SEMI-GROUPS AND REPRESENTATIONS OF LIE GROUPS †

ABSTRACT

With every Lie semi-group, Π, possessing certain regularity properties, there is associated a Lie algebra, A; and with every strongly continuous representation of Π in a Banach space there is associated a representation $A(a)$ of A. Certain theorems regarding this representation are established.

The above theorems are valid for a representation of a Lie group also. In this case, it is shown that it is possible to extend the representation to elliptic elements of the universal enveloping algebra. It is also shown that the representatives of the strongly elliptic elements of the universal enveloping algebra are the infinitesimal generators of holomorphic semi-groups. Integral representations of these semi-groups are given.

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INTRODUCTION

The study of Lie semi-groups and their representations was initiated by E. Hille in [6]. For a survey of the basic problems and results the reader is referred to that paper and to Chapter XXV of [7]. This thesis is a continuation of work begun there; we summarize briefly the results it contains.

In Chapter I, the "Dense Graph Theorems" suggested in [6] are proved and it is shown that linear combinations of the infinitesimal generators form, in the precise sense of Theorems 4 and 6, a representation of a Lie algebra canonically associated with the semi-group.

In Chapter II the study of the infinitesimal generators is continued. For the work of this chapter it is necessary to assume that the semi-group is a full Lie group. It is shown (Theorem 7) that the representation of the Lie algebra can be extended, in a natural manner, to a representation of the elliptic elements of the universal enveloping algebra. Then the spectral properties of operators corresponding to strongly elliptic elements are discussed; in particular it is shown (Theorem 8) that they are the infinitesimal generators of semi-groups holomorphic in a sector of the complex plane. Canonical representations of these semi-groups as integrals are given in Theorem 9.

The reader interested in other work to which that of Chapter II is related is referred to [9], [13], [19], and a forthcoming paper by E. Nelson.

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CHAPTER I

1. Lie semi-groups have been defined in [6] and [7]. We shall be concerned with semi-groups, Π, whose underlying topological space is $\bar{E}_n^+ = \{(x_1, \dots, x_n) | x_i \geq 0,\, i = 1, \dots, n\}$, a subset of real Euclidean n-space. We denote the semi-group operation by either $F(p, q)$ or $p \circ q$. The following conditions, numbered as in [7], are supposed satisfied.

P_2. $F(a, 0) = F(0, a)$

P_3. $F\big(a, F(b, c)\big) = F\big(F(a, b), c\big)$

P_5. There exists a fixed positive constant B such that for all points a_1, a_2 and b in Π

$$\max\{|F(a_1, b) - F(a_2, b)|,\, |F(b, a_1) - F(b, a_2)|\} \leq (1 + B|b|)|a_1 - a_2|$$

P_6. There exists a positive, monotone increasing continuous function $\omega(t), 0 < t < \infty$, tending to zero with t such that

$$|F(a, b) - a - b| \leq r\omega(s) \quad r = \min\{|a|, |b|\},\, s = |a| + |b|$$

P_{11}. At every point of $\bar{E}_n^+ \times \bar{E}_n^+$ the n coordinates of $F(p, q)$ have continuous partial derivatives with respect to the coordinates of p and q up to and including the third order.

Then, by Theorem 25.3.1 of [7], there is a continuous function $f(a)$ from Π into Π such that $f\big((\rho + \sigma)a\big) = f(\rho a) \circ f(\sigma a)$ for $a \in \Pi, \rho, \sigma \geq 0$.

Let $T(p)$ be a representation of Π in a Banach space X, which is strongly continuous in a neighborhood of the origin, then for $a \in \Pi, \rho \geq 0,\, \rho \to T\big(f(\rho a)\big)$ is a strongly continuous one-parameter semi-group. Denote its infinitesimal generator by $A(a)$. In this chapter we investigate the relations among the $A(a)$ and their adjoints $A^*(a)$. For the purposes of Chapter II, we remark that similar theorems are valid for a representation of a Lie group.

We first construct a common domain for the operators, $A(a)$, $a \in \Pi$, which is large enough for our purposes. We use the following notation: $\frac{\partial F^k}{\partial p^j}(p, q) = F^k_{j;}(p, q)$; $\frac{\partial F^k}{\partial q^j}(p, q) = F^k_{;j}(p, q)$; $\frac{\partial^2 F^k}{\partial q^i \partial p^j} = F^k_{j;i}(p, q)$; $F^k_{i;j}(0, 0) - F^k_{j;i}(0, 0) = \gamma^k_{ij}$. $F(p, q)$ may be extended to a twice continuously differentiable function defined on $E_n \times E_n$.[1] Denote some fixed extension by $F(p, q)$. Since $F^k_{j;}(0, 0) = F^k_{;j}(0, 0) = \delta^k_j$

[1] Cf. the construction on p. 12 of [12].

(the Kronecker delta), there are open spheres $N_1, N_2 \subseteq N$ about the origin and three times continuously differentiable functions $\psi(q,h)$ and $\chi(q,h)$ defined on $N_1 \times N_1$ such that $\psi(0,0) = \chi(0,0) = 0$, $F\big(h, \psi(q,h)\big) = q$, and $F\big(\chi(q,h), h\big) = q$. Moreover if $F(h,p) = q$ $[F(p,h) = q]$ with $p, h \in N_2$, then $q \in N_1$ and $\psi(q,h) = p$ $[\chi(q,h) = p]$. We may also suppose that all derivatives of $\psi(q,h)$ and $\chi(q,h)$ up to the third order are bounded in N_1, that $T(p)$ is strongly continuous in $N_1 \cap \Pi$, and that det $\big(F^k_{j;}(p,0)\big) \geq 1/2$ and det $\big(F^k_{;j}(p,0)\big) \geq 1/2$ for p in N_1. If $N \subseteq N_1$ is an open sphere about the origin, set

$$E(N) = \{y = \int_\Pi K(q)T(q) \times dq | x \in X,\ K(q) \in C_2(N \cap \Pi)\}.$$

$C_2(N \cap \Pi)$ is the set of twice continuously differentiable functions which are zero outside of $N \cap \Pi$. We refer the reader to [7] for a proof that $E(N)$ is dense in X.

Proposition 1. *Let N_3 be an open sphere about the origin with $F(N_3, N_3) \subseteq N_2$. If $y \in E(N_3)$ then $T(p)y$ is a twice continuously differentiable function of p in $N_3 \cap \Pi$.*

Proof. We understand that some derivatives at the boundary will be one-sided. If $y \in E(N_3)$ and $e_j = (\delta^1_j, \ldots, \delta^n_j)$ we have, recalling that $K(q)$ is zero outside of $N_3 \cap \Pi$,

$$\begin{aligned}
&\lim_{s\to 0} s^{-1}\big(T(p+se_j)y - T(p)y\big) \\
&= \lim_{s\to 0} s^{-1} \int_{N_3\cap\Pi} K(q)\big(T\big((p+se_j)\circ q\big) - T(p\circ q)\big)x\, dq \\
&= \lim_{s\to 0} s^{-1} \int_{N_2\cap\Pi} \left(K\big(\psi(q,r)\big)\det\left(\frac{\partial\psi^k}{\partial q^i}(q,r)\right)\right)\Bigg|_{r=p}^{r=p+se_j} T(q)x\, dq \\
&= \int_{N_2\cap\Pi} \frac{\partial}{\partial p^j}\left(K\big(\psi(q,p)\big)\det\left(\frac{\partial\psi^k}{\partial q_i}(q,p)\right)\right)T(q)x\, dq + \lim_{s\to 0}\int_{N_2\cap\Pi} G(q,p,s)\big)dq \\
&= \int_{N_2\cap\Pi} \frac{\partial}{\partial p_j}\left(K\big(\psi(q,p)\big)\det\left(\frac{\partial\psi^k}{\partial q^i}(q,p)\right)\right)T(q)x\, dq
\end{aligned}$$

since $G(q,p,s)$ converges boundedly to 0 with s. The final integral is a continuous function of p. In a similar manner we show that it is once continuously differentiable. We remark the following formulae, valid for $y \in E(N_3)$, $p \in N_3 \cap \Pi$:

(i) $\lim_{s\to 0} s^{-1}\big(T\big(f(sa)\big) - I\big)T(p)y$

$$\begin{aligned}
&= \lim_{s\to 0} s^{-1}\big(T\big(f(sa)\circ p\big)y - T(p)y\big) \\
&= \lim_{s\to 0}\left[\textstyle\sum_{j=1}^n s^{-1}\big(F^j\big(f(sa)\circ p\big) - p^j\big)\frac{\partial}{\partial p^j}T(p)y + s^{-1}o\big(|f(sa)\circ p - p|\big)\right]
\end{aligned}$$

which equals

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n}F_{i;}^{j}(0,p)a^{i}\right)\frac{\partial}{\partial p^{j}}T(p)y. \tag{1.1}$$

So $T(p)y \in D\big(A(a)\big)$, and $A(a)T(p)y$ is given by (1.1).

(ii) $T(p)A(a)y = \lim_{s\to 0} s^{-1}\big(T\big(p\circ f(sa)\big)y - T(p)y\big)$, which equals

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{n}F_{;k}^{j}(p,0)a^{k}\right)\frac{\partial^{j}}{\partial p^{j}}T(p)y \tag{1.2}$$

(iii) Setting $\big(F_{;k}^{j}(p,0)\big)^{-1} = (\gamma_{k}^{j}(p))$, we have

$$\frac{\partial}{\partial p_{j}}T(p)y = \sum_{k=1}^{n}\gamma_{k}^{j}(p)T(p)A(e_{k})y \tag{1.3}$$

(iv) Setting $\sum_{j=1}^{n}F_{i;}^{j}(0,p)\gamma_{j}^{k}(p) = \beta_{i}^{k}(p)$, we have

$$A(a)A(b)T(p)y = \sum_{k,j=1}^{n}\left(\sum_{i=1}^{n}\beta_{i}^{k}(p)b^{i}\right)\left(\sum_{m=1}^{n}F_{m;}^{j}(0,p)a^{m}\right)\frac{\partial}{\partial p^{j}}T(p)A(e_{k})y.$$

(v) (α) $A(a+b)y = A(a)y + A(b)y$

(β) $A(e_i)A(e_j)y - A(e_j)A(e_i)y = \sum_{k=1}^{n}\gamma_{ij}^{k}A(e_k)y.$

For a proof of the latter relation, see [7], p. 758.

2. The first theorem is known as a "Dense Graph Theorem" and has been suggested by E. Hille in [6] and [7].

Theorem 1. *Let* $\{a_1,\ldots,a_p\} \subseteq \Pi$. *If* G_o *is the closure in the product topology on* $X\times\ldots\times X$ *($p+1$ factors) of* $\{\big(x, A(a_1)x,\ldots,A(a_p)x\big)|x \in E(N_3)\}$ *and* $G = \{(x, A(a_1)x,\ldots,A(a_p)x)|x \in \cap_{j=1}^{p}D(A(a))\}$, *then* $G = G_o$.

Proof. $G \supseteq G_o$ since an infinitesimal generator is a closed operator. We show that $G_o \supseteq G$. Let $\{b_{r+1},\ldots,b_n\}$ be a maximal linearly independent subset of $\{a_1,\ldots,a_p\}$; it is sufficient to prove the theorem for the former set. Let $\{b_1,\ldots,b_n\} \subseteq \Pi$ be a basis for E_n. If $t=(t^1,\ldots,t^n)\in\Pi$, set $p(t) = f(t^1b_1)\circ\ldots\circ f(t^nb_n)$. $p(t)$ is a twice continuously differentiable map of Π into Π and may be extended to a twice continuously differentiable map of E_n into E_n. Denote some fixed extension by $p(t)$. The above process is analogous to the introduction of canonical coordinates of the second kind on a Lie group.

Since $\frac{\partial p^k}{\partial t^j}(0) = b^k_j$, $p(t)$ has a twice continuously differentiable inverse defined in a sphere N_4 about the origin. We may suppose that $F(N_4, N_4) \subseteq N_3$ and that all derivatives of the inverse function up to the second order are bounded in N_4. If $y \in E(N_4)$ and $p \in N_4 \cap \Pi$, then $T(p)y \in E(N_3)$. For $y \in E(N_4)$. set

$$u(y,\, s) = \int_{R(s)} S(t)y\,dt$$

where $s = (s^1, \ldots, s^n)$, $S(t) = T\big(p(t)\big)$, $R(s)$ is the rectangle with sides $[0,\; s^j e_j]$, and $R(s)$ is contained in the image of N_4 under the inverse map. By (1.1),

$$A(b_k)u(y,s) = \int_{R(s)} A(b_k)S(t)y\,dt = \int_{R(s)} \sum_{i=1}^{n} \zeta^i_k(t)\frac{\partial}{\partial t^i}S(t)y\,dt$$

where $\zeta^i_k(t) = \sum_{j,m=1}^n F^j_{m;}\big(0,\, p(t)\big)b^m_k\,\frac{\partial t^i}{\partial p^j}$ is once continuously differentiable. Integrate by parts to obtain

$$A(b_k)u(y,\, s) = \sum_{i=1}^{n}\int_{R(\hat{s}^i)} \zeta^i_k(t)S(t)y\Big|_{(\hat{t}^i,0)}^{(\hat{t}^i,\, s^i)} d\hat{t}^i - \int_{R(s)}\sum_{i=1}^{n}\frac{\partial\varphi k^i}{\partial t^i}S(t)y\,dt. \tag{1.4}$$

Since the integral of a function with values lying in a closed subspace of a Banach space is contained in that subspace,

$$\big(u(y,s),\; A(b_{r+1})u(y,s), \ldots, A(b_n)u(y,s)\big) \in G_o. \tag{1.5}$$

