phi(\cdot),\Psi(\cdot))$. It is easily verified that

$$(f\Phi(\cdot),\Psi(\cdot)) = (\Phi(\cdot), f^*\Psi(\cdot))$$

if $f^*(\cdot) = (f_1^*(\cdot),\dots,f_r^*(\cdot))$ and $f_i^*(\cdot)$ is defined by $f_i^*(\Lambda) = \overline{f_i(-\bar{\Lambda})}$. Consequently $(f^*f\Phi(\cdot),\Phi(\cdot)) \geqq 0$. If $\ell > k$ there is a function $g(\cdot)$ satisfying the same conditions as $f(\cdot)$ such that $\ell^2 - f_i^*(\Lambda)f_i(\Lambda) = g_i^*(\Lambda)g_i(\Lambda), 1 \leqq i \leqq r$. Consequently

$$\ell^2(\Phi(\cdot),\Phi(\cdot)) - (f\Phi(\cdot), f\Phi(\cdot)) = (g\Phi(\cdot), g\Phi(\cdot)) \geqq 0.$$

The lemma is an easy consequence of this inequality. In particular take

$$f_i(\Lambda) = (\mu - (\Lambda, \Lambda))^{-1}$$

with $\mu > (\rho, \rho)$. Then $\lambda(f)$ is self-adjoint with a dense range; consequently the operator $A = \mu - \lambda(f)^{-1}$ is a self-adjoint operator, usually unbounded, on $L(\{P\}, \{V\}, W)$. If $\Psi_i(\Lambda) = (\Lambda, \Lambda)\Phi_i(\Lambda), 1 \leqq i \leqq r$, then $A\hat{\phi} = \hat{\psi}$. The resolvent $R(z, A) = (z - A)^{-1}$ is an analytic function of z off the infinite interval $(-\infty, (\rho, \rho)]$.

5. A theorem.

Theorem. *For each i and each j and each s in $\Omega(\mathfrak{a}^i, \mathfrak{a}^j)$ the function $M(s, \Lambda)$ is meromorphic on the dual of $\mathfrak{a}^i_{\mathbf{C}}$. For each Φ in $\xi(V^i, W)$ the function $E(\cdot, \Phi, \Lambda)$ with values in the space of continuous functions on $\Gamma\backslash G$ is meromorphic on the dual of $\mathfrak{a}^i_{\mathbf{C}}$. If $s \in \Omega(\mathfrak{a}^i, \mathfrak{a}^j), t \in \Omega(\mathfrak{a}^j, \mathfrak{a}^k)$ and $\Phi \in \mathcal{E}(V^i, W)$ the functional equations*

$$M(ts, \Lambda) = M(t, s\Lambda)M(s, \Lambda),$$
$$E(g, M(s, \Lambda)\Phi, s\Lambda) = E(g, \Phi, \Lambda)$$

are satisfied.

The first, and most difficult, step in the proof of this theorem is to show that it is true when $\dim a^i = 1$ for one, and hence all, i. Most of the important ideas in this case have been described by Selberg in his talk at the International Congress.

6. In which the number variables is one. If $\dim a^i = 1$ then r is 1 or 2. If z is a complex number let $\Lambda^i(z)$ be such that $(\alpha^i, \Lambda^i(z)) = z(\alpha^i, \alpha^i)^{\frac{1}{2}}$ if α^i is the unique simple root of a^i. Let $\mathcal{E} = \mathcal{E}(V^1, W)$ or $\mathcal{E}(V^1, W) \oplus \mathcal{E}(V^2, W)$ according as r is 1 or 2. If $r = 1$, there is an s in $\Omega(\mathfrak{a}^1, \mathfrak{a}^1)$ different from the identity; let $M(z) = M(s, \Lambda^1(z))$. If $r = 2$ and s is in $\Omega(\mathfrak{a}^1, \mathfrak{a}^2)$ then $s\Lambda^1(z) = -\Lambda^2(z)$. In this case let

$$M(z) = \begin{pmatrix} 0 & M(x^{-1}, \Lambda^2(z)) \\ M(s, \Lambda^1(z)) & 0 \end{pmatrix}.$$

In both cases $M(z)$ is a linear transformation of $\mathcal{E}$. If $\Phi = (\Phi_1)$ or (Φ_1, Φ_2) belongs to $\mathcal{E}$ let

$$E(g, \Phi, z) = \sum_i E(g, \Phi_i, \Lambda^i(z)).$$

The theorem may be restated as:

Theorem. (i) $E(\cdot, \Phi, z)$ *and* $M(z)$ *are meromorphic in the complex plane,*

(ii) $M(z)M(-z) = I$,

(iii) $E(g, M(z)\Phi, -z) = E(g, \Phi, z)$.

If (i) and (ii) are true and P is any maximal standard cuspidal subgroup then

$$\int_{\Gamma\cap N\backslash N} E(ng, M(z)\Phi, -z) - E(ng, \Phi, z)dn = 0.$$

It follows from this that the integrand is a cusp form. Since on the other hand it is by construction orthogonal to the cusp forms it must vanish identically. Thus (iii) is also true.

The space $\mathcal{H}$ may be regarded as a space of functions, each of which is defined on some strip of the form $|\mathrm{Re} z| < (\rho, \rho)^{\frac{1}{2}} + \epsilon, \epsilon > 0$, by setting

$$\Phi(z) = \sum_i \oplus \Phi_i(\Lambda^i(z)).$$

$\Phi(\cdot)$ takes values in $\mathcal{E}$. If c is close to but greater than $(\rho, \rho)^{\frac{1}{2}}$

$$(\hat{\phi}, \hat{\psi}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\Phi(z), \Psi(-\bar{z})) + (M(z)\Phi(z), \Psi(\bar{z}))dz.$$

If $c_1 > \mathrm{Re}\lambda > c$ then $(R(\lambda^2, \Lambda)\hat{\phi}\hat{\psi})$ is the sum of

$$\frac{1}{2\lambda}\{(\Phi(\lambda), \Psi(-\bar{\lambda})) + (M(\lambda)\Phi(\lambda), \Psi(\bar{\lambda}))\} \tag{3}$$

and

$$\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{1}{\lambda^2 - z^2}\{(\Phi(z), \Psi(-\bar{z})) + (M(z)\Phi(z)\Psi(\bar{z}))\}dz. \tag{4}$$

If $\Phi(z) = \exp z^2\Phi$ and $\Psi(z) = \exp z^2\Psi$ with Φ and Ψ in $\mathcal{E}$ then (4) is an entire function of λ and (3) is equal to

$$(\exp 2\lambda^2/2\lambda)\{(\Phi, \Psi) + (M(\lambda)\Phi, \Psi)\}.$$

Consequently $M(\lambda)$ is analytic wherever $(R(\lambda^2, A)\hat{\phi}, \hat{\psi})$ is. In particular it is analytic for $\mathrm{Re}\lambda > 0, \lambda \notin (0, (\rho, \rho)^{\frac{1}{2}}]$.

