The Dimension of Spaces of Automorphic Forms*

R.P. Langlands

1. The trace formula of Selberg reduces the problem of calculating the dimension of a space of automorphic forms, at least when there is a compact fundamental domain, to the evaluation of certain integrals. Some of these integrals have been evaluated by Selberg. An apparently different class of definite integrals has occurred in Harish-Chandra's investigations of the representations of semi-simple groups. These integrals have been evaluated. In this paper, after clarifying the relation between the two types of integrals, we go on to complete the evaluation of the integrals appearing in the trace formula. Before the formula for the dimension that results is described let us review Harish-Chandra's construction of bounded symmetric domains and introduce the automorphic forms to be considered.

If $\bar{G}$ is the connected component of the identity in the group of pseudo-conformal mappings of a bounded symmetric domain then $\bar{G}$ has a trivial centre and a maximal compact subgroup of any simple component has non discrete centre. Conversely if $\bar{G}$ is a connected semi-simple group with these two properties then $\bar{G}$ is the connected component of the identity in the group of pseudo-conformal mappings of a bounded symmetric domain [2(d)]. Let $\mathfrak{g}$ be the Lie algebra of $\bar{G}$ and $\mathfrak{g}_c$ its complexification. Let G_c be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}_c$; replace $\bar{G}$ by the connected subgroup G of G_c with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$; then $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Fix once and for all an order on $\mathfrak{h}$. This order is to be so chosen that $\mathfrak{g}_c$ is the direct sum of $\mathfrak{k}_c, \mathfrak{p}_+$ and $\mathfrak{p}_-$; $\mathfrak{p}_+$ is spanned by the root vectors belonging to the totally positive roots and $\mathfrak{p}_-$ by the root vectors belonging to the totally negative roots. Moreover $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian and $[\mathfrak{k}_c, \mathfrak{p}_+] \subseteq \mathfrak{p}_+$ and $[\mathfrak{k}_c, \mathfrak{p}_-] \subseteq \mathfrak{p}_-$. Let P_+, P_-, and K_c be the connected subgroups of G_c with Lie algebras $\mathfrak{p}_+, \mathfrak{p}_-$, and $\mathfrak{k}_c$ respectively. The exponential mapping of $\mathfrak{p}_+$ into P_+ is bijective; thus P_+ is provided with the structure of a complex vector space. Moreover $G \subseteq P_+K_cP_-$ and $P_+ \cap K_cP_- = \{1\}$. Then $P_+K_cP_-/K_cP_-$ which is identified with $\mathfrak{p}_+$ is a complex vector space and the image of G is a bounded symmetric domain B. Finally it should be observed that $G \cap K_cP_- = K$ and that $\mathfrak{p}_+$ is an open subset of the space G_c/K_cP_-. Now identify $\mathfrak{p}_+$ with complex coordinate space and let z be the column of coordinates. If $g \in G_c, z \in \mathfrak{p}_+$, and $z' = g(z) \in \mathfrak{p}_+$ (in the space G_c/K_cP_-) let $dz' = \mu(g, z)dz$. Before defining the automorphic forms it is necessary to establish a lemma.

---

* First appeared in Am. J. Math., vol. 85 (1963)

Lemma 1. *Let $\bar{K}_c$ be the restriction of K_c to $\mathfrak{p}_+$ then $\mu(g,z) \in \bar{K}_c$.*

Suppose $X \in \mathfrak{p}_+$ and $f(\cdot)$ is holomorphic in a neighborhood of z' on $\mathfrak{p}_+$ or P_+ since they are identified. Set $h(p_+kp_-) = f(p_+), p_+ \in P_+, k \in K_c, p_- \in P_-$. Then $h(\cdot)$ is a holomorphic function on part of G_c. Let $z = p \in P_+$ and $z' = p' \in P_+$ then, at $t = 0$,

$$\begin{aligned}\frac{d}{dt}h(g\exp(tX)p) &= \frac{d}{dt}h(\exp(tg(X))p')\\ &= \frac{d}{dt}h(p'\exp(tp'^{-1}g(X)))\\ &= L(X_1)h(p') + L(X_2)h(p') + L(X_3)h(p')\\ &= L(X_1)h(p') = R(X_1)h(p').\end{aligned}$$

Here $p'^{-1}g(X) = X_1+X_2+X_3, X_1 \in \mathfrak{p}_+, X_2 \in \mathfrak{k}_c, X_3 \in \mathfrak{p}_-$. $L(X_i)$ and $R(X_i)$ denote the obvious left or right invariant differential operators. It is necessary to verify that the map $X \longrightarrow X_1$ is given by an element of K_c. But $gp = p'kp_-$; so $p'^{-1}g = kp_-p^{-1}$ and $kp_-p^{-1}(X) = kp_-(X) \equiv k(X)(\mathrm{mod}\ \mathfrak{k}_c+\mathfrak{p}_-)$; thus $X_1 = k(X)$. Finally it should be remarked that $\mu(g,z)$ is a holomorphic function of g and z.

Suppose that σ is an irreducible, holomorphic matrix representation of $\bar{K}_c$ of degree d which is unitary on $\bar{K}$. Then, since $\mu(g_1g_2,z) = \mu(g_1,g_2z)\mu(g_2,z)$, it is easily seen that the action of G on the space $H(\sigma)$ of holomorphic functions on B, whose values are column vectors of length d, defined by $g^{-1}f(z) = \sigma^{-1}(g,z)f(gz)$, with $\sigma(g,z) = \sigma(\mu(g,z))$, is a representation of G. If Γ is a discrete subgroup of G define an (unrestricted) automorphic form of type σ to be a function f in $H(\sigma)$ such that $\gamma f = f$ for all γ in Γ. For subgroups of the symplectic group this definition is essentially the same as that of [7]. As is shown there the dimension of the space, $H(\Gamma,\sigma)$, of automorphic forms of type σ is finite if G/Γ is compact. For a large class of representations σ the calculations of this paper lead to the following formula for the dimension

$$N(\Gamma,\sigma) = \sum_{\{\gamma\}} \nu(G_\gamma/\Gamma_\gamma)\chi(\gamma). \tag{1}$$

The sum is over a set of representatives of those conjugacy classes of Γ that have a fixed point in B. G_γ is the centralizer of γ in G and Γ_γ is the centralizer of γ in Γ. If the Haar measure ν on G_γ is appropriately normalized then $\chi(\gamma)$ equals

$$\frac{(-1)^{b_\gamma}}{v(B_\gamma)}\frac{\sum_{w_\gamma\backslash w}\epsilon(s)\prod_{\alpha\in P_\gamma}(s\Lambda(H_\alpha)+s_\rho(H_\alpha))e^{s\Lambda(H)+s\rho(H)}}{[G_\gamma:G_\gamma^0]\prod_{\alpha\in P_\gamma}\rho_\gamma(H_\alpha)\prod_{\substack{\alpha\in P\\ \alpha\notin P_\gamma}}(e^{\frac{1}{2}\alpha(H)}-e^{-\frac{1}{2}\alpha(H)})}$$

The various symbols will be explained in the course of the proof. This formula agrees with those presented in [3] and [6].

2. In this paragraph and the next the trace formula is reviewed in our special context and a first connection with the work of Harish-Chandra is established. The end result is formula (1) with the numbers $\chi(\gamma)$ expressed as integrals.

Since $\mu(k, z) = \bar{k} \in K_c$, the measure

$$dz = |\det(\mu(g, 0))|^{-2} \prod_i dx_i dy_i,$$

with $z = g(0)$, is well defined on B and invariant under G. The invariant measure on G is to be so normalized that

$$\int_B f(z) dz = \int_G f(g(0)) dg.$$

Set $G(z) = \sigma^{*-1}(g, 0)\sigma^{-1}(g, 0)$ with $z = g(0)$. $G(z)$ is well defined and $G(g(z)) = \sigma^{*-1}(g, z)G(z)$ $\sigma^{-1}(g, z)$. Introduce the space $H^2(\sigma)$ of functions f in $H(\sigma)$ for which $\int f^*(z)G(z)f(z)dz$ is finite. The action of G on $H^2(\sigma)$ is easily seen to be unitary. The functional $f \to f_j(z)$, where $f_j(z)$ is the j-th coordinate of $f(z)$, is bounded on $H^2(\sigma)$; let

$$\int_B g_j^*(z_1, z_2)G(z_2)f(z_2)dz_2 = f_j(z_1)$$

and set $K(z_1, z_2) = (g_1(z_1, z_2), \ldots, g_d(z_1, z_2))^*$. Observe that $K(gz_1, gz_2) = \sigma(g, z_1)K(z_1, z_2)\sigma^*$ (g, z_2). If $L^2(\sigma)$ is the space of measurable functions f on B for which $\int f^*(z)G(z)f(z)dz$ is finite then

$$g(z_1) = \int_B K(z_1, z_2)G(z_2)f(z_2)dz_2$$

defines the orthogonal projection of $L^2(\sigma)$ onto $H^2(\sigma)$. Consequently

$$K^*(z_2, z_1) = K(z_1, z_2)$$

and

$$\int_B K(z_3, z_2)G(z_2)K(z_2, z_1)dz_2 = K(z_3, z_1).$$

Although not necessary it is convenient to verify now that the representation of G in $H^2(\sigma)$ is equivalent to a representation investigated by Harish-Chandra [2(c)]. Let W be the inverse image of B under the map $G_c \longrightarrow G_c^{-1} \longrightarrow G_c^{-1}/K_cP_-$ ($W = P_-K_cB^{-1}$ if B is considered a subset of P_+); then if $g \in W$ and $f \in H(\sigma)$ set $f(g) = \sigma^{-1}(g^{-1}, 0)f(g^{-1}(0))$. Then $f(g)$ satisfies: $(\alpha_0) f(pkg) = \sigma(k)f(g)$ if $p \in P_-$ and $k \in K_c$; moreover $f(g)$ is holomorphic on W and if $f(z)$ is in $H^2(\sigma)$ then

$$\|f(\cdot)\|^2 = \int_B f^*(z)G(z)f(z)dz = \int_G \|f(g)\|^2 dg.$$

So the mapping is an isometry on $H^2(\sigma)$. The kernel is replaced by

$$K(g_1, g_2) = \sigma^{-1}(g_1^{-1}, 0)K(g_1^{-1}(0), g_2^{-1}(0))\sigma^{*-1}(g_2^{-1}, 0).$$

Observe that (i) $K(k_1g_1, k_2g_2) = \sigma(k_1)K(g_1, g_2)\sigma^*(k_2)$ if $k_1, k_2 \in K_c$, (ii) $K(pg_1, g_2) = K(g_1, pg_2) = K(g_1, g_2)$ if $p \in P_-$, (iii) $K(g_1g, g_2g) = K(g_1, g_2)$ if $g \in G$, (iv) $K^*(g_2, g_1) = K(g_1, g_2)$, and (v) $\int K(g_1, g_2)K(g_2, g_3)dg_2 = K(g_1, g_3)$.

