BASE CHANGE FOR GL(2)†

by

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FOREWORD

These are the notes from a course of lectures given at The Institute for Advanced Study in the fall of 1975. Following a suggestion of A. Borel, I have added a section (§2) with an outline of the material and have discussed the applications to Artin L-functions in more detail (§3), including some which were discovered only after the course was completed. I have also made corrections and other improvements suggested to me by him, and by T. Callahan, A. Knapp, and R. Kottwitz. But on the whole I have preferred to leave the notes in their original, rude form, on the principle that bad ideas are best allowed to languish, and that a good idea will make its own way in the world, eventually discovering that it had so many fathers it could dispense with a mother.

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Table of Contents

1. INTRODUCTION

The problem of base change or of lifting for automorphic representation can be introduced in several ways. It emerges very quickly when one pursues the formal principles expounded in the article [20] which can in fact be reduced to one, viz., the functoriality of automorphic forms with respect to what is now referred to as the L-group. This is not the place to rehearse in any generality the considerations which led to the principle, or its theoretical background, for which it is best to consult [4]; but it is useful to review them briefly in the form which is here pertinent.

Suppose that F is a non-archimedean local field and G is $GL(2)$. If O is the ring of integers of F the Hecke algebra $\mathcal{H}$ of compactly supported functions on the double cosets of $G(F)//G(O)$ has a known structure. It is in particular isomorphic to the algebra of functions on $GL(2,\mathbf{C})$ obtained by taking linear combinations of characters of finite-dimensional analytic representations. According to the definitions of [20], the L-group of G over F is the direct product

$$^LG = {}^LG^o \times \mathfrak{G}(K/F).$$

Here $^LG^o$, the connected component of ^LG, is $GL(2,\mathbf{C})$, and K is simply a finite Galois extension of F, large enough for all purposes at hand.

If K/F is unramified the Frobenius element Φ in $\mathfrak{G}(K/F)$ is defined and the Hecke algebra $\mathcal{H}$ is also isomorphic to the algebra of functions on

$$^LG^o \times \Phi \subseteq {}^LG$$

obtained by restriction of linear combinations of characters of analytic representations of the complex Lie group ^LG.

Suppose E is a finite separable extension of F. The group $\overline{G}$ obtained from G by restriction of scalars from E to F is so defined that $\overline{G}(F) = G(E)$. As a group over F it has an associated L-group, whose connected component $^L\overline{G}^o$ is

$$\prod\nolimits_{\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/F)} GL(2,\mathbf{C}).$$

The group $\mathfrak{G}(K/F)$ operates on $^L\overline{G}^o$ via its action on coordinates. The L-group $\overline{G}$ is a semi-direct product

$$^L\overline{G} = {}^L\,\overline{G}^o \times \mathfrak{G}(K/F).$$

If E/F and K/F are unramified the Hecke algebra $\mathcal{H}_E$ of $G(E)$ with respect to $G(O_E)$ is isomorphic to the algebra of functions on

$$^L\overline{G}^o \times \Phi \subseteq {}^L\,\overline{G}$$

obtained by the restriction of linear combinations of characters of finite-dimensional analytic representations of ${}^L\overline{G}$.

At first this is a little baffling for Hecke algebras on $G(E)$ and $\overline{G}(F)$ are the same, while the first is isomorphic to the representation ring of $GL(2,\mathbf{C})$ and the second to an algebra of functions on ${}^L\overline{G}^o \times \Phi$. If $f^\vee$ and $\overline{f}^\vee$ represent the same element of the Hecke algebra then

$$\overline{f}^\vee((g_1,\cdots,g_\ell)\times\Phi) = f^\vee(g_\ell\cdots g_2g_1) \quad \ell=[E:F]. \tag{1.1}$$

The homomorphism

$$\varphi : (g\times\tau) \to (g,\cdots,g)\times\tau$$

of ${}^LG$ to ${}^L\overline{G}$ takes ${}^LG^o\times\Phi$ to ${}^L\overline{G}^o\times\Phi$. It allows us to pull back functions from ${}^L\overline{G}^o\times\Phi$ to ${}^LG^o\times\Phi$, and yields especially a homomorphism $\varphi^* : \mathcal{H}_E \to \mathcal{H}$. To give an irreducible admissible representation π of $G(F)$ which contains the trivial representation of $G(O)$ is tantamount to giving a homomorphism λ of $\mathcal{H}$ onto $\mathbf{C}$, and to give an irreducible representation Π of $G(E)$ which contains the trivial representation of $G(O_E)$ is tantamount to giving a homomorphism λ' of $\mathcal{H}_E$ onto $\mathbf{C}$. We say that Π is a lifting of π if $\lambda'=\lambda\circ\varphi^*$.

The notion of lifting may also be introduced when E is simply a direct sum of finite separable extensions. For example if $E=F\oplus\cdots\oplus F$ then

$$\overline{G}(F)=G(E)=G(F)\times\cdots\times G(F)$$

and ${}^L\overline{G}$ is the direct product

$$GL(2,\mathbf{C})\times\cdots\times GL(2,\mathbf{C})\times\mathfrak{G}(K/F).$$

We may define φ as before. The algebra $\mathcal{H}_E$ is $\mathcal{H}\otimes\cdots\otimes\mathcal{H}$. It is easily verified that if $f_1\otimes\cdots\otimes f_\ell$ lies in $\mathcal{H}_E$ then $\varphi^*(f_1\otimes\cdots\otimes f_\ell)$ is the convolution $f_1*\cdots*f_\ell$, and so the lifting of π, defined by the same formal properties as before, turns out to be nothing but $\pi\otimes\cdots\otimes\pi$.

Thus when E is a direct sum of several copies of F, the concept of a lifting is very simple, and can be extended immediately to all irreducible, admissible representations. However when E is a field, it is not at all clear how to extend the notion to cover ramified π. Nonetheless class field theory suggests not only that this might be possible but also that it might be possible to introduce the notion of a lifting over a global field.

The principal constraint on these notions will be the compatibility between the local and the global liftings. If F is a global field and E is a finite separable extension of F then for each place v of F we define E_v to be $E\otimes_F F_v$. If $\pi=\otimes_v\pi_v$ is an automorphic representation of $G(\mathbf{A})$ ([3]), where $\mathbf{A}$ is the adèle ring of F, then the automorphic representation Π of $G(\mathbf{A}_E)$ will be a lifting of π if and only if Π_v is a lifting of π_v for all v. Since π_v

is unramified for almost all v and the strong form of the multiplicity one theorem implies that, in general, Π is determined when almost all Π_v are given, this is a strong constraint.

Proceeding more formally, we may define the L-groups ${}^LG$ and ${}^L\overline{G}$ over a global field F too.

$$ {}^LG = GL(2, \mathbf{A}) \times \mathfrak{G}(K/F) $$

and

$$ {}^L\overline{G} = \left(\prod\nolimits_{\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/F)} GL(2, \mathbf{A}) \right) \rtimes \mathfrak{G}(K/F). $$

We also introduce

$$ \varphi : (g, \tau) \to (g, \cdots, g) \times \tau $$

once again. If v is a place of F we may extend it to a place of K. The imbedding $\mathfrak{G}(K_v/F_v) \hookrightarrow \mathfrak{G}(K/F)$ yields imbeddings of the local L-groups

$$ {}^LG_v = GL(2, \mathbf{C}) \times \mathfrak{G}(K_v/F_v) \hookrightarrow^L G, \qquad {}^L\overline{G}_v \hookrightarrow^L \overline{G}. $$

The restriction φ_v of φ to ${}^LG_v$ carries it to ${}^L\overline{G}_v$ and is the homomorphism we met before. If $\Pi(G/F)$ is the set of automorphic representations of $G(\mathbf{A})$ and $\Pi(\overline{G}/F)$ is the same set for $\overline{G}(\mathbf{A}) = G(\mathbf{A}_E)$, the global form of the principle of functoriality in the associate group should associate to φ a map

$$ \Pi(\varphi) : \Pi(G/F) \to \Pi(\overline{G}/F). $$

The lifting Π of π would be the image of π under $\Pi(\varphi)$. Since the principle, although unproved, is supported by all available evidence we expect Π to exist.