Now we want to show that $E(\cdot, \Phi, z)$ is analytic in this region also. If $f(g)$ is a continuous function on G with compact support such that $f(kgk^{-1}) = f(g)$ for all k in K there is an entire function $\pi(f, z)$ with values in the space of linear transformations of $\mathcal{E}$ such that the convolution of $E(g, \Phi, z)$ and $f(g)$

is $E(g,\pi(f,z)\Phi,z)$. As a consequence it is enough to show that if $\psi(g)$ is any continuous function on $\Gamma\backslash G$ with compact support then

$$\int_{\Gamma\backslash G} E(g,\Phi,z)\bar{\psi}(g)dg$$

is analytic in this region. In doing this we are free to modify $E(g,\Phi,z)$ outside of the support of ψ. If $\Phi=\sum_i \oplus\Phi_i$ then

$$E(g,\Phi,z)=\sum_i \sum_{\Gamma\cap P_i\backslash\Gamma} F(\gamma g,\Phi_i,\Lambda^i(z))$$

with

$$F(g,\Phi_i,\Lambda^i)=\exp(\Lambda^i(H^i(g))+\rho(H^i(g)))\Phi_i(g).$$

According to a principal stated by Borel in his lectures on reduction there is a number x such that, for $1\leqq i\leqq r$, the inverse image in G of the support of ψ is contained in $\{g \mid \alpha^i(H^i(g))<x(\alpha^i,\alpha^i)^{\frac{1}{2}}\}$. Let $F''(g,\Phi_i,z)$ equal $F(g,\Phi_i,\Lambda^i(z))$ if $\alpha^i(H^i(g))<x(\alpha^i,\alpha^i)^{\frac{1}{2}}$ and let it equal $-F(g,\Phi_i(z),-\Lambda^i(z))$ otherwise. Here $\Phi_i(z)$ is defined by

$$M(z)\Phi=\sum_i \oplus\Phi_i(z).$$

Set

$$E''(g,\Phi,z)=\sum_i \sum_{\Gamma\cap P^i\backslash\Gamma} F''(\gamma g,\Phi_i,z).$$

The functions $E(g,\Phi,z)$ and $E''(g,\Phi,z)$ are equal on the support of ψ.

It is easy to compute the Fourier transform of $F''(g,\Phi_i,z)$. The argument of §4 allows us to show that $E''(g,\Phi,z)$ is in $L^2(\Gamma\backslash G)$ and that the inner product $(E''(\cdot,\Phi,\lambda),E''(\cdot,\Phi,\lambda))$ is equal to

$$\begin{aligned}(\lambda+\bar{\mu})^{-1}\{\exp x(\lambda+\bar{\mu})(\Phi,\Psi)-\exp(-x(\lambda+\bar{\mu}))(M(\lambda)\Phi,M(\mu)\Psi)\}\\ +(\lambda-\bar{\mu})^{-1}\{\exp x(\lambda-\bar{\mu})(\Phi,M(\mu)\Psi)-\exp x(\bar{\mu}-\lambda)(M(\lambda)\Phi,\Psi)\}.\end{aligned}$$

Call this expression $\omega(\lambda,\bar{\mu};\Phi,\Psi)$. Suppose $E''(g,\Phi,\lambda)$ is defined at $\lambda=\lambda_0$ and that $\omega(\lambda,\bar{\mu};\Phi,\Phi)$ is analytic in λ and $\bar{\mu}$ for $|\lambda-\lambda_0|<R, |\bar{\mu}-\bar{\lambda}_0|<R$. Since

$$\left|\frac{\partial^n}{\partial\lambda^n}E''(\cdot,\Phi,\lambda_0)\right|^2=\frac{\partial^{2n}}{\partial\lambda^n\partial\bar{\mu}^n}\omega(\lambda_0,\bar{\lambda}_0;\Phi,\Phi)$$

we easily show that

$$\sum_{n=0}^{\infty}\frac{(\lambda-\lambda_0)^n}{n!}\frac{\partial^n}{\partial\lambda^n}E''(\cdot,\Phi,\lambda_0)$$

converges for $|\lambda - \lambda_0| < R$ so that $E''(\cdot, \Phi, \lambda)$ is an analytic function of λ in this region with values in $L^2(\Gamma\backslash G)$. It is easy to convince oneself that if $M(\lambda)$ is a meromorphic function of λ satisfying $M(\lambda)M(-\lambda) = I$ then $\omega(\lambda, \mu; \Phi, \Psi)$ is a meromorphic function of λ and $\bar{\mu}$ whose only singularities are on the lines $\lambda = \lambda_0$ or $\bar{\mu} = \bar{\lambda}_0$ where λ_0 is a singularity of $M(\lambda)$. In verifying this use the relation $M^*(\lambda) = M(\bar{\lambda})$. Because of this remark our only responsibility is to show that $M(\lambda)$ is meromorphic in the entire complex plane and satisfies the stated functional equation. However the functions $E''(g, \Phi, z)$ will still be used in an auxiliary role.