Now we introduce a third space of functions. Suppose $\mathfrak{k}'_c$ is the semi-simple part of $\mathfrak{k}_c$ and $\mathfrak{c}$ is the centre of $\mathfrak{k}_c$ then $\mathfrak{k}_c = \mathfrak{k}'_c + \mathfrak{c}$ and $\mathfrak{h}_c = \mathfrak{h}_c \cap \mathfrak{k}'_c + \mathfrak{c}$. Any linear functional on $\mathfrak{h}_c \cap \mathfrak{k}'_c$ may be extended to $\mathfrak{h}_c$ by setting it equal to zero on $\mathfrak{c}$. Then the given order on the real linear functions on $\mathfrak{h}_c$ induces an order on the real linear functions on $\mathfrak{h}_c \cap \mathfrak{k}'_c$. The representation σ restricted to $\mathfrak{k}'_c$ is irreducible; let ψ_0 be a unit vector belonging to the highest weight with respect to the above order. Then, for all $h \in \mathfrak{h}_c, h\psi_0 = \Lambda(h)\psi_0$ where Λ is a linear functional on $\mathfrak{h}_c$. Extending the customary language call Λ the highest weight of σ. If $f(g)$ is a holomorphic function on W satisfying (α_0) above set $h(g) = (f(g), \psi_0)$. Then (α) if $p \in P_-, h(pg) = h(g)$; (β) if $n \in N'$, the connected group with Lie algebra $\mathfrak{n}' = \Sigma \mathbf{C} X_{-\alpha}$, the sum being over the positive roots α for which $X_{-\alpha} \in \mathfrak{k}_c$, then $h(ng) = h(g)$; and (γ) if a is in the Cartan subgroup A of G_c with algebra $\mathfrak{h}_c$ then $h(ag) = \xi(a)h(g)$ with $\xi(a) = e^{\Lambda(H)}$ if $a = \exp(H)$. Conversely given a holomorphic function on W satisfying $(\alpha), (\beta)$, and (γ) there is a holomorphic function $f(g)$ such that $h(g) = (f(g), \psi_0)$. Indeed for fixed g the function $h'(k) = h(kg)$ on K satisfies (i) $h'(ak) = \xi(a)h'(k)$ if $a \in A \cap K$ and (ii) $R(X)h'(k) = 0$ if $X \in \mathfrak{n}'$. Let ℓ be an index for the classes of inequivalent irreducible representations of K and let $(\psi^\ell_{ij}(k))$ be the matrices of the representations chosen with respect to a basis $(\phi_1, \ldots, \phi_{d_\ell})$ consisting of eigenvectors of $\mathfrak{h}$; moreover suppose ϕ_1 belongs to the highest weight. Then $h'(k) \sim \sum_\ell \sum_{i,j} \alpha^\ell_{ij}\psi^\ell_{ij}(k)$. Using (i) and (ii) it is easily seen that, first of all, $\alpha^\ell_{ij} = 0$ unless $i = 1$ and then that $\alpha^\ell_{ij} = 0$ unless $\ell = \ell(\sigma)$. So $h'(k) = \sum_j \alpha_{1j}\sigma_{1j}(k)$. Set $f(g) = \sum \alpha_{ij}\phi_j$ then $h(g) = (f(g), \psi_0)$; moreover $f(g)$ is a holomorphic function of g satisfying (α_0) above. Finally the Schur orthogonality relations imply that

$$\begin{aligned}\int_G |h(g)|^2 dg &= \int_G |(f(g), \psi_0)|^2 dg \\ &= \int_K dk \int_G |(f(kg), \psi_0)|^2 dg \\ &= d^{-1} \int_G \|f(g)\|^2 dg.\end{aligned}$$

This shows that the representation of G on $H^2(\sigma)$ is equivalent to the representation π_Λ studied by Harish-Chandra [2(c)].

Now set $\psi(g_1, g_2) = (K(g_1, g_2)\psi_0, \psi_0)$. This function satisfies (i) $\psi(ng, 1) = \psi(g, 1)$ if $n \in N'$, (ii) $\psi(pg, 1) = \psi(g, 1)$ if $p \in P_-$, (iii) $\psi(ag, 1) = \xi(a)\psi(g, 1)$ if $a \in A$, (iv) $\psi(ga, 1) = \overline{\xi(a)}^{-1}\psi(g, 1)$ if $a \in A \cap K$, and (v) $\psi(g, 1)$ is a holomorphic function on W. But Harish-Chandra ([2(c)], p. 22) has shown that there is essentially only one function with these properties; so $\psi(g, 1) = \delta\psi_\Lambda(g)$. $\delta = \psi(1, 1)$ and $\psi_\Lambda(g) = (\zeta(g)\phi_0, \phi_0)e^{\lambda(\Lambda(g))}$. Here ζ is a representation of G_c with highest weight Λ_0 and $\lambda = \Lambda - \Lambda_0$; moreover Λ_0 is so chosen that λ vanishes on $\mathfrak{h}_c \cap \mathfrak{k}'_c$. ϕ_0 is a unit vector belonging to the weight Λ_0. The function $\mu^{-1}(g^{-1}, 0)$ is a holomorphic function on W with values in $\bar{K}_c$. It may be lifted to a function on $\tilde{W}$, the universal covering space of W, with values in $\tilde{K}_c$, the universal covering group of K_c. $\tilde{K}_c$ is the product of a simply connected, complex abelian group C with Lie algebra $\mathfrak{c}$ and a semi-simple group. Mapping $\tilde{W}$ into $\tilde{K}_c$, projecting on C, and then taking the logarithm one obtains $\Gamma(g)$ which lies in $\mathfrak{c}$. Thus $\Gamma(g)$ is a single-valued function on $\tilde{W}$ but a multiple-valued function on W.

Certainly $\delta \neq 0$ if $H^2(\sigma) \neq \{0\}$. In particular, if X_β is a root vector belonging to the positive root β, if $X_{-\beta}$ belongs to $-\beta$ and $H_\beta = [X_\beta, -X_{-\beta}]$ then ([2(d)], p. 612) $H^2(\sigma) \neq \{0\}$ if $2\beta^{-1}(H_\beta)(\Lambda(H_\beta) + \rho(H_\beta)) < 0$ for every totally positive root β. ρ is one-half the sum of the positive roots.

3. It will now be supposed that $2\beta^{-1}(H_\beta)(\Lambda(H_\beta) + \rho(H_\beta) + 2\rho_+(H_\beta)) < 1$ for every totally positive root β; ρ_+ is one-half the sum of the totally positive roots. Then ([2(d)], p. 610) $\psi_\Lambda(g)$ is integrable and, since σ is irreducible, $K(g, 1)$ is integrable. Let $H^\infty(\sigma)$ be the space of functions in $H(\sigma)$ such that $f^*(z)G(z)f(z)$ is bounded. Then, if $f(z)$ is in $H^\infty(\sigma)$,

$$\int_B K(z_1, z_2)G(z_2)f(z_2)dz_2$$

converges. To verify that it equals $f(z_1)$ it is sufficient to show that $H^2(\sigma)$ contains all polynomials for then the argument of Godement in [7] applies. To do this it is sufficient to show that $G(z)$ is integrable over B. This is the same as showing that $\|\sigma^{-1}(g^{-1}, 0)\|^2$ is integrable over G. Let A_+ be the connected group with Lie algebra $\mathfrak{a}_{\mathfrak{p}_0}$ ([2(d)], p. 583). Then every element of G may be written as $k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A_+$. Moreover

$$\|\sigma^{-1}(g^{-1}, 0)\|^2 = \|\sigma^{-1}(a^{-1}, 0)\|^2 = \|\sigma(h(a))\|^2$$

([2(d)], p. 599). But $\|\sigma(h(a))\|^2 = \operatorname{tr}(\sigma(h^2(a)))$ which is known to be integrable. Let $L^2(\Gamma, \sigma)$ be the space of measurable functions on B, whose values are column vectors of length d, such that

$\sigma^{-1}(\gamma, z)f(\gamma z) = f(z)$ for all z in B and all γ in Γ and $\int_F f^*(z)G(z)f(z)dz$ is finite, where F is a fundamental domain for Γ in B. Then

$$f(\cdot) \longrightarrow \int_B K(\cdot, z)G(z)f(z)dz$$

defines the orthogonal projection of $L^2(\Gamma, \sigma)$ onto $H(\Gamma, \sigma)$. Now

$$\int_B K(z_1, z_2)G(z_2)f(z_2)dz_2 = \int_F \sum_\Gamma K(z_1, \gamma z_2)\sigma^{*-1}(\gamma, z_2)G(z_2)f(z_2)dz_2$$

provided $\sum_\Gamma K(z_1, \gamma a_2)\sigma^{*-1}(\gamma, z_2)$ is uniformly absolutely convergent for z_1 and z_2 in F. To verify this it is sufficient to show that $\sum_\Gamma K(g_1, g_2\gamma)$ converges uniformly absolutely in some neighborhood of each point (g_1', g_2') in $G \times G$. Since σ is irreducible it is enough to consider the series $\sum_\Gamma |\psi_\Lambda(g_1\gamma g_2^{-1})|$. Writing $g = k_1ak_2$ we have ([2(d)], pp. 598-600)

$$\begin{aligned} |\psi_\Lambda(g)| &\leqq |(\zeta(k_1^{-1})\phi_0, \zeta(a)\zeta(k_2)\phi_0)e^{\lambda(\Gamma(a))}| \\ &\leqq e^{\lambda(\Gamma(a))}\chi_{\Lambda_0}(h(a)). \end{aligned}$$

Let ϕ be the mapping of G onto the symmetric space G/K and let r be the metric on G/K. Every element of G may be written as a product $g = k_1ak_2$ with $k_1, k_2 \in K$ and with $a \in A_+$ such that $\log a \in \mathfrak{a}_{\mathfrak{p}_0}^+ = \{X \in \mathfrak{a}_{\mathfrak{p}_0} | \alpha(X) \geqq 0 \text{ for all positive roots } \alpha\}$. To be more precise one introduces an order on the linear functions on $\mathfrak{a}_{\mathfrak{p}_0}$, extends $\mathfrak{a}_{\mathfrak{p}_0}$ to a Cartan subalgebra of $\mathfrak{g}$, extends the ordering, and takes the positive roots of this subalgebra with respect to the resulting order. Although it is not *a priori* uniquely determined by g we set $a(g) = a$. We want to show that if $\epsilon > 0$ is given it is possible to choose ϵ_1 so that if h_1 and h_2 are in G and $r(\phi(h_1), \phi(h_2)) < \epsilon_1$ then $|\nu(\log(a(h_1)) - \log(a(h_2)))| \leqq \epsilon\|\nu\|$ for any linear functional ν on $\mathfrak{a}_{\mathfrak{p}_0}$. It is enough to establish this for a basis of the space of linear functionals which may be supposed to consist of the highest weights of certain representations of G restricted to $\mathfrak{a}_{\mathfrak{p}_0}$. Let π be such a representation which may be supposed to satisfy $\pi(\theta(g)) = \pi^{*-1}(g)$ if θ is the Cartan involution of G leaving K fixed. Then $g \longrightarrow \pi(g)\pi^*(g)$ defines an imbedding of G/K in a manner which we shall pretend is isometric in the space of positive definite Hermitian matrices with the Riemannian metric $d^2Y = \text{tr}(Y^{-1}dYY^{-1}dY)$. So it is enough to show that if P_1 and P_2 are positive definite matrices with maximum eigenvalues λ_1, λ_2 then $|\log \lambda_1/\lambda_2| < \epsilon$ if $r(P_1, P_2) < \epsilon$ (cf. [2(i)], p. 280). If $P_1 = AA^*$ and $P_2 = A(\delta_{ij}e^{\alpha_i})A^*,\ \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$, then $r(P_1, P_2) = (\sum_{i=1}^n \alpha_i^2)^{\frac{1}{2}}$. Let $\|x\| = 1$ and $\|A^*x\|^2 = \lambda_1$; then, if $y = A^*x, \lambda_2 \geqq \sum_{i=1}^n e^{\alpha_i}y_i^2 \geqq e^{\alpha_n}\lambda_1$. Similarly $\lambda_1 \geqq e^{-\alpha_1}\lambda_2$; so $-\alpha_1 \leqq \log \lambda_1/\lambda_2 \leqq -\alpha_n$.

As a consequence there are positive numbers c and $\delta < 1$ such that if $U(g') = \{g \mid r(\phi(g), \phi(g')) < \delta\}$ then

$$e^{\lambda(\Gamma(a(g')))}\chi_{\Lambda_0}(h(a(g'))) \leqq c \int_{U(g')} e^{\lambda(\Gamma(a(g)))}\chi_{\Lambda_0}(h(a(g)))dg.$$

Let U_1 and U_2 be compact neighborhoods of g_1' and g_2' respectively. There is an integer N such that for $g_1 \in U_1$ and $g_2 \in U_2$ any point in G belongs to at most N of the sets $U(g_1\gamma g_2^{-1}), \gamma \in \Gamma$. Finally given $\epsilon > 0$ there is a positive number M such that if $V_M = \{g \mid r(\phi(g), \phi(1)) \geqq M\}$ then

$$\int_{V_M} e^{\lambda(\Gamma(a(g)))}\chi_{\Lambda_0}(h(a(g)))dg < \epsilon.$$

For all but a finite set Γ_1 of elements of $\Gamma, U_1\gamma U_2^{-1} \subseteq G - V_{M+1}$. Thus

$$\sum_{\gamma \notin \Gamma_1} \exp(\lambda(\Gamma(a(g_1\gamma g_2^{-1}))))\chi_{\Lambda_0}(h(a(g_1\gamma g_2^{-1}))) \leqq cN\epsilon$$

which was to be shown.