The local form of the principle should associate to φ_v a map $\Pi(\varphi_v)$ from $\Pi(G/F_v)$, the set of classes of irreducible admissible representations of $G(F_v)$, to $\Pi(\overline{G}/F_v)$ and hence should give a local lifting. Whatever other properties this local lifting may have it should be compatible with that defined above when E_v/F_v is unramified and π_v contains the trivial representation of $G(O)$. Moreover, as observed already, local and global lifting should be compatible so that if $\pi = \otimes\pi_v$ lifts to $\Pi = \otimes\Pi_v$ then Π_v should be a lifting of π for each v.

The main purpose of these notes is to establish the existence of a lifting when E/F is cyclic of prime degree. It is worthwhile, before stating the results, to describe some other paths to the lifting problem. If H is the group consisting of a single element then the associate group ${}^LH$ is just $\mathfrak{G}(K/F)$ and a homomorphism

$$ \varphi :{}^L H \to {}^L G $$

compatible with the projections of the two groups on $\mathfrak{G}(K/F)$ is simply a two-dimensional representation ρ of $\mathfrak{G}(K/F)$. Since $\Pi(H/F)$ consists of a single element, all $\Pi(\varphi)$ should do now is select a particular automorphic representation $\pi = \pi(\rho)$ in $\Pi(G/F)$.

The local functoriality should associate to φ_v a representation $\pi_v = \pi(\rho_v)$, where ρ_v is the restriction of ρ to the decomposition group $\mathfrak{G}(K_v/F_v)$. We let Φ_v be the Frobenius at a place v at which K_v is unramified and suppose

$$\varphi_v : \Phi_v \to t_v \times \Phi_v.$$

The associated homomorphism φ_v^* of the Hecke algebra $\mathcal{H}_v$ of G at v into that of H at v, namely, to $\mathbf{C}$ is obtained by identifying $\mathcal{H}_v$ with the representation ring of $GL(2,\mathbf{C})$ and evaluating at t_v. For such a v, $\pi_v = \pi(\rho_v)$ is defined as the representation corresponding to this homomorphism. We may define $\pi = \pi(\rho)$ globally by demanding that $\pi = \otimes\pi_v$ with $\pi_v = \pi(\rho_v)$ for almost all v. This of course does not prove that it exists. It is also possible to characterize $\pi(\rho_v)$ for all v (§12 of [14]), although not in a truly satisfactory manner. Nonetheless $\pi(\rho_v)$ can now be shown to exist ([17]), but by purely local methods quite different from those of these notes, where the emphasis is on the existence of $\pi(\rho)$ globally.

These considerations can be generalized. If ρ is a continuous two-dimensional representation of the Weil group $W_{K/F}$ by semi-simple matrices we may define ρ_v as the restriction of ρ to W_{K_v/F_v}. For almost all v, ρ_v factors through

$$W_{K_v/F_v} \to \mathbf{Z} \to GL(2,\mathbf{C}).$$

If t_v is the image of a Frobenius element, that is, of $1 \in \mathbf{Z}$, and π_v the representation of $G(F_v)$ which contains the trivial representation of $G(O_v)$ and yields the homomorphism of $\mathcal{H}_v$ into $\mathbf{C}$ defined by evaluation at t_v, we say that $\pi(\rho_v) = \pi_v$. We may, at least for irreducible ρ, define $\pi = \pi(\rho)$ globally by the demand that $\pi = \otimes\pi_v$ and $\pi_v = \pi(\rho_v)$ for almost all v. If $\pi(\rho)$ exists its local factors π_v can be characterized in terms of ρ_v.

If ρ is reducible the existence of $\pi(\rho)$ is proved in the theory of Eisenstein series. If ρ is dihedral, by which I shall mean, in spite of justified reproofs, induced from a quasi-character of the Weil group of a quadratic extension, the existence of $\pi(\rho)$ is implicit in the work of Hecke and of Maass. But nothing more was known when, late in 1966 or early in 1967, the principle of functoriality, and hence the existence of $\pi(\rho)$, was first suggested by the general theory of Eisenstein series. It was desirable to test a principle with so many consequences – for example, the existence of $\pi(\rho)$ implies the Artin conjecture for the Artin L-function $L(s,\rho)$ – as thoroughly as possible. Weil's elaboration of the Hecke theory, which had been completed not long before, together with a careful analysis ([21]) of the factors appearing in the functional equation of the Artin L-functions, enabled one to show that the existence of $\pi(\rho)$ was implied by Weil's form of the Artin conjecture ([14]), and to obtain at the same time a much better understanding of the local maps $\rho_v \to \pi(\rho_v)$.

In retrospect it was clear that one could argue for the existence of $\pi(\rho)$ by comparing the form of the functional equation for the Artin L-functions on one hand and of the Euler products associated by Hecke and Maass to automorphic forms on the other. This is especially so when $F = \mathbf{Q}$ and ρ_∞ factors through

$$W_{\mathbf{C}/\mathbf{R}} \longrightarrow \mathfrak{G}(\mathbf{C}/\mathbf{R}) \xrightarrow{\rho_\infty^\circ} GL(2,\mathbf{C})$$

with the second homomorphism taking complex conjugation to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This argument is simple, can be formulated in classical terms, and resembles closely the argument which led Weil to his conjecture relating elliptic curves and automorphic forms, and thus has the sanction of both tradition and authority, and that is a comfort to many. The emphasis on holomorphic forms of weight one is misleading, but the connection with elliptic curves is not, for, as Weil himself has pointed out ([33]), the consequent pursuit of his conjecture leads ineluctably to the supposition that $\pi(\rho)$ exists, at least when F is a function field.

Once the conjecture that $\pi(\rho)$ existed began to be accepted, the question of characterizing those automorphic representations π which equal $\pi(\rho)$ for some two-dimensional representation of the Galois group arose. It seems to have been generally suspected, for reasons which are no longer clear to me, that if F is a number field then π is a $\pi(\rho)$ if and only if, for each archimedean place v, $\pi_v = \pi(\rho_v)$, where ρ_v is a representation of $\mathfrak{G}(\overline{F}_v/F_v)$; but there was no cogent argument for giving any credence to this suspicion before the work of Deligne and Serre ([6]) who established that it is correct if $F = \mathbf{Q}$ and $\pi_\infty = \pi(\rho^o_\infty)$.

This aside, it was clear that one of the impediments to proving the existence of $\pi(\rho)$ was the absence of a process analogous to composition with the norm, which in class field theory enables one to pass from a field to an extension, that is, to effect a lifting or a base change. The expectation that there will be a close relation between automorphic L-functions on one hand and motivic L-functions on the other entails the existence of such a process, for it implies that to any operation on motives must correspond an analogous operation on automorphic representations, and one of the simplest operations on motives is to pass to a larger field of definition, or, as one says, to change the base. For motives defined by a representation of a Galois group or a Weil group over F, base change is simply restriction to the Galois or Weil group over E.