Since (1.4) is a continuous function of y and $E(N_4)$ is dense in X, for any $y \in X$, $u(y,s) \in \cap_{j=1}^n D\big(A(b_j)\big)$ and (1.4) and (1.5) hold. To complete the proof it is sufficient to show

$$\lim_{\sigma\to 0} \sigma^{-n}u\big(y, s(\sigma)\big) = y \tag{1.6}$$

$$\lim_{\sigma\to 0} A(b_k)\sigma^{-n}u\big(y, s(\sigma)\big) = A(b_k)y \tag{1.7}$$

for $k \geq r+1$, $y \in \cap_{k=r+1}^{n} D\big(A(b_k)\big)$, and $s(\sigma) = (\sigma, \dots, \sigma)$. (1.6) is clear; to prove (1.7) we expand $\zeta_k^i(t)$ in a Taylor's series and consider

$$\begin{aligned}
&\lim_{\sigma \to 0} \sigma^{-n} \int_{R\left(\hat{s}^i(\sigma)\right)} \zeta_k^i(t) S(t) y \Big|_{(\hat{t}^i, 0)}^{(\hat{t}^i, \sigma)} d\hat{t}^i \\
&= \lim_{\sigma \to 0} \sigma^{-n+1} \int_{R(\hat{s}^i)} \delta_k^i \sigma^{-1} \big(S(\hat{t}^i, \sigma) y - S(\hat{t}^i, 0) y\big)\, d\hat{t}^i \\
&+ \sigma^{-n+1} \int_{R(\hat{s}^i)} \frac{\partial \zeta_k^i}{\partial t^i}(0) S(\hat{t}^i, 0) y \, d\hat{t}^i \\
&+ \sigma^{-n+1} \int_{R(\hat{S}^i)} \left(\sum_{j \neq i} \sigma^{-1} t^j \frac{\partial \zeta_k^i}{\partial t^j}(0) \right) \dots \big(S(\hat{t}^i, \sigma) y - S(\hat{t}^i, 0) y\big) d\hat{t}^i \\
&= \delta_k^i A(b_k) y + \frac{\partial \zeta_k^i}{\partial t^i}(0) y
\end{aligned}$$

provided

$$\lim_{\sigma \to 0} \sigma^{-1} \big(S(t^k, \sigma) y - S(t^k, 0) y\big) = A(b_k) y. \tag{1.8}$$

But the left side is

$$\prod_{j=1}^{k-1} T\big(f(t^j b_j)\big)$$

applied to

$$\sigma^{-1}\big(T(f(\sigma b_k)) y - y\big) + \big(T(f(\sigma b_k)) - I\big) \left(\sum_{i=k+1}^{n} \Big(\prod_{m=k+1}^{i-1} T\big(f(t^m b_m)\big)\big) \sigma^{-1} \big(T(f(t^i b_i)) - y\big) \right)$$

and (1.8) follows if we recall that $t^i \leq \sigma$ and that $y \in D\big(A(b_i)\big)$ for $i \geq k \geq r+1$. Summing over i and taking the last term of (1.4) into account we obtain (1.7).

The following theorem is not of so much interest as the one just proved but we want to use it to establish the analogue of a theorem of [7]. We merely sketch the proof.

Theorem 2. *If* F_o *is the closure in the product topology of* $\{\big(y, A(e_1)y, \dots, A(e_n)y, A(e_i)A(e_j)y\big) | y \in E(N_3)\}$ *and if* $F = \{(y, A(e_1)y, \dots, A(e_n)y, A(e_i)A(e_j)y\big) | y \in \cap_{k=1}^{n} D\big(A(e_k)\big) \cap D\big(A(e_i)A(e_j)\big)\}$, *then* $F = F_o$.

Proof. F is a closed set and thus $F \supseteq F_o$. We show $F_o \supseteq F$. Taking $b_k = e_k$ we use the notation of the proof of Theorem 1. For $y \in E(N_4)$,

$$A(e_i)A(e_j)u(y, s) = \int_{R(s)} A(e_i)A(e_j)S(t) y \, dt = \int_{R(s)} \sum_{k,m=1}^{n} \delta_m^k(t) \frac{\partial}{\partial t^m} \big(S(t)A(e_k) y\big) \, dt$$

where $\delta^k_m(t) = \sum_{r=1}^n \beta^k_j\big(p(t)\big)F^r_{i;}\big(0,p(t)\big)\frac{\partial t^m}{\partial p^r}$ is once continuously differentiable. Integrating by parts, we obtain the following relation (1.9)

$$A(e_i)A(e_j)u(y,s) = \sum_{m=1}^n \int_{R(\hat{s}^m)} \sum_{k=1}^n \delta^k_m(t)S(t)A(e_k)y\Big|_{(\hat{t}^m,0)}^{(\hat{t}^m,s^m)} d\hat{t}^m - \int_{R(s)} \sum_{k,\,m=1}^n \frac{\partial \delta^k_m}{\partial t^m}(t)S(t)A(e_k)y\,dt.$$

Theorem 1 implies that (1.9) holds for $y \in \cap_{k=1}^n D\big(A(e_k)\big)$. The proof is now completed as above.

3. We now consider the adjoints of the infinitesimal generators and prove the corresponding dense graph theorem. If $y^* \in X^*$, the dual space of X, we denote the value of y^* at $y \in X$ by (y,y^*). If $N \subseteq N_1$, set

$$E(N) = \{y^* \in X^* | (y,y^*) = \int_\Pi \big(y, K(q)T^*(q)x^*\big)dq\}$$

with $x^* \in X^*,\ K(q) \in C^2\big(N \cap \Pi\big)$, and for all $y \in X.\ E^*(N)$ is dense in X^* in the weak-$*$ topology.

Proposition 2. *If $y^* \in E^*(N_3)\,T^*(p)y^*$ is twice continuously differentiable in the weak-$*$ topology, for p in $N_3 \cap \Pi$.*

Proof. We merely sketch the calculations since the proof is essentially the same as that of Proposition 1.

$$\begin{aligned}
&\lim_{s\to 0} s^{-1}\int_\Pi \big(y, K(q)\big(T^*(p+se_j) - T^*(p)\big)T^*(q)x^*\big)dq \\
&\quad= \lim_{s\to 0} s^{-1}\int_\Pi \big(y, K(q)\big(T^*(q\circ(p+se_j)x^* - T^*(q)x^*\big)\big)dq \\
&\quad= \int_{N^2\cap\Pi} \left(y, \frac{\partial}{\partial p^j}\left(K(\chi(q,p))\det\left(\frac{\partial \chi^k}{\partial q^i}(q,p)\right)\right)\right)T^*(q)x^*dq.
\end{aligned}$$

The last integral is again a continuously differentiable function of p.

We remark the following, valid for $y^* \in E^*(N_3)$ and $p \in N_3 \cap \Pi$.

(i)

$$\lim_{s\to 0} s^{-1}\big(y, \big(T^*(f(sa)) - I\big)T^*(p)y^*\big) = \sum_{j=1}^n \left(\sum_{m=1}^n F^j_{;m}(p,0)a^m\right)\frac{\partial}{\partial p^j}\big(y, T^*(p)y^*\big). \tag{1.10}$$

This implies that $T^*(p)y^* \in D\big(A^*(a)\big)$ and that $\big(y, A^*(a)T^*(p)y^*\big)$ is given by the right side of (1.10).

(ii) As in the remarks following Proposition 1 we may show, for $y^* \in E^*(N_3)$,

$$(\alpha')\ A^*(a+b)y^* = A^*(a)y^* + A^*(b)y^*$$

$$(\beta')\ A^*(e_i)A^*(e_j)y^* - A^*(e_j)A^*(e_i)y^* = -\sum_{k=1}^{n} \gamma_{ij}^k A^*(e_k)y^*.$$

Theorem 3. *Let* $\{a_1, \ldots, a_p\} \subseteq \Pi$. *If* H_o *is the closure (in the product of the weak-∗ topologies) of* $\{(y^*, A^*(a_1)y^*, \ldots, A^*(a_p)y^*) | y^* \in E^*(N_3)\}$ *and* $H = \{(y^*, A^*(a_1)y^*, \ldots, A^*(a_p)y^*) | y^* \in \cap_{j=1}^p D(A^*(a_j))\}$, *then* $H = H_o$.

Proof. $H \supseteq H_o$ since $A^*(a)$ is closed in the weak-∗ topology. We show $H_o \supseteq H$. Let $\{b_1, \ldots, b_r\}$ be a maximal linearly independent subset of $\{a_1, \ldots, a_p\}$; it is sufficient to prove the theorem for the former set. Let $\{b_1, \ldots, b_n\}$ be a basis for E_n. Again we use the notation of the proof of Theorem 1. If $y^* \in E^*(N_4)$, define $u(y^*,\ s)$ by

$$\big(y, u(y^*, s)\big) = \int_{R(s)} \big(y, S^*(t)y^*\big)\,dt$$

with $S^*(t) = T^*\big(p(t)\big)$. As above

$$\begin{aligned} &\big(y, A^*(b_k)u(y^*, s)\big) \\ &= \sum_{i=1}^{n} \int_{R(\hat{s}_i)} \xi_k^i(t)\big(y, S^*(t)y^*\big)\Big|_{(\hat{t}^i,0)}^{(\hat{t}^i,s^i)} d\hat{t}^i - \int_{R(s)} \sum_{i=1}^{n} \frac{\partial \xi_k^i}{\partial t^i}(t)\big(y, S^*(t)y^*\big)\,dt \end{aligned} \tag{1.11}$$

with $\xi_k^i(t) = \sum_{j,m=1}^{n} F_{;j}^m\big(p(t), 0\big) b_k^j \frac{\partial t^i}{\partial p^m}$. As above $u(y^*, s) \in \cap_{k=1}^n D\big(A^*(b_k)\big)$ for all $y^* \in X^*$ and $A^*(b_k)y^*$ is given by (1.11). Moreover,

$$\big(u(y^*, s)A^*(b_1)u(y^*, s), \ldots, A^*(b_r)u(y^*, s)\big) \in H_o.$$

The proof may be completed as before if we show that

$$\lim_{\sigma \to 0} \sigma^{-1}\big(y, \big(S^*(\hat{t}^k, \sigma) - S^*(\hat{t}^k, 0)\big)y^*\big) = \big(y, A^*(b_k)y^*\big) \tag{1.12}$$

for $1 \le k \le r$, $t^j \le \sigma$, and $y^* \in \cap_{i=1}^r D\big(A^*(b_i)\big)$. But the expression on the left equals

$$\begin{aligned} &\Bigg(\prod_{j=k+1}^{n} T\big(f(t^j b_j)\big)y, \sigma^{-1}\big(T^*(f(\sigma b_k)) - I\big)y^*\Bigg) \\ &+ \sum_{i=1}^{k-1} \Bigg(\prod_{m=i+1}^{k-1} T\big(f(t^m b_m)\big)\big(T(f(\sigma b_k)) - I\big) \prod_{j=k+1}^{n} T\big(f(t^j b_j)\big)y, \sigma^{-1}\big(T^*(f(t^i b_i)) - I\big)y^*\Bigg), \end{aligned}$$

and (1.12) follows since, see [11], $\sigma^{-1}\big(T^*(f(t^i b_i)) - I\big)y^*$ is uniformly bounded and $\sigma^{-1}\big(T^*\big(f(\sigma b_k)\big) - I\big)y^*$ converges in the weak-$*$ topology to $A^*(b_k)y^*$.

4. If $a = (a^1, \ldots, a^n) \in E_n$, $A(a) = \sum_{j=1}^n a^j A(e_j)y$ is defined for $y \in E(N_3)$. By the remarks after Proposition 2, $E^*(N_3)$ is contained in the domain of its adjoint so that $A(a)$ has a least closed extension which we again denote by $A(a)$. By Theorem 1, this notation is consistent with that used previously for a in Π.

Lemma 1. *$A^*(a)$, the adjoint of $A(a)$, is the weak-$*$ closure of the operator $\sum_{j=1}^n a^j A^*(e_j)$ with domain $E^*(N_3)$.*

Proof. Suppose $(y, x_1^*) = \big(A(a)y, x_2^*\big)$ for all $y \in E(N_3)$. Then, using Theorem 1 and the notation of its proof with $b_j = e_j$, for $y \in X$

$$\sigma^{-n}\int_{R(s(\sigma))}\big(S(t)y, x^*\big)\,dt$$
$$= \sigma^{-n}\sum_{j=1}^n a^j\left[\sum_{i=1}^n\int_{R(\hat{s}^i)}(\zeta_j^i(t)S(t)y, x_2^*)\Big|_{(\hat{t}^i,0)}^{(\hat{t}^i,\sigma)}d\hat{t}^i - \int_{R(s)}\left(\sum_{i=1}^n\frac{\partial\zeta_i^i}{\partial t^i}(t)S(t)y, x_2^*\right)dt\right].$$

Transposing and taking limits, we have

$$\lim_{\sigma\to 0}\sigma^{-n}\sum_{j=1}^n a_j\int_{R(\hat{s}^j)}\big(y, (S^*(\hat{t}^j,\sigma) - S^*(\hat{t}^j,0))x_2\big)d\hat{t}^j = (y, x_1^*).$$

Then using (1.11), we obtain

$$\lim_{\sigma\to 0}\sigma^{-n}\left(y, \sum_{j=1}^n a^j A^*(e_j)u(x_2^*, s(\sigma))\right) = (y, x_1^*). \tag{1.13}$$

Theorem 3 implies that $u\big(x_2^*, s(\sigma)\big)$ is in the domain of the weak-$*$ closure of $\sum a^j A^*(e_j)$ and (1.13) then shows that x_2^* is also. By Theorem 25.8.1 of [7] the γ_{ij}^k, as defined in Paragraph 1, may be used as the structural constants of a Lie algebra A over E_n. Denoting the Lie product, in this algebra, of a and b by $[a, b]$, we have $[a, b]^k = \sum_{i,j=1}^n \gamma_{ij}^k a^i b^j$. We can now prove the following theorem.

Theorem 4. *I. The function $a \to A(a)$ defined on A has the properties*

(i) If $x \in D\big(A(a)\big) \cap D\big(A(b)\big)$ then $x \in D\big(A(sa + tb)\big)$ and $A(sa + tb)x = sA(a)x + tA(B)x$.

(ii) If $x \in D\big(A(a)A(b)\big) \cap D\big(A(b)A(a)\big)$ then $x \in D\big(A([a, b])\big)$ and $A([a, b])x = A(a)A(b)x - A(b)xA(a)x$.

II. The function $a \to A^*(a)$ has the properties *(i) If* $x^* \in D\big(A^*(a)\big) \cap D\big(A^*(b)\big)$ *then* $x^* \in D\big(A^*([sa+tb])\big)$ *and* $A^*([sa+tb])x^* = sA^*(a)x^* + tA^*(b)x^*$.

(ii) If $x^* \in D\big(A^*(a)A^*(b)\big) \cap DA^*(b)A^*(a)\big)$ *then* $x^* \in D(A^*[a,b])$ *and* $A^*([a,b])x^* = A^*(b)A^*(a)x^* - A^*(a)A^*(b)x^*$.