The kernel $\sum_\Gamma K(z_1, \gamma z_2)\sigma^{*-1}(\gamma, z_2)$ is continuous; thus if $\{\omega^i\}, i = 1, \cdots, N(\Gamma, \sigma)$, is an orthonormal basis for $H(\Gamma, \sigma)$,

$$\sum K(z_1, \gamma z_2)\sigma^{*-1}(\gamma, z_2) = \sum_{i=1}^{N(\Gamma,\sigma)} \omega^i(z_1)\omega^{i*}(z_2).$$

But

$$N(\Gamma, \sigma) = \sum_{i=1}^{N(\Gamma,\sigma)} \int_F \sum_{j,\ell} \bar{\omega}_\ell^i(z)\ G_{ij}(z)\omega_j^i(z)dz;$$

consequently

$$N(\Gamma, \sigma) = \int_F \operatorname{tr}\{\sum_\gamma K(z, \gamma z)\sigma^{*-1}(\gamma, z)G(z)\}dz.$$

Following Selberg [5] this may be written

$$\sum_{\{\gamma\}} \sum_{\Gamma_\gamma \backslash \Gamma} \int_F \operatorname{tr}\{K(z, \beta^{-1}\gamma\beta z)\sigma^{*-1}(\beta^{-1}\gamma\beta, z)G(z)\}dz.$$

The outer sum is over a set of representatives of the conjugacy classes of Γ; the inner sum is over a set of coset representatives of the centralizer Γ_γ of γ in Γ. Rewrite the last sum as

$$\sum_{\{\gamma\}} \sum_{\Gamma_\gamma \backslash \Gamma} \int_F \operatorname{tr}\{\sigma^{-1}(\beta, z)K(\beta z, \gamma\beta z)\sigma^{*-1}(\gamma\beta, z)G(z)\}dz = \sum_{\{\gamma\}} \int_{F_\gamma} \operatorname{tr}\{K(z, \gamma z)\sigma^{*-1}(\gamma, z)G(z)\}dz.$$

F_γ is a fundamental domain for Γ_γ in B. Replace these integrals by integrals over fundamental domains F'_γ for Γ_γ acting on G to the right to obtain

$$\sum_{\{\gamma\}}\int_{F'_\gamma} \operatorname{tr}\{K(g^{-1}(0),\gamma g^{-1}(0))\sigma^{*-1}(\gamma,g^{-1}(0))G(g^{-1}(0))\}dg = \sum_{\{\gamma\}}\int_{F'_\gamma} \operatorname{tr}\{K(g\gamma g^{-1},1)\}dg.$$

According to [5] this equals

$$\sum_{\{\gamma\}}\nu(G_\gamma/\Gamma_\gamma)\int_{S_\gamma} \operatorname{tr}\{K(g\gamma g^{-1},1)\}ds_\gamma;$$

G_γ is the centralizer of γ in G; $S_\gamma = G/G_\gamma$; and the measures are so normalized that $dg = ds_\gamma dg_\gamma$. $\nu(G_\gamma/\Gamma_\gamma)$ is the volume of a fundamental domain for Γ_γ acting to the left (or right) in G_γ. It is of some importance to observe that every integral appearing is absolutely convergent. If γ is in Γ set

$$\chi(\gamma) = \int_{S_\gamma} \operatorname{tr}\{K(g\gamma g^{-1},1)ds_\gamma.$$

Apart from the arbitrariness of the invariant measure on S_γ, $\chi(\gamma)$ depends only on the conjugacy class of γ in G. By the Schur orthogonality relations

$$\begin{aligned}\chi(\gamma) &= d\int_{S_\gamma}\int_K (K(kg\gamma g^{-1}k^{-1},1)\psi_0,\psi_0)dk ds_\gamma \\ &= d\delta\int_{S_\gamma}\int_K \psi_\Lambda(kg\gamma g^{-1}k^{-1})dk ds_\gamma.\end{aligned}$$

4. The first step in the evaluation of these integrals is to calculate $d\delta$. Now

$$\begin{aligned}\|(K(g,1)\psi_0,\psi_0)\|^2 &= \int_G |(K(g,1)\psi_0,\psi_0)|^2 dg \\ &= \int_G\int_K |(\sigma(k)K(g,1)\psi_0,\psi_0)|^2 dk dg \\ &= d^{-1}\int_G \|K(g,1)\psi_0\|^2 dg \\ &= d^{-1}\int_G |K^*(g,1)K(g,1)\psi_0,\psi_0)dg \\ &= d^{-1}(K(1,1)\psi_0,\psi_0) = d^{-1}\delta\end{aligned}$$

which shows that $d\delta = \|\psi_\Lambda(g)\|^{-2}$ since $\|(K(g,1)\psi_0,\psi_0)\|^2 = \delta^2\|\psi_\Lambda(g)\|^2$. But $\|\psi_\Lambda(g)\|^{-2}$ has been calculated by Harish-Chandra ([2(d)], p. 608); it equals

$$c(G)\prod_{\beta\in P}|(\Lambda(H_\beta)+\rho(H_\beta))/\rho(H_\beta)|,$$

where P is the set of positive roots and $c(G)$ is a constant independent of Λ. To calculate $c(G)$ take $\sigma(\bar{k}) = (\det\, \bar{k})^{-1}, \bar{k} \in \bar{K}_c$, so that $\Lambda = -2\rho_+$; this is permissible since, as will be seen in a moment, $2\beta^{-1}(H_\beta)(-2\rho_+(H_\beta) + \rho(H_\beta)) < 0$ for every totally positive root β. Then

$$\|\psi_\Lambda(g)\|^{-2} = c(G) \prod_{\beta \in P} |(-2\rho_+(H_\beta) + \rho(H_\beta))/\rho(H_\beta)| = c(G)$$

since $\prod_{\beta\in P} | - 2\rho_+(H_\beta) + \rho(H_\beta)| = \prod_{\beta\in P} |\rho(H_\beta)|$. To see this observe that ([2(b)], p. 749) one could choose as a set of positive roots the positive roots with root vectors in $\mathfrak{k}_c$ and the negatives of the totally positive roots. Let ρ' be one-half the sum of the positive roots in this new order. Then $\rho' = \rho - 2\rho_+$ and $2\beta^{-1}(H_\beta)\rho'(H_\beta) = -2(-\beta(H_{-\beta}))^{-1}\rho'(H_{-\beta}) < 0$ since $-\beta$ is positive in this new order if β is totally positive. There is an element s in the normalizer of $\mathfrak{h}_c$ in G_c which takes the positive roots in the original order into the positive roots in the new order; in particular $\rho'(H) = \rho(s^{-1}(H))$. Now

$$[H, s(X_\alpha)] = s([s^{-1}(H), X_\alpha]) = \alpha(s^{-1}(H))s(X_\alpha) = s(\alpha)(H)s(X_\alpha)$$

and

$$H'_{s(\alpha)} = [X'_{s(\alpha)}, X'_{-s(\alpha)}] = [s(X_\alpha), s(X_{-\alpha})] = s(H_\alpha).$$

Now $[H'_{s(\alpha)}, X'_{s(\alpha)}] = s[H_\alpha, X_\alpha] = \alpha(H_\alpha)X'_{s(\alpha)}$. So if all X_α are so normalized that $\alpha(H_\alpha) = 2$ then $H'_{s(\alpha)} = H_{s(\alpha)}$. Consequently

$$|\prod_{\beta\in P} \rho'(H_\beta)| = |\prod_{\beta\in P} \rho'(H'_{s(\beta)})| = |\prod_{\beta\in P} \rho'(s(H_\beta))| = |\prod_{\beta\in P} \rho(H_\beta)|.$$

On the other hand $\psi(g) = \det(\mu(g^{-1}, 0))$ satisfies (i) $\psi(pg) = \psi(g)$ if $p \in P_-$, (ii) $\psi(gk) = \psi(kg) = \det^{-1}(\bar{k})\psi(g)$ if $k \in K_c$, (iii) $\psi(g)$ is holomorphic on W, and (iv) $\psi(1) = 1$. This is enough to ensure that $\psi(g) = \psi_\Lambda(g)$. Thus

$$\begin{aligned}
\|\psi_\Lambda(g)\|^2 &= \int_G |\psi(g)|^2 dg \\
&= \int_B |\det(\mu(g,0))|^2 |\det(\mu(g,0))|^{-2} \prod_i dx_i dy_i \\
&= v(B).
\end{aligned}$$

$v(B)$ is the Euclidean volume of B. In conclusion

$$\chi(1) = (-1)^b/v(B) \prod_{\beta\in P} ((\Lambda(H_\beta) + \rho(H_\beta))/\rho(H_\beta)),$$

with b equal to the complex dimension of B.

It will be useful at this point to establish some notation. The universal covering groups of G and K_c have been denoted by $\tilde{G}$ and $\tilde{K}_c$. If G_1 is a subgroup of G then $\tilde{G}_1$ is the group of all elements in $\tilde{G}$ lying over G_1. Elements of $\tilde{G}$ will be denoted by $\tilde{g}$ and $\tilde{\gamma}$ and their projection in G by g and γ; similarly $\tilde{k} \in \tilde{K}_c$ projects on k. If N is a simply-connected subgroup of G then N is isomorphic to the connected component of the identity in $\tilde{N}$ so the same symbol will be used for corresponding elements in the two groups. Finally $\tilde{g}\tilde{\gamma}\tilde{g}^{-1}$ will be written $\tilde{g}\tilde{\gamma}\tilde{g}^{-1}$.

Every element of Γ is semi-simple [1]; this implies in particular that G^0_γ, the connected component of the identity in G_γ, is of finite index in G_γ. The measure on G_γ will be so normalized that, on $G^0_\gamma, dg_\gamma = dg^0_\gamma$. Then if the measure on $S^0_\gamma = G/G^0_\gamma$ is normalized in the usual manner

$$\chi(\gamma) = d_\Lambda [G_\gamma : G^0_\gamma]^{-1} \int_{S^0_\gamma} \int_K \psi_\Lambda(kg\gamma g^{-1}k^{-1}) dk ds^0_\gamma$$

with $d_\Lambda = d\delta$. But ψ_Λ may be lifted to a function on $\tilde{G}$ and $\chi(\gamma)$ may be written as

$$d_\Lambda [G_\lambda : G^0_\gamma]^{-1} \int_{S^0_\gamma} \int_K \psi_\Lambda(kg\tilde{\gamma}g^{-1}) dk ds^0_\gamma. \tag{3}$$

Recall that $\psi_\Lambda(\tilde{g}) = (\zeta(g)\phi_0, \phi_0) e^{\lambda(\Gamma(\tilde{g}))}$. Revising the notation slightly denote the linear functions λ and Λ associated to the representation σ by λ' and Λ' and let λ be an arbitrary linear function on $\mathfrak{h}_c$ vanishing on $\mathfrak{h}_c \cap \mathfrak{k}'_c$ and, accordingly, let $\Lambda = \Lambda_0 + \lambda$. d_Λ and $\psi_\Lambda(\tilde{g})$, but not $\psi_\Lambda(g)$, are still defined. Let us now see for which functions λ the integral converges.

The function $\mu(g, z)$ on $G \times B$ may be lifted to a function $\mu(\tilde{g}, z)$ on $\tilde{G} \times B$ with values in $\tilde{K}_c$ which satisfies $\mu(\tilde{g}_1, g_2 z)\mu(\tilde{g}_2, z) = \mu(\tilde{g}_1\tilde{g}_2, z)$. Perhaps the simplest way to see this is to observe that if $z = p, p \in P_+$, then p^{-1} is in $\tilde{W} = P_-\tilde{K}_c B^{-1}$ ([2(c)], p. 5) and so is $p^{-1}g^{-1} = p_-\tilde{k}^{-1}p_+^{-1}, p_+ \in B$; then $\mu(\tilde{g}, z) = \tilde{k}$. In particular $\mu(\tilde{k}_1\tilde{g}\tilde{k}_2, 0) = \tilde{k}_1\mu(\tilde{g}, 0)\tilde{k}_2$ so that

$$\Gamma(\tilde{k}_1\tilde{g}\tilde{k}_2) = \Gamma(\tilde{k}_1) + \Gamma(\tilde{g}) + \Gamma(\tilde{k}_2).$$

Now write $\tilde{g} = \tilde{k}_1 a \tilde{k}_2$ with $a = \exp(\sum_{i=1}^{s} t_i(X_{\gamma_i} + X_{-\gamma_i}))$ ([2(d)], p. 599).