If F is a local field and $\rho : W_{K/F} \to GL(2, \mathbf{C})$ an unramified two-dimensional representation of the local Weil group we have already defined $\pi(\rho)$. It must be observed that if E/F is unramified and σ the restriction of ρ to $W_{K/E}$ then $\pi(\sigma)$ is the lifting of π. Otherwise, base change for automorphic forms would be incompatible with base change for motives. That $\pi(\sigma)$ is the lifting of π follows from formula (1.1) and the definition of $\pi(\sigma)$ and $\pi(\rho)$.

Although the lifting problem emerges from the general principle of functoriality in the L-group, some of its historical roots and most of the sources of progress lie elsewhere. The initial steps were taken for $F = \mathbf{Q}$ and E quadratic by Doi and Naganuma. It is instructive to review their early work ([7],[8]). We first recall the relevant facts about L-functions associated to automorphic forms.

If ρ is any analytic representation of ${}^LG$ and π an automorphic representation it is possible ([20]) to introduce an Euler product

$$L(s, \pi, \rho) = \prod\nolimits_v L(s, \pi_v, \rho).$$

To be frank it is at the moment only possible to define almost all of the factors on the right. For a few ρ it is possible to define them all; for example, if ρ is the projection ρ^o of ${}^LG$ on its first factor $GL(2,\mathbf{C})$ then $L(s,\pi,\rho)$ is the Hecke function $L(s,\pi)$ studied in [14]. One basic property of these Euler products is that

$$L(s,\pi,\rho_1\oplus\rho_2)=L(s,\pi,\rho_1)\,L(s,\pi,\rho_2).$$

If $\overline{\rho}$ is a representation of ${}^L\overline{G}$ and Π an automorphic representation of $\overline{G}(\mathbf{A})$ we may also introduce $L(s,\Pi,\overline{\rho})$ ([20]). These functions are so defined that if $\varphi:{}^L G\rightarrow {}^L\overline{G}$ is defined as above and Π is the lifting of π then

$$L(s,\Pi,\overline{\rho})=L(s,\pi,\overline{\rho}\circ\varphi).$$

If $\overline{\rho}\circ\varphi$ is reducible the function on the right is a product. An automorphic representation for $\overline{G}(\mathbf{A})$ is also one for $G(\mathbf{A}_E)$, because the two groups are the same. However, ${}^LG_E$, the associate group of $GL(2)$ over E, is $GL(2,\mathbf{C})\times\mathfrak{G}(K/E)$. Given a representation ρ_E of ${}^LG_E$ we may define a representation $\overline{\rho}$ of ${}^L\overline{G}$ so that

$$L(s,\Pi,\overline{\rho})=L(s,\Pi,\rho_E).$$

Choose a set of representations $\tau_1,\cdots,\tau_\ell$ for $\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/F)$ and let $\tau_i\tau=\sigma_i(\tau)\tau_{j(i)}$, with $\sigma_i(\tau)\in\mathfrak{G}(K/E)$. Set

$$\overline{\rho}(g_1,\cdots,g_\ell)=\rho_E(g_1)\oplus\cdots\oplus\rho_E(g_\ell)$$

and let

$$\overline{\rho}(\tau):\oplus_{v_i}\rightarrow\oplus\rho_E(\sigma_i(\tau))v_{j(i)}\qquad\tau\in\mathfrak{G}(K/F).$$

The role played by passage from ρ to $\overline{\rho}$ is analogous to, and in fact an amplification of, that played by induction in the study of Artin L-functions. Suppose for example that $\rho=\rho_E^o$ is the standard two-dimensional representation of ${}^LG_E$, obtained by projection on the first factor, and Π is the lifting of π. If E/F is cyclic of prime degree, let ω be a non-trivial character of $\mathfrak{G}(E/F)$ and hence of $\mathfrak{G}(K/F)$ and let ρ_i be the representation of ${}^LG$ defined by

$$\rho_i(g\times\tau)=\omega^i(\tau)\rho^o(g).$$

Then

$$\overline{\rho}\circ\varphi=\bigoplus\nolimits_{i=0}^{\ell-1}\rho_i$$

and

$$L(s,\Pi,\rho_E^o)=L(s,\Pi,\overline{\rho})=L(s,\pi,\overline{\rho}\circ\varphi)=\prod\ L(s,\pi,\rho_i).$$

However, ω may also be regarded as a character of $F^\times\backslash I_F$ and

$$L(s,\pi,\rho_i)=L(s,\omega^i\otimes\pi).$$

Take F to be $\mathbf{Q}$ and E to be a real quadratic field. Suppose G_1 is the multiplicative group of a quaternion algebra over E which splits at only one of its two infinite places. The L-groups of G_1 and G over E are the same. There is also associated to G_1 a family of algebraic curves S which are defined over E and called Shimura curves. The Hasse-Weil zeta function of S can be written as a quotient of products of the L-functions $L(s,\Pi_1) = L(s,\Pi_1,\rho_E^o)$ corresponding to automorphic representations of $G_1(\mathbf{A}_E)$. It can happen that S not only is connected and elliptic, so that the non-trivial part of its zeta-function is exactly $L(s,\Pi_1)$ for a certain Π_1, but also has a model defined over $\mathbf{Q}$ ([7]). Then the conjecture of Taniyama as refined by Shimura and Weil ([32]) affirms that there is an automorphic representation π of $G(\mathbf{A}_\mathbf{Q})$ such that the interesting part of the zeta-function of the model is $L(s,\pi)$. Hence

$$L(s,\Pi_1) = L(s,\pi)\, L(s,\omega\otimes\pi)$$

if ω is the character of $\mathbf{Q}^\times\backslash I_\mathbf{Q}$ defined by E. This equation is tantamount to the assertion that Π_1 is a lifting of π; and the problem of lifting as posed by Doi and Naganuma was not from π to Π but from π to Π_1, where G_1 was some quaternion algebra over E. However, if Π_1 is any automorphic representation of $G_1(\mathbf{A}_E)$ there is always (cf. [14], and especially the references therein to the work of Shimizu) an automorphic representation Π of $G(\mathbf{A}_E)$ such that

$$L(s,\Pi) = L(s,\Pi_1)$$

and the problem of lifting from π to Π_1 becomes the problem of lifting from π to Π.

Following a suggestion of Shimura they were able to establish the existence of Π for a large number of π by combining an idea of Rankin with the theory of Hecke ([8]), at least when $F = \mathbf{Q}$ and E is a real quadratic field. Their idea was pursued by Jacquet ([13]) who removed the restriction on F as well as the restrictions on π which are inevitable when working in the context of holomorphic automorphic forms. However, the method was limited to quadratic extensions, and could establish the existence of a lifting, but could not characterize those Π which were liftings.

The next step was taken by Saito ([27]), who applied what one can refer to as the twisted trace formula to establish the existence of a lifting and to characterize them when E/F is cyclic of prime degree. This is in fact not what he did, for he worked with holomorphic forms in the customary sense, without any knowledge of representation theory; and the language of holomorphic forms seems to be inadequate to the statement of a theorem of any generality much less to its proof. It is not simply that one can only deal with $\pi = \otimes\pi_v$ for which π_v belongs to the discrete series at each infinite place, although this alone precludes the applications of these lectures, but rather that one is in addition confined to forms of low level. But Saito certainly does establish the usefulness of the twisted trace formula, the application of which may have been suggested by some computations of Busam and Hirzebruch.