Proof. If $x \in D\big(A(a)\big) \cap D\big(A(b)\big)$ there is a sequence $\{x_n\} \subseteq E(N_3)$ such that $x_n \to x$, $A(a)x_n \to A(a)x$ and $A(b)x_n \to A(b)x$; but then, using formula (α), $A(sa+tb)x_n = sA(a)x_n + tA(b)x_n \to sa(a)x + tA(b)x$. Since $A(sa+tb)$ is a closed operator, $x \in D\big(A(sa+tb)\big)$ and $A(sa+tb)x = sA(a)x + tA(b)x$.

If $x \in D\big(A(a)A(b)\big) \cap D\big(A(b)A(a)\big)$, then for $x \in E^*(N_3)$

$$\big(A(a)A(b)x - A(b)A(a)x, x^*\big) = \big(x, A^*(b)A^*(a)x^* - A^*(a)A^*(b)x^*\big).$$

So, using formula (β'),

$$\big(A(a)A(b)x - A(b)A(a)x, x^*\big) = \big(x, A^*([a,b])x^*\big). \tag{1.14}$$

The lemma implies that (1.14) holds for $x^* \in D\big(A^*([a,b])\big)$. In other words, the vector $u = \big(A(a)A(b)x - A(b)A(a)x, x\big)$ in $X \oplus X$ is annihilated by the annihilator of the subspace $U = \{(A([a,b])y, y) | y \in D\big(A([a,b])\big)\}$. So $u \in U$, or $x \in D\big(A([a,b])\big)$ and $A([a,b])x = A(a)A(b)x - A(b)A(a)x$. The remainder of the theorem is proved in a similar manner.

Recalling that if a sequence of once continuously differentiable functions and the sequences of first order derivatives converge uniformly on some domain then the limit function is once continuously differentiable and its partial derivatives are the limits of the sequences of partial derivatives, we have, using (1.3) and Theorem 1, the following result.

Theorem 5. *If* $y \in \cap_{j=1}^n D\big(A(e_j)\big)$ *then* $T(p)y$ *is once continuously differentiable in a neighborhood in* Π*, of the origin and (1.3) holds. Consequently,* $T(p)y \in D\big(A(a)\big)$ *for* $a \in E_n$ *and* p *in this neighborhood and (1.1) and (1.2) hold for* $a \in \Pi$.

The following theorem, analogous to Theorem 10.9.4 of [7], is an immediate consequence of Theorem 2.

Theorem 6. *If* $y \in \cap_{k=1}^n D\big(A(e_k)\big) \cap D\big(A(e_j)A(e_j)\big)$ *then* $y \in D\big(A(e_j)A(e_i)\big)$.

The only properties of $E(N_3)$ used in the proof of Theorem 1 were that $T(p)E(N_4) \subseteq E(N_3)$ for p in a neighborhood of the origin, that $E(N_4)$ was dense in X, and that equation (1.1) was valid. Thus, using Theorem 5, we could repeat the proof of Theorem 1 to obtain

Theorem 1′. *Let $F \subseteq E \subseteq X$ be the two dense subspaces of X contained in $\cap_{a\in\Pi} D\big(A(a)\big)$ and let $T(p)F \subseteq E$ for p in a neighborhood of the origin, then Theorem 1 is valid with $E(N_3)$ replaced by E.*

In the next chapter we shall consider strongly continuous representations of Lie groups only. The group will be denoted by G and its Lie algebra by A. A little care is necessary in the definition of A in order that the formulae above remain valid. A is taken as an algebra isomorphic to the algebra of left-invariant infinitesimal transformations with the multiplication $XY - YX$ (Cf. [2]). Then if $e(a)$ denotes the exponential map of A into G, and the representation is $T(p)$, $A(a)$ is the infinitesimal generator of the one parameter group $T\big(e(ta)\big)$. With a we associate the following left- and right-invariant infinitesimal transformations

$$L_a f(p) = \lim_{t\to 0} t^{-1}\big(f(pe(ta)) - f(p))\big)$$
$$R_a f(p) = \lim_{t\to 0} t^{-1}\big(f\big(e(-ta)p\big) - f(p)\big).$$

These mappings are isomorphisms of the Lie algebras involved. Formulae (1.1) and (1.2) may now be written very simply.

$$(1.1') \qquad A(a)T(p)y = -R_a T(p)y,$$
$$(1.2') \qquad T(p)A(a)y = L_a T(p)y.$$

The adjoint representation $p \to d\alpha_p$ of G is defined in [2]. With respect to a fixed basis $\{e_i, \ldots, e_n\}$ of A let the matrix of the representation be $\big(\alpha^i_j(p)\big)$ so that $d\alpha_p\big(\sum_{j=1}^n a^j e_j\big) = \sum_{i=1}^n \big(\sum_{j=1}^n \alpha^i_j(p)a^j\big)e_i$. We state formally the following simple lemma.

Lemma 2. *If $x \in D\big(A(a)\big)$, then $T(p)x \in D\big(A(d\alpha_p(a))\big)$ and*

$$(1.15) \qquad A\big(d\alpha_p(a)\big)T(p)x = T(p)A(a)x.$$

Proof. $x \in D\big(A(a)\big)$ if and only if

$$\lim_{t\to 0} t^{-1}\big(T(e(ta)\big)x - x) = A(a)x$$

exists, or

$$\lim_{t\to 0} t^{-1}T(p)\big(T(e(ta)\big)x - x\big) = T(p)A(a)x$$

exists, or

$$\lim_{t\to 0} t^{-1}T(p)\big(T\big(e(ta)\big) - I\big)T(p^{-1})T(p)x = \lim_{t\to 0} t^{-1}T\big(e\big(td\alpha_p(a)\big)\big)T(p)x - T(p)x = A\big(d\alpha_p(a)\big)T(p)x$$

exists.

This proves the lemma. We may write (1.15) as $A\big(d\alpha_p(a)\big)x = T(p)A(a)T(p^{-1})x$. Formula (1.15) is implicit in formulae (1.1), (1.2), and (1.3).

Using the basis of A previously introduced we set $A_i = A(e_i)$. If $\{X_i\}$, $i = 1, \dots, n$ is a set of n indeterminates and $\alpha = (\alpha_1, \dots, \alpha_m)$, is an m-tuple of integers $1 \le \alpha_i \le n$, we write $X_\alpha = X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_m}$. The absolute value of α, $|\alpha|$, is equal to m. This notation is slightly unorthodox but it is necessary to allow for the fact that the A_i's do not commute. We shall be interested in forms $\sum_{|\alpha| \le m} a_\alpha A_\alpha$ in the set $\{A_i\}$.

Let E be the set of vectors y in x which can be written in the form

$$y = \int_G K(p)T(p)x\mu(dp)$$

with μ a left-invariant Haar measure, x in X, and $K(p)$ an infinitely differentiable function with compact support in G. E satisfies the conditions of Theorem $1'$. Similarly E^* is the set of y^* in X^* such that for $x \in X$

$$(x, y^*) = \int_G K(p)\big(x, T^*(p)x^*\big)\mu(dp).$$

With any form $\sum_{|\alpha| \le m} a_\alpha X_\alpha$ we may associate the operator B_o, with domain E, defined by $B_o x = \sum_{|\alpha| \le m} a_\alpha A_\alpha x$ and the operator B_o^*, with domain E^*, defined by $B_o^* x^* = \sum_{|\alpha| \le m} a_\alpha A^*_{\alpha *} x^*$. If $\alpha = (\alpha_1, \dots, \alpha_{|\alpha|})$ then $\alpha^* = (\alpha_{|\alpha|}, \dots, \alpha_1)$.

The following simple proposition is of some interest. A special case has been considered in [17].

Proposition 3. *If, for x in E, $B_o T(p)x = T(p)B_o x$, then the adjoint of B_o is the weak-$*$ closure of B_o^*.*

Proof. Suppose that for all x in E

$$(B_o x, x_1^*) = (x, x_2^*).$$

Then, for x in E,

$$\left(\int_G K(p)T(p)x, x_2^*\right)\mu(dp) = \left(\int_G K(p)B_o T(p)x\mu(dp), x_1^*\right)$$
$$= \left(\int_G K(p)T(p)B_o x\mu(dp), x_1^*\right).$$

We may write this as

$$\left(x, \int_G K(p)T^*(p)x_2^*\mu(dp)\right) = \left(x, B_o^* \int_G K(p)T^*(p)x_1^*\mu(dp)\right).$$

The integrals in the final formula are taken in the weak-$*$ topology. We now let $K(p)$ approach the δ-function and obtain $\int_G K(p)T^*(p)x_1^*\mu(dp) \to x_1^*$ and $B_o^* \int_G K(p)T^*(p)x_1^*\mu(dp) = \int_G K(p)T^*(p)x_2^*\mu(dp) \to x_2^*$ in the weak-$*$ topology.

CHAPTER II

1. Before proving the principal theorems of this chapter we must establish some estimates for the fundamental solutions of strongly elliptic differential equations and a differeintability property of weak solutions of elliptic equations. The estimates are deduced from familiar ones for the fundamental solutions of parabolic equations (cf. [3], [15], [18]). Since we are unable to refer the reader to complete proofs of the latter estimates we establish them below. Although the required property of weak solutions of elliptic equations is known (cf. [1]) we have included a proof.

2. A differential operator, $\sum_{|\alpha|\leq m}(-i)^{|\alpha|}a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$, with constant coefficients, is called strongly elliptic if for any real n-vector ξ, Re $\{\sum_{|\alpha|=m} a_\alpha \xi^\alpha\} \geq \rho|\xi|^m$, with a fixed $\rho > 0$. A fundamental solution for the operator $\sum_{|\alpha|\leq m}(-i)^{|\alpha|}a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} + \lambda$ is

$$G(x,\lambda) \sim \frac{1}{(2\pi)^n}\int_{E_n} \frac{e^{ix\cdot\xi}}{\sum_{|\alpha|\leq m} a_\alpha \xi^\alpha + \lambda} d\xi.$$

If

$$g(x,t) = \frac{1}{(2\pi)^n}\int_{E_n} e^{-(\sum a_\alpha \xi^\alpha)t}\, e^{ix\cdot\xi} d\xi$$

then

$$G(x,\lambda) = \int_0^\infty e^{-\lambda t} g(x,t)\, dt.$$

This is a basic observation since it allows us to obtain estimates for $G(x,\lambda)$ from those for $g(x,t)$.

We shall be interested in the case that $a_\alpha = a_\alpha(y)$, $|\alpha| \leq m$, depends on a parameter, y, varying in a region, U, of n-dimensional real space. We shall suppose that $a_\alpha(y)$, $|\alpha| \leq m$, is m times continuously differentiable in U and that, in U,

$$(i) \quad \left|\frac{\partial^\gamma}{\partial y^\gamma} a_\alpha(y)\right| \leq M;\ |\gamma| \leq m$$

$$(ii) \quad \mathrm{Re}\left\{\sum_{|\alpha|=m} a_\alpha(y)\xi^\alpha\right\} \geq \rho|\xi|^m.$$

We want to estimate the mixed partial derivatives of $g(x, y, t)$ and $G(x, y, \lambda)$ up to the order m. We notice that, for $t > 0$,

$$\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\beta}{\partial x^\beta}\, g(x, y, t) = \frac{1}{(2\pi)^n}\int_{E_n} i^{|\beta|^{\xi^\beta}}\, e^{-(\sum a_\alpha \xi^\alpha)t} M_\gamma(\xi, t)\, e^{ix\cdot\xi} d\xi$$

$$= \frac{1}{(2\pi)^n t^{|\beta|/m}}\int_{E_n} i^{|\beta|}\xi^\beta t^{|\beta|/m}\, e^{-(\sum a_\alpha \xi^\alpha)t} M_\gamma(\xi, t)\, e^{ix\cdot\xi} d\xi$$

and that the integrand in the final integral (with the factor $e^{ix\cdot\xi}$ removed) as a function of the complex n-vector $\xi = \sigma_1 + \sigma + i\tau, \sigma_1, \sigma$, and τ real, is dominated by an expression $c_1 e^{-\rho_1|\sigma_1|^m t} e^{a_1 t|\sigma + i\tau|^m}\, e^{a_2 t}$ when $|\gamma + \beta| \le m$. The constants c_1, ρ_1, a_1, a_2 depend on n, m, M, and ρ only. Consequently

$$\begin{aligned}
&\left|\frac{\partial^k}{\partial \xi_i^k}\{\xi^\beta t^{|\beta|/m} M_\gamma(\xi, t) e^{-(\sum a_\alpha \xi^\alpha)t}\}\right| \\
&= \left|\frac{k!}{(2\pi)^n}\int_{|\zeta_1 - \xi_1| = r} d\zeta_1 \cdots \int_{|\zeta_n - \xi_n| = r} d\zeta_n \{\zeta^\beta t^{|\beta|/m} M_\gamma(\zeta, t) e^{-(\sum a_\alpha \varphi^\alpha)t} \Big/ \prod(\varphi_i - \xi_i)\}\right| \\
&\le c_2 k!\, e^{a_2 t} e^{-\rho_1|\xi|^m t}\, \frac{e^{a_3 t r^m}}{r^k}.
\end{aligned}$$

Here and in the following all constants, unless the contrary is mentioned, depend only on n, m, M, and ρ. Since r, in the above inequality, is arbitrary we choose it to be $(k/t)^{1/m}$ and obtain

$$\left|\frac{\partial^k}{\partial \xi_i^k}\left\{\xi^\beta t^{|\beta|/m} M_\gamma(\xi, t) e^{-(\sum a_\alpha \xi^\alpha)} t\right\}\right| \le c_2 k!\, e^{a_2 t}\, e^{-\rho_1|\xi|^m t} c_3^k \left(\frac{t}{k}\right)^{\frac{k}{m}}.$$

Then

$$\begin{aligned}
|x|^{2k}\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\beta}{\partial x^\beta}\, g(x, y, t)\right| &\le n^{2k-1}\left(\sum_{i=1}^n x_i^{2k}\right)\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\beta}{\partial x^\beta} g(x, y, t)\right| \\
&= \left|\frac{n^{2k-1}}{(2\pi)^n}\,\frac{1}{t^{|\beta|/m}}\int_{E_n}\sum_{i=1}^n \frac{\partial^{2k}}{\partial \xi_i^{2k}}\{\xi^\beta t^{|\beta|/m} M_\gamma(\xi, t) e^{-(\sum a_\alpha \xi^\alpha t)}\} e^{ix\cdot\xi} d\xi\right| \\
&\le c_4 \frac{e^{a_2 t}}{t^{|\beta|/m}} (c_5)^{2k} (2k)! \left(\frac{t}{2k}\right)^{2k/m} \int_{E_n} e^{-\rho_1|\xi|^m t} d\xi \\
&\le c_7 \frac{e^{a_2 t}}{t^{(n+|\beta|)/m}} (c_6 t)^{2k/m} (2k)^{2k(\frac{m-1}{m}) + 1/2} e^{-2k}.
\end{aligned}$$