It is possible to choose a basis $\{c_1, \cdots, c_k\}$ for $\mathfrak{c}$ so that the coordinates of $\Gamma(\tilde{k}_1)$ and $\Gamma(\tilde{k}_2)$ are purely imaginary and those of $\Gamma(a)$ are of the form $\sum_i \log(\cosh t_i) a_{ij}$ with $a_{ij} \geqq 0$. If the basis is chosen from $i(\mathfrak{c} \cap \mathfrak{k})$ the first condition is satisfied. The second will be satisfied if we choose a basis so that the projection of $H_{\gamma_i}, i = 1, \cdots, s$, on the centre has positive coordinates ([2(d)], p. 600). It will be enough to show that this can be done when the group is simple and $\mathfrak{c}$ has dimension 1. But

$2\rho_+(H_{\gamma_i}) > 0, i = 1, \cdots, s$, and $2\rho_+(H)$ is determined solely by the projection of H on $\mathfrak{c}$ since it is the trace of the representation of $\mathfrak{k}_\mathfrak{c}$ on $\mathfrak{p}_+$. Since $H_{\gamma_i} \in i\mathfrak{k}$ the assertion is proved. It will be shown below that for fixed $\tilde{\gamma}$ the imaginary parts of the coordinates of $\Gamma(g\tilde{\gamma}g^{-1})$ remain bounded as g varies over G; consequently the integral over S^0_γ converges absolutely if $\mathrm{Re}\{\lambda(c_i) - \lambda'(c_i)\} \leqq 0$ and represents a function of λ which is continuous on this set and holomorphic in its interior. Thus it will be sufficient to evaluate the integral (3) when $\lambda(c_i)$ is real and very much less than zero.

We now establish the unproved assertion. The notation of [2(d)] will be used. In showing that the imaginary parts of the coordinates of $\Gamma(\tilde{g}\tilde{\gamma}g^{-1})$ are bounded we may suppose that $\tilde{g} = \tilde{k}_1 \exp(X)\tilde{k}_2$ with $X \in \mathfrak{a}_{\mathfrak{p}_0}$ and $\tilde{k}_1$ and $\tilde{k}_2$ in some fixed compact subset of $\tilde{K}$. Suppose $\pi(\tilde{k})$ is the result of projecting $\tilde{k}$ on the centre of $\tilde{K}_c$ and then taking the logarithm. Then

$$\begin{aligned}\Gamma(g\tilde{\gamma}g^{-1}) &= \pi(\mu^{-1}(\exp(X)\tilde{k}_2\gamma^{-1}\tilde{k}_2^{-1}\exp(-X), 0\\ &= -\pi(\mu(\exp(-X), 0)) - \pi(\mu(\gamma, k_2^{-1}\exp(-X)(0))) - \pi(\mu(\exp(X), k_2\gamma^{-1}k_2^{-1}\exp(-X)(0))).\end{aligned}$$

The first term gives no contribution to the imaginary part. $\mu(\gamma, z)$ is defined for z in an open subset of $\mathfrak{p}_+$ containing the closure of B; so it is possible to define $\mu(\tilde{\gamma}, z)$ on the same set. Since it is continuous it takes the closure of B into a compact set. Thus only the third term causes trouble. So we consider $\mu(\exp(X), z)$ letting z vary over B.

The calculations will be simplified if we first prove a lemma. Every element X of $\mathfrak{p}_+$ determines a linear transformation from $\mathfrak{p}_-$ to $\mathfrak{k}_c$, namely $T(X)Y = [X, Y]$ if $Y \in \mathfrak{p}_-$. Introduce on $\mathfrak{p}_-$ and $\mathfrak{k}_c$ the Hermitian inner product $-B(Y_1, \tilde{\theta}(Y_2))$ then

Lemma 2. *B is the set of vectors X in $\mathfrak{p}_+$ for which $2I - T^*(X)T(X)$ is positive definite.*

If k is in K then

$$T(k(X)) = \mathrm{Ad}(k)T(X)\,\mathrm{Ad}(k^{-1})$$

and

$$T^*(k(X))T(k(X)) = \mathrm{Ad}(k)T^*(X)T(X)\,\mathrm{Ad}(k^{-1});$$

moreover X is in B if and only if $k(X)$ is in B. So in proving the lemma we may replace X by any element equivalent to it under the adjoint action of K. Suppose X is in B then X may be supposed equal to $\sum_{i=1}^{s} a_i X_{\gamma_i}$ with $-1 < a_i < 1$. Any element of $\mathfrak{p}_-$ may be written as

$$Y = \sum_{i=1}^{s} b_i X_{-\gamma_i} + \sum_i \sum_{\alpha \in P_i} b_\alpha X_{-\alpha} + \sum_{i<j} \sum_{\alpha \in P_{ij}} b_\alpha X_{-\alpha}.$$

Then

$$[X,Y]=\sum_{i=1}^{s}a_ib_i[X_{\gamma_i},X_{-\gamma_i}]+\sum_i\sum_{\alpha\in P_i}a_ib_\alpha[X_{\gamma_i},X_{-\alpha}]+\sum_{i,j}\sum_{\alpha\in P_{ij}}a_ib_\alpha[X_{\gamma_i},X_{-\alpha}].$$

It is easily seen that $B(X_\alpha,\tilde\theta(X_\beta))=0$ unless $\alpha=\beta$ and that $[X_{\gamma_i},Y]$ is orthogonal to $[X_{\gamma_j},Y]$ if $i\neq j$. Moreover $\tilde\theta([X_{\gamma_i},X_{-\alpha}])=[X_{-\gamma_i},X_\alpha]$ and

$$\begin{aligned}-B([X_{\gamma_i},X_{-\alpha}],[X_{-\gamma_i},X_\beta])&=-B([[X_{\gamma_i},X_{-\alpha}],X_{-\gamma_i}],X_\beta)\\&=-B([H_{\gamma_i},X_{-\alpha}],X_\beta)\\&=\alpha(H_{\gamma_i})B(X_{-\alpha},X_\beta).\end{aligned}$$

Since $\alpha(H_{\gamma_i})=0,1$, or 2 it follows that $\|T(X)Y\|^2<2\|Y\|^2$. Conversely, suppose $X\in\mathfrak{p}_+$ and $\|T(X)Y\|^2<2\|Y\|^2$ for every Y in $\mathfrak{p}_-$. If $X=\sum_{i=1}^{s}a_iX_{\gamma_i}$ with a_i real, as may be assumed, then $\|[X,X_{-\gamma_i}]\|^2=2a_i^2\|X_{-\gamma_i}\|^2$, so that $|a_i|<1$. It follows that X is in B.

Similar calculations now show that if

$$X_0=\sum_{i=1}a_iX_{\gamma_i}+\sum_i\sum_{\alpha\in P_i}b_\alpha X_\alpha+\sum_{i<j}\sum_{\alpha\in P_{ij}}b_\alpha X_\alpha$$

is in B then $|a_i|<1, i=1,\cdots,s$.

The original assertion will be proved if we show that the imaginary coordinates of $\pi(\mu(\exp(t(X_{\gamma_i}+X_{-\gamma_i})),z))$ remain bounded as z varies over B. Let $z=\exp(X_0)$ with X_0 as above and set $g(t)=\exp(t(X_{\gamma_i}+X_{-\gamma_i}))$. Write

$$X_0=\sum_{\alpha\in S_i}a_\alpha X_\alpha+a_iX_{\gamma_i}+\sum_{\substack{\alpha\notin S_i\\ \alpha\neq\gamma_i}}a_\alpha X_\alpha=X_1+X_2+X_3$$

where S_i is the set of roots which vanish on H_{γ_i}. Then

$$g(t)\exp(X_0)=\exp(X_1)g(t)\exp(X_2)\exp(X_3).$$

Here $g(t)$ and $\exp(X_2)$ belong to the complex group whose Lie algebra is spanned by H_i, X_{γ_i}, and $X_{-\gamma_i}$. A simple calculation in $SL(2,\mathbf{C})$ shows that $g(t)=\exp(a(t)X_{\gamma_i})\exp(b(t)H_{\gamma_i})\exp(c(t)X_{-\gamma_i})$ with

$$\begin{aligned}a(t)&=(a_i\,\mathrm{cosh}t+\,\mathrm{sinh}t)(a_i\,\mathrm{sinh}t+\,\mathrm{cosh}t)^{-1}\\b(t)&=-\log(a_i\,\mathrm{sinh}t+\,\mathrm{cosh}t)\\c(t)&=\,\mathrm{sinh}t(a_i\,\mathrm{sinh}t+\,\mathrm{cosh}t)^{-1}.\end{aligned}$$

Finally

$$\exp(c(t)X_{-\gamma_i})\exp(X_3) = \exp(X_3)\exp(X_4)$$

with $X_4 = \operatorname{Ad}(\exp(-X_3))(c(t)X_{-\gamma_i})$; so $X_4 = c(t)X_{-\gamma_i} + \sum c_\alpha X_\alpha$. The sum is over the positive compact roots. This implies that $\exp(X_4)$ is the product of an element in K_c', the semi-simple component of $\tilde{K}_c$, and an element in P_-. So

$$\pi(\mu(g(t), z)) = \pi(\exp(b(t)H_{\gamma_i})) = -\log(a_i \sinh t + \cosh t)\pi(\exp(H_{\gamma_i})).$$

The coordinates of $\pi(\exp(H_{\gamma_i}))$ are real and, since $|a_i| < 1$,

$$\operatorname{Re}(a_i \sinh t + \cosh t) > 0;$$

so

$$-\frac{\pi}{2} < \operatorname{Im}(\log(a_i \sinh t + \cosh t)) < \frac{\pi}{2}.$$

It will be seen that for $\lambda(c_i) \ll 0$ and $\tilde{\gamma}$ semi-simple the double integral in (3) is absolutely convergent; consequently in our analysis the integral over K may be omitted and γ need not belong to Γ.

5. It will be convenient in the evaluation of the integrals (3) to omit at first any detailed estimates. These will be discussed in the next paragraph. Suppose that γ is a regular element in G and let γ belong to the centralizer B of the Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$. According to [2(e)] it may be supposed that $\theta(\mathfrak{j}) = \mathfrak{j}$. Hence $\mathfrak{j} = \mathfrak{j}_1 + \mathfrak{j}_2$ with $\mathfrak{j}_1 = \mathfrak{j} \cap \mathfrak{k}$ and $\mathfrak{j}_2 = \mathfrak{j} \cap \mathfrak{p}$. The case that $\mathfrak{j}_2 = \{0\}$ will be treated first. Then $B \subseteq K$ and it may be supposed that $\mathfrak{j} = \mathfrak{h}$. $G_\gamma \subseteq K$ and the measure on G_γ is so normalized that the total measure of G_γ^0 is 1. The integration over S_γ^0 in (3) may then be replaced by an integration over G; as will be seen below the integral is then a continuous function of $\tilde{\gamma}$ as $\tilde{\gamma}$ varies over the regular elements in $\tilde{K}$. Harish-Chandra has shown that if T_Λ is the character of the representation π_Λ then

$$T_\Lambda(f) = d_\Lambda \int_G dg\{\int_{\tilde{G}} f(g_1)\psi_\Lambda(g\tilde{g}_1 g^{-1})d\tilde{g}_1\}$$

when f is an infinitely differentiable function with compact support. If the support of f is contained in the set of regular elements in GKG^{-1} the order of integration may be reversed. Another formula for $T_\Lambda(f)$ is implicit in the papers [2(c)] and [2(e)]. However before introducing this it must be observed that $\tilde{B}$ is connected and thus every element of $\tilde{B}$ can be written as the exponential of an element in $\mathfrak{h}$.