To carry over an idea in the theory of automorphic forms from a function-theoretic to a presentation-theoretic context is seldom straightforward and usually demand new insight. What was needed to give suppleness and power to the idea of Saito was the correct notion of a local lifting. This was supplied by Shintani, who sketched his ideas during the U.S.-Japan seminar on number theory held at Ann Arbor in 1975, and has now published them in more detail in [30]. It was Shintani who fired my interest in the twisted trace formula. It soon became clear* that his ideas, coupled with those of Saito, could, when pursued along lines which he had perhaps foreseen, be applied in a striking, but after this lengthy introduction no longer surprising, fashion to the study of Artin L-functions. Before giving the applications, I describe the results on lifting yielded by a fully developed – but only for $GL(2)$ and only for cyclic extensions of prime degree! – theory. Moreover, only fields of characteristic zero will be considered. This is largely a result of indolence.

* when reflecting upon these matters not long after the seminar at our cabin in the Laurentians

2. PROPERTIES OF BASE CHANGE

So far all applications of the trace formula to the comparison of automorphic representations of two different groups have been accompanied by local comparison theorems for characters, the typical example being provided by twisted forms of $GL(2)$ ([14]). Base change for cyclic extensions is no exception, and, following Shintani, local liftings can be defined by character relations.

Suppose F is a local field, and E a cyclic extension of prime degree ℓ. The Galois group $\mathfrak{G} = \mathfrak{G}(E/F)$ acts on $G(E)$, and we introduce the semi-direct product

$$G'(E) = G(E) \times \mathfrak{G}.$$

The group $\mathfrak{G}$ operates on irreducible admissible representations of $G(E)$, or rather on their classes,

$$\Pi^\tau : g \to \Pi(\tau(g))$$

and Π can be extended to a representation Π' of $G'(E)$ on the same space if and only if $\Pi^\tau \sim \Pi$ for all τ. Fix a generator σ of $\mathfrak{G}$. Then $\Pi^\tau \sim \Pi$ for all τ if and only if $\Pi^\sigma \sim \Pi$. The representation Π' is not unique, but any other extension is of the form $\omega \otimes \Pi'$, where ω is a character of $\mathfrak{G}$. There are ℓ choices for ω. It will be shown in §7 that the character of Π' exists as a locally integrable function.

If g lies in $G(E)$, we form

$$Ng = g\sigma(g)\cdots\sigma^{\ell-1}(g).$$

This operation, introduced by Saito, is easy to study. Its properties are described in §4. It is not the element Ng which is important, but rather its conjugacy class in $G(E)$, and indeed the intersection of that conjugacy class with $G(F)$, which is then a conjugacy class in $G(F)$. We also denote an element of that class by Ng. The class of Ng in $G(F)$ depends only on the class of $g \times \sigma$ in $G'(E)$.

The representation Π of $G(E)$ is said to be a lifting of the representation π of $G(F)$ if one of the following two conditions is satisfied:

(i) Π is $\pi(\mu', \nu')$, π is $\pi(\mu, \nu)$, and $\mu'(x) = \mu(N_{E/F}x)$, $\nu'(x) = \nu(N_{E/F}x)$ for $x \in E^\times$.

(ii) Π is fixed by $\mathfrak{G}$ and for some choice of Π' the equality

$$\chi_{\Pi'}(g \times \sigma) = \chi_\pi(h)$$

is valid whenever $h = Ng$ has distinct eigenvalues.

The representation $\pi(\mu, \nu)$ associated to two characters of $F^\times$ is defined on p. 103 of [14]. The characters $\chi_{\pi'}$, $\chi_{\Pi'}$ of π and Π' are well-defined functions where Ng has distinct eigenvalues, so that the equality of (ii) is meaningful. It should perhaps be underlined that it is understood that π and Π are irreducible and admissible, and that they are sometimes representations, and sometimes classes of equivalent representations. It is at first sight dismaying the liftings cannot be universally characterized by character identities, but it is so, and we are meeting here a particular manifestation of a widespread phenomenon.

We shall prove the following results on local lifting for fields of characteristic zero.

a) Every π has a unique lifting.

b) Π is a lifting if and only if $\Pi^\tau \simeq \Pi$ for all $\tau \in \mathfrak{G}$.

c) Suppose Π is a lifting of π and of π'. If $\pi = \pi(\mu, \nu)$ then $\pi' = \pi(\mu', \nu')$ where $\mu^{-1}\mu'$ and $\nu^{-1}\nu'$ are characters of $NE^\times \backslash F^\times$. Otherwise $\pi' \simeq \omega \otimes \pi$ where ω is a character of $NE^\times \backslash F^\times$. If ω is non-trivial then $\pi \simeq \omega \otimes \pi$ if and only if ℓ is 2 and there is a quasi-character θ of $E^\times$ such that $\pi = \pi(\tau)$ with

$$\tau = \mathrm{Ind}(W_{E/F}, W_{E/E}, \theta).$$

d) If $k \subset F \subset E$ and $E/k, F/k$ are Galois and $\tau \in \mathfrak{G}(E/k)$ then the lifting of π^τ is Π^τ if the lifting of π is Π.

e) If ρ is reducible or dihedral and $\pi = \pi(\rho)$ then the lifting of π is $\pi(P)$ if P is the restriction of ρ to $W_{K/E}$.

f) If Π is the lifting of π and Π and π have central characters ω_Π and ω_π respectively then $\omega_\Pi(z) = \omega_\pi(N_{E/F}z)$.

g) The notion of local lifting is independent of the choice of σ.

The assertion (e) cries out for improvement. One can, without difficulty, use the results of §3 to extend it to tetrahedral ρ, but it is not clear that the methods of these notes can, unaided, establish it for octahedral ρ. I have not pursued the question.

Many of the properties of local liftings will be proved by global means, namely, the trace formula. For this it is important that the map on characters $\chi_\pi \to \chi_{\Pi'}$ which appears in the definition of local liftings is dual to a map $\phi \to f$ of functions. It is only the values of $\chi_{\Pi'}$ on $G(E) \times \sigma$ which matter, and thus ϕ will be a function on $G(E) \times \sigma$, or, more simply, a function on $G(E)$. Since the χ_π are class functions, it is not necessary – or possible – to specify f uniquely. It is only its orbital integrals which are relevant, and these must be specified by the orbital integrals of ϕ. But these will be integrals over conjugacy classes on $G(E) \times \sigma$, a subset of $G'(E)$. As a step preliminary to the introduction of the trace formula, the map $\phi \to f$ will be defined and introduced in §6. Objections can be made to the arrow, because the map is in fact only a correspondence, but the notation is convenient, and not lightly to be abandoned.

There are other local problems to be treated before broaching the trace formula, but before describing them it will be best to recall the function of the trace formula. Let F be for now a global field and E a cyclic extension of prime degree ℓ. Let Z be the group of scalar matrices, and set

$$Z_E(\mathbf{A}) = Z(F)N_{E/F}Z(\mathbf{A}_E).$$

Let ξ be a unitary character of $Z_E(\mathbf{A})$ trivial on $Z(F)$.

We introduce the space $L_s(\xi)$ of measurable functions φ on $G(F)\backslash G(\mathbf{A})$ which satisfy

(a)

$$\varphi(zg) = \xi(z)\varphi(g) \quad \text{for all} \quad z \in Z_E(\mathbf{A})$$

(b)

$$\int_{Z_E(\mathbf{A})G(F)\backslash G(\mathbf{A})} |\varphi(g)|^2 dg < \infty.$$

$G(\mathbf{A})$ acts on $L_s(\xi)$ by left translations. The space $L_s(\xi)$ is the direct sum of two mutually orthogonal invariant subspaces: $L_{sp}(\xi)$, the space of square-integrable cusp forms; and $L_{se}(\xi)$, its orthogonal complement. The theory of Eisenstein series decomposes $L_{se}(\xi)$ further, into the sum of $L^0_{se}(\xi)$, the span of the one-dimensional invariant subspaces of $L_s(\xi)$, and $L^1_{se}(\xi)$. We denote by r the representation of $G(\mathbf{A})$ on the sum of $L_{sp}(\xi)$ and $L^0_{se}(\xi)$.