If $|x|/(c_6 t)^{1/m} > 2$, set

$$k = \left[\frac{1}{2}\,\frac{|x|^{m/m-1}}{(c_6 t)^{1/m-1}}\right]$$

to obtain

$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\beta}{\partial x^\beta}\, g(x, y, t)\right| \le c_8 \frac{e^{a_2 t}}{t^{(n+|\beta|)/m}} e^{-\rho_2(\frac{|x|}{t^{1/m}})^{\frac{m}{m-1}}}.$$

If $|x| \le 2(c_6 t)^{1/m}$, then
$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\,\frac{\partial^\beta}{\partial x^\beta}\,g(x,y,t)\right| \le c_9\,\frac{e^{a_2 t}}{t^{(n+|\beta|)/m}}.$$

Now we observe that

$$\mathrm{Re}\,\{e^{i\varphi}\big(\sum_{|\alpha|=m} a_\alpha(y)\xi_\alpha\big)\} \ge (\cos\varphi\rho - k\sin\varphi)|\xi|^m.$$

So there exists ϕ_1 and ϕ_2 with $\pi/2 < \phi_1 < 0 < \phi_2 < \pi/2$ such that, for $\phi_1 \le \varphi \le \phi_2$,

$$\mathrm{Re}\,\{e^{i\varphi}\sum_{|\alpha|=m} a_\alpha(y)\xi_\alpha)\} \ge \rho/2|\xi|^m.$$

Consequently we have proved

Lemma 3. *Let all the above conditions be fulfilled. Then for $\phi_1 \le \arg t \le \phi_2$ and $|\gamma+\beta| \le m$, the following inequalities are valid.*

(i) If $|x|/|t|^{1/m} \ge b_1$, then
$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\,\frac{\partial^\beta}{\partial x^\beta}\,g(x,y,t)\right| \le b_2\,\frac{e^{b_3|t|}}{|t|^{(n+|\beta|)/m}}\;e^{-\rho_3\left(\frac{|x|}{|t|^{1/m}}\right)^{\frac{m}{m-1}}}.$$

(ii) If $|x|/|t|^{1/m} \le b_1$, then
$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\,\frac{\partial^\beta}{\partial x^\beta}\,g(x,y,t)\right| \le b_4\,\frac{e^{b_3|t|}}{|t|^{(n+|\beta|)/m}}.$$

The constants depend only on n, m, ρ, and M.

As a consequence, if $|\gamma+\beta| \le m$,
$$\int_U \left|\frac{\partial^\gamma}{\partial y^\gamma}\,\frac{\partial^\beta}{\partial x^\beta}\,g(x-z,x,t)\right|dx$$

is at most

$$b_4\,\frac{e^{b_3|t|}}{t^{|\beta|/m}}\int_{|x-z|\le b_1|t|^{1/m}}\frac{1}{|t|^{n/m}}dx + b_2\,\frac{e^{b_3|t|}}{t^{|\beta|/m}}\int_{|x-z|\ge b_1|t|^{1/m}}\frac{e^{-\rho_1\left(\frac{|x|}{|t|^{1/m}}\right)^{\frac{m}{m-1}}}}{|t|^{n/m}}dx \le b_5\,\frac{e^{b_3|t|}}{|t|^{|\beta|/m}}.$$

Let S be the sector in the complex plane defined by

$$S = \{z|\,\mathrm{Re}\,(ze^{i\phi_1}) \le b_3 \text{ and } \mathrm{Re}\,(ze^{i\phi_2}) \le b_3\}.$$

If λ is not in S we can find a φ, $\phi_1 \le \varphi \le \phi_2$, such that Re $(\lambda e^{i\varphi}) - b_3 \ge \rho(\lambda, S)$, the distance from λ to S.

Lemma 4. *If λ is not in S, then for $|\beta| < m$ and $|\gamma + \beta| \leq m$*

$$\int_U \left| \frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\beta}{\partial x^\beta} G(x - z, x, \lambda) \right| dx \leq \frac{C}{\big(\rho(\lambda, S)\big)^{1-|\beta|/m}}.$$

Proof. Choose φ as above; then

$$\begin{aligned}
\int_U \left| \frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\beta}{\partial x^\beta} G(x - z, x, \lambda) \right| dx &\leq \int_U \int_0^{e^{i\varphi}\infty} |e^{-\lambda t}| \left| \frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\beta}{\partial x^\beta} g(x - z, x, t) \right| |dt| \\
&\leq b_5 \int_0^\infty \frac{e^{-(\mathrm{Re}(\lambda \cdot e^{i\varphi}) - b_3)t}}{t^{|\beta|/m}} \, dt \\
&\leq \frac{C}{\{\mathrm{Re}\,(\lambda \cdot e^{i\varphi}) = b_3\}^{1-|\beta|/m}} \\
&\leq \frac{C}{\rho(\lambda, S)^{1-|\beta|/m}}.
\end{aligned}$$

We must now estimate $\frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\beta}{\partial x^\beta} G(x - z, x, \lambda)$ pointwise for $|x| \leq R, R > 0$. Choose y as above; then

$$\left| \frac{\partial^\gamma}{\partial y^\gamma} \frac{\partial^\beta}{\partial x^\beta} G(x - y, x, \lambda) \right|,$$

is at most

$$\begin{aligned}
&\leq b_4 \int_{t \geq \left(\frac{|x-z|}{b_1}\right)^m} \frac{e^{-(\mathrm{Re}\,(\lambda e^{i\varphi}) - b_3)t}}{t^{(n+|\beta|)/m}} dt + b_2 \int_{t \leq \left(\frac{|x-z|}{b_1}\right)^m} \frac{e^{-(\mathrm{Re}\,(\lambda e^{i\varphi}) - b_3)t}}{t^{(n+|\beta|)/m}} \, e^{-\rho_3 \left(\frac{|x|}{t^{1/m}}\right)^{\frac{m}{m-1}}} \, dt \\
&= b_4 I_1 + b_2 I_2.
\end{aligned}$$

We estimate the two terms separately for $\mathrm{Re}\,(\lambda e^{i\varphi}) - b_3 = \omega \geq \delta_0 > 0$. For simplicity we replace $x - z$ by x.

(i) $0 < \delta_1 \leq |x| \leq R$.

$$\begin{aligned}
I_1 &\leq e^{-\omega\left(\frac{|x|}{b_1}\right)^m} \int_0^\infty e^{-\omega t} \left| \left(\frac{|x|}{b_1}\right)^m + t \right|^{-(n+|\beta|)/m} dt \\
&\leq e^{-\omega\left(\frac{|x|}{b_1}\right)^m} M(\delta_0, \delta_1, R).
\end{aligned}$$

(ii) $|x| \leq \delta_1$.

$$I_1 \leq \int_1^\infty \frac{e^{-\omega t}}{t^{(n+|\beta|)/m}} \, dt + \int_{\left(\frac{|x|}{b_1}\right)^m}^1 \frac{1}{t^{(n+|\beta|)/m}} \, dt$$

$$\leq \begin{cases} c_1 + c_2 |x|^{m-n-|\beta|} & m - n - |\beta| \neq 0 \\ c_1 + c_2 |\log|x|\,| & m - n - |\beta| = 0. \end{cases}$$

We remark that, for $|\beta| = m-1, m-n-|\beta| = 1-n = 0$ only if $n=1$, in which case $\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\beta}{\partial x^\beta}G(x-z,x,\lambda)\right| \le C(\lambda)$ for $|x-z|\le\delta$. This is a simple fact about Fourier transforms in one variable and we do not prove it.

(iii) $0<\delta_1=|x|\le R$.

$$I_2 \le \omega^{\frac{n+|\beta|}{m}-1}\left\{\int_0^{(\omega|x|^m)^{\frac{1}{m}}} \frac{e^{-\rho_3\left(\frac{\omega|x|^m}{t}\right)^{\frac{1}{m-1}}}}{t^{(n+\beta)/m}}\,dt + \int_{(\omega|x|^m)^{\frac{1}{m}}}^\infty \frac{e^{\rho_3 t}}{t^{(n+|\beta|)/m}}\,dt\right\} = \omega^{\frac{n+|\beta|}{m}-1}\{J_1+J_2\}.$$

$$\begin{aligned}
J_2 &= e^{-\omega^{\frac{1}{m}}|x|}\int_0^\infty \frac{e^{-\rho_3 t}}{(\omega|x|^m+t)^{(n+|\beta|)/m}}\,dt \le c_3 e^{-\omega^{\frac{1}{m}}|x|}.\\
J_1 &= (\omega|x|^m)^{1-\frac{n+|\beta|}{m}}\int_{(\omega|x|^m)^{\frac{m-1}{m}}}^\infty e^{-\rho_3 t^{\frac{1}{m-1}}}\,t^{\frac{n+|\beta|}{m}-2}\,dt\\
&\le c_4(\omega|x|^m)^{1-\frac{n+|\beta|}{m}}\int_{\omega^{\frac{1}{m}}|x|}^\infty e^{-\rho_3 t}t^{(m-1)\left(\frac{n+|\beta|}{m}-2\right)+m-2}dt\\
&\le c_5 e^{-\rho_3\omega^{\frac{1}{m}}|x|}\{(\omega|x|^m)^{q_1}+(\omega|x|^m)^{q_2}\}.
\end{aligned}$$

with certain exponents q_1 and q_2.

(iv) $|x|\le\delta_1$.

$$\begin{aligned}
I_2 &= \int_0^{\left(\frac{|x|}{b_1}\right)^m} \frac{e^{-\rho_3\left(|x|/t^{1/m}\right)^{\frac{m-1}{m}}}}{t^{(n+|\beta|)/m}}\,dt\\
&= |x|^{m-n-|\beta|}\int_0^{b_5}\frac{e^{-\rho_3/t^{\frac{1}{m-1}}}}{t^{(n+|\beta|)/m}}\,dt\\
&\le c_6|x|^{m-n-|\beta|}.
\end{aligned}$$

The estimates are precise enough for our purposes. All we need to know is that $G(x-y,x,\lambda)$ goes to zero uniformly as ω increases provided $|x-y|$ remains between two fixed positive constants and that the derivatives of order $m-1$ go to infinity like $|x-y|^{1-n}$ as $|x-y|$ goes to zero.

3. The differential operator, $\sum_{|\alpha|=m} a_\alpha\frac{\partial^\alpha}{\partial x^\alpha}$, with constant coefficients, is elliptic if $Q(\xi) = \sum_{|\alpha|=m} a_\alpha\xi^\alpha \ne 0$ for any real n-vector ξ. F. John, in [10], has constructed fundamental solutions for such differential operators. These are, for n, the number of variables, odd and even respectively,

$$K(x-y) = \frac{1}{4(2\pi i)^{n-1}(m-1)!}\Delta_x^{\frac{n-1}{2}}\int_{\Omega_\xi}\frac{((x-y)\cdot\xi)^{m-1}\mathrm{sgn}\left((x-y)\cdot\xi\right)}{Q(\xi)}d\omega_\xi \tag{2.1}$$

$$K(x-y) = \frac{-1}{(2\pi i)^n m!}\,\Delta_x^{n/2}\int_{\Omega_\xi}\frac{((x-y)\cdot\xi)^m\log|(x-y)\cdot\xi|}{Q(\xi)}d\omega_\xi \tag{2.2}$$

where Ω_ξ is the unit sphere and Δ is the Laplacian. Actually John was concerned only with the case that the coefficients are real; however, a repetition of his argument shows that (2.1) and (2.2) are fundamental solutions when the coefficients are complex. In order to use these fundamental solutions we must perform the indicated differentiations. Let $L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$; then

$$\Delta = \frac{1}{2r^2} \sum_{i,j=1}^{n} L_{ij}^2 + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \tag{2.3}$$

$$\frac{\partial}{\partial x_k} = \frac{1}{2r^2} \sum_{i,j=1}^{n} (\delta_{kj} x_i - \delta_{ki} x_j) L_{ij} + \frac{x_i}{r} \frac{\partial}{\partial r}. \tag{2.4}$$

With a suitable skew-symmetric matrix, A_{ij},

$$\begin{aligned} L_{ij} \int_{\Omega_\xi} g(x \cdot \xi) f(\xi) d\xi &= \lim_{t \to 0} \frac{1}{t} \left[\int_{\Omega_\xi} g(e^{tA_{ij}} x \cdot \xi) f(\xi) d\omega_\xi - \int_{\Omega_\xi} g(x \cdot \xi) f(\xi) d\omega_\xi \right] \\ &= \lim_{t \to 0} \frac{1}{t} \left[\int_{\Omega_\xi} g(x \cdot \xi) \{ f(e^{tA_{ij}} \xi) - f(\xi) \} d\omega_\xi \right] \\ &= \int_{\Omega_\xi} g(x \cdot \xi) L_{ij} f(\xi) d\omega_\xi. \end{aligned}$$

Of course, in the last integrand, x has been replaced by ξ in the operator L_{ij}. Setting $\tilde{x} = \frac{x}{|x|}$, we have for n odd

$$K(x) = \frac{1}{4(2\pi i)^{n-1}(m-1)!} r^{m-n} \int_{\Omega_\xi} (\tilde{x} \cdot \xi)^{m-1} \operatorname{sgn}(\tilde{x} \cdot \xi) \frac{P(\xi)}{\{Q(\xi)\}^n} d\omega_\xi. \tag{2.5}$$

$P(\xi)$ is a polynomial in ξ. A similar formula is valid for n-even. We may also show that, for n odd,

$$\frac{\partial^\alpha}{\partial x^\alpha} K(x) = \frac{1}{4(2\pi i)^{n-1}(m-1)!} r^{m-n-|\alpha|} \int_{\Omega_\xi} (\tilde{x} \cdot \xi)^{m-1} \operatorname{sgn}(\tilde{x} \cdot \xi) \frac{P(\tilde{x}, \xi)}{\{Q(\xi)\}^{n+|\alpha|}} d\omega_\xi$$

$P(\tilde{x}, \xi)$ is a polynomial in $\tilde{x}$ and ξ. Again, a similar formula is valid for n even. For $x \neq 0$, $\sum a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} K(x) = 0$. If $\varphi(y)$ is an infinitely differentiable function with compact support and $\varphi(x) \equiv 1$ in a neighborhood of 0, then (cf. p. 57 of [10])

$$\begin{aligned} 1 = \varphi(0) &= \int_{E_n} K(-x) \sum_{|\alpha|=m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} \varphi(x) \\ &= \lim_{\varepsilon \to 0} \int_{|x|=\varepsilon} \left\{ \sum a_\alpha \left(\frac{\partial^{\hat{\alpha}}}{\partial x^{\hat{\alpha}}} K \right) (-x) x_{\alpha_{|\alpha|}} \right\} \frac{\varphi(x)}{\varepsilon} d\omega \end{aligned}$$

with $\hat{\alpha} = (\alpha_1, \ldots, \alpha_{|\alpha|-1})$; or

$$1 = \lim_{\varepsilon \to 0} \int_{|x|=\varepsilon} \sum a_\alpha \left(\frac{\partial^{\hat{\alpha}}}{\partial x^{\hat{\alpha}}} K \right)(-x) x_{\alpha_{|\alpha|}} / r d\omega. \tag{2.6}$$

(2.6) is also valid for the fundamental solutions discussed in Paragraph 1.