In particular, let $\tilde{\gamma} = \exp(H)$. Then $T_\Lambda(f)$ is obtained by integrating f against a continuous function whose value at $\tilde{\gamma}$ is

$$\{\prod_{\alpha \in P} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})\}^{-1} \sum_{s \in w} \epsilon(s) e^{s\Lambda(H)+s\rho(H)}.$$

P is the set of positive roots; w is the Weyl group of K_c; and $\epsilon(s) = \pm 1$ according as s is the product of an even or odd number of reflections. It should be observed that the second hypothesis of Section 10 of [2(e)] does not hold here. So it is necessary to prove Lemma 43 using Fourier integrals rather than series. The value of $\chi(\gamma)$ obtained agrees with (2) since B_γ is reduced to a point with $v(B_\gamma) = 1$, P_γ is empty , and w_γ is reduced to $\{1\}$.

Retaining the assumption that $\tilde{\gamma}$ is regular it will now be supposed that $\mathfrak{j}_2 \neq \{0\}$. Define the subgroups M and N as on page 212 of [2(g)] with $\mathfrak{j}$ replacing $\mathfrak{h}_0$ then ([2(g)], p. 216)

$$\int_{G/G_\gamma^0} \psi_\Lambda(g\tilde{\gamma}g^{-1}) ds_\gamma^0 = \int_{K\times M\times N} \psi_\Lambda(kmn\tilde{\gamma}n^{-1}m^{-1}k^{-1}) dkdmdn,$$

if the Haar measures on M and N are suitably normalized. It should be observed that, contrary to the assertion in [2(e)], the centralizer $\tilde{B}$ in $\tilde{G}$ of a Cartan subalgebra is not always commutative. Thus, if $\tilde{\gamma}$ belong to $\tilde{B}$ one must consider $\int_{G/B^0} f(g\tilde{\gamma}g^{-1}) d\bar{g}$ and not $\int_{G/B} f(g\tilde{\gamma}g^{-1}) d\bar{g}$; B is the projection of $\tilde{B}$ on G and B^0 is the connected component of the identity in B. The theorems of [2(h)] used later must be interpreted with this observation in mind. It is not difficult (cf. [2(a)], p. 509) to see that the above integral equals

$$\xi(X_i)^{-1}(\gamma) \int_{K\times M\times N} \psi_\Lambda(knm\tilde{\gamma}m^{-1}) dkdmdn$$

with $\xi(X_i)(\gamma)$ equal to the determinant of the restriction of $I - ad(\gamma)$ to $\mathfrak{n}$, the Lie algebra of N. It can be assumed that $\mathfrak{j}_2$ is contained in $\mathfrak{a}_{\mathfrak{p}_0}$. Then, in the notation of [2(d)], for some ℓ either $\nu^{-1}(X_{\gamma_\ell})$ or $\nu^{-1}(X_{-\gamma_\ell})$ is in $\mathfrak{n}_c$. Since the order on $\mathfrak{j}_2$ is arbitrary suppose that $\nu^{-1}(X_{\gamma_\ell})$ is in $\mathfrak{n}_c$.

$$\nu^{-1}(X_{\gamma_\ell}) = \frac{1}{2}(X_{\gamma_\ell} - X_{-\gamma_\ell} - H_{\gamma_\ell})$$

and $2i_\nu^{-1}(X_{\gamma_\ell}) = X$ is in $\mathfrak{g}$ and thus in $\mathfrak{n}$. Let $N_1 = \{\exp(tX) | -\infty < t < \infty\}$; N_1 is a closed subgroup of N, so that the above integral may be written

$$\xi^{-1}(\gamma) \int_{K\times M\times N_1\backslash N} \{\int_{-\infty}^{\infty} \psi_\Lambda(k\exp(tX)nm\tilde{\gamma}m^{-1}k^{-1}) dkdmd\bar{n}.$$

To show that $\chi(\gamma) = 0$ it is sufficient to show that the inner integral is identically zero; this will be done using Cauchy's integral theorem. Recall that

$$\psi_\Lambda(k\exp(tX)nm\tilde{\gamma}m^{-1}k^{-1}) = (\zeta(k)\zeta(\exp(tX))\zeta(nm\gamma m^{-1})\phi_0, \phi_0) e^{\lambda(\Gamma(g))}$$

with $g = \exp(tX)nm\tilde{\gamma}m^{-1}k^{-1}$. The first term is clearly an entire function of t.

$$\Gamma(g) = -\pi(\mu(m\tilde{\gamma}m^{-1}n^{-1}, \exp(-tX)(0))) - \pi(\mu(\exp(-tX), 0)).$$

For $m, \tilde{\gamma}$, and n fixed the first term is defined, bounded, and analytic in t so long as $\exp(-tX)(0)$ is in B. If it is observed that the subgroup of G_c whose Lie algebra is spanned by $H_{\gamma_\ell}, X_{\gamma_\ell}$, and $X_{-\gamma_\ell}$ is the homomorphic image of $\mathrm{SL}(2, \mathbf{C})$ then the calculations may be performed in this group. Now

$$X = \begin{pmatrix} -i & i \\ -i & i \end{pmatrix} \qquad \exp(tX) = \begin{pmatrix} 1-ti & ti \\ -ti & 1+ti \end{pmatrix}$$

and

$$\begin{pmatrix} 1+ti & -ti \\ ti & 1-ti \end{pmatrix} = \begin{pmatrix} 1 & -it(1-ti)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1-ti)^{-1} & 0 \\ 0 & (1-ti) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ it(1-ti)^{-1} & 1 \end{pmatrix}.$$

Thus $\exp(-tX)(0) = -it(1-it)^{-1}X_{\gamma_\ell}$ is in B if $|it(1-ti)^{-1}| < 1$ or $\mathrm{Im}(t) > -\frac{1}{2}$ and

$$-\pi(\mu(\exp(-tX), 0)) = \log(1-ti)\pi(\exp(H_{\gamma_\ell}))$$

is analytic in the half-plane $\mathrm{Im}(t) > -\frac{1}{2}$. Moreover, in this region

$$\begin{aligned} |\psi_\Lambda(k\exp(tX)nm\tilde{\gamma}m^{-1}k^{-1})| &\leqq c(1+|t|)^n|1-ti|^{\lambda(H_{\gamma_\ell})} \\ &\leqq c(1+|t|)^{-2} \end{aligned}$$

if $\lambda(H_{\lambda_\ell}) \ll 0$. n is a positive integer. Here and in what follows c is used as a generic symbol for a positive constant. Cauchy's integral theorem may now be applied.

Suppose $\tilde{\gamma}$ is singular. $\tilde{\gamma}$ belongs to the centralizer of at least one Cartan subgroup $\mathfrak{j}$ of $\mathfrak{g}$; $\mathfrak{j}$ may be taken such that $\theta(\mathfrak{j}) = \mathfrak{j}$. Let $\mathfrak{g}_\gamma$ be the centralizer of $\tilde{\gamma}$ in $\mathfrak{g}$, then $\theta(\mathfrak{g}_\gamma) = \mathfrak{g}_\gamma$. Consequently $\mathfrak{g}_\gamma$ is the direct sum of an abelian algebra $\mathfrak{a}$ and a semi-simple algebra $\mathfrak{g}_1$. Let $\mathfrak{j}_1$ be a fundamental Cartan subalgebra of $\mathfrak{g}_1$ ([2(g)], p. 236). Then $\mathfrak{j}_1 + \mathfrak{a}$ is a Cartan sub-algebra of $\mathfrak{g}$ which may be supposed to equal $\mathfrak{j}$. Let B^0 be the connected component of the centralizer of $\mathfrak{j}$ in G. Then

$$\begin{aligned} \int_{G/B^0} f(g^*)dg^* &= \int_{G/G_\gamma^0} ds_\gamma^0 \int_{G_\gamma^0/B^0} f((gg_0)^*)dg_0^* \\ &= \int_{G/G_\gamma^0} ds_\gamma^0 \int_{G_1/B_1} f((gg_1)^*)dg_1^* \end{aligned}$$

if G_1 is the connected group with Lie algebra $\mathfrak{g}_1$ and $B_1 = B^0 \cap G_1$. The measures are so normalized that $dg = dg^*db$ and $dg_\gamma^0 = dg_0^*db$, db being the Haar measure on B^0. Moreover G_γ^0 is the homomorphic image of $G_1 \times A$ where A is the connected group with Lie algebra $\mathfrak{a}$. The measure on G^0_γ is so normalized that

$$\int_{G^0_\gamma} f(g^0_\gamma) dg^0_\gamma = \int_{G_1 \times A} f(g_1 a) dg_1 da.$$

Finally let B^0_1 be the connected component of the identity in B_1 and write the above integral as

$$[B_1 : B^0_1]^{-1} \int_{G/G^0_\gamma} ds^0_\gamma \int_{G^0_1/B^0_1} f((gg_1)^*) dg^*_1.$$

Choose $\tilde{\gamma}_1$ close to the identity in B so that $\tilde{\gamma}\tilde{\gamma}_1$ is regular then

$$(4) \qquad [B_1 : B^0_1] \int_{G/B^0} f(g\tilde{\gamma}\tilde{\gamma}_1 g^{-1}) dg^* = \int_{G/G^0_\gamma} ds^0_\gamma \int_{G_1/B^0_1} f(g\tilde{\gamma} g_1 \tilde{\gamma}_1 g_1^{-1} g^{-1}) dg^*_1.$$

If G_1 is any connected semi-simple group with finite center, B_1 the centralizer of a Cartan subalgebra $\mathfrak{j}_1$ of $\mathfrak{g}_1$, and $m(\tilde{g}_1)$ a function on $\tilde{G}_1$, the universal covering group of G_1, then a function can be defined on $\tilde{B}_1$ by

$$\phi_m(\tilde{\gamma}_1) = \Delta_1(\tilde{\gamma}_1) \int_{G^0_1/B^0_1} m(g_1 \tilde{\gamma}_1 g_1^{-1}) dg^*_1$$

when the integral exists. To obtain $\Delta_1(\tilde{\gamma}_1)$ map $\tilde{G}_1$ into the simply connected complex group whose Lie algebra is the complexification of $\mathfrak{g}_1$; let $\tilde{\gamma}_1$ go into γ_1 and set $\gamma_1 = \exp(H_1)$ with H_1 in the complexification of $\mathfrak{j}_1$. Then $\Delta_1(\tilde{\gamma}_1) = e^{-\rho_1(H_1)} \prod_{\alpha \in P} (e^{\alpha(H_1)} - 1)$; P_1 is the set of positive roots with respect to some order on $\mathfrak{j}_1$ and ρ_1 is one-half the sum of the roots in P_1. For every $\alpha \in P_1, H_\alpha$ defines an invariant differential operator D_α on $\tilde{B}_1$; set $D_1 = \prod_{\alpha \in P} D_\alpha$. Harish-Chandra [2(h)] has shown that if $m(\tilde{g}_1)$ is infinitely differentiable with compact support then

$$\lim_{\gamma_1 \to 1} D_1 \phi_m(\tilde{\gamma}_1) = am(1).$$

a is a constant independent of m and $a \neq 0$ if $\mathfrak{j}_1$ is fundamental. To be more precise $\phi_m(\tilde{\gamma}_1)$ is defined if $\tilde{\gamma}_1$ is regular and the limit is taken on the set of regular elements.