Suppose we have a collection of functions f_v, one for each place v of F, satisfying the following conditions.

i) f_v is a function on $G(F_v)$, smooth and compactly supported modulo $Z(F_v)$.

ii) $f_v(zg) = \xi^{-1}(z)f_v(g) \quad \text{for} \quad \mathrm{z} \in \mathrm{N_{E_v/F_v}Z(E_v)}$.

iii) For almost all v, f_v is invariant under $G(O_{F_v})$, is supported on the product $G(O_{F_v})N_{E_v/F_v}$ $Z(E_v)$, and satisfies

$$\int_{N_{E_v/F_v}Z(E_v)\backslash G(O_{F_v})N_{E_v/F_v}Z(E_v)} f_v(g)dg = 1.$$

Then we may define a function f on $G(\mathbf{A})$ by

$$f(g) = \prod\nolimits_v f_v(g_v),$$

where $g = (g_v)$. The operator

$$r(f) = \int_{N_{E/F}Z(\mathbf{A}_E)\backslash G(\mathbf{A})} f(g)r(g)dg$$

is defined and of trace class.

Let ξ_E be the character $z \to \xi(N_{E/F}z)$ of $Z(\mathbf{A}_E)$. We may also introduce the space $L_s(\xi_E)$ of measurable functions φ on $G(E)\backslash G(\mathbf{A}_E)$ satisfying

(a)

$$\varphi(zg) = \xi_E(z)\varphi(g) \quad \text{for all} \quad z \in Z(\mathbf{A}_E)$$

(b)

$$\int_{Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} |\varphi(g)|^2 dg < \infty.$$

Once again we have a representation r of $G(\mathbf{A}_E)$ on the sum of $L_{sp}(\xi_E)$ and $L^0_{se}(\xi_E)$. But r now extends to a representation of the semi-direct product

$$G'(\mathbf{A}_E) = G(\mathbf{A}_E) \times \mathfrak{G}.$$

An element τ of $\mathfrak{G}$ sends φ to φ' with

$$\varphi'(h) = \varphi(\tau^{-1}(h)).$$

We will consider functions ϕ on $G(\mathbf{A}_E)$ defined by a collection ϕ_v, one for each place of F, satisfying

i) ϕ_v is a function on $G(E_v)$, smooth and compactly supported modulo $Z(E_v)$.

ii) $\phi_V(zg) = \xi_E^{-1}(z) f_v(g) \quad \text{for} \quad \mathrm{z} \in \mathrm{Z(E_v)}.$

iii) For almost all v, ϕ_v is invariant under $G(O_{E_v})$, is supported on $Z(E_v)G(O_{E_v})$, and satisfies

$$\int_{Z(E_v)\backslash Z(E_v)G(O_{E_v})} \phi(g)dg = 1.$$

Then $\phi(g) = \prod_v \phi_v(g_v)$, and

$$r(\phi) = \int_{Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g) r(g) dg$$

is defined and of trace class.

We now introduce another representation, R, of $G'(\mathbf{A}_E)$. If ℓ is odd, then R is the direct sum of ℓ copies of r. The definition of R for ℓ even is best postponed to §11. The function of the trace formula is to show that, for compatible choices of ϕ and f,

$$\operatorname{trace} R(\phi)R(\sigma) = \operatorname{trace} r(f). \tag{2.1}$$

Here σ is the fixed generator of $\mathfrak{G}(E/F)$. The trace formula for the left side is somewhat different than the usual trace formula, and is usually referred to as the twisted trace formula. It will be reviewed in §10.

The condition of compatibility means that $\phi_v \to f_v$ for all v. As we observed, the meaning of the arrow will be explained in §6 for those v which remain prime in E. Its meaning for v which split will be explained later, in the very brief §8. It is very important that when v does not ramify in E, ϕ_v lies in $\mathcal{H}_{E_v}$, and f_v is its image in $\mathcal{H}_{F_v}$ under the homomorphism introduced in §1, then the relation $\phi_v \to f_v$ is satisfied. This was verified by Saito [27], who had no occasion to mention that the homomorphism from $\mathcal{H}_{E_v}$ to $\mathcal{H}_{F_v}$ was just one of many provided by the general theory of spherical functions and the formalism of the L-group. In §5 another verification is given; it exploits the simplest of the buildings introduced by Bruhat–Tits.

The definition of the arrow $\phi_v \to f_v$ and the structure of the trace formula together imply immediately that the two sides of (2.1) are almost equal. The difference is made up of terms contributed to the trace formula by the cusps. There is a place for insight and elegance in the proof that it is indeed zero, but in these notes the proof is regarded as a technical difficulty to be bashed through somehow or other. The local information accumulated in §5 and in §9, which is primarily technical and of interest only to specialists, allows us to put the difference of the two sides of (2.1) in a form sufficiently tractable that we can exploit the fact that we are dealing with a difference of two traces to establish equality.

This is the first step taken in §11. The equality (2.1) available, one chooses a finite set of places, V, including the infinite places and the places ramifying in E, and for each $v \notin V$ an unramified representation Π_v of $G(E_v)$ such that $\Pi_v^\sigma \sim \Pi_v$. Let $\mathfrak{A}$ be the set of irreducible constituents Π of R, counted with multiplicity, such that Π_v is the given Π_v outside of V. By the strong form of multiplicity one, $\mathfrak{A}$ is either empty or consists of a single repeated element, and if $\Pi \in \mathfrak{A}$ then $\Pi^\sigma \sim \Pi$. If $\Pi^\sigma \sim \Pi$ then $G'(\mathbf{A}_E)$ leaves the space of Π invariant, and so we obtain a representation Π' of $G'(\mathbf{A}_E)$, as well as local representations Π'_v. Set

$$A = \sum\nolimits_{\Pi \in \mathfrak{A}} \prod\nolimits_{v \in V} \operatorname{trace} \Pi_v(\phi_v) \Pi'_v(\sigma).$$

Let $\mathfrak{B}$ be the set of constituents $\pi = \otimes_v \pi_v$ of r such that Π_v is a lifting of π_v for each v outside of V. Set

$$B = \sum\nolimits_{\pi \in \mathfrak{B}} \prod\nolimits_{v \in V} \operatorname{trace} \pi_v(f_v).$$

Elementary functional analysis enables us to deduce from (2.1) that $A = B$. This equality is local, although the set V may contain more than one element, and we have no control on the size of B. Nonetheless, when combined with some local harmonic analysis, it will yield the asserted results on local lifting.

The necessary harmonic analysis is carried out in §7. Some of the results are simple; none can surprise a specialist. They are proved because they are needed. The last part of §7, from Lemma 7.17 on, contains material that was originally intended for inclusion in [18], and found its way into these notes only because they were written first. It is joint work with J.-P. Labesse, and it was he who observed Lemma 7.17. Although [18] was written later, the work was carried out earlier, and the methods are less developed than those of these notes. At the time, one hesitated to strike out on a global expedition without providing in advance for all foreseeable local needs. One could probably, reworking [18], dispense with some of the computations of §7. But little would be gained.

A word might be in order to explain why the last part of §7 and the more elaborate definition of R when $\ell = 2$ are called for. When $[E : F] = 2$ there are two-dimensional representations ρ of the Weil group W_F induced from characters of the Weil group W_E. These representations have several distinctive properties, which we must expect to be mirrored by the $\pi(\rho)$. For example, ρ can be irreducible but its restriction to W_E will be reducible. If F is global this means that the cuspidal representation $\pi(\rho)$ becomes Eisensteinian upon lifting, and this complicates the proofs.