We can now prove the lemma of this paragraph. We consider a differential operator $B = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$ which is defined and uniformly elliptic in a domain V of Euclidean n-space; that is, for any real n-vector ξ and any $x \in V, |\sum_{|\alpha|=m} a_\alpha(x) \xi^k| \geq \rho |\xi|^m$ with some fixed constant ρ. We suppose that $a_\alpha(x)$ is $|\alpha|$-times continuously differentiable in V and that its derivatives up to the $|\alpha|^{\text{th}}$ order are bounded in V. $C_c^\infty(V)$ is the set of infinitely differentiable functions with compact support in V. Then we have:

Lemma 5. *Suppose $u(x)$ and $f(x)$ are two continuous functions in V such that*

$$\int_V B\varphi(x) u(x) dx = \int_V \varphi(x) f(x) dx \tag{2.7}$$

for all functions φ in $C_c^\infty(V)$. Then $u(x)$ is $m-1$ times continuously differentiable in V and the modulus of continuity of any $(m-1)$st order derivative is $0(\delta \log 1/\delta)$ uniformly in any compact subset, U, of V.

Proof. By the usual arguments it can be shown that (2.7) holds for $\varphi(x)$ m-times continuously differentiable with compact support in V. Let $K(x-z, y)$ be the fundamental solution of the operator $\sum_{|\alpha|=m} a_\alpha(y) \frac{\partial^\alpha}{\partial x^\alpha}$; let $\psi(y)$ be infinitely differentiable with compact support in V and be identically 1 in a neighborhood, W, of U. Let $j_k(y)$ be infinitely differentiable; $j_k(y) \geq 0$; $\int_{E_n} j_k(y) dy = 1$; and $j_k(y) = 0$ if $|y| > \frac{1}{k}$. Then, for large k, (2.7) is valid with

$$\varphi(y) = \int_{E_n} j_k(y-z) \psi(z) K(x-z, y) dz.$$

We calculate

$$\begin{aligned}
&\sum_{|\alpha| \leq m} a_\alpha(y) \frac{\partial^\alpha}{\partial y^\alpha} \int_{E_n} j_k(y-z) \psi(z) K(x-z, y) dz \\
&= \int_{E_n} \sum_{|\alpha| \leq m} a_\alpha(y) \sum_{\alpha_1 + \alpha_2 = \alpha} (-1)^{|\alpha_1|} \frac{\partial^\alpha 1}{\partial z^{\alpha_1}} j_k(y-z) \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} K(x-z, y) \psi(z) dz \\
&= \int_{E_n} (-1)^m \sum_{|\alpha|=m} a_\alpha(y) \frac{\partial^\alpha}{\partial z^\alpha} j_k(y-z) \psi(z) K(x-z, y) dz \\
&+ \sum_{|\alpha| \leq m} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ |\alpha_1| < m}} \int_{E_n} a_\alpha(y) j_k(y-z) \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \{\psi(z) \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} K(x-z, y)\} dz.
\end{aligned}$$

With our unorthodox notation the symbol $\alpha_1 + \alpha_2 = \alpha$ is a little difficult to explain. It means that α_1 and α_2 are subsequences of the sequence α whose union exhausts α. Integrate the first term by parts to obtain

$$\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{|x-z|=\varepsilon}\sum_{|\alpha|=m} a_\alpha(y)j_k(y-z)\frac{\partial^{\hat\alpha}}{\partial z^{\hat\alpha}}K(x-z,y)\frac{(x-z)_{\alpha_{|\alpha|}}}{|x-z|}d\omega\\
&\qquad+\int_{V-W} j_k(y-z)\sum_{|\alpha|=m} a_\alpha(y)\frac{\partial^\alpha}{\partial z^\alpha}\{K(x-z,y)\psi(z)\}dz\\
&\qquad= j_k(y-x)+\int_{V-W} j_k(y-z)\sum_{|\alpha|=m}\frac{\partial^\alpha}{\partial z^\alpha}\{\psi(z)K(x-z,y)\}dz.
\end{aligned}$$

Substituting these formulae into (2.7) and letting $k\to\infty$ we obtain, for $x\in W$,

$$\begin{aligned}
u(x) &= \int_V \psi(y)K(x-y,y)f(y)dy\\
&\quad-\int_{V-W}\sum_{|\alpha|=m} a_\alpha(y)\frac{\partial^\alpha}{\partial z^\alpha}\{\psi(z)K(x-z,y)\}u(y)dy\big|_{z=y}\\
&\quad-\sum_{|\alpha|=m}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ |\alpha_1|<m}}\int_V a_\alpha(y)\frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\left\{\psi(z)\frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}}K(x-z,y)\right\}u(y)dy.\big|_{z=y}
\end{aligned}\tag{2.8}$$

We use this representation of $u(x)$ to prove the lemma. We first show that if $\omega(\delta)$ is the modulus of continuity, in a compact subset of W, of a typical term of the right hand side, as a function of x, then $\omega(\delta)=0(\delta\log 1/\delta)$. This is obvious for the second term since it is an infinitely differentiable function of x. The only terms which give trouble are those which contain derivatives of K, with respect to z, of order $m-1$. Consider then

$$L(x)=\int_V u(y)a_\alpha(y)\frac{\partial^{\alpha_1}}{\partial y^{\alpha_1}}\psi(y)\frac{\partial^{\alpha_2}}{\partial z^{\alpha_2}}\,\frac{\partial^{\alpha_3}}{\partial y^{\alpha_3}}K(x-y,y)dy$$

with $|\alpha_2|=m-1$. We estimate $L(x+w)-L(x)$, which is

$$\int_{E_n} u(y+x)a_\alpha(y+x)\frac{\partial^{\alpha_1}}{\partial y^{\alpha_1}}\psi(y+x)\left\{\frac{\partial^{\alpha_2}}{\partial z^{\alpha_2}}\frac{\partial^{\alpha_3}}{\partial y^{\alpha_3}}K(w-y,y+x)-\frac{\partial^{\alpha^2}}{\partial z^{\alpha_2}}\,\frac{\partial^{\alpha_3}}{\partial y^{\alpha_3}}K(-y,y+x)\right\}dy.$$

We remark that $\frac{\partial^{\alpha_2}}{\partial z^{\alpha_2}}\,\frac{\partial^{\alpha_3}}{\partial y^{\alpha_3}}K(w-y,y+x)$ is $\frac{\partial^{\alpha_2}}{\partial z_1^{\alpha_2}}\,\frac{\partial^{\alpha_3}}{\partial y_1^{\alpha_3}}K(x_1-z_1,y_1)$ evaluated at $x_1-z_1=w-y$ and $y_1=y+x$. This notation is perhaps a little confusing; but it is desirable to keep the number of letters and subscripts introduced to a minimum. Also $K(w-y,y+x)$ is defined only for $y+x\in V$;

but since we are multiplying by a derivative of $\psi(y+x)$ there is no difficulty in taking the integral over E_n. Now

$$\frac{\partial^{\alpha_2}}{\partial z^{\alpha_2}}\frac{\partial^{\alpha_3}}{\partial y^{\alpha_3}}K(w-y,y+x)=|y-w|^{1-n}\int g(\widetilde{w-y}\cdot\xi)P(\widetilde{w-y},\xi,y+x)dw_\xi$$

$$=|y-w|^{1-n}\,G(w-y,y+x).$$

$G(w-y,y+x)$ is once continuously differentiable, with respect to w, when $w\neq y$ and

$$\left|\frac{\partial}{\partial w_i}G(w-y,y+x)\right|\leq\frac{K}{|w-y|}$$

for $y+x$ in V.

Write $L(x+w)-L(x)$ as

$$\begin{aligned}&\int_{E_n}u(y+x)a_\alpha(y+x)\frac{\partial^{\alpha_1}}{\partial y^{\alpha_1}}\psi(y+x)\left\{\frac{G(w-y,y+x)-G(-y,y+x)}{|y|^{n-1}}\right\}dy\\&+\int_{E_n}u(y+x)a_\alpha(y+x)\frac{\partial^{\alpha_1}}{\partial y^{\alpha_1}}\psi(y+x)G(w-y,y+x)\left\{\frac{1}{|y-w|^{n-1}}-\frac{1}{|y|^{n-1}}\right\}dy\\&=I_1+I_2.\end{aligned}$$

If $|w|$ is small enough

$$\begin{aligned}|I_1|&\leq\int_{|y|\leq|w|\log 1/|w|}+\int_{|y|\geq|w|\log 1/|w|}\\&\leq 0(|w|\log 1/|w|)+K_1\int_{R\geq|y|\geq|w|\log 1/|w|}\frac{|G(w-y,y+x)-G(-y,y+x)|}{|y|^{n-1}}dy.\end{aligned}$$

The integrand in the second term is dominated by

$$\frac{K|w|}{|y|^{n-1}|y-\theta w|}\leq\frac{K}{|y|^n}\frac{|w|}{|\tilde{y}-\frac{\theta w}{|y|}|}\leq\frac{K_0|w|}{|y|^n},$$

$0\leq\theta\leq 1$. Integrating

$$\begin{aligned}I_1&=O\left(|w|\log\frac{1}{|w|}\right)+O\left(|w|\log\left(\frac{1}{|w|}\log\frac{1}{|w|}\right)\right)\\&=O\left(|w|\log\frac{1}{|w|}\right)\\|I_2|&\leq K_2\int_{|y|\leq R}\left|\frac{1}{|y-w|^{n-1}}-\frac{1}{|y|^{n-1}}\right|dy\\&=O\left(|w|\log\frac{1}{|w|}\right).\end{aligned}$$

If $m = 1$, there is nothing more to prove. We suppose $m > 1$. Now we observe that the equations

$$\frac{d^\beta}{dy^\beta}\left[\frac{\partial^\gamma}{\partial y^\gamma}K(w-y,y)\right] = \sum_{\beta_1+\beta_2=\beta}\frac{\partial^{\beta_1}}{\partial z^{\beta_1}}\frac{\partial^{\beta_2+\gamma}}{\partial y^{\beta_2+\gamma}}K(x-y,y)$$

allow us to replace, in (2.8), partial derivatives of $K(x-y,y)$ by sums of total derivatives of terms $\frac{\partial^\gamma}{\partial y^\gamma}K(x-y,y)$. To avoid confusion, we explain this in detail.

Until now we have when differentiating the function $K = K(x-z,y)$ regarded it as a function of the three variables x, y, and z and only after taking derivatives have we substituted y for z. However, in the following it will be necessary to integrate by parts. To do this it is necessary to replace the function

$$\left.\frac{\partial^\beta}{\partial z^\beta}\frac{\partial^\gamma}{\partial y^\gamma}K(x-z,y)\right|_{z=y}$$

by partial derivatives of some function of y. The above formula is the means to do this. The right hand side is obtained by taking $\frac{\partial^\gamma}{\partial y^\gamma}$ of $K(x-z,y)$; setting $z = y$; and then taking $\frac{\partial^\beta}{\partial y^\beta}$ of the resulting function of x and y. We have indicated this by writing the sign for a total derivative.

We wish to invoke the lemmas of E. Hopf [8]. First we must observe that if we replace $u(y)$ by 1 in the terms of (2.8) containing partial derivatives of $K(x-z,y)$ with respect to z of order $m-1$ we may replace partial derivatives by total derivatives and integrate by parts, for the $a_\alpha(y)$ involved in these expressions will be once continuously differentiable. This lowers the order of the singularity of the integrand so that we may now differentiate with respect to x to obtain a continuous function. The lemmas just mentioned now imply that $u(x)$ is once continuously differentiable in a neighborhood of U.

Now that we know $u(x)$ is once continuously differentiable in a neighborhood of U we return to the expression (2.8). We replace $\psi(y)$, which has served its purpose, by another infinitely differentiable function which has its support in a neighborhood of U in which we know $u(x)$ to be once continuously differentiable. We write all partial derivatives as sums of total derivatives; integrate those terms involving total derivatives of order $m-1$ by parts; and then take the derivative, with respect to x, of the integrand in every integral on the right hand side of (2.8). This gives us an expression similar to (2.8) for $u'(x)$. The lemma is now established by induction. It is only necessary to observe that the derivatives of the coefficients and of $K(x-z,y)$, with respect to y, which are taken in the proof all exist. For the purposes of this thesis it may be assumed that the coefficients are infinitely differentiable; then this difficulty does not arise.

4. We return now to the study of representations of Lie groups. We use the same notation as before. Set $W_k = \{x \in X | x \in \cap_{a_1,\dots,a_k \in A} D(A(a_1)\dots A(a_k))\}$ and set $W_k^* = \{x^* \in X^* | x^* \in \cap_{a_1,\dots,a_k \in A} D(A^*(a_1)\dots A^*(a_k))\}$.