Apply this result formally to equation (4) with $f(g) = d_\Lambda \psi_\Lambda(g)$ and $m(\tilde{g}_1) = d_\Lambda \psi_\Lambda(\tilde{g}\tilde{\gamma}\tilde{g}_1 g^{-1})$. If B^0 is not compact the left side is 0 and one obtains

$$\int_{G/G^0_\gamma} d_\Lambda \psi_\Lambda(g\tilde{\gamma} g^{-1}) ds^0_\gamma = 0,$$

so that $\chi(\gamma) = 0$ if γ has no fixed point in B. If B^0 is compact it may be supposed that $\mathfrak{j} = \mathfrak{h}$ and that $\mathfrak{j}_1 = \mathfrak{j} \cap \mathfrak{g}_1$. In this case B_1 is connected. If $\tilde{\gamma}\tilde{\gamma}_1 = \exp(H + H_1)$ then

$$\begin{aligned} a &\int_{G/G_\gamma^0} d_\Lambda \psi_\Lambda(g\tilde{\gamma}g^{-1}) ds_\gamma^0 \\ = &\lim_{H_1 \to 0} D_1\{(\prod_{\substack{\alpha \in P \\ \alpha \notin P_1}} (e^{\frac{1}{2}\alpha(H+H_1)} - e^{-\frac{1}{2}\alpha(H+H_1)}))^{-1} \sum_{s \in w} \epsilon(s) e^{s\Lambda(H+H_1)+s\rho(H+H_1)}\}, \end{aligned}$$

if the total measure of B^0 is 1 as will be assumed and if an order on $\mathfrak{j}_1$ is so chosen that the positive roots are just the positive roots of $\mathfrak{j}$ whose root vectors lie in $\mathfrak{g}_{1,c}$. The denominator is regular at $H_1 = 0$ and is invariant under the Weyl group of $\mathfrak{g}_{1,c}$. Thus, as on page 159 of [2(h)] the right side equals

$$(\prod_{\substack{\alpha \in P \\ \alpha \notin P}} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}))^{-1} \lim_{H_1 \to 0} D_1\{\sum_{s} \epsilon(s) e^{s\Lambda(H+H_1)+s\rho(H+H_1)}\}.$$

The second term equals

$$\omega_1 \sum_{w_1/w} \epsilon(s) \prod_{\alpha \in P_1} (s\Lambda(H_\alpha) + s\rho(H_\alpha)) e^{s\Lambda(H)+s\rho(H)};$$

w_1 is the Weyl group of $\mathfrak{k}_c \cap \mathfrak{g}_{1,c}$ and ω_1 is its order. The sum is over a set of representatives of cosets of w_1 in w. It remains to calculate a. Since, as is easily seen, every non-compact positive root of $\mathfrak{g}_{1,c}$ is totally positive, G_1 is locally isomorphic to the product of a compact group and the group of pseudo-conformal mappings of a bounded symmetric domain B_γ. Λ_1 and $\psi_{\Lambda_1}(\tilde{g}_1)$ may be defined in the same manner as Λ and $\psi_\Lambda(g)$; the compact component causes no difficulty [2(c)]. Apply the limit formula of Harish-Chandra to $d_{\Lambda_1}\psi_{\Lambda_1}(\tilde{g}_1)$ to obtain

$$\begin{aligned} ad_{\Lambda_1} &= \lim_{H_1 \to 0} D_1\{\sum_{s \in w_1} \epsilon(s) e^{s\Lambda_1(H_1)+s\rho_1(H_1)}\} \\ &= \omega_1 \prod_{\alpha \in P_1} (\Lambda_1(H_\alpha) + \rho_1(H_\alpha)) \end{aligned}$$

if the total measures of B_1^0 is 1 as may be assumed. If the measure on G_1 is normalized in the same way as that on G, then

$$d_{\Lambda_1} = (-1)^{b_\gamma}/v(B_\gamma) \prod_{\alpha \in P_1} ((\Lambda_1(H_\alpha) + \rho_1(H_\alpha))/\rho_1(H_\alpha))$$

if B_γ is equal to the complex dimension of B_γ. The constant a is now easily determined. Setting $P_1 = P_\gamma, \rho_1 = \rho_\gamma$, and $w_1 = w_\gamma$ the value of (3) is found to be

$$\frac{(-1)^{b_\gamma}}{v(B_\gamma)} \frac{\sum_{w_\gamma/w} \epsilon(s) \prod_{\alpha\in P_\gamma} (s\Lambda(H_\alpha) + s\rho(H_\alpha))e^{s\Lambda(H)+s\rho(H)}}{[G_\gamma : G_\gamma^0]^{-1} \prod_{\alpha\in P_\gamma} \rho_\gamma(H_\alpha) \prod_{\substack{\alpha\in P\\ \alpha\notin P_\gamma}} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})}$$

and (2) is established. It should be observed that since the total measure of both B^0 and B_1^0 must be 1 the measure on A, and thus on G_γ^0, is completely determined.

6. The prime task of this section is to justify the above application of the limit formula of Harish-Chandra. The truth of the other unproved statements above will become evident in the course of the justification so there is no need to mention them explicitly again. G will now denote a connected semi-simple group with finite centre and $\tilde{G}$ will denote its universal covering group. All the other standard symbols will also refer to G.

In particular $\mathfrak{a}_\mathfrak{p}$ will be a maximal abelian subalgebra of $\mathfrak{p}$. If $\{H_i\}$ is a basis for $\mathfrak{a}_\mathfrak{p}$ and if $\tilde{g} = \tilde{k}_1 \exp(H)\tilde{k}_2$ with $H = \sum_{i=1}^{s} t_i H_i$ set $t_i = t_i(g)$. The family $(t_i(g))$ is not uniquely determined by g. Let $\omega(k)$, for $k \in K$, be the determinant of the restriction of $I - Ad(k)$ to $\mathfrak{p}$; then

Lemma 3. *There are positive constants ϵ, c, and q such that*

$$\exp(\sum_{i=1}^{s} |t_i(gkg^{-1})|) \geqq c|\omega(k)|^q \exp(\epsilon \sum_{i=1}^{s} |t_i(g)|).$$

If $A = (A_{ij})$ is any matrix set $\|A\| = (\sum_{i,j} |A_{ij}|^2)^{\frac{1}{2}}$ and if $g \in G$ let $\|g\| = \|Ad(g)\|$ where $Ad(g)$ is the matrix of the adjoint of g with respect to a basis of $\mathfrak{g}$ orthonormal with respect to the inner product $-B(X, \theta(Y))$. It is easy to verify that

$$c_1 \exp(\beta_1 \sum_{i=1}^{s} |t_i(g)|) \geqq \|g\| \geqq c_2 \exp(\beta_1 \sum_{i=1}^{s} |t_i(g)|) \tag{5}$$

for some positive constants c_1, c_2, β_1, and β_2. Now it is sufficient to verify the lemma for $g = a = \exp(H)$. If $Ad(k) = (s_{ij})$ with respect to a basis which diagonalizes $\mathfrak{a}_\mathfrak{p}$ then $\|aka^{-1}\|^2 = \sum_{i,j} e^{\lambda_{ij}(H)} s_{ij}^2$. The λ_{ij} are linear functions on $\mathfrak{a}_\mathfrak{p}$. For a fixed k with $\omega(k) \neq 0$ this must approach infinity with $\sum_{i=1}^{s} |t_i|$ ([2(h)], p. 743). Let $S(H)$ be the set of pairs (ij) for which $\lambda_{ij}(H) > 0$ then

$$\sum_{(ij)\in S(H)} s_{ij}^2 \neq 0$$

unless $\omega(k) = 0$. Moreover, for some small positive number $\epsilon(H), (ij) \in S(H)$ implies $\lambda_{ij}(H) > 3\epsilon(H) \sum_{i=1}^{s} |t_i|$. Let $M = \{H = \sum_{i=1}^{s} t_i H_i | \sum_{i=1}^{s} |t_i| = 1\}$. If H is in M there is a neighbourhood $U(H)$ of H in M such that if H' is in $U(H)$ and $t > 0$ then $\| \exp(tH')k \exp(-tH')\|^2 \geqq \exp(\epsilon(H)t \sum_{i=1}^{s} |t'_i|)\{\sum_{(ij)\in S(H)} s_{ij}^2\}$. Since $\sum_{(ij)\in S(H)} s_{ij}^2$ vanishes only when $\omega(k)$ vanishes the theorem of Łojasiewicz [4] implies that there are positive constants $c(H)$ and $q(H)$ such that

$$\Big(\sum_{(ij)\in S(H)} s_{ij}^2\Big)^{\frac{1}{2}} \geqq c(H)|\omega(k)|^{q(H)}$$

for all k. All that is left is to observe that M is compact.

Suppose $\tilde{\gamma}$ is a semi-simple element of $\tilde{G}$. Define $G_\gamma, G_\gamma^0, G_1$, and so on, as before. It is no longer necessary, however, to suppose that $\mathfrak{j}_1$ is fundamental. Then

$$[B_1 : B_1^0] \int_{G/B^0} f(g^*)dg^* = \int_{G/G_\gamma^0} ds_\gamma^0 \int_{G_1/B_1^0} f((gg_1)^*)dg_1^*.$$

If $\Delta_1(\tilde{\gamma}_1)$ and D_1 are defined as above we are to show that

$$\begin{aligned} \lim_{\gamma_1 \to 1} D_1\Delta_1(\tilde{\gamma}_1) \int_{G/G_\gamma^0} ds_\gamma^0 \int_{G_1/B_1^0} (g\tilde{\gamma}g_1\tilde{\gamma}_1 g_1^{-1} g^{-1}) dg_1^* \\ = a \int_{G/G_\gamma^0} \psi(g\tilde{\gamma}g^{-1}) ds_\gamma^0. \end{aligned} \tag{6}$$

$\tilde{\gamma}_1$ is chosen so that $\tilde{\gamma}\tilde{\gamma}_1$ is regular and the limit is taken in the manner previously indicated. Of course it will be necessary to impose some conditions on the function ψ. If ψ is infinitely differentiable with compact support then for $\tilde{\gamma}_1$ in some compact neighbourhood of the identity the inner integral on the left, a function on G/G_γ^0, vanishes outside some fixed compact set U ([2(h)], Thm. 1). Moreover

$$D_1\Delta_1(\tilde{\gamma}_1) \int_{G_1/B_1^0} \psi(g\tilde{\gamma}g_1\tilde{\gamma}_1 g_1^{-1} g^{-1}) dg_1^*$$

converges uniformly on U to $a\psi(g\tilde{\gamma}g^{-1})$ ([2(h)], Thms. 2 and 4). This shows the validity of (6) for functions with compact support. To establish it for another function ψ it would be sufficient to show that for any $\epsilon > 0$ there is a sequence $\{\psi_i(\tilde{g})\}$ of infinitely differentiable functions with compact support such that

(i) $$\lim_{i\to\infty} D_1\Delta_1(\tilde{\gamma}_1) \int_{G/B^0} \psi_i(g\tilde{\gamma}\tilde{\gamma}_1 g^{-1}) dg^* = D_1\Delta_1(\tilde{\gamma}_1) \int_{G/B^0} \psi(g\tilde{\gamma}\tilde{\gamma}_1 g^{-1}) dg^*$$

uniformly in $\tilde\gamma_1$ and

(ii) $$\lim_{i\to\infty}\int_{G/G_\gamma^0}\psi_i(g\tilde\gamma g^{-1})ds_\gamma^0=\int_{G/G_\gamma^0}\psi(g\tilde\gamma g^{-1})ds_\gamma^0.$$

$\tilde\gamma_1$ is, of course, to lie in a fixed compact neighbourhood of the identity and be such that $\tilde\gamma\tilde\gamma_1$ is regular.

In order to establish the existence of $\{\psi_i\}$ it is sufficient to assume that ψ is infinitely differentiable and that there is a sufficiently large constant α such that, for any left-invariant differential operator D on $\tilde G$, $|D\psi(\not{g})|\leqq c(D)\|g\|^{-\alpha}$.