In the course of proving the results on local lifting, we also obtain the existence of global liftings, at least for a cyclic extension of prime degree ℓ. If Π is an automorphic representation of $G(\mathbf{A}_E)$ then, for each place v of F,

Π determines a representation Π_v of $G(E_v)$, and Π is said to be a lifting of π if Π_v is a lifting of π_v for each v. The first properties of global liftings are:

A) Every π has a unique lifting.

B) If Π is isobaric in the sense of [24], in particular cuspidal, then Π is a lifting if and only if $\Pi^\tau \sim \Pi$ for all $\tau \in \mathfrak{G}(E/F)$.

C) Suppose π lifts to Π. If $\pi = \pi(\mu, \nu)$ with two characters of the idèle class group ([14]), then the only other automorphic representations lifting to Π are $\pi(\mu_1\mu, \nu_1\nu)$, where μ_1, ν_1 are characters of $F^\times N_{E/F} I_E \backslash I_F$. If π is cuspidal then π' lifts to Π if and only if $\pi' = \omega \otimes \pi$ where ω is again a character of $F^\times N_{E/F} I_E$. The number of such π' is ℓ unless $\ell = 2$ and $\pi = \pi(\tau)$ where τ is a two-dimensional representation of $W_{E/F}$ induced by a character of $E^\times \backslash I_E$, when it is one, for $\pi \sim \omega \otimes \pi$ in this case.

D) Suppose $k \subset F \subset E$ and $F/k,\ E/k$ are Galois. If $\tau \in \mathfrak{G}(E/k)$ and Π is a lifting of π then Π^τ is a lifting of π^τ.

E) The central character ω_π of π is defined by $\pi(z) = \omega_\pi(z) I,\ z \in Z(\mathbf{A}) = I_F$, and ω_Π is defined in a similar fashion. If Π is a lifting of π then

$$\omega_\Pi(z) = \omega_\pi(N_{E/F} z).$$

If Π is cuspidal then Π is said to be a quasi-lifting of π if Π_v is a lifting of π_v for almost all v. A property of global liftings that has considerable influence on the structure of the proofs is:

F) A quasi-lifting is a lifting.

It is worthwhile to remark, and easy to verify, that the first five of these properties have analogues for two-dimensional representations of the Weil group W_F of F if lifting is replaced by restriction to W_E. The central character is replaced by the determinant.

3. APPLICATIONS TO ARTIN L-FUNCTIONS

Suppose F is a global field and ρ is a two-dimensional representation of the Weil group $W_{K/F}$, K being some large Galois extension. There are two possible definitions of $\pi(\rho)$. If $\pi(\rho_v)$ is characterized as in §12 of [14], we could say that $\pi = \pi(\rho)$ if $\pi = \otimes\pi_v$ and $\pi_v = \pi(\rho_v)$ for all v. On the other hand we could say that $\pi = \pi(\rho)$ if π is isobaric, in the sense of [24], and $\pi_v = \pi(\rho_v)$ for almost all v. The second definition is easier to work with, for it does not presuppose any elaborate local theory, while for the first the relation

$$L(s, \pi) = L(s, \rho)$$

is clear. It will be useful to know that they are equivalent. The first condition is easily seen to imply the second. To show that the second implies the first, we use improved forms of results of [5] and [14] which were communicated to me by T. Callahan.

He also provided a proof of the following strong form of the multiplicity one theorem.

Lemma 3.1 *Suppose π and π' are two isobaric automorphic representations of $GL(2, \mathbf{A})$. If $\pi_v \sim \pi'_v$ for almost all v then $\pi \sim \pi'$.*

If π is isobaric and not cuspidal then $\pi = \pi(\mu, \nu)$, where μ, ν are two idèle class characters. An examination of the associated L-functions $L(s, \omega \otimes \pi)$ and $L(s, \omega \otimes \pi')$ shows easily that if $\pi_v \sim \pi'_v$ for almost all v and $\pi = \pi(\mu, \nu)$ then $\pi' = \pi(\mu', \nu')$. Thus the lemma is quickly reduced to the case that π and π' are cuspidal. It is stronger than the theorem of Casselman ([5]) because it does not assume that $\pi_v \sim \pi'_v$ for archimedean v, but the proof is similar.

One has to observe that if v is archimedean and if for every character ω_v of $F_v^\times$ and some fixed non-trivial character ψ_v of F_v the function of s given by

$$\varepsilon'(s, \omega_v \otimes \pi_v, \psi_v) = \frac{L(1 - s, \omega_v^{-1} \otimes \pi_v)\varepsilon(s, \omega_v \otimes \pi_v, \psi_v)}{L(s, \omega_v \otimes \pi_v)}$$

is a constant multiple of $\varepsilon'(s, \omega_v \otimes \pi'_v, \psi_v)$ then $\pi_v \sim \pi'_v$. This is an archimedean analogue of Corollary 2.19 of [14], and is a result of the formulae for $\varepsilon'(s, \omega_v \otimes \pi_v, \psi_v)$ given in the proofs of Lemma 5.18 and Corollary 6.6 of [14].

One needs in addition the following variant of Lemma 12.5 of [14].

Lemma 3.2 *Suppose that we are given at each infinite place v of F a character χ_v of $F_v^\times$ and, in addition, an integral ideal $\mathfrak{A}$ of F. Then there exists an idèle-class character ω which is such that ω_v is close to χ_v for each archimedean v and whose conductor is divisible by $\mathfrak{A}$.*

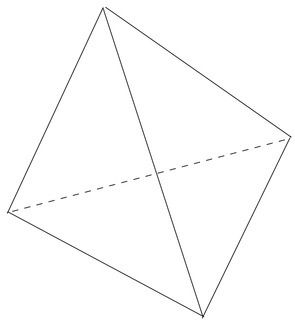
This lemma, whose proof will not be given, can also be used, in conjunction with the methods of §12 of [14], to show that the second definition of $\pi(\rho)$ implies the first. Again, if $\pi(\rho)$ is not cuspidal then it is $\pi(\mu,\nu)$ and ρ must be the direct sum of the one-dimensional representations μ and ν.

There is one further property of liftings which is now clear.

G) If E/F is cyclic of prime degree, if $\pi=\pi(\rho)$, Π is the lifting of π, and P the restriction of ρ to the Weil group over E, then $\Pi=\pi(P)$.

Of course the definition of $\pi(\rho)$ does not imply that it always exists. If ρ is irreducible and $\pi=\pi(\rho)$ then π is necessarily cuspidal and, since $L(s,\pi)=L(s,\rho)$, the Artin L-function attached to ρ is entire, as it should be. In this paragraph we take the results of global liftings announced in the previous paragraph for granted, and see what can be deduced about the existence of $\pi(\rho)$. The representation ρ is of course to be two-dimensional, and we may as well assume that it is neither reducible nor dihedral. If ρ is a representation of $W_{K/F}$ the image of $K^\times\backslash I_K$ will then consist of scalar matrices, and passing to $PGL(2,\mathbf{C})\simeq SO(3,\mathbf{C})$ we obtain a finite group which will be tetrahedral, octahedral, or icosahedral. About the last I can say nothing. I consider the other two in turn.