If $\lambda = \sigma + i\tau$ then $\omega(\lambda, \bar{\lambda}; \Phi, \Psi)$ which equals

$$(1/2\sigma)\{\exp 2x\sigma(\Phi, \Psi) - \exp(-2x\sigma)(M(\lambda)\Phi, M(\lambda)\Psi)\}$$
$$+ (1/2i\tau)(\{\exp 2ix\tau(\Phi, M(\lambda)\Psi) - \exp(-2ix\tau)(M(\lambda)\Phi, \Psi)\}.$$

is a positive semidefinite form in Φ and Ψ. As a consequence

$$\|M(\lambda)\| \leqq \max\left\{\sqrt{2}\exp 2x\sigma, \frac{4\sigma}{|\tau|}\exp 2x\sigma\right\}.$$

We conclude first of all that if U is a set of the form $a \leqq \tau \leqq b, 0 < \sigma \leqq c$, with $ab > 0$, then $\|M(\lambda)\|$ is bounded uniformly for λ in U. This allows us to estimate $E(g, \Phi, \lambda)$ for λ in U and then, utilizing the close relation between $E(g, \Phi, \lambda)$ and $E''(g, \Phi, \lambda)$, to show that $\|E''(\cdot, \Phi, \lambda)\|$ is uniformly bounded for λ in U. Unfortunately the analysis required for these two steps is rather elaborate and cannot be reproduced here. It may be found in §5 of my notes on Eisenstein series. To continue we observe that this implies, by the very definition of $\omega(\lambda, \bar{\lambda}, \Phi, \tau)$, that, for each Φ and $\Psi, \omega(\lambda, \bar{\lambda}; \Phi, \Psi)$ is bounded in U. This can only be so if

$$\lim_{\sigma\downarrow 0} M^*(\sigma + i\tau)M(\sigma + i\tau) = M(\sigma - i\tau)M(\sigma + i\tau) = I$$

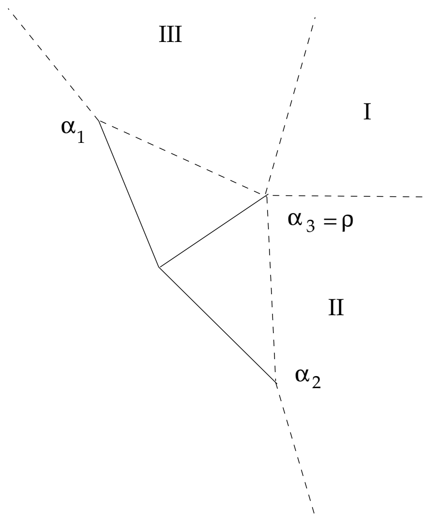
and

$$\lim_{\sigma\downarrow 0} M^{-1}(\sigma - i\tau) - M(\sigma + i\tau) = 0$$

uniformly for $t \in [a, b]$. Roughly speaking this means that $M(i\tau) = M^{-1}(-i\tau)$ for τ real. In any case, by an appropriate variant of the Schwarz reflection principle we can show that if we set $M(\lambda) = M^{-1}(-\lambda)$ for $\mathrm{Re}\lambda < 0, \lambda \notin [-(\rho, \rho)^{\frac{1}{2}}, 0]$ then $M(\lambda)$ can be extended across the imaginary axis to be meromorphic everywhere but in the interval $[-(\rho, \rho)^{\frac{1}{2}}, (\rho, \rho)^{\frac{1}{2}}]$.

Finally it must be shown that $M(\lambda)$ is also meromorphic in the interval $[-(\rho, \rho)^{\frac{1}{2}}, (\rho, \rho)^{\frac{1}{2}}]$. Since the proof of this is also based on §5 of my notes I shall not present it here.

7. In which the number of variables is usually two. In the proof of the functional equations for Eisenstein series in one variable there are two main points: to show that the function $M(z)$ is meromorphic and satisfies the stated functional equation and to construct the functions $E''(g, \Phi, z)$ and find the expression $\omega(\lambda, \bar{\mu}; \Phi, \Psi)$ for the inner product of two such functions. In the general case the first step is to show that the functions $M(s, \Lambda)$ are meromorphic everywhere and satisfy the equations of the theorem. After this one can proceed in two ways. Either one can find the analogues of the function $E''(g, \Phi, z)$ and the expression $\omega(\lambda, \bar{\mu}; \Phi, \Psi)$ as we shall do now or one can proceed in a more direct fashion to continue analytically the functions $E(g, \Phi, \Lambda)$ as is done at the end of §6 of the notes referred to before. Since in proceeding the first way I work from rather rough notes you may prefer the second upon which a little more reliance can be placed. I present the first because it introduces a number of ideas and formulas likely to be of use in the attempt to obtain in the general case a trace formula in the sense of Selberg.



The first step is based on familiar ideas. It will probably be easier to understand if we discuss it in a very simple case. Let $G = SL(3, R)$, let $\Gamma = SL(3, Z)$, and let $\{P\}$ consist of one group, the group P of upper triangular matrices in G. In the diagram α_1 and α_2 are the simple roots of $\mathfrak{a}$, $\alpha_3 = \rho = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)$ is the other positive root, and I is the region $(\Lambda, \alpha_i) > (\rho, \alpha_i), i = 1, 2$. The union of I and II is the convex hull of I and its reflection in the line $(\alpha_1, \Lambda) = 0$. The region III plays the same role as II with the line $(\alpha_1, \Lambda) = 0$ replaced by $(\alpha_2, \Lambda) = 0$. Let A be the tube over I, B the tube over the union of I and II, and C the tube over the union of I and III. The functions $M(s, \Lambda)$ are at first defined only in A.

Let $s_i, i = 1, 2$, be the reflection corresponding to the root α_i. For reasons to be discussed later $M(s_i, \Lambda)$ depends only on the projection of Λ on the orthogonal complement of the line $(\Lambda, \alpha_i) = 0$

and is a meromorphic function of Λ. Suppose we could show that, for all s, $M(s,\Lambda)$ is meromorphic in B and satisfies there the relation

$$M(ss_1,\Lambda)=M(s,s_1\Lambda)M(s_1,\Lambda). \tag{5}$$

Suppose we could also show the analogous facts for s_2. Then, for example,

$$M(s_1s_2,\Lambda)=M(s_1,s_2\Lambda)M(s_2,\Lambda)$$

in A. Since the right side is meromorphic in the entire two-dimensional complex plane so is the left. An easy induction can be used to show that $M(s,\Lambda)$ is meromorphic everywhere for each s and that the functional equations are satisfied.