In analogy to the terminology in the theory of partial differential equations, we call the form $\sum_{|\alpha|\le m} a_\alpha X_\alpha$ elliptic if when we substitute a real non-zero n-vector ξ for X, $\sum_{|\alpha|=m} a_\alpha \xi_\alpha \neq 0$. With an elliptic form, $\sum_{|\alpha|\le m} a_\alpha X_\alpha$, we associate the operator B_0 with domain W_m, defined by $B_0 x = \sum_{|\alpha|\le m} a_\alpha A_\alpha x$. We shall need to consider also the operator B_0^*, with domain W_m^*, defined by $B_0^* x^* = \sum_{|\alpha|\le m} a_\alpha A_{\alpha*}^* x^*$. Since the domain of B_0 is dense and that of B_0^* is dense in the weak-* topology and since they are adjoint, the closure B and the weak-* closure, B^*, of B_0 and B_0^*, respectively, are well defined. The following theorem shows this notation to be justified.

Theorem 7. *B^* is the adjoint of B.*

Proof. Suppose that for all $x \in W_m$

$$\left(\sum_{|\alpha|\le m} a_\alpha A_\alpha x, x_1^*\right) = (x, x_2^*).$$

We shall show that $x_1^* \in W_{m-1}^*$. Let μ be a left-invariant Haar measure on G and set $R_i = R(e_i)$. If $K(p)$ is infinitely differentiable with compact support in G,

$$\sum a_\alpha A_\alpha \left\{ \int_G K(p) T(p) x \mu(dp) \right\} = \int_G \left\{ \sum a_\alpha R_\alpha K(p) \right\} T(p) x \mu(dp).$$

Consequently

$$\int_G \sum a_\alpha R_\alpha K(p) \big(T(p)x, x_1^*\big) \mu(dp) = \int_G K(p) \big(T(p)x, x_2^*\big) \mu(dp).$$

Let $\{t_i\}$ be an analytic coordinate system of the second kind [14] corresponding to the basis $\{e_i\}$, in a neighborhood, V, of the identity; then, assuming that K has support in V,

$$\int_V \left\{ \sum_{|\alpha|\le m} b_\alpha(t) \frac{\partial^\alpha}{\partial t^\alpha} K\big(p(t)\big) \right\} \{ \big(T(p(t))x, x_1^*\big) \} F(t)\, dt = \int_V K\big(p(t)\big) \{ (T(p(t))x, x_2^*) \} F(t)\, dt.$$

Here $F(t)$ and $b_\alpha(t)$ are analytic functions; $F(t)$ is nowhere zero; and $\sum_{|\alpha|\le m} b_\alpha(t) \frac{\partial^\alpha}{\partial t^\alpha}$ is elliptic in a neighborhood $U \subseteq V$ of the origin since $b_\alpha(0) = a_\alpha$. It is then a consequence of Lemma 5 that $\big(T(p)x, x_1^*\big)$ is $m-1$ times continuously differentiable. This implies that $x_1^* \in W_{m-1}^*$. If $x \in W_m$, $(x, x_2^*) = \sum a_\alpha (A_\alpha x, x_1^*) = \sum a_\alpha (A_{\alpha_{|\alpha|}} x, A_{\hat\alpha^*}^* x_1^*)$. Since $E \subseteq W_m$, Theorem $1'$ implies that

$$(x, x_2^*) = \sum a_\alpha (A_{\alpha_{|\alpha|}} x, A_{\hat\alpha^*}^* x_1^*) \tag{2.9}$$

for all $x \in W_1$. Since $\{t_i\}$ is a canonical coordinate system of the second kind we may infer as in the proof of Theorem 1 that $\int_{R(s(\sigma))} S(t)x\,dt$ is in W_1 for all $x \in X$. The notation is the same as in the proof of that theorem; in particular, $S(t) = T\big(\rho(t)\big)$. Also

$$A_i \int_{R(s)} S(t)x\,dt = \int_{R(\hat{s}^i)} S(\hat{t}^i, \sigma)x - S(\hat{t}^i, 0)x\,d\hat{t}^i + G(\sigma)$$

with $\lim_{\sigma\to 0} \frac{G(\sigma)}{\sigma^n} = 0$. Then, using (2.9),

$$\lim_{\sigma\to 0} \left\{ \sum_\alpha a_\alpha \frac{1}{\sigma^n} \int_{R(\hat{s}^{\alpha_{|\alpha|}})}, \big(S(\hat{t}^{\alpha_{|\alpha|}}, \sigma)x - S(\hat{t}^{\alpha_{|\alpha|}}, 0)x, A^*_{\hat{\alpha}^*} x_1^*)d\hat{t}^{\alpha_{|\alpha|}} + \frac{G_1(\sigma)}{\sigma^n} \right\} =$$
$$\lim_{\sigma\to 0} \frac{1}{\sigma^n} \int_{R(s)} \big(S(t)x, x_2^*\big)\,dt.$$

Here $\frac{G_1(\sigma)}{\sigma^n} \to 0$ as $\sigma \to 0$ for all $x \in X$. Consequently

$$\lim_{\sigma\to 0} \frac{1}{\sigma^n} \left[\sum_\alpha a_\alpha \int_{R(\hat{s}^{\alpha_{|\alpha|}})} \big(S(\hat{t}^{\alpha_{|\alpha|}}, \sigma)x - S(\hat{t}^{\alpha_{|\alpha|}}, 0)x A^*_{\hat{\alpha}^*} x_1^* d\hat{t}^{\alpha_{|\alpha|}} \right] = (x, x_2^*). \tag{2.10}$$

Now $\int_{R(s)} S^*(t)x_1^* dt$ (the integral is taken in the weak-$*$ topology) is in W_m^*; and by Lemma 2 and formula (1.2′) we have, for $x \in W_m$,

$$\begin{aligned}
\int_{R(s)} (A_{\alpha_1} \dots A_{\alpha_{|\alpha|}} x, S^*(t)x_1^*)\,dt &= \int_{R(s)} \big(S(t)A_{\alpha_1} \dots A_{\alpha_{|\alpha|}} x, x_1^*)\,dt \\
&= \int R(s) \sum_{|\beta|=|\alpha|} c_{\alpha\beta}(t)(A_\beta S(t)x, x_1^*)\,dt \\
&= \int_{R(s)} \sum c_{\alpha\beta}(t)(A_{\beta_{|\beta|}} S(t)x, A^*_{\hat{\beta}^*} x_1^*)\,dt \\
&= \int_{R(s)} \sum_\beta c_{\alpha\beta}(t) \sum_j \zeta^j_{\beta_{|\beta|}} \frac{\partial}{\partial t_j} \big(S(t)x, A^*_{\hat{\beta}^*} x_1^*\big)\,dt.
\end{aligned}$$

We may choose the $c_{\alpha\beta}(t)$ so that $c_{\alpha\beta}(0) = 0$ unless $\alpha = \beta$ and $c_{\alpha\alpha}(0) = 1$. Also $\zeta_i^j(0) = \delta_i^j$. Integrate by parts to obtain

$$\int_{R(\hat{s}^{\alpha_{|\alpha|}})} (S(\hat{t}^{\alpha_{|\alpha|}}, \sigma)x - S(\hat{t}^{\alpha_{|\alpha|}}, 0)x, A^*_{\hat{\alpha}^*} x_1^*)d\hat{t}^{\alpha_{|\alpha|}} + G_2(\sigma, x).$$

We observe that $G_2(\sigma, x)$ is a linear function of x which is uniformly bounded as $\sigma \to 0$. Since it clearly converges to 0 for $x \in W_m$ it converges to 0 for all x. Consequently, summing over α and using (2.10),

$$\lim_{\sigma\to 0} (x, \sum a_\alpha A^*_{\alpha^*} \int_{R(s)} S^*(t)x_1^*\,dt) = (x, x_2^*).$$

This completes the proof of the theorem.

The form $\sum_{|\alpha|\leq m} a_\alpha x_\alpha$ is called strongly elliptic if

$$\mathrm{Re}\left\{\sum_{|\alpha=m} a_\alpha \xi_\alpha\right\} \geq \rho|\xi|^m, \rho > 0,$$

for any real n-vector ξ. Let $\sum_{|\alpha|\leq m} a_\alpha x_\alpha$ be strongly elliptic and let B be the operator associated, by the previous theorem, to the form $-\sum_{|\alpha|=m}(-i)^{|\alpha|}a_\alpha x_\alpha$. Then we have the following theorem.

Theorem 8. *B is the infinitesimal generator of a semi-group, $U(t)$, of class $H(\phi_1, \phi_2)$ [7].*

Proof. If $x \in W_m$ and λ is a complex number

$$\begin{aligned}
\big(Bx - \lambda x, T^*(p)x^*\big) &= \Big(-\sum(-i)^{|\alpha|}a_\alpha A_\alpha x - \lambda x, T^*(p)x^*)\\
&= -\sum(-i)^{|\alpha|}a_\alpha\big(T(p)A_\alpha x, x^*) - \lambda\big(T(p)x, x^*)\\
&= -\sum(-i)^{|\alpha|}a_\alpha L_\alpha\big(T(p)x, x^*\big) - \lambda\big(T(p)x, x^*\big).
\end{aligned}$$

Let $t = (t_1, \ldots, t_n)$ be a canonical coordinate system of say, the first kind associated with $\{e_1, \ldots, e_n\}$, in a neighborhood, V, of the identity and let

$$-\sum_{|\alpha|\leq m}(-i)^{|\alpha|}a_\alpha L_\alpha = -\sum_{|\alpha|\leq 2m}(-i)^{|\alpha|}b_\alpha(t)\frac{\partial^\alpha}{\partial t^\alpha}$$

in this coordinate system. Since we may choose the $b_\alpha(t)$ in such a manner that $b_\alpha(0) = a_\alpha$, the right hand side is uniformly strongly elliptic in a neighborhood $U \subset V$ of 0. Let $K(s-t, r, \lambda)$ be the fundamental solution of $\sum(-i)^{|\alpha|}b_\alpha(r)\frac{\partial^\alpha}{\partial t^\alpha} + \lambda$ considered in Section 1. We have established estimates for $K(s-t, r, \lambda)$ for $\rho(\lambda, S) \geq \delta > 0$, with S a certain sector in the complex plane. Let $\varphi(t)$ be an infinitely differentiable function with support in U and with $\varphi(t) = 1$ if $|t| \leq \delta_1$ for some small δ_1. Then, if $|s| \leq \delta_1/2$,

$$\begin{aligned}
&\int_U \varphi(t)K(s-t, t, \lambda)\ \big(Bx - \lambda x, S^*(t)x^*\big)\, dt\\
&\quad = -\int_U \varphi(t)K(s-t, t, \lambda)\bigg\{\Big(\sum(-i)^{|\alpha|}b_\alpha(t)\frac{\partial^\alpha}{\partial t^\alpha} + \lambda\bigg\}(x, S^*(t)x^*)\, dt\\
&\quad = -\lim_{\varepsilon\to 0}\int_{|s-t|=\varepsilon}\sum(-i)^{|\alpha|}\, b_\alpha(t)\frac{\partial^{\hat\alpha}}{\partial s^{\hat\alpha}}K(s-t, t, \lambda)\frac{(s-t)_{\alpha_{|\alpha|}}}{|s-t|}(x, S^*(t)x^*)d\omega\\
&\quad\quad -\int_{|s-t|\geq\delta_1}\varphi(t)\sum(-i)^{|\alpha|}b_\alpha(t)\frac{\partial^\alpha}{\partial s^\alpha}K(s-t, t, \lambda)(x, S^*(t)x^*\big)\, dt
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\alpha}\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\|\alpha_1|<|\alpha|}}\int i^{|\alpha|}\frac{\partial^{\alpha_3}}{\partial t^{\alpha_3}}b_\alpha(t)\frac{\partial^{\alpha_2}}{\partial r^{\alpha_2}}\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}K(s-t,t,\lambda)\big(x,S^*(t)x^*\big)\,dt\\
&=-\lim_{\varepsilon\to 0}\int_{|s-t|=\varepsilon}\sum(-i)^{|\alpha}b_\alpha(s)\frac{\partial^{\hat\alpha}}{\partial s^{\hat\alpha}}K(s-t,s,\lambda)\frac{(s-t)_{\alpha_{|\alpha|}}}{|s-t|}\big(x,S^*(t)x^*\big)d\omega-\cdots\\
&=-\big(x,S^*(x)x^*\big)-\cdots
\end{aligned}
$$

Here, as before, $S^*(t)=T^*\big(p(t)\big)$. Also we have used our usual convention regarding partial derivatives of the function $K(s-t,r,\lambda)$. Since $|b_\alpha(s)-b_\alpha(t)|\le M_0|s-t|$ and

$$\left|\frac{\partial^{\hat\alpha}}{\partial s^{\hat\alpha}}K(s-t,s,\lambda)-\frac{\partial^{\hat\alpha}}{\partial s^{\hat\alpha}}K(s-t,t,\lambda)\right|\le\frac{M_1}{|s-t|^{n-2}},$$

we could replace s by t in the appropriate places in the surface integral. We now set $s=0$, choose an x^* such that $\|x^*\|=1,(x,x^*)=\|x\|$, and make use of the estimates of Paragraph 2 to obtain

$$\frac{N_1}{\rho(\lambda,S)}\|Bx-\lambda x\|\ge\|x\|-\frac{N_2\|x\|}{\rho(\lambda,S)}-\frac{N_3\|x\|}{\rho(\lambda,S)^{\frac{1}{m}}}.$$

Consequently, for $\rho(\lambda,S)\ge N_4$,

$$\|x\|\le\frac{N_5}{\rho(\lambda,S)}\|Bx-\lambda x\|.$$

This inequality remains valid for $x\in D(A)$. For $x^*\in W_m^*$, consider

$$
\begin{aligned}
\big(T(p)x,B^*x^*-\lambda x^*\big)&=\big(T(p)x,-\sum(-i)^{|\alpha|}a_\alpha A^*_{\alpha^*}x^*-\lambda x^*\big)\\
&=-\sum(i)^{|\alpha|}a_\alpha R_{\alpha^*}\big(T(p)x,x^*\big)-\lambda\big(T(p)x,x^*\big).
\end{aligned}
$$

Change into local coordinates and perform the same calculations as above to obtain

$$\int_{U_1}\varphi_1(t)K_1(s-t,t,\lambda)\big(S(t)x,B^*x^*-\lambda x^*\big)\,dt=-\big(S(s)x,x^*\big)-\cdots \tag{2.11}$$

By the proof of the previous theorem, if $x^*\in D(B^*)$ we can choose a sequence $\{x_n^*\}\in W_m^*$ such that $(x,x_n^*)\to(x,x^*)$ and $(x,B^*x_n^*)\to(x,B^*x^*)$ for all $x\in X$. By the principle of uniform boundedness, $\|x_n^*\|$ and $\|B^*x_n^*\|$ are uniformly bounded. Consequently, in u_1, $\big(S(t)x,B^*x_n^*\big)\to\big(S(t)x,B^*x^*-\lambda x^*\big)$ boundedly and $\big(S(t)x,x_n^*\big)\to\big(S(t)x,x^*\big)$ boundedly. The dominated convergence theorem now allows us to assert the validity of (2.11) for all $x^*\in D(B^*)$. Now, given an $x^*\in D(B^*)$, we choose an $x\in X$ such that $\|x\|\le 1$, $(x,x^*)\ge\frac{\|x^*\|}{2}$, and set $s=0$ in (2.11) to obtain the inequality

$$\frac{N_1'}{\rho(\lambda,S')}\|B^*x^*-\lambda x^*\|\ge\frac{\|x^*\|}{2}-\frac{N_2'}{\rho(\lambda,S')}\|x^*\|-\frac{N_3'}{\rho(\lambda,S')^{\frac{1}{m}}}\|x^*\|.$$

Here we make use of the estimates for the function $K_1(s-t,r,\lambda)$ established in Paragraph 2. Consequently, for $\rho(\lambda,S')) \geq N_4'$,

$$\|x^*\| \leq \frac{N_5'}{\rho(\lambda,S')}\|B^*x^* - \lambda x^*\|.$$

Thus the resolvent $R(\lambda,B)$ exists for $\rho(\lambda,S') \geq N_4'$ and $\|R(\lambda,B)\| \leq N_5$ if $\rho(\lambda,S) \geq N_4$. The theorem is now a consequence of Theorem 12.8.1 of [7].