i) *Tetrahedral type*



There are three pairs of opposite edges so that we obtain a map of $\mathfrak{G}(K/F)$ into S_3. Since we only obtain proper motions of the tetrahedron the image must in fact be $\mathbf{A}_3\simeq\mathbf{Z}_3$. The kernel defines a cyclic extension E of degree 3. The restriction P of ρ to $W_{K/E}$ must be dihedral, and so $\Pi=\pi(P)$ exists as an automorphic representation of $G(\mathbf{A}_E)$. If $\tau\in\mathfrak{G}(E/F)$ has a representative u in $W_{K/F}$ then Π^τ is clearly $\pi(P^\tau)$ if P^τ is the representation, or rather the class of representations, defined by

$$P^\tau(w)=P(uwu^{-1}).$$

However, P and P^τ are equivalent so $\Pi^\tau\simeq\Pi$ and Π is a lifting of an automorphic representation π of $G(\mathbf{A})$. If $\omega_\rho=\det\rho$ then ω_ρ and ω_π pull back to the same quasi-character of $E^\times\backslash I_E$. Thus there is a character ω of

$F^\times NI_E \setminus I_F$ such that $\omega_\rho = \omega^2\omega_\pi$. Replacing π by $\omega \otimes \pi$, we can arrange that $\omega_\rho = \omega_\pi$ and that π lifts to $\pi(P)$. This determines π uniquely. We write $\pi = \pi_{\mathrm{ps}}(\rho)$, which is to be read $\pi_{\mathrm{pseudo}}(\rho)$.

$(D.1)$

$$\begin{tikzcd} & PGL(2,\mathbf{C}) \arrow[dr] & \\ GL(2,\mathbf{C}) \arrow[ur] \arrow[rr] \arrow[drr, "\varphi"'] & & SL(3,\mathbf{C}) \arrow[d] \\ & & GL(3,\mathbf{C}) \end{tikzcd}$$

It follows from (C) and (G) that if $\pi(\rho)$ exists then it must be $\pi_{\mathrm{ps}}(\rho)$, but at the moment all we have in our hands is $\pi_{\mathrm{ps}}(\rho)$, and the problem is to show that it is in fact $\pi(\rho)$. This will be deduced from results of Gelbart, Jacquet, Piatetskii-Shapiro, and Shalika (cf. [11]).

Consider the commutative diagram (D.1) in which the skewed arrow on the right is given by the adjoint representation. Taking the product with $\mathfrak{G}(K/F)$, we obtain a diagram of L-groups with $G_1 = SL(2),\ H_1 = PGL(3),\ H = GL(3)$.

The representation $\sigma = \varphi \circ \rho$ is a representation of $\mathfrak{G}(K/F)$. Each of the one-dimensional subspaces defined by an axis passing through opposite edges of the tetrahedron is fixed by $\mathfrak{G}(K/E)$ and thus defines a character θ of $\mathfrak{G}(K/E)$. It is easy to see that $\sigma = \mathrm{Ind}(\mathfrak{G}(K/F), \mathfrak{G}(K/E), \theta)$.

$(D.2)$

$$\begin{tikzcd} & {}^LG_1 \arrow[dr] & \\ {}^LG \arrow[ur] \arrow[rr] \arrow[drr, "\varphi"'] & & {}^LH_1 \arrow[d] \\ & & {}^LH \end{tikzcd}$$

For each finite place v at which σ_v is unramified one attaches a conjugacy class in $GL(3,\mathbf{C})$ to σ_v, namely that of $\sigma_v(\Phi)$ if Φ is the Frobenius at v. Moreover one also attaches a conjugacy class $\{A(\pi_v^1)\}$ in $GL(3,\mathbf{C})$ to each unramified representation π_v^1 of $GL(3,F_v)$ (cf. [3], [20], [26]). The representation is determined by the conjugacy class, and one says that $\pi_v^1 = \pi(\sigma_v)$ if $\{A(\pi_v^1)\} = \{\sigma_v(\Phi)\}$. The following instance of the principle of functoriality is due to Piatetskii-Shapiro ([16]):

1) *There is a cuspidal representation* π^1 *of* $GL(3,\mathbf{A})$ *such that* $\pi_v^1 = \pi(\sigma_v)$ *for almost all* v.

There is another instance of the principle due to Gelbart-Jacquet ([12]):

2) *Let* $\pi = \pi_{\mathrm{ps}}(\rho)$. *Then there is a cuspidal representation* π^2 *of* $GL(3,\mathbf{A})$ *such that* $\{A(\pi_v^2)\} = \{\varphi(A(\pi_v))\}$.
Recall that evaluation at the class $\{A(\pi_v)\}$ defines the homomorphism of the Hecke algebra into $\mathbf{C}$ associated to π_v.

It is to be expected that π^1 and π^2 are equivalent, and this can indeed be established, using a criterion of Jacquet–Shalika ([15]). Let π^{-1} be the contragredient of π^1. All that need be verified is, in the notation of [15], that

$$L(s, \pi_v^1 \times \tilde{\pi}_v^1) = L(s, \pi_v^2 \times \tilde{\pi}_v^{-1})$$

for almost all v. The left side is

$$\det{}^{-1}(1 - |\varpi_v|^s A(\pi_v^1) \otimes {}^t A^{-1}(\pi_v^1)), \tag{3.1}$$

and the right side is

$$\det{}^{-1}(1 - |\varpi_v|^s A(\pi_v^2) \otimes {}^t A^{-1}(\pi_v^1)). \tag{3.2}$$

In general, if $\pi_v^1 = \pi(\sigma_v)$ and σ_v is unramified then

$$\det(1 - |\varpi_v|^s B \otimes {}^t A^{-1}(\pi_v^1)) = \prod\nolimits_{w|v} \det(1 - |\varpi_v|^{n(w) s_B n(w)})$$

if $n(w)$ is the degree $[E_w : F_v]$.

If v splits completely in E then $\rho_v(\Phi)$ is conjugate to $A(\pi_v)$. Since

$$\{A(\pi_v^1)\} = \{\varphi(\rho_v(\Phi))\}$$

and

$$\{A(\pi_v^2)\} = \{\varphi(A(\pi_v))\}$$

the equality of (3.1) and (3.2) is clear. If v does not split in E then $n(w) = 3$, and, by definition,

$$\{A^3(\pi_v)\} = \{\rho_v^3(\Phi)\}.$$

The equality is again clear.

To show that $\pi_{\mathrm{ps}}(\rho)$ is $\pi(\rho)$ we have to show that

$$\{A(\pi_v)\} = \{\rho_v(\Phi)\}$$

even when v does not split in E. We have so chosen π that both sides have the same determinant. Thus we may suppose that

$$\{\rho_v(\Phi)\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}$$

and that

$$\{A(\pi_v)\} = \left\{ \begin{pmatrix} \xi a & 0 \\ 0 & \xi^2 b \end{pmatrix} \right\}$$

with $\xi^3 = 1$. We need to show that ξ may be taken to be 1. Since π^1 and π^2 are equivalent,

$$\left\{\varphi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})\right\} = \left\{\varphi(\begin{pmatrix} \xi a & 0 \\ 0 & \xi^2 b \end{pmatrix})\right\}$$

in $GL(3, \mathbf{C})$. This implies either that $\xi = 1$, and then we are finished, or that

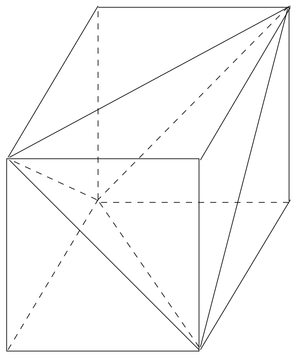
$$a^2 = \xi b^2.$$

From this equation we conclude that either $a = \xi^2 b$, which also leads to the desired conclusion, or $a = -\xi^2 b$, which implies that $\varphi(\rho_v(\Phi))$ has order 6 if ξ is not 1. Since the tetrahedral group contains no element of order 6, the last possibility is precluded.