How then do we continue $M(s,\Lambda)$ over B and prove (5). Suppose that for any Φ in $\mathcal{E}(V,W)$ we could analytically continue, $E(\cdot,\Phi,\Lambda)$ over all of B (except perhaps for some poles) and show that

$$E(\cdot,M(s_1,\Lambda)\Phi,s_1\Lambda)=E(\cdot,\Phi,\Lambda) \tag{6}$$

in this region. If N is the group of upper triangular unipotent matrices and Ω is the Weyl group of G

$$\int_{\Gamma\cap N\backslash N}E(ng,\Phi,\Lambda)dn=\sum_{s\in\Omega}\exp(s\Lambda(H(g))+\rho(H(g)))(M(s,\Lambda)\Phi)(g)$$

and

$$\int_{\Gamma\cap N\backslash N}E(ng,M(s_1,\Lambda)\Phi,s_1\Lambda)dn=\sum_{s\in\Omega}\exp(ss_1\Lambda(H(g))+\rho(H(g)))(M(s,s_1\Lambda)M(s_1,\Lambda)\Phi)(g).$$

The left-hand sides of these equations are meromorphic and equal in B. As a consequence the functions $M(s,\Lambda)$ are all meromorphic in the same region and the equations (5) are satisfied.

As a further simplification we shall in proving (6) assume that $\mathcal{E}(V,W)$ is the space of constant functions. The space $\mathfrak{a}$ is the set of diagonal matrices $D(x_1,x_2,x_3)$ of trace zero. Suppose α_1 is the linear function x_1-x_2. Let ${}^*P$ be the group of all matrices in G of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ 0 & 0 & x_{33}\end{pmatrix}.$$

${}^*N$ is the group of all such matrices with $x_{12}=x_{21}=0$ and $x_{11}=x_{22}=x_{33}=1$. ${}^*M$ is the group of all such matrices with $x_{13}=x_{23}=0$ and $x_{33}=\pm1$ and ${}^*\Theta=\Gamma\cap{}^*N\backslash\Gamma\cap{}^*P$ is an arithmetic subgroup of ${}^*M$. Moreover

$${}^{\dagger}P={}^*N\backslash P\cap{}^*N{}^*M$$

is a percuspidal subgroup of ${}^*M$. We can choose ${}^\dagger V$ and ${}^\dagger W$ bearing the same relation to ${}^\dagger P$ as V and W bear to P so that $\mathcal{E}({}^\dagger V, {}^\dagger W)$ is also the space of constant functions. There is a natural map $\Phi \to {}^\dagger\Phi$ of $\mathcal{E}(V, W)$ onto $\mathcal{E}({}^\dagger V, {}^\dagger W)$. $\mathfrak{a}$ is the direct sum of ${}^*\mathfrak{a} = \{D(x, x, -2x)\}$ and ${}^\dagger\mathfrak{a} = \{D(x, -x, 0)\}$ and ${}^\dagger\mathfrak{a}$ may be regarded as the split component of ${}^\dagger P$. The restriction ${}^\dagger s_1$ of s_1 to ${}^\dagger\mathfrak{a}$ belongs to the Weyl group of ${}^\dagger\mathfrak{a}$. Corresponding to ${}^\dagger s_1$ there is a function $M({}^\dagger s_1, {}^\dagger\Lambda)$ on the dual of ${}^\dagger\mathfrak{a}_{\mathbf{C}}$ with values in the space of linear transformations of $\mathcal{E}({}^\dagger V, {}^\dagger W)$. Because the dimension of $\mathfrak{a}$ is one we know that $M({}^\dagger s_1, {}^\dagger\Lambda)$ is meromorphic everywhere in the dual space of ${}^\dagger\mathfrak{a}_{\mathbf{C}}$. The dual space of $\mathfrak{a}_{\mathbf{C}}$ is of course isomorphic to the sum of the dual spaces of ${}^*\mathfrak{a}_{\mathbf{C}}$ and ${}^\dagger\mathfrak{a}_{\mathbf{C}}$. Thus we may decompose a general Λ as a sum ${}^*\Lambda + {}^\dagger\Lambda$. A careful study of the computations following the statement of Lemma 3 reveals that if Φ corresponds to ${}^\dagger\Phi$ then $M(s_1, \Lambda)\Phi$ corresponds to $M({}^\dagger s_1, {}^\dagger\Lambda){}^\dagger\Phi$. This is the fact with which we started.

By definition

$$\begin{aligned} E(g, \Phi, \Lambda) &= \sum_{\Gamma\cap P\backslash\Gamma} \exp(\Lambda(H(\gamma g)) + \rho(H(\gamma g)))\Phi(\gamma g) \\ &= \sum_{\Gamma\cap {}^*P\backslash\Gamma} \left\{ \sum_{\Gamma\cap P\backslash\Gamma\cap {}^*P} \exp(\Lambda(H(\delta\gamma g)) + \rho(H(\delta\gamma g)))\Phi(\delta\gamma g) \right\}. \end{aligned}$$

Consider the inner sum with the argument γg replaced by g and let $g = namk, n \in {}^*N, m = m(g) \in {}^*M, a \in {}^*A$, and k in K. It equals

$$\exp({}^*\Lambda({}^*H(g)) + \rho({}^*H(g))) \left\{ \sum_{{}^*\Theta\cap{}^\dagger P\backslash{}^*\Theta} \exp({}^\dagger\Lambda({}^\dagger H(\theta m)) + \rho({}^\dagger H(\theta m))){}^\dagger\Phi(\theta m) \right\}$$

or

$$\exp({}^*\Lambda({}^*H(g)) + \rho({}^*H(g)))E(m, {}^\dagger\Phi, {}^\dagger\Lambda).$$

Consequently

$$E(g, \Phi, \Lambda) = \sum_{\Gamma\cap{}^*P\backslash\Gamma} \exp({}^*\Lambda({}^*H(\gamma g)) + \rho({}^*H(\gamma g)))E(m(\gamma g), {}^\dagger\Phi, {}^\dagger\Lambda).$$