Once it has been verified that this condition is satisfied by $\psi_\Lambda(\tilde g)$ when λ is real and $\lambda(H_{\gamma_i})\ll 0$ there will no longer be any need to refer specifically to this function. For convenience, if $X\in\mathfrak{g}$ we denote the differential operator $\frac{d}{dt}f(\tilde g\exp(tX))|_{t=0}$ by X. It may be supposed that $D=\prod_{i=1}^k X_i$ so that

$$D\psi_\Lambda(\tilde g)=D\{(\zeta(g)\phi_0,\phi_0)e^{\lambda(\Gamma(g))}\}=\sum_\sigma D_{\sigma'}(\zeta(g)\phi_0,\phi_0)D_\sigma e^{\lambda(\Gamma(g))};$$

σ runs over the subsets of $\{1,\cdots,k\}$ and $D_\sigma=\prod_{i\in\sigma}X_i$ with the order of the X_i's left unchanged. σ' is the complement of σ. Now $D_{\sigma'}(\zeta(g)\phi_0,\phi_0)=(\zeta(g)\prod_{i\in\sigma'}\zeta(X_i)\phi_0,\phi_0)$, so that there is no doubt that

$$|D_{\sigma'}(\zeta(g)\phi_0,\phi_0)|\leqq c(D'_\sigma)\|g\|^{\alpha_1}$$

with some constant α_1. To find $D_\sigma e^{\lambda(\Gamma(g))}$ we must differentiate

$$\exp(\lambda(\Gamma(\tilde g\prod_{i\in\sigma}\exp(t_iX_i)))$$

with respect to each of the variables and evaluate the result at the origin. But

$$\Gamma(\tilde g\prod_{i\in\sigma}\exp(t_iX_i))=-\pi(\mu({\prod_{i\in\sigma}}'\exp(-t_iX_i)g^{-1},0)),$$

the prime indicating that the order of the factors is reversed, and this equals

$$-\pi(\mu(\not{g}^{-1},0))-\pi(\mu({\prod_{i\in\sigma}}'\exp(-t_iX_i),g^{-1}(0)).$$

There is an open neighbourhood U of the identity in G_c and an open neighbourhood V of $\bar B$, the closure of B, such that $h\in U$ implies $h^{-1}(V)\subseteq\mathfrak{p}_+$. Consequently $\mu(h^{-1},z)$ is defined and analytic on $U\times V$. So is $\pi(\mu(h^{-1},z))$ and its derivatives at the identity are bounded functions on B. Thus

$$|D_\sigma e^{\lambda(\Gamma(\tilde g))}|\leqq c(D_\sigma)|e^{\lambda(\Gamma(\tilde g))}|;$$

but $|e^{\lambda(\Gamma(\tilde{g}))}| \leqq c(\lambda)\exp(\sum_{i=1}^{s}|t_i(g)|\lambda(H_{\gamma_i}))$ ([2(d)], p. 600). Thus, if $\lambda(H_{\gamma_i}) \ll 0, i = 1, \cdots, s$,

$$|D_\sigma e^{\lambda(\Gamma(\tilde{g}))}| \leqq c(D_\sigma)\exp(-\alpha_2\sum_{i=1}^{s}|t_i(g)|)$$

with α_2 large and positive. These remarks and formula (5) show that the assumption is satisfied.

We shall need a non-decreasing sequence $\{\phi_i(\tilde{g})\}$ of infinitely differentiable functions on $\tilde{G}$ with compact support satisfying conditions: (α) $\lim_{i\to\infty}\phi_i(\tilde{g}) = 1$, (β) there is a non-decreasing sequence $\{U_i\}$ of open sets which exhausts $\tilde{G}$ such that $\phi_i(\tilde{g}) \equiv 1$ on U_i, (γ) if D is a left-invariant differential operator on $\tilde{G}$ then $|D\phi_i(\tilde{g})| \leqq c(D)$ for all i and $\tilde{g}$. We write, after Iwasawa, $\tilde{G} = KHN$. If $\{X_i\}$ is a basis for $\mathfrak{g}$ and if (a_{ij}) is the matrix of $Ad(hn)$ with respect to this basis then

$$\begin{aligned} X_i f(\tilde{k}hn) &= \sum_j a_{ji}\frac{d}{dt}f(\tilde{k}\exp(tX_j)hn) \\ &= \sum_j q_{ji} f(\tilde{k}hn; Y_j, Z_j) \end{aligned}$$

if $X_j = Y_j + Z_j, Y_j \in \mathfrak{k}$ and $Z_j \in \mathfrak{h} + \mathfrak{n}$. If D_1 is a left-invariant differential on $\tilde{K}$ and D_2 a right-invariant differential operator on HN then $f(\tilde{g}; D_1, D_2)$ is the result of the successive applications of these two operators to f considered as a function on $\tilde{K} \times HN$. In particular,

$$f(\tilde{k}hn; 1, Z_j) = \frac{d}{dt}f(\tilde{k}\exp(tZ_j)hn)|_{t=0}.$$

Iterating we obtain

$$Df(\tilde{k}hn) = \sum_j g_i(hn) f(\tilde{k}hn; D_1^i, D_2^i);$$

$g_i(hn)$ is a polynomial in the coefficients of $Ad(hn)$. If $X \in \mathfrak{h} + \mathfrak{n}$ and $X = X_1 + X_2, X_1 \in \mathfrak{h}, X_2 \in \mathfrak{n}$, then

$$\frac{d}{dt}f(\exp(tX)hn) = \frac{d}{dt}f(\exp(tX_1)hn) + \frac{d}{dt}f(h\exp(th^{-1}(X_2))n).$$

Consequently

$$Df(\tilde{k}hn) = \sum_i g_i(hn, h^{-1}) f(\tilde{k}hn; D_1^i, D_2^i, D_3^i);$$

D_1^i acts on $\tilde{K}$, D_2^i on H, and D_3^i on N. Moreover $g_i(hn, h^{-1})$ is a polynomial in the coefficients of $Ad(hn)$ and $Ad(h^{-1})$. The functions $\phi_i(\tilde{g})$ are to be constructed as products $\phi_i(\tilde{k}hn) = \phi_i^1(\tilde{k})\phi_i^2(h)\phi_i^3(n)$. Since the coefficients $g_j(hn, h^{-1})$ are independent of $\tilde{k}$ we need only require that $\{\phi_i^1(\tilde{k})\}$ satisfy $(\alpha), (\beta)$, and (γ). This requirement is easily satisfied since $\tilde{K}$ is the product of a vector group and a compact group. H is a vector group and the coefficients $g_i(nh, h^{-1})$ are exponential polynomials on H so we need only require that $\{\phi_i^2(h)\}$ satisfy (α) and (β) and that the derivatives of $\phi_i^2(h)$ go to zero faster than any exponential polynomial uniformly in i. N is a closed subset of the space of endomorphisms of $\mathfrak{g}$ and the functions $g_j(hn, h^{-1})$ are polynomials in the coefficients of $Ad(n)$. Thus the functions $\phi_i^3(n)$ can be obtained as the restriction to N of a sequence of functions on a vector group which satisfies (α) and (β) and is such that the derivatives of the functions go to zero faster than any inverse polynomial uniformly in i.

Now set $\psi_i(\tilde{g}) = \phi_i(\tilde{g})\psi(\tilde{g})$. To establish (i) it is sufficient to show that for any invariant differential operator v on $\tilde{B}$

$$\lim_{i\to\infty} v\Delta(\tilde{\gamma}) \int_{G/B^0} \psi_i(g\tilde{\gamma}g^{-1})dg^* = v\Delta(\tilde{\gamma}) \int_{G/B^0} \psi(g\tilde{\gamma}g^{-1})dg^* \tag{7}$$

uniformly in $\tilde{\gamma}$ on any bounded subset (i.e. a subset with compact closure in $\tilde{B}$) of the set of regular elements in $\tilde{B}$. $\Delta(\tilde{\gamma})$ is defined in the same manner as $\Delta_1(\tilde{\gamma}_1)$. To obtain (i) from this relation it is sufficient to set $v = D_1$ and to observe that $\Delta_1(\tilde{\gamma}_1)\Delta^{-1}(\tilde{\gamma}\tilde{\gamma}_1)$ is regular at $\tilde{\gamma}_1 = 1$.

If M and N are the groups introduced on p. 212 of [2(g)] then

$$\Delta(\tilde{\gamma}) \int_{G/B^0} \psi(g\tilde{\gamma}g^{-1})dg^* = \Delta(\tilde{\gamma})\xi(X_i)^{-1}(\tilde{\gamma}) \int_{K\times M\times N} \psi(knm\tilde{\gamma}m^{-1}k^{-1})dkdmdn.$$

Let S be a finite set of invariant differential operators on B and let ℓ be the maximum degree of the operators in S. Let δ belong to S and let D be a left-invariant differential operator on $\tilde{G}$. Then δ determines in an obvious fashion a left invariant differential operator on $\tilde{G}$ which will be denoted δ'. Then

$$\begin{aligned} |\delta(D\psi(knm\tilde{\gamma}m^{-1}k^{-1}))| &= |Ad(km)(\delta')D\psi(knm\tilde{\gamma}m^{-1}k^{-1})| \\ &\leqq \|km\|^{\ell} c(D)\|nm\gamma m^{-1}\|^{-\alpha} \end{aligned}$$

and

$$|\delta(D\psi(knm\tilde{\gamma}m^{-1}k^{-1})) - \delta(D\psi_i(knm\tilde{\gamma}m^{-1}k^{-1}))| \leqq \|km\|^{\ell} c(D)\|nm\gamma m^{-1}\|^{-\alpha}.$$

Moreover there is an increasing sequence $\{V_i\}$ of open sets in $N \times M$ which exhaust $N \times M$, so that the left side of the latter inequality is zero if $(n, m) \in V_i$.

Recall that B^0 is the connected component of the identity of the centralizer in G of a Cartan subalgebra $\mathfrak{j}$ and that $\theta(\mathfrak{j}) = \mathfrak{j}$. An examination of the form of the matrices of $Ad(m\gamma m^{-1})$ and $Ad(n)$ with respect to a basis which diagonalizes $\mathfrak{j} \cap \mathfrak{p}$ shows that $\|nm\gamma m^{-1}\| \geqq \|m\gamma m^{-1}\|$. Thus

$\|nm\gamma m^{-1}\|^{-\alpha} \leqq \|m\gamma m^{-1}\|^{-\alpha_1}\|nm\gamma m^{-1}\|^{-\alpha_2}$ if $\alpha_1+\alpha_2=\alpha$. Now $\tilde{\gamma}$ may be written as $\tilde{\gamma}_-\tilde{\gamma}_+$ with $\tilde{\gamma}_- \in \tilde{B}\cap\tilde{K}$ and $\tilde{\gamma}_+ \in \exp(\mathfrak{j}\cap\mathfrak{p})$. Then

$$\|m\gamma m^{-1}\| \geqq \|\gamma_+^{-1}\|^{-1}\|m\gamma_- m^{-1}\| \geqq c|\omega_-(\gamma_-)|^q\|\gamma_+^{-1}\|^{-1}\|m\|^{\epsilon}.$$

Here $\omega_-(\gamma_-)$ is the determinant of the restriction of $I-Ad(\gamma_-)$ to $\mathfrak{m}\cap\mathfrak{p}$ and c,q, and ϵ are positive constants. Thus

$$\|km\|^{\ell}\|nm\gamma m^{-1}\|^{-\alpha} \leqq c|\omega_-(\gamma_-)|^{q(\alpha_2-\alpha_1)}\|\gamma_+^{-1}\|^{\alpha_1-\alpha_2}\|m\|^{\epsilon(\alpha_2-\alpha_1)+\ell}\|n\|^{-\alpha_2}.$$

Consequently

$$\int_{K\times M\times N}|\delta(D\psi(knm\tilde{\gamma}m^{-1}k^{-1}))|dkdmdn \tag{8}$$

is at most

$$c(D)|\omega_-(\gamma_-)|^{q(\alpha_2-\alpha_1)}\|\tilde{\gamma}_+^{-1}\|^{\alpha_1-\alpha_2}\int_M\|m\|^{\epsilon(\alpha_2-\alpha_1)+\ell}dm\int_N\|n\|^{-\alpha_2}dn$$

and

$$\int_{K\times M\times N}|\delta(D\psi(knm\tilde{\gamma}m^{-1}k^{-1}))-\delta(D\psi_i(knm\tilde{\gamma}m^{-1}k^{-1}))|dkdmdn \tag{9}$$

is at most

$$c(D)|\omega_-(\gamma_-)|^{q(\alpha_2-\alpha_1)}\|\gamma_+^{-1}\|^{\alpha_1-\alpha_2}\int_{V_i'}\|m\|^{\epsilon(\alpha_2-\alpha_1)+\ell}\|n\|^{-\alpha_2}dmdn.$$

Now it can be shown (cf. [2(g)], Cor. 1 to Lemma 6) that the integral over N in (8) converges if α_2 is sufficiently large. Then, fixing α_2, we can choose α_1 so large that the first integral converges. Moreover, by the dominated convergence theorem, the integrals in (8) converge to zero as i approaches infinity. We conclude first of all that

$$\phi_\psi(\tilde{\gamma})=\Delta(\tilde{\gamma})\int_{G/B^0}\psi(g\tilde{\gamma}g^{-1})dg^*$$

is defined on $\tilde{B}'=\{\tilde{\gamma}\in\tilde{B}|\omega_-(\gamma_-)\neq 0\}$ and is the uniform limit on compact subsets of $\tilde{B}'$ of the sequence $\{\phi_{\psi_i}(\tilde{\gamma})\}$.