5. In this paragraph the strongly elliptic form $\sum a_\alpha x_\alpha$ will be fixed. We denote the operator associated with $-\sum_{|\alpha|\leq m}(-i)^{|\alpha|}a_\alpha x_\alpha$ by B and the semi-group it generates by $U(t)$. Since the space, X, on which the group G acts will vary in the course of the proof, we shall specify the space by writing $B(X)$ and $U(t,X)$ when there is a danger of confusion.

Let μ be left-invariant Haar measure on G and let $L_1(\mu)$ be the Banach space of functions on G integrable with respect to μ. Two representations of G in $L_1(\mu)$ of particular interest are $\{L(p)f\}(q) = f(p^{-1}q)$ and $\{R(p)f\}(q) = f(qp)$. It is easily shown that these representations are strongly continuous. We may call them, respectively, the representation by left translations and by right translations. A linear operator on $L_1(\mu)$ is said to commute with right translations if it commutes with all operators $R(p)$. We shall need the following lemma, proved in the general case just as it is for the line [7].

Lemma 6. *Let S be a bounded linear operator on $L_1(\mu)$ which commutes with right translations. Then there is a finite, countably additive Borel set function, ν, such that*

$$Sf(p) = \int_G f(q^{-1}p)\nu)dq) \tag{2.12}$$

for almost all p. Moreover $\operatorname{var}(\nu) = \|S\|$.

Proof. Let $\{g_k(p)\}$ be an approximation to the identity on G and let f be a function in $L_1(\mu)$ with compact support. Set

$$\begin{aligned} h_k(p) &= \int_G f(q^{-1}p)g_k(q)\mu(dq) \\ &= \int_G f(q^{-1})g_k(pq)\mu(dq). \end{aligned}$$

Then

$$\begin{aligned} Th_k(p) &= \int_G f(q^{-1})(Tg_k)(pq)\mu(dq) \\ &= \int_G f(q^{-1}p)(Tg_k)(q)\mu(dq) \\ &= \int_G f(q^{-1}p)\nu_k(dq) \end{aligned} \tag{2.13}$$

with $\nu_k(dq) = (Tg_k)(q)\mu(dq)$. Since $\|g_k\|_{L_1(\mu)} = 1$, $\mathrm{var}\,(\nu_k) \leq \|T\|$. Let ν be an accumulation point of the sequence $\{\nu_k\}$ in the space of bounded, countably additive set functions with its weak-$*$ topology as the conjugate space of C_0, the space of continuous functions on G vanishing at infinity. For any f in $L_1(\mu)$, h_k is defined and (2.13) is valid. Moreover $h_k \to f$ as $k \to \infty$; and, then, $Th_k \to Tf$. But if f is continuous with compact support, $\int_G f(q^{-1}p)\nu(dq)$ is an accumulation point of $(Th_k)(p)$, as given by (2.13). Consequently, for all $f \in L_1(\mu)$,

$$(Tf)(p) = \int_G f(q^{-1}p)\nu(dq)$$

for almost all p. Clearly $\mathrm{var}\,(\nu) \leq \|T\|$ and $\|T\| \leq \mathrm{var}\,\nu$.

We remark that the ν satisfying (2.12) is unique. We may now state the theorem of this paragraph.

Theorem 9. *There exist finite, countably additive Borel set functions, $\mu(t,\cdot)$, depending only on the form $\sum a_\alpha X_\alpha$, and G such that*

$$U(t)x = \int_G T(p)x\mu(t,dp) \tag{2.14}$$

at least for $\psi_1 \leq \arg t \leq \psi_2$; $\psi_1 < 0 < \psi_2$.

The integral is, of course, a Bochner integral. As the theorem is stated ψ_1 and ψ_2 may vary with the representation. It is true, however, that ψ_1 and ψ_2 may be taken to depend only on the form and on G. To establish this we have only to observe that the angles of the sector, outside of which the estimates for $R(\lambda, B)$ were established, depend only on the form and on G.

Proof. Consider first the representation $L(p)$ of G in $L_1(\mu)$. The semi-group $U\big(t, L_1(\mu)\big)$ generated by the operator $B\big(L_1(\mu)\big)$ associated with the form $-\sum(-i)^{|\alpha|}a_\alpha X_\alpha$ in this representation commutes with right translations and, consequently, is given by

$$U\big(t, L_1(\mu)\big)f(p) = \int f(q^{-1}p)\mu(t,dq). \tag{2.15}$$

This establishes the theorem in this case. We next establish it for the case of the representation by left translations in C_0. If f is in $L_1(\mu)$ and g is in C_0, the function

$$h(p) = \int_G f(pg)g(q^{-1})\mu(dq)$$

is in C_0 and $\|h\|_{C_0} \leq \|f\|_{L_1(\mu)}\|g\|_{C_0}$. Let $f_t = u\big(t, L_1(\mu)\big)f$ and set

$$h_t(p) = \int_G f_t(pq)g(q^{-1})\mu(dq).$$

We assert that $h_t = U(t, C_0)h$. To prove this we notice that

$$(i) \qquad \begin{aligned} \|h_t\|_{C_0} &\le \|f_t\|_{L_1(\mu)}\|g\|_{C_0} \\ &\le |U\big(t, L_1(\mu)\big)\| \; \|f\|_{L_1(\mu)}\|g\|_{C_0} \\ &\le Ke^{\omega t}\|f\|_{L_1(\mu)}\|g\|_{C_0}. \end{aligned}$$

Here ω and K are some constants and t is greater then or equal to zero.

$$(ii) \qquad \|h_t - h\|_{C_0} \le \|f_t - f\|_{L_1(\mu)}\|g\|_{C_0} \to 0 \text{ as } t \to 0.$$

$$(iii) \qquad \begin{aligned} \frac{d}{dt}h_t &= \frac{d}{dt}\int_G f_t(\cdot\, q)g(q^{-1})\mu(dq) \\ &= \int_G \frac{d}{dt}f_t(\cdot\, q)g(q^{-1})\mu(dq) \\ &= \int_G B\big(L_1(\mu)\big)f_t(\cdot\, q)g(q^{-1})\mu(dq) \\ &= B(C_0)h_t. \end{aligned}$$

The derivatives are taken in the strong topology.

For $t \ge 0$ the asserted equality now follows from Theorem 23.7.1 of [7]. By analytic continuation $h_t = u(t, C_0)h$ in the domain common to the two sectors in which they are defined. We may now write

$$(2.16) \qquad \begin{aligned} u(t, C_0)h(p) &= \int_G \left\{ \int_G f(r^{-1}pq)\mu(t, dr) \right\} g(q^{-1})\mu(dq) \\ &= \int_G h(r^{-1}p)\mu(t, dr). \end{aligned}$$

Since functions, h, of the above form are dense in C_0 the theorem is established for C_0. In order to complete the proof we must introduce two new spaces of functions. These function spaces are closely related to the given representation, $T(p)$, of G in X. Let Y be the space of continuous functions, f, on G satisfying

$$(a)\ \|f\|_Y = \sup_q \frac{|f(q)|}{\lambda(q)} < \infty$$

$$(b)\ \|f(p^{-1}\cdot) - f(\cdot)\|_Y \to 0 \text{ as } p \to 1.$$

For brevity, we have set $\|T(q)\| + \|T(q^{-1})\| = \lambda(q)$. Y is a Banach space and the representation by left translations of G in Y is strongly continuous. In particular

$$\begin{aligned} \|L(p)f\|_Y &= \sup_q \frac{|f(p^{-1}q)|}{\lambda(q)} \\ &= \sup_q \frac{|f(p^{-1}q)|}{\lambda(p^{-1}q)}\ \frac{\lambda(p^{-1}q)}{\lambda(q)} \\ &\le \lambda(p)\|f\|_Y \end{aligned}$$

for $\lambda(p) = \lambda(p^{-1})$ and $\lambda(pq) \le \lambda(p)\lambda(q)$. It is important to notice that if x is in X and x^* is in X^* then $\big(T(p^{-1})x, x^*\big)$ and $\|T(p^{-1})x\|$ are functions in Y. Moreover, if x is in $W_1(x)$ and a is in A, then

$$\sup_q \frac{|t^{-1}\{\big(T(q^{-1}e(ta))x, x^*\big) - \big(T(q^{-1})x, x^*\big)\} - \big(T(q^{-1})A(a)x, x^*\big)|}{\lambda(q)}$$
$$\le \|x^*\|\ \|t^{-1}\{T\big(e(ta)\big)x - x\} - \big(A(a)x\| \to 0$$

as $t \to 0$. Consequently $\big(T(p^{-1})x, x^*\big)$ is in $W_1(Y)$ and $A(a,Y)(T(p^{-1})x, x^*\big) = \big(T(p^{-1})A(a)x, x^*\big)$. The same relation holds between $W_k(X)$ and $W_k(Y)$. The converse statement is weaker. If $f_{x^*}(p) = \big(T(p^{-1})x, x^*\big)$ is in $W_1(Y)$ for every x^* in x^* and $\big(L(e(ta)) - I\big)A(a,Y)f_{x^*} = 0(t^\alpha)$ as $t \to 0$ for some $\alpha > 0$, then x is in $W_1(x)$. First of all $A(a,Y)f_{x^*}(0) = x_0(x^*)$ defines a bounded linear functional x_0 on X^*. But

$$t^{-1}\{\big(L(e(ta))x - x, x^*\big) - (x_0, x^*)\} = \frac{1}{t}\int_0^t \big(L(e(ta)) - I\big)A(a,Y)f_{x^*}(0)\,dt = 0(t^\alpha).$$

Consequently

$$\left\|\frac{L\big(e(ta)\big)x - x}{t} - x_0\right\|_{X^{**}} = 0(t^\alpha).$$

Thus x_0 is in X and $x_0 = A(a)x$. The same relation holds between $W_k(Y)$ and $W_k(X)$.

The second space, Z, to be introduced is, in a certain sense, dual to Y. It is the space of measurable functions, f, on G satisfying

$$(c) \qquad \int_G |f(q)|\lambda(q)\mu(dq) = \|f\|_Z < \infty.$$

It is essential to observe that $\lambda(q)$ is lower semi-continuous and therefore measurable. The representation by left translations of G in Z is strongly continuous. Z is a subset of $L_1(\mu)$ and $\|f\|_Z \ge \|f\|_{L_1(\mu)}$. Moreover, if $f \in D\big(B(Z)\big)$, then $f \in D\big(B(L_1(\mu))\big)$ and $B(Z)f = B\big(L_1(\mu)\big)f$. Thus a solution of normal type of the abstract Cauchy problem for $B(Z)$ is a solution of normal type of the abstract Cauchy problem for $B\big(L_1(\mu)\big)$. Again, Theorem 23.7.1 of [7] allows us to assert that $U(t,z)f = U\big(t, L_1(\mu)\big)f$. We make use of (2.15) to write

$$(2.17) \qquad U(t,Z)f(p) = \int_G f(q^{-1}p)\mu(t,dq).$$

This is a weaker assertion, in this case, than that of the theorem. We have not yet shown that $\int_G f(q^{-1}.)\mu(t,dq)$ exists as a Bochner integral. Let f be in Z and g be in Y. Consider

$$h(p) = \int_G f(pq)g(q^{-1})\mu(dq).$$

Then

$$\begin{aligned}|h(p)| &\le \int_G |f(pq)|\,|g(q^{-1})|\mu(dq)\\ &\le \|g\|_Y \int_G |f(pq)|\lambda(q)\mu(dq)\\ &\le \|g\|_Y \int_G |f(q)|\lambda(p^{-1}q)\mu(dq)\\ &\le \lambda(p)\|g\|_Y\|f\|_Z.\end{aligned}$$

In other words, $\|h\|_Y \le \|g\|_Y\|f\|_Z$. We remark another simple fact, which allows us to assert that functions, h, of the above form are dense in Y. If f has compact support and $\int_G f(p)\mu(sp) = 1$ then

$$\begin{aligned}\frac{|h(p)-g(p)|}{\lambda(p)} &= \frac{1}{\lambda(p)}\left|\int_G f(pq)\{g(q^{-1}) - g(p)\}\mu)(dq)\right|\\ &\le \frac{1}{\lambda(p)}\int_G |f(q)|\,|g(q^{-1}p) - g(p)|\mu(dq)\\ &\le \sup_{q\in\,\mathrm{supp}\, f} \|g(q^{-1}\cdot) - g(\cdot)\|_Y \int_G |f(q)|\mu(dq).\end{aligned}$$

Using the same technique as before, we set $f_t = U(t,Z)f$ and then set

$$h_t(p) = \int_G f_t(pq)g(q^{-1})\mu(dq).$$

Again the uniqueness theorem for the abstract Cauchy problem assures us that $h_t = U(t,Y)h$. Making use of (2.17) we may write

$$U(t,Y)h(p) = \int_G \left\{\int_G f(r^{-1}pq)\mu(t,dr)\right\}g(q^{-1})\mu(dq). \tag{2.18}$$

Formally changing the order of integration, we obtain

$$U(t,Y)h(p) = \int_G h(r^{-1}p)\mu(t,dr).$$

However, we have not yet provided that the integral in (2.18) is absolutely convergent and we are, consequently, unable to justify the change in the order of integration.