It can be shown that the series on the right converges at any point of B at which it is defined and that it represents a meromorphic function in B. The relation (6) is an immediate consequence of the known relation

$$E(m, M({}^\dagger s_1, {}^\dagger\Lambda){}^\dagger\Phi, {}^\dagger s_1{}^\dagger\Lambda) = E(m, {}^\dagger\Phi, {}^\dagger\Lambda).$$

8. A combinatorial lemma. Before defining the functions $E''(g,\Phi,\Lambda)$ we had best discuss a simple combinatorial lemma. V will be a Euclidean space; V' will be its dual; $\{\lambda^1,\ldots,\lambda^p\}$ will be a basis of V' such that $(\lambda^i,\lambda^j)\leqq 0$ if $i\neq j$; and $\{\mu^1,\ldots,\mu^p\}$ will be a basis of V' dual to $\{\lambda^1,\ldots,\lambda^p\}$. Suppose $\mathfrak{p}$ is an ordered partition of $\{1,\ldots,p\}$ into $r=r(\mathfrak{p})$ nonempty subsets $F_u, 1\leqq u\leqq r$. If $i\in F_u$ let $\mu^i_{\mathfrak{p}}$ be the projection of μ^i on the orthogonal complement of the space spanned by $\{\mu^j \mid j\in F_v, v<u\}$ and let $\lambda^i_{\mathfrak{p}}, 1\leqq i\leqq p$, be such that $(\lambda^i_{\mathfrak{p}},\mu^j_{\mathfrak{p}})=\delta_{ij}$. A point Λ in V' will be called singular if, for some i and some $\mathfrak{p}$, $(\Lambda,\mu^i_{\mathfrak{p}})=0$ or $(\Lambda,\lambda^i_{\mathfrak{p}})=0$ and a point H in V will be called singular if $\lambda^i_{\mathfrak{p}}(H)=0$ for some i and some $\mathfrak{p}$. Suppose Λ in V' is not singular. Define the function $\phi^\Lambda_{\mathfrak{p}}$ on V by the condition that $\phi^\Lambda_{\mathfrak{p}}(H)=0$ unless $\lambda^i_{\mathfrak{p}}(H)(\mu^i_{\mathfrak{p}},\Lambda)<0$ for all i when $\phi^\Lambda_{\mathfrak{p}}(H)=1$. Define the function $\psi^\Lambda_{\mathfrak{p}}$ by the condition that $\psi^\Lambda_{\mathfrak{p}}(H)=0$ unless $\lambda^i_{\mathfrak{p}}(H)>0$ for i in F_1 and $\lambda^i_{\mathfrak{p}}(H)(\mu^i_{\mathfrak{p}},\Lambda)<0$ for i not in F_1 when $\psi^\Lambda_{\mathfrak{p}}(H)=1$. Let $a^u_{\mathfrak{p}}$ be the number of elements in F_u; let $b^\Lambda_{\mathfrak{p}}$ be the number of i such that $(\mu^i_{\mathfrak{p}},\Lambda)<0$, and let $c^\Lambda_{\mathfrak{p}}$ be the number of i in $\cup_{u=2}^r F_u$ such that $(\mu^i_{\mathfrak{p}},\Lambda)<0$. Set

$$\alpha^\Lambda_{\mathfrak{p}}=b^\Lambda_{\mathfrak{p}}+\sum_{u=1}^r(a^u_{\mathfrak{p}}+1)\ ,\beta^\Lambda_{\mathfrak{p}}=1+c^\Lambda_{\mathfrak{p}}+\sum_{u=2}^r(a^u_{\mathfrak{p}}+1).$$

Lemma 5. *If H is not singular then*

$$\sum_{\mathfrak{p}}(-1)^{\alpha^{\Lambda\mathfrak{p}}}\phi^\Lambda_{\mathfrak{p}}(H)=\sum_{\mathfrak{p}}(-1)^{\beta^\Lambda_{\mathfrak{p}}}\psi^\Lambda_{\mathfrak{p}}(H)$$

if $(\lambda^i,\Lambda)<0$ for some i and

$$\sum_{\mathfrak{p}}(-1)^{\alpha^\Lambda_{\mathfrak{p}}}\phi^\Lambda_{\mathfrak{p}}(H)=1+\sum_{\mathfrak{p}}(-1)^{\beta^\Lambda_{\mathfrak{p}}}\psi^\Lambda_{\mathfrak{p}}(H)$$

if $(\lambda^i,\Lambda)>0$ for all i.

It is a pleasant exercise to prove this lemma.

9. $L^2(\Gamma\backslash G)$ as the bed of Procrustes. Suppose $\mathfrak{a}=\mathfrak{a}^{i_0}$ and $\Phi\in\mathcal{E}(V^{i_0},W)$ (the notation is that of §4). Suppose Λ in the dual of $\mathfrak{a}_{\mathbf{C}}$ is such that for all i and all s in $\Omega(\mathfrak{a},\mathfrak{a}^i)$ the point $\mathrm{Re}(s\Lambda)$ is not singular in the sense of the previous paragraph. Take V to be $\mathfrak{a}^i$ and $\lambda^1,\ldots,\lambda^p$ to be the simple roots of $\mathfrak{a}^i$. Suppose also that $\mathrm{Re}(\Lambda,\alpha)>(\rho,\alpha)$ if α is a positive root of $\mathfrak{a}$. Choose a point H_0 in the split component of the standard percuspidal subgroup such that $\alpha(H_0)$ is very large for every positive root and let H^i_0 be its projection on $\mathfrak{a}^i$. For each i let $F''_i(g,\Phi,\Lambda)$ be the function

$$\sum_{s\in\Lambda(\mathfrak{a},\mathfrak{a}^i)}\sum_{\mathfrak{p}}(-1)^{\alpha_{\mathfrak{p}}^{\mathrm{Re}(s\Lambda)}}_{\mathfrak{p}}\phi_{\mathfrak{p}}^{\mathrm{Re}(s\Lambda)}(H^i(g)-H^i_0)\exp(s\Lambda(H^i(g))+\rho(H^i(g)))((M(s,\Lambda)\Phi)(g)).$$