There is a finite set $\{v_1,\cdots,v_w\}$ of invariant differential operators on $\tilde{B}$ such that any other v, may be written as $v=\sum_{j=1}^{w}v_ju_j$ where the u_j are invariant under the Weyl group of $\mathfrak{g}_c$ ([2(f)], p. 101). For each u_j there is a left-invariant and right-invariant differential operator D_j on $\tilde{G}$ so that $u_j\phi\psi_i(\tilde{\gamma})=\phi_{D_j}\psi_i(\tilde{\gamma})$ ([2(h)], p. 155). Then

$$v\phi\psi_i(\tilde{\gamma})=\sum_{j=1}^{w}v_j\phi_{D_j}\psi_i(\tilde{\gamma}).$$

The right side is a sum of terms of the form

$$\phi(\tilde{\gamma}) \int_{K\times M\times N} \delta(D\psi_i(knm\tilde{\gamma}m^{-1}k^{-1}))dkdmdn.$$

$\phi(\tilde{\gamma})$ is a regular function on $\tilde{B}$; δ is one of a finite set of invariant differential operators on $\tilde{B}$; and D is a left-invariant and right-invariant differential operator on $\tilde{G}$. As a consequence of the estimates above the sequence $\{v\phi\psi_i(\tilde{\gamma})\}$ converges uniformly on compact subsets of $\tilde{B}'$ and it must converge to $v\phi\psi(\tilde{\gamma})$. So

$$|v\phi\psi(\tilde{\gamma}) - v\phi\psi_i(\tilde{\gamma})| \leqq |\omega_-(\gamma_-)|^{q(\alpha_2-\alpha_1)}c(v,i)$$

on any fixed bounded subset of $\tilde{B}'$. Moreover $\lim\limits_{i\to\infty} c(v,i) = 0$. The proof of (7) can now be completed by an argument essentially the same as that on pp. 208-211 of [2(g)]. There is no point in reproducing it. If we show that

$$\int_{G/G_\gamma^0} \psi(g\tilde{\gamma}g^{-1})ds_\gamma^0 \tag{10}$$

is absolutely convergent then a simple application of the dominated convergence theorem suffices to establish (ii). Choose a maximal abelian subspace of $\mathfrak{g}_1 \cap \mathfrak{p}$ and extend it to a Cartan subalgebra $\mathfrak{j}_1$ of $\mathfrak{g}_1$, then $\mathfrak{j} = \mathfrak{j}_1 + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\theta(\mathfrak{j}) = \mathfrak{j}$. We again introduce the groups M and N. If $\mathfrak{n}$ is the Lie algebra of N let $\mathfrak{n}_1 = \mathfrak{n} \cap \mathfrak{g}_1$ and let N_1 be the connected group with Lie algebra $\mathfrak{n}_1$. If $\mathfrak{n}_2 = (I - Ad(\gamma))\mathfrak{n}$ then, according to Lemma 7.0 of [1], every element of N may be written uniquely in the form $\exp(Y_2)n_1$ with $n_1 \in N_1$ and $Y_2 \in \mathfrak{n}_2$. Then if B_+ is the connected group with Lie algebra $\mathfrak{j} \cap \mathfrak{p}$ every element of G may be written uniquely as $g = k\exp(X)\exp(Y_2)n_1b_+$ with $X \in \mathfrak{m} \cap \mathfrak{p}$, $\mathfrak{m}$ being the Lie algebra of M, and $b_+ \in B_+$ (cf. [2(h)], p. 215). Let $\phi_0(n_1b_+)$ be a non-negative function on N_1B_+ such that

$$\int_{N_1B_+} \phi_0(n_1b_+)dn_1db_+ = 1,$$

and set $\phi(g) = \|\exp(X)\exp(Y_2)\|^\beta \phi_0(n_1b_+)$ if $g = k\exp(X)\exp(Y_2)n_1b_+$. Here β is a suitably chosen non-negative constant. Then a function may be defined on G/G_γ^0 by

$$\phi(s_\gamma^0) = \int_{G_\gamma^0} \phi(gg_\gamma)dg_\gamma;$$

s^0_γ is the coset containing g. We shall show that if β is sufficiently large then $\phi(s^0_\gamma)$ is greater than some fixed positive constant for all s^0_γ. It may be assumed that $g = k\exp(X)\exp(Y_2)$. If K_γ is the connected group with Lie algebra $\mathfrak{k}\cap\mathfrak{g}_\gamma$ and if $gu = k'\exp(X')\exp(Y_2')n_1'b_+'$, $u \in K_\gamma$, then

$$\begin{aligned}\phi(s^0_\gamma) &= \int_{K_\gamma\times N_1\times B_+}\phi(gun_1b_+)du dn_1 db_+\\ &= \int_{K_\gamma}\|\exp(X')\exp(Y_2')\|^\beta e^{-2\rho_1(\log b_+')}du.\end{aligned}$$

Suppose $f_1(z_1)$ and $f_2(z_2)$ are two non-negative functions of the variables z_1 and z_2 and z_1 and z_2 are subject to some relation. We shall write $f_1 \succ f_2$ if there is a positive constant c and a non-negative constant β such that $cf_1^\beta(z_1) \geqq f_2(z_2)$ for all pairs (z_1, z_2) satisfying the given relation. The assertion will be proved if it is shown that $\|\exp(X')\exp(Y_2')\| \succ e^{2\rho_1(\log b_+')}$. If $H_+ \in \mathfrak{j}\cap\mathfrak{p}$ then $2\rho_1(H^+)$ is the trace of the restriction of $Ad(H_+)$ to $\mathfrak{n}_1$.

Since $\|\exp(X)\exp(Y)\| = \|\exp(X')\exp(Y_2')n_1'b_+'\|$ it is easily seen, after choosing a basis of $\mathfrak{g}$ which diagonalizes $\mathfrak{j}\cap\mathfrak{p}$, that $\|\exp(X)\exp(Y)\| \geqq \|\exp(X')b_+'\|$. If $\mathfrak{a}_-$ is a maximal abelian subalgebra of $\mathfrak{m}\cap\mathfrak{p}$ then $\mathfrak{a}_- + (\mathfrak{j}\cap\mathfrak{p})$ is a maximum abelian subalgebra $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$. Let $X' = k_-(H_-)$ with $H_- \in \mathfrak{a}_-$ and $k_- \in M\cap K$ and let $b_+' = \exp(H_+)$ with $H_+ \in \mathfrak{j}\cap\mathfrak{p}$. If α is the restriction of a root to $\mathfrak{a}_\mathfrak{p}$ then

$$\log\|\exp(X)b_+'\| \geqq |\alpha(H_- + H_+)|;$$

since the restrictions of the roots to $\mathfrak{a}_\mathfrak{p}$ span the space of linear functions on $\mathfrak{a}_\mathfrak{p}$, there is a constant c such that for any linear function λ

$$c\|\lambda\|\log\|\exp(X')b_+'\| \geqq |\lambda(H_- + H_+)|.$$

Since $\mathfrak{a}_-\cap(\mathfrak{j}\cap\mathfrak{p}) = \{0\}$ it is now clear that $\|\exp(X)\exp(Y_2)\| \succ b_+'\|$ and $\|\exp(X)\exp(Y_2)\| \succ \|\exp(X')\|$. From this one easily deduces that $\|\exp(X)\exp(Y_2)\| \succ \|\exp(Y_2')n_1'\|$. If $n_1' = \exp(Y_1')$ and $\exp(Y_2')\exp(Y_1') = \exp(Y')$ with $Y_1' \in \mathfrak{n}_1$ and $Y' \in \mathfrak{n}$ then the four variables $(Y_1', Y_2'), Y', (Ad(\exp(Y_1')), Ad(\exp(Y_2'))$ and $Ad(\exp(Y'))$ are polynomial functions of each other (cf.[2(h)], pp. 737-738 and the reference cited there.) Consequently $\|\exp(Y')\| \succ \|\exp(Y_1')\|$ and $\|\exp(Y')\| \succ \|\exp(Y_2')\|$, so that $\|\exp(X)\exp(Y_2)\| \succ \|\exp(Y_1')\|$ and $\|\exp(X)\exp(Y_2)\| \succ \|\exp(Y_2')\|$. However, if $n_1'b_+'u^{-1} = u'b_+^{-1}n_1^{-1}$ then

$$k\exp(X)\exp(Y_2)n_1b_+ = k'\exp(X')\exp(Y_2')u'$$

and the argument may be reversed. Consequently $\|\exp(X')\exp(Y')\| \succ \|b_+'\| \succ e^{2\rho_1(\log b_+')}$.

The absolute convergence of (10) will be established if it is shown that

$$\int_G |\psi(g\tilde{\gamma}g^{-1})|\phi(g)dg$$

converges. However this integral equals

$$\int_{K\times M\times N\times B_+} |\psi(kmn\tilde{\gamma}n^{-1}m^{-1}k^{-1})|\phi(kmnb_+)dkdmdndb_+.$$

If $n = \exp(Y_2)n_1$ as before then $dn = dY_2 dn_1$ where dY_2 is the Euclidean measure on $\mathfrak{n}_2$ and this integral equals

$$\int_{K\times M\times \mathfrak{n}_2} |\psi(km\exp(Y_2)\tilde{\gamma}\exp(-Y_2)m^{-1}k^{-1})|\ \|m\exp(Y_2)\|^{\beta}dkdmdY_2.$$

The integrand is less than or equal to

$$c\|(m\exp(Y_2)\gamma\exp(-Y_2)\gamma^{-1}m^{-1})m\gamma m^{-1}\|^{-\alpha}\|m\|^{\beta}\|\exp(Y_2)\|^{\beta}$$

which is at most

$$c\|\exp(Y_2)\gamma\exp(-Y_2)\gamma^{-1}\|^{-\alpha_1}\|m\|^{2\alpha_1+\beta}\|m\gamma m^{-1}\|^{-\alpha_2}\|\exp(Y_2)\|^{\beta}$$

if $\alpha_1+\alpha_2 = \alpha, \alpha_1, \alpha_2 \geqq 0$. It follows from Lemma 8 of [2(h)] that $\|m\gamma m^{-1}\| \succ \|m\|$ and from Lemma 2 of that paper that

$$\|\exp(Y_2)\gamma\exp(-Y_2)\gamma^{-1}\| \succ 1+\|Y_2\|.$$

Thus if α is sufficiently large the integrand is less than or equal to a multiple of $(1+\|Y_2\|)^{-\beta_1}\|m\|^{-\beta_2}$ with β_1 and β_2 large. Consequently the integral converges. It should be observed that $\tilde{\gamma}$ is fixed so that uniform estimates like that of Lemma 3 are not necessary.

References

1. A. Borel and Harish-Chandra, *Annals of Mathematics*, vol. 75 (1962), pp. 485-535.
2. Harish-Chandra, (a) *Transactions of the American Mathematical Society*, vol. 76 (1954), pp. 485-528.
   (b) *American Journal of Mathematics*, vol 77 (1955), pp. 743-777.
   (c) *American Journal of Mathematics*, vol 78 (1956), pp. 1-41.
   (d) *American Journal of Mathematics*, vol. 78 (1956), pp. 564-628.
   (e) *Transactions of the American Mathematical Society*, vol. 83 (1956), pp. 98-163.
   (f) *American Journal of Mathematics*, vol 79 (1957), pp. 87-120.
   (g) *American Journal of Mathematics*, vol 79 (1957), pp. 193-257.
   (h) *American Journal of Mathematics*, vol 79 (1957), pp. 733-760.
   (i) *American Journal of Mathematics*, vol. 80 (1958), pp. 241-310.
3. F. Hirzebruch, *Symposium Internacional de Topologia Algebraica*, Universidad de Mexico (1958), pp. 129-143.
4. A. Łojasiewicz, *Studia Mathematica*, vol. 18 (1959), pp. 87-136.
5. A. Selberg, (a) *Journal of the Indian Mathematical Society*, vol. 20 (1956), pp. 47-87.
   (b) *Seminars on Analytic Functions*, Institute for Advanced Study, Princeton, vol. 2, pp. 152-161.
6. H. Cartan, *Séminaire*, Paris, 1957/1958.