C_0 is a subset of Y and $\|g\|_Y = \|g\|_{C_0}$. Consequently, $U(t,C_0)g$ is a solution of normal type of the abstract Cauchy problem for $B(Y)$. The uniqueness theorem again implies that $U(t,C_0)g = U(t,Y)g$. Making use of (2.16), we write

$$U(t,Y)g(p) = \int_G g(q^{-1}p)\mu(t,dq).$$

Then

$$\left| \int_G \frac{g(q^{-1}p)\mu(t,dq)}{\lambda(p)} \right| \leq \|U(t,Y)\| \, \|g\|_Y.$$

By the usual argument it follows that

$$\int_G \frac{|g(q^{-1}p)| \, |\mu|(t,dq)}{\lambda(p)} \leq \|U(t,Y)\| \, \|g\|_Y.$$

But if $f(q)$ is in Y we can find a sequence $\{g_n(q)\}$ in C_0 such that $g_n \to |f|$. Consequently

$$\int_G |f(q^{-1}p)| \, |\mu|(t,dq) \leq \lambda(p)|U(t,Y)\| \, \|f\|_Y.$$

In particular, setting $f(q) = \|T(q^{-1})x\|$ and setting $p = 1$, we obtain

$$\int_G \|T(q)x\| \, |\mu|(t,dq) \geq 2\|U(t,Y)\| \, \|x\|.$$

We are now able to justify the inversion of the order of integration in (2.18). We apply the last inequality to the space Z and to the representation $L(p)$ of G in Z.

$$\begin{aligned}
\int_G \int_G |f(r^{-1}pq)| \, |g(q^{-1})| \, |\mu|(t,dr)\mu(dq) \\
&\leq \int_G \int_G |f(r^{-1}q)|\lambda(p)\lambda(q)|\mu|(t,dr)\mu(dq) \\
&= \lambda(p) \int_G \|L(r)f\|_Z |\mu|(t,dr) \\
&< \infty.
\end{aligned}$$

We now show that if $x(t) = \int_G T(p)x\mu(t,dp)$ then $x(t) = U(t,X)x$. We first observe that

$$\begin{aligned}
\big(T(q^{-1})x(t), x^*\big) &= \int_G \big(T(q^{-1})T(p)x, x^*\big)\mu(t,dp) \\
&= \int_G \big(T(q^{-1}p)x, x^*\big)\mu(t,dp) \\
&= U(t,Y)\big(T(q^{-1})x, x^*\big).
\end{aligned}$$

We know that $\|x(t)\| \leq 2\|U(t,Y)\| \, \|x\| \leq K_1 e^{\omega_1 t}\|x\|$, with some constants c_1 and ω_1 when $t \geq 0$. If $x \in W_m(x)$ then $\big(T(q^{-1}x, x^*\big)$ is in $W_m(Y)$ and, taking $q = 1$ in the above equality, it follows that $t^{-1}\{\big(x(t), x^*\big) - (x, x^*)\}$ converges to (Bx, x^*) as $t \to 0$. In particular, applying the principle of uniform boundedness, $\|x(t) - x\| \to 0$ as $t \to 0$. Since $U(t,Y)$ is a holomorphic semi-group, $\big(x(t), x^*\big)$ is a holomorphic function and $x(t)$ is a holomorphic function. Moreover $\big(T(q^{-1})x(t), x^*\big)$ is in $D\big(B^k(Y)\big)$ for any k; the work of the next paragraph shows that $\big(T(q^{-1})x(t), x^*\big)$ is in $W_k(Y)$ for any k. Consequently $x(t)$ is in $W_k(x)$ for any k. We observe finally that $\frac{d}{dt}\big(x(t), x^*\big) = \big(Bx(t), x^*\big)$

and, thus, $\frac{d}{dt}x(t) = Bx(t)$. Another application of the uniqueness theorem for the abstract Cauchy problem shows that $x(t) = U(t, X)x$ when x is in $W_m(X)$. Since $W_m(X)$ is dense in x, the equation is valid for all x in X.

6. In this paragraph we establish the basic analytical properties of $U(t)x$ and of $\mu(t, dp)$. $U(t)x$ is an analytic function of t and

$$B^k U(t)x = \frac{d^k}{dt^k} U(t)x = \frac{k!}{2\pi i} \int_{|\rho - t| = r(t)} \frac{U(\zeta)x}{(\zeta - t)^{k+1}} d\zeta.$$

We observe that B^k, as a power of B, is the operator associated, by Theorem 7, with the elliptic form $(-1)^k \big(\sum (-i)^{|\alpha|} a_\alpha x_\alpha \big)^k$ for it is equal to that operator on W_{mk} and its adjoint is equal to that operator's adjoint on W^*_{mk}. Let ν be a right-invariant Haar measure on G and let $K(p)$ be an infinitely differentiable function on G with compact support. If x is in W_{mk}, then

$$\begin{aligned}\int_G K(p) \big(B^k x, T^*(p) x^*\big) \nu(dp) &= \int_G K(p) \sum_{|\alpha| \le mk} b_\alpha \big(A_\alpha x, T^*(p) x^*\big) \nu(dp) \\ &= \int_G \Big\{ -\sum (-1)^{|\alpha|} b_\alpha L_\alpha K(p) \Big\} \big(x, T^*(p) x^*\big) \nu(dp).\end{aligned}$$

$\{L_i\}$ is the set of left-invariant differential operators introduced in Chapter I. This formula remains valid for x in $D(B^k)$. As above, by Lemma 5, if x in $D(B^k)$ then x is in W_{mk-1}. In particular, $U(t)x$ is in $\cap_k W_k$ and $T(p)U(t)x$ is an infinitely differentiable function of p. $A_\alpha U(t)x$ is defined for all x in X; we show that it is a bounded linear function of x. If $|\alpha| = 1, A_\alpha U(t)$ is a closed, everywhere defined linear operator on X; consequently, it is bounded. By induction, it is apparent that $A_\alpha U(t)$ is a bounded linear operator. Consequently $\|A_\alpha S(t)x\| \le N_\alpha(t)\|x\|$. $T(p)U(t)x$ is infinitely differentiable as a function of p and t and

$$\left\| \frac{d^k}{dt^k} A_\alpha U(t)x \right\| = \left\| \frac{K!}{2\pi i} \int_{|\zeta - t| = r(t)} \frac{A_\alpha U(\zeta)x}{(\zeta - t)^{k+1}} d\zeta \right\| \le N(k, \alpha, t)\|x\|.$$

The equation

$$\begin{aligned}\frac{\partial}{\partial t} \big(T(p)U(t)x, x^*\big) &= \big(T(p)BU(t)x, x^*\big) \\ &= -\sum (-i)^{|\alpha|} a_\alpha L_\alpha \big(T(p)U(t)x, x^*\big)\end{aligned}$$

when written in an analytic coordinate system, $\{s_i\}$, about the identity is a parabolic equation with analytic coefficients. We now apply the results of [3]. The facts which we need from this paper are not explicitly stated as theorems and the proofs are not given in complete detail. However, since the proofs are quite complicated and the assertions to be derived from these facts ancillary to the rest of the thesis, we prefer not to perform the calculations in detail here.

The work in the paper shows that $\big(T(p(s))U(t)x, x^*\big) = u(s,t)$ may be extended to an analytic function in a complex neighborhood, $N(t)$ of the origin in s-space. $N(t)$ may be taken locally in t, to be independent of t; and the upper bound of $|u(s,t)|$ in $N(t)$ depends only on upper bounds for the absolute value of $u(s,t)$ and a certain number of its derivatives for real s. Thus $u(s,t)$ may be extended to an analytic function of s and t in a certain open set, M, of complex (s,t)-space, which contains all the points (s,t) with t in the sector in which $U(t)$ was shown to exist and s real and close to the origin. In a neighborhood of any point (s_0, t_0), $|u(s,t)|$ is bounded by an expression $K(s_0,t_0)\|x\|\,\|x^*\|$. For fixed x and varying x^*, $u(s,t)$ defines a bounded linear functional $v(s,t,x)$ on X^*. $v(s,t,x)$ is an analytic map of M into X^{**}. But $v(s,t,x)$ is in X for s real and close to the origin; so $v(s,t,x)$ is in X for all (s,t) in M. In particular, $U(t)x$ is a well-behaved vector, in the sense of [5], in the interior of the sector in which $U(t)$ was shown to exist. Since $U(t)x \to x$ as $t \to 0$, we have

Theorem 10. *The well-behaved vectors are dense for any strongly continuous representation of G.*

We now show that there is a function, $h(t,p)$, analytic to t and p such that $\mu(t,dp) = h(t,p)\mu(dp)$. μ is a left-invariant Haar measure on G. If $f(x)$, in $L_1(\mu)$, is infinitely differentiable with compact support and $\{s_i\}$ is an analytic coordinate system in a neighborhood of the identity, then there are analytic functions, $a_{ij}(s)$, independent of f such that, for small s,

$$\frac{\partial}{\partial s_i} f(s) = \sum_{j=1}^{n} a_{ij}(s) L_j f(s).$$

Consequently, for small δ,

$$\begin{aligned}\int_{|s|\le\delta} \left|\frac{\partial}{\partial s_i} f(s)\right| ds &\le \sum_{j=1}^{n} k \int_{|s|\le\delta} |L_j f(s)| ds \\ &\le K_1 \sum_{j=1}^{n} \|L_j f\|_{L_1(\mu)}.\end{aligned}$$

Theorem $1'$ implies that if f is in $W_1\big(L_1(\mu)\big)$ then it may be approximated by a sequence $\{f_n\}$ of infinitely differentiable functions with compact support in such a manner that $L_j f_n \to L_j f$ in $L_1(\mu)$. Thus, if f is in $W_1\big(L_1(\mu)\big)$, its distribution derivatives, with respect to $\{s_i\}$, in a neighborhood, N, of the origin are in $L_1(\mu, N)$ and

$$\int_{|s|\le\delta} \left|\frac{\partial}{\partial s_i} f_t(s)\right| ds \le K_1 \sum_{j=1}^{n} \|L_j f\|_{L_1}(\mu).$$

Similar remarks apply to the higher-order derivatives. Since, when f is in $L_1(\mu)$, $f_t = U(t, L_1(\mu))f$ is in $W_k(L_1(\mu))$ for any k, we have

$$\int_{|s|\leq\delta} \left|\frac{\partial^\alpha}{\partial s^\alpha} f_t(s)\right| ds \leq C_\alpha(t)\|f\|.$$

It is well-known [1] that this implies that f_t may be taken as an infinitely differentiable function in a neighborhood O, of the origin and that

$$\begin{array}{ll} (i) & |f_t(s)| \leq D_1(t)\|f\| \\ (ii) & \left|\dfrac{\partial}{\partial s_i f_t}(s)\right| \leq D_2(t)\|f\| \end{array}$$

in O. Consequently, for every $p = p(s)$, s in O, there is a bounded measurable function $g(t, p, q)$ such that

$$f_t(p) = \int_G f(q)g(t,p,q)\mu(dq).$$

Moreover $\|g(t, p, \cdot) - g(t, 1, \cdot)\|_{L_\infty}(\mu) \to 0$ as $p \to 1$. If f is continuous with compact support

$$\begin{aligned} f_t(p) &= \int_G f(q^{-1}p)\mu(t, dq) \\ &= \int_G f(q)\mu(t, pdq^{-1}). \end{aligned}$$

Consequently $\mu(t, pdq^{-1}) = g(t, p, q)\mu(dq)$. In particular (cf. [4], p. 265)

$$\mu(t, dq) = g(t, 1, q^{-1})\Delta(q^{-1})\mu(dq) = h(t, q)\mu(dq)\mu(dq \cdot r) = \Delta(r)\mu(dq).$$

Then

$$\mu(t, pdq) = h(t, pq)\mu(dq) = g(t, p, q^{-1})\Delta(q^{-1})\mu(dq),$$

so that $h(t, pq) = g(t, p, q)^{-1}\Delta(q^{-1})$. $h(t, p)$ satisfies the following two conditions.

(i) $\|h(t, \cdot)\|_V = \text{ess}-\sup_q|\Delta(q)h(t, q)| = \text{ess}-\sup_q|g(t, 1, q^{-1})| < \infty$.

(ii) $\text{ess}-\sup_q|\Delta(q)\{h(t, p^{-1}q) - h(t, q)\}| = \text{ess}-\sup_q|g(t, p^{-1}, q^{-1}) - g(t, 1, q^{-1})| \to 0$ as $p \to 1$.

As anticipated in the notation, we call the Banach space of functions satisfying (i) and (ii), with the norm given by (i), V. The functions in V are equivalent to continuous functions so we take V to be a space of continuous functions. The representation, $L(p)$, by left-translations of G in V is strongly continuous. In order to use this fact we must observe that

$$\int_G h(t_1, q^{-1}p)h(t_2, q)\mu(dq) = h(t_1 + t_2, p).$$

To prove this we notice that for f in C_0

$$\begin{aligned}\int_G f(q^{-1}r)h(t_1+t_2,q)\mu(dq) &= u(t_1+t_2,C_0)f(r)\\ &= u(t_1,C_0)u(t_2,C_0)f(r)\\ &= \int_G \{f(p^{-1}q^{-1}r)h(t_1,p)\mu(dp)\}h(t_2,q)\mu(dq)\\ &= \int_G \left\{ \int_G f(p^{-1}r)h(t_1,q^{-1}p)\mu(dp)\right\}h(t_2,q)\mu(dq)\\ &= \int_G f(p^{-1}r)\left\{ \int_G h(t_1,q^{-1}p)h(t_2,q)\mu(dq)\right\}\mu(dp).\end{aligned}$$

However, setting $u(t_2,V)h(t_1,\cdot) = h_{t_2}(t_1,\cdot)$, we also have $h_{t_2}(t_1,p) = \int_G h(t_1,q^{-1}p)h(t_2,q)\mu(dq)$. Consequently $h(t_1+t_2,p) = h_{t_2}(t_1,p)$. Then $h(t_1+t_2,q^{-1}\cdot) = L(q)h_{t_2}(t_1,\cdot)$ is an analytic function of t_2 and q with values in Z. Applying the linear functional which evaluates a function at the identity we see that $h(t,p)$ is an analytic function of t and p.

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