Since the functions $\psi_{\mathfrak{p}}^{\mathrm{Re}(s\Lambda)}(H^i(g) - H_0^i)$ are zero on

$$\{g \in G \mid \mu^j(H^i(g) - H_0^i) < 0, 1 \leqq j \leqq p\}$$

the lemma shows that $F_i''(g, \Phi, \Lambda)$ is zero almost everywhere on this set unless $i = i_0$ and that

$$F_{i_0}''(g, \Phi, \Lambda) - \exp(\Lambda(H^{i_0}(g)) - \rho(H^{i_0}(g)))\Phi(g)$$

is zero almost everywhere on this set. Set

$$E''(g, \Phi, \Lambda) = \sum_{i=1}^{r} \sum_{\Gamma \cap P^i \backslash \Gamma} F_i''(\gamma g, \Phi, \Lambda).$$

It is a consequence of the above remarks and the minimum principle stated by Borel in his lectures on reduction theory that if U is any compact set in $\Gamma \backslash G$ the point H_0 may be so chosen that

$$E''(g, \Phi, \Lambda) = E(g, \Phi, \Lambda)$$

almost everywhere on U.

It is an easy matter to compute the Fourier transform of the functions $F_i''(g, \Phi, \Lambda)$. The arguments of §4 may be used to show that $E''(g, \Phi, \Lambda)$ is square integrable. The relation (2) may be used to evaluate

$$(E''(g, \Phi, \Lambda), E''(g, \Psi, M))$$

if Ψ lies in $\mathcal{E}(V^{i_0'}, W)$ and M in the dual of $\mathfrak{A}_{\mathbf{C}}' = \mathfrak{A}_{\mathbf{C}}^{i_0'}$ satisfies the same conditions as Λ. If $\alpha_{\mathfrak{p}} = \sum_{u=1}^{r}(a^u + \mathfrak{p} + 1)$ the result is

$$\sum_{j=1}^{r} \sum_{s \in \Omega(\mathfrak{a}, \mathfrak{a}^j)} \sum_{t \in \Omega(\mathfrak{a}^i, \mathfrak{a}^j)} \sum_{\mathfrak{p}} (-1)^{\alpha_{\mathfrak{p}}} \frac{\exp(t\Lambda + s\bar{M})(H_0^j)}{\prod_{m=1}^{p} (\mu_{\mathfrak{p}}^m, t\Lambda + s\bar{M})} (M(t, \Lambda)\Phi, M(s, M)\Psi).$$

The notation is poor because the linear functions $\mu_{\mathfrak{p}}^m$ depend, of course, on j. Since it can be shown that the functional equations for the functions $M(t, \Lambda)$ imply that this expression is an analytic function of Λ and $\bar{M}$ wherever all the functions $M(t, \Lambda)$ and $M^*(s, M)$ are, we can proceed as in the rank one case to complete the proof of the theorem.

10. More Eisenstein series. Once one knows that the functions $E(g, \Phi, \Lambda)$ and $M(s, \Lambda)$ are meromorphic everywhere one can try to use the formula

$$(\hat{\phi}, \hat{\psi}) = \frac{1}{(2\pi)^q} \int_{\mathrm{Re}\Lambda = \Lambda_0} \sum (M(s, \Lambda)\Phi(\Lambda), \Psi(-s\bar{\Lambda}))|d\Lambda|$$

to analyze the space $L(\{P\},\{V\},W)$. In order to get some idea of what actually happens let us look at a particular case. We shall study the case that $G = SL(3,\mathbf{R}), \Gamma = SL(3,\mathbf{Z}), P$ is the percuspidal subgroup introduced in §7, and V and W, and hence $\mathcal{E}(V,W)$, are the space of constant functions. As a preliminary let us look at the same situation with $SL(3,\mathbf{R})$ replaced by $SL(2,\mathbf{R})$ and with the other objects of our attention modified accordingly. Godement has already done this in his first lecture. However, he was not concerned with the discrete spectrum in $L(\{P\},\{V\},W)$ and we shall be.

To remind you of the notation:

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \big| x \in \mathbf{R} \right\};$$

$$A_{\mathbf{R}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \big| \alpha \in \mathbf{R}^{\times} \right\};$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \big| \theta \in \mathbf{R} \right\}.$$

Take $dn = dx, da = |\alpha|^{-1}d\alpha, dk = d\theta/2\pi$, and take dg to be such that

$$\int_G \phi(g)dg = \int_N dn \int_{A_{\mathbf{R}}} da \int_K dk|\alpha|^{-2}\phi(nak).$$

Then the inner product of $\hat{\phi}$ is equal to

$$(a) \qquad \frac{1}{2\pi i}\int_{\mathrm{Re}\ z=z_0} \Phi(z)\bar{\Psi}(-\bar{z}) + \frac{\xi(z)}{\xi(1+z)}\Phi(z)\bar{\Psi}(\bar{z})dz \quad (z_0 > 1).$$

Here $\Phi(z) = \Phi(\Lambda(z))$ where $\Lambda(z)$ is the linear function such that $\Lambda(H_\alpha) = z$ if

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the present situation $\Phi(\cdot)$ is a scalar-valued function so inner products are replaced by products and if s is the nontrivial element of the Weyl group $M(s,\Lambda(z))$ is a scalar-valued function equal to $\xi(z)/\xi(1+z)$ if

$$\xi(z) = \pi^{-z/2}\Gamma(z/2)\xi(z).$$

Using the residue theorem we see that the expression (a) is the sum of two terms

$$(b) \qquad \frac{1}{2\pi i}\int_{\mathrm{Re} z=0} \Phi(z)\bar{\Psi}(-\bar{z}) + \frac{\xi(z)}{\xi(z+1)}\Phi(z)\bar{\Psi}(\bar{z})dz$$

and

$$(c) \qquad \frac{1}{\xi(2)}\Phi(1)\bar{\Psi}(1).$$

The esimates of §6 justify this application of the residue theorem. We immediately see that $L(\{(P\},\{V\},W)$ is the direct sum of two subspaces $L_i(\{P\},\{V\},W), i=0,1$. $L_0(\{P\},\{V\},W)$ is the space of constant functions and the inner product of the projection of $\hat{\phi}$ and $\hat{\psi}$ on this space is given by (c). The inner product of the projection of $\hat{\phi}$ and $\hat{\psi}$ on $L_1(\{P\},\{V\},W)$ is given by (b) which equals

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{2}\left\{\Phi(iy)+\frac{\xi(-iy)}{\xi(1-iy)}\Phi(-iy)\right\}\cdot\frac{1}{2}\overline{\left\{\Psi(iy)+\frac{\xi(-iy)}{\xi(1-iy)}\Psi(-iy)\right\}}dy.$$

As a consequence $L_1(\{P\},\{V\},W)$ is isometric to the space of all functions Υ square integrable on the imaginary axis with respect to the measure dy/π, which satisfy

$$\Upsilon(-iy)=\frac{\xi(iy)}{\xi(1+iy)}\Upsilon(iy).$$

The term (c) comes from the pole of $\xi(z)/\xi(1+z)$ at $z=1$. As it happens $E(g,\Phi,z)$ also has a pole at $z=1$; to see what the residue is we observe that

$$(a)\qquad \int_{\Gamma\cap N\backslash N}\mathrm{Res}\ E(ng,\Phi,z)dn=\operatorname*{Res}_{z=1}\int_{\Gamma\cap N\backslash N}E(ng,\Phi,z)dn.$$

This of course is equal to

$$\operatorname*{Res}_{z=1}\left\{\exp((\Lambda(z)+\rho)(H(g)))+\frac{\xi(z)}{\xi(1+z)}\exp((-\Lambda(z)+\rho)(H(g)))\right\}\Phi=\frac{1}{\xi(2)}\Phi$$

if $\Phi(g)\equiv\Phi$. Thus

$$\operatorname*{Res}_{z=1}E(g,\Phi,z)-\frac{1}{\xi(2)}\Phi$$

is a cusp form. Since it is also orthogonal to all cusp forms it must be zero.

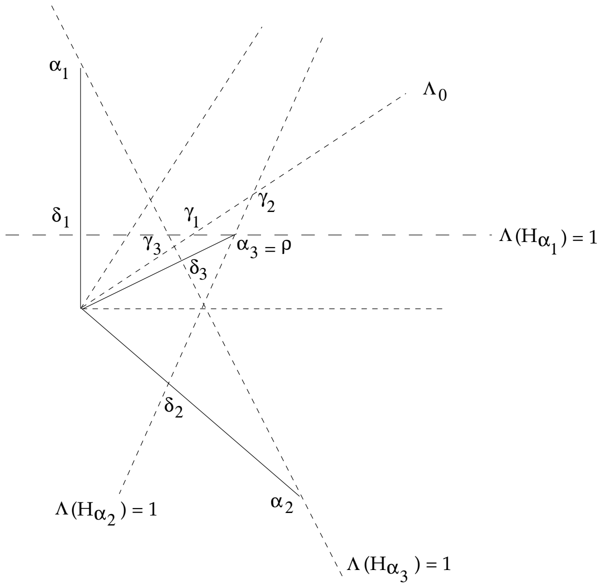
The analogue of the expression (a) when $G=SL(3,\mathbb{R})$ is

$$(d)\qquad \frac{1}{(2\pi)^2}\int_{\mathrm{Re}\Lambda=\Lambda_0}\sum_{s\in\Omega}M(s,\Lambda)\Phi(\Lambda)\bar{\Psi}(-s\bar{\Lambda})|d\Lambda|$$

with

$$M(s,\Lambda)=\prod_{\alpha>0;s\alpha<0}\frac{\xi(\Lambda(H_\alpha))}{\xi(1+\Lambda(H_\alpha))}$$

(for notation, see my lecture on the volume of fundamental domains).



The only singularities of the functions $M(s,\Lambda)$ which meet the tube over the positive Weyl chamber are simple poles on the lines $\Lambda(H_{\alpha_i}) = 1, i = 1,2,3$. If $\Phi(\Lambda)$ vanishes on these three lines then (d) is equal to

$$(e) \qquad \frac{1}{(2\pi)^2}\int_{\mathrm{Re}\Lambda=0}\sum_{s\in\Omega} M(s,\Lambda)\Phi(\Lambda)\bar{\Psi}(-s\bar{\Lambda})|d\Lambda|.$$

Call the closed subspace generated by the functions $\hat{\phi}$ corresponding to $\Phi(\cdot)$ of this sort $L_2(\{P\},\{V\},W)$. As before, $L_2(\{P\},\{V\},W)$ is isometric to the space of square-integrable functions on the (real) plane, $\mathrm{Re}\Lambda = 0$, which satisfy certain functional equations. Under this isometry convolution by K-invariant functions on G becomes multiplication by scalar-valued functions. The inner product of the projection of any $\hat{\phi}$ and $\hat{\psi}$ on $L_2(\{P\},\{V\},W)$ is given by (e).

The difference between (d) and (e) is nothing but the inner product of the projection of $\hat{\phi}$ and $\hat{\psi}$ on the orthogonal complement of $L_2(\{P\},\{V\},W)$. If $\mathfrak{s}_i$ is the complex line $\Lambda(H_{\alpha_i}) = 1, 1 \leqq i \leqq 3$, then by the residue theorem the difference will be a sum of three integrals taken respectively over the real lines $\mathrm{Re}\Lambda = \gamma_i$ in $\mathfrak{s}_i$. To describe the exact form of the integrals we need a little notation. Let $\Omega(\mathfrak{s}_i,\mathfrak{s}_j)$ be the set of distinct affine transformations from $\mathfrak{s}_i$ to $-\mathfrak{s}_j$ obtained by restricting those elements of Ω which take $\mathfrak{s}_i$ to $-\mathfrak{s}_j$. The difference we spoke of can be written as

$$(f) \qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s\in\Omega(\mathfrak{s}_i,\mathfrak{s}_j)}\frac{1}{2\pi}\int_{\mathrm{Re}\Lambda=\gamma_i} M(s,\Lambda)\Phi(\Lambda)\bar{\Psi}(-s\Lambda)|d\Lambda|.$$

Here $M(s;\Lambda)$ is a certain scalar valued function on $\mathfrak{s}_i$. In a moment I shall give the explicit form of these functions. First we observe that $\Omega(\mathfrak{s}_1,\mathfrak{s}_2)$ contains exactly one element ρ, the restriction to $\mathfrak{s}_1$ of the reflection in $\Lambda(H_{\alpha_1})=0$, that $\Omega(\mathfrak{s}_1,\mathfrak{s}_2)$ contains exactly one element σ, the restriction to $\mathfrak{s}_1$ of the reflection in $\Lambda(H_{\alpha_3})=0$, and that $\Omega(\mathfrak{s}_1,\mathfrak{s}_3)$ contains exactly one element τ, the restriction to $\mathfrak{s}_1$ of the rotation through an angle of $2\pi/3$. From these three elements we can obtain for each i and j the unique element of $\Omega(\mathfrak{s}_i,\mathfrak{s}_j)$. For example, the unique element of $\Omega(\mathfrak{s}_3,\mathfrak{s}_2)$ is $\sigma\rho\tau^{-1}$. Observe that, for example, $\tau\rho$ takes $\mathfrak{s}_1$ to $\mathfrak{s}_3$. If Λ is in $\mathfrak{s}_1$ and $\Lambda(H_{\beta_3})+\frac{1}{2}=z$ the number in the second row and third column of the following table is $M(\sigma\rho\tau^{-1},\tau\rho\Lambda)$. The other entries are interpreted accordingly.

	ρ	σ	τ
ρ	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\frac{\xi\left(-z-\frac{1}{2}\right)}{\xi\left(-z+\frac{3}{2}\right)}$	$\frac{1}{\xi(2)}\frac{\xi\left(\frac{1}{2}-z\right)}{\xi\left(\frac{3}{2}-z\right)}$
σ	$\frac{1}{\xi(2)}\frac{\xi\left(z-\frac{1}{2}\right)}{\xi\left(z+\frac{3}{2}\right)}$	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\frac{\xi\left(\frac{1}{2}+z\right)}{\xi\left(\frac{3}{2}+z\right)}$
τ	$\frac{1}{\xi(2)}\frac{\xi\left(z+\frac{1}{2}\right)}{\xi\left(z+\frac{3}{2}\right)}$	$\frac{1}{\xi(2)}\frac{\xi\left(-z+\frac{1}{2}\right)}{\xi\left(-z+\frac{3}{2}\right)}$	$\frac{1}{\xi(2)}\frac{\xi\left(\frac{1}{2}-z\right)}{\xi\left(\frac{3}{2}-z\right)}\frac{\xi\left(\frac{1}{2}+z\right)}{\xi\left(\frac{3}{2}+z\right)}$

The matrix defined by this table is of rank one.

The integral (f) is the sum of

$$(g)\qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s\in\Omega(\mathfrak{s}_i,\mathfrak{s}_j)}\frac{1}{2\pi}\int_{\mathrm{Re}\Lambda=\delta_i}M(s,\Lambda)\Phi(\Lambda)\bar{\Psi}(-s\bar{\Lambda})|d\Lambda|$$

and

$$(h)\qquad \frac{1}{\xi(2)\xi(3)}\Phi(\rho)\bar{\Psi}(\rho).$$

The points δ_i are shown on the diagram. Correspondingly, the orthogonal complement of $L_2(\{P\},\{V\},W)$ in $L(\{P\},\{V\},W)$ is the direct sum of $L_1(\{P\},\{V\},W)$ and $L_0(\{P\},\{V\},W)$ and the inner product of the projections of $\hat{\phi}$ and $\hat{\psi}$ on these two spaces are given respectively by (g) and (h). $L_0(\{P\},\{V\},W)$ is just the space of constant functions. There is an isometry of $L_1(\{P\},\{V\},W)$ with a subspace of the direct sum of the spaces of square-integrable functions on $\mathrm{Re}\Lambda_1=\delta_1$ and $\mathrm{Re}\Lambda_1=\delta_2$ which is such that convolution by K-invariant functions corresponds to multiplication by scalar valued functions.

The functions $E(g,\Phi,\Lambda)$ also have poles on the lines $\mathfrak{s}_i$. To compute the residue of $E(g,\Phi,\Lambda)$ on the line $\mathfrak{s}_1$ we combine our earlier result for $SL(2,\mathbf{R})$ with the formula of §7. The result is

$$\frac{1}{\xi(2)}\sum_{\Gamma\cap{}^*P\backslash\Gamma}\exp({}^*\Lambda({}^*H(\gamma g))+\rho({}^*H(\gamma g)))\Phi=\frac{1}{\xi(2)}E'(g,\Phi,{}^*\Lambda),$$

an Eisenstein series belonging to the cuspidal subgroup ${}^*P$. The Eisenstein series on the left is, unlike those we have dealt with up to now, not an Eisenstein series associated to a cusp form. An automatic consequence of the above is that the function defined by the sum on the left is everywhere meromorphic.

Denote the residue of $E(g,\Phi,\Lambda)$ on $\mathfrak{s}_i$, by $E_i(g,\Phi,\Lambda)$. Then

$$\int_{\Gamma\cap N\backslash N} E_i(ng,\Phi,\Lambda)dn = \sum_{j=1}^{3}\sum_{s\in\Omega(\mathfrak{s}_i,\mathfrak{s}_j)} \exp(s\Lambda(H(g))+\rho(H(g)))(M(s,\Lambda)\Phi)(g).$$

Since the matrix introduced above is of rank one this implies that

$$\begin{aligned} E_2(g,\Phi,\sigma\rho\Lambda) &= \frac{\xi(-z-\frac{1}{2})}{\xi(-z+\frac{3}{2})}E_1(g,\Phi,\Lambda), \\ E_3(g,\Phi,\tau\rho\Lambda) &= \frac{\xi(\frac{1}{2}-z)}{\xi(\frac{3}{2}-z)}E_1(g,\Phi,\Lambda). \end{aligned}$$

In the general case one can show that $L(\{P\},\{V,\},W)$ is a direct sum

$$\bigoplus_{i=1}^{g} L_i(\{P\},\{V\},W)$$

with g equal to the rank of the elements of $\{P\}$. In the course of doing this one sees that all Eisenstein series define functions which are everywhere meromorphic and satisfy functional equations of the expected type. The spectrum of $L_i(\{P\},\{V\},W)$ is again continuous of dimension i. Beyond this, however, the situation is very foggy.