We have proved the following theorem.

Theorem 3.3 *If F is a number field and ρ a two-dimensional representation of the Weil group of F of tetrahedral type then the L-function $L(s, \rho)$ is entire.*

ii) *Octahedral type.* Rather than an octahedron I draw a cube in which I inscribe a tetrahedron. The subgroup of $\mathfrak{G}(K/F)$ which takes the tetrahedron to itself defines a quadratic extension E of F.



The restriction P of ρ to $W_{K/E}$ is of tetrahedral type; so $\Pi = \pi(P)$ exists. If $\tau \in \mathfrak{G}(E/F)$ then $\Pi^\tau = \pi(P^\tau)$. Since $P^\tau \simeq P$ we conclude that $\Pi^\tau \simeq \Pi$. Hence Π is the lifting of exactly two automorphic representations π, π' of $G(\mathbf{A})$, one of which can be obtained from the other by tensoring with the non-trivial character ω of $F^\times N I_E \backslash I_F$. We are no longer able to define $\pi_{\mathrm{ps}}(\rho)$ uniquely; we take it to be either of the two representations π, π'.

We are only able to show that one of the $\pi_{\mathrm{ps}}(\rho)$ is in fact $\pi(\rho)$ in very special cases. We will exploit a result of Deligne-Serre. There is a general observation to be made first. Suppose E is a cyclic extension of arbitrary prime degree ℓ and ρ a two-dimensional representation of the Weil group of F. Suppose in addition that the restriction P of ρ to the Weil group of E is irreducible and that $\Pi = \pi(P)$ exists. Let π lift to Π, and suppose that π is $\pi(\rho')$ for some ρ', perhaps different from ρ.

If P' is the restriction of ρ' to the Weil group of E then $P'_w = P_w$ for almost all places of E and thus (see, for example, Lemma 12.3 of [14]) $P' = P$. Consequently, $\rho = \omega \otimes \rho'$ and $\pi(\rho) = \omega \otimes \pi$ exists, so that $L(s, \rho)$ is entire. Here ω is a characer of $F^\times N_{E/F} I_E \backslash I_F$.

Thus, for ρ of tetrahedral type and E the associated quadratic extension, we can conclude that one of π or π' is $\pi(\rho)$ if we can show that π is $\pi(\rho')$ for some ρ'. By the result of Deligne-Serre ([6]), this will be so if $F = \mathbf{Q}$ and the infinite component π_∞ of π is $\pi(\rho'_\infty)$ where $\rho'_\infty = \mu \oplus \nu,\ \mu,\ \nu$ being two characters of $\mathbf{R}^\times$ with

$$\mu(x) = \nu(x)\mathrm{sgn}\, x.$$

This is the condition that guarantees that the tensor product of π with some idèle class character is the automorphic representation defined by a holomorphic form of weight one. We will not be able to show that π_∞ has this form unless we assume that ρ_∞, the infinite component of ρ, has the same form as ρ'_∞. Interpreted concretely this means that the image of complex conjugation in the octahedral group is rotation through an angle of 180° about some axis.

This axis passes either through the center of a face of the cube or through the center of an edge. If it passes through the center of a face then complex conjugation fixes E, which is therefore a real quadratic field. If v is either of the infinite places of E, then π_∞ is equivalent to Π_v and $\Pi_v = \pi(P_v)$. Since $P_v = \rho_\infty$, the representation π_∞ satisfies the condition which allows us to apply the theorem of Deligne–Serre.

Theorem 3.4 *Suppose ρ is a two-dimensional representation of the Weil group of $\mathbf{Q}$ which is of octahedral type. If the image of complex conjugation is rotation through an angle of 180° about an axis passing through a vertex of the octahedron or, what is the same, the center of a face of the dual cube, then $L(s, \rho)$ is entire.*

There is one other condition which allows us to conclude that π_∞ is of the desired type. We continue to suppose that the image of complex conjugation is rotation through an angle of 180°. If ω_π is the central character of π and ω_ρ the determinant of ρ then $\eta = \omega_\pi \omega_\rho^{-1}$ is of order two. Since its local component is trivial at all places which split in E, it is either trivial itself or the quadratic character associated to the extension E. π_∞ has the desired form if and only if η_∞ is trivial. If E is a real quadratic field then η_∞ is necessarily trivial, and so we obtain the previous theorem. If E is an imaginary quadratic field then η_∞ is trivial if and only if η is; and η is trivial if and only if η_v is trivial for some place of F which does not split in E.

Theorem 3.5 *Suppose ρ is a two-dimensional representation of the Weil group of $\mathbf{Q}$ which is of octahedral type. Suppose the image of complex conjugation is rotation through 180° about an axis passing through the center of an edge. If for some place v which does not split in E, the quadratic field defined by the tetrahedral subgroup, the local representation ρ_v is dihedral then $L(s, \rho)$ is entire.*

It is clear that $\eta_v = \omega_{\pi_v}\omega^{-1}_{\pi(\rho_v)}$. However π_v and $\pi(\rho_v)$ have the same lifting to $G(E_v)$. Thus, by property (c) of local liftings,

$$\pi(\rho_v) = \omega \otimes \pi_v$$

with ω of order two. We conclude that $\omega_{\pi(\rho_v)} = \omega_{\pi_v}$.

4. σ–CONJUGACY

Suppose F is a field and E is a cyclic extension of prime degree ℓ. Fix a generator σ of $\mathfrak{G} = \mathfrak{G}(E/F)$. If x and y belong to $G(E)$ we say that they are σ-conjugate if for some $h \in G(E)$

$$y = h^{-1} x \sigma(h).$$

Then

$$y\sigma(y)\cdots\sigma^{\ell-1}(y) = h^{-1}x\sigma(x)\cdots\sigma^{\ell-1}(x)h.$$

We set

$$Nx = x\sigma(x)\cdots\sigma^{\ell-1}(x).$$

If $u = Nx$ and $v = h^{-1}uh$ then $v = Ny$.

Lemma 4.1 *If $u = Nx$ then u is conjugate in $G(E)$ to an element of $G(F)$.*

Let $\overline{F}$ be an algebraic closure of F containing E. It is sufficient to verify that the set of eigenvalues of Nx, with multiplicities, is invariant under $\mathfrak{G}(\overline{F}/F)$, or even under those $\sigma' \in \mathfrak{G}(\overline{F}/F)$ with image σ in $\mathfrak{G}$. Acting on the set with σ' we obtain the eigenvalues of $\sigma(u)$, and

$$\sigma(u) = x^{-1}ux.$$

The invariance follows.

Suppose $u = Nx$ lies in $G(F)$. Let G_u be the centralizer of u and let $G_x^\sigma(E)$ be the set of all g in $G(E)$ for which

$$x = g^{-1}x\sigma(g).$$

The matrix x belongs to $G_u(E)$ and $y \to x\sigma(y)x^{-1}$ is an automorphism of $G_u(E)$ of order ℓ. It therefore defines a twisted form G_u^σ of G_u. Clearly

$$G_u^\sigma(F) = G_x^\sigma(E).$$

If M is the algebra of 2×2 matrices and M_u the centralizer of u, we may also introduce the twisted form M_u^σ of M_u. Then G_u^σ is the group of invertible elements in M_u^σ, and it follows readily from the exercise on p. 160 of [28] that

$$H^1(\mathfrak{G}, G_u^\sigma(E)) = \{1\}.$$

Lemma 4.2 *If Nx and Ny are conjugate then x and y are σ-conjugate.*

We reduce ourselves immediately to the case that $u = Nx$ lies in $G(F)$ and $Nx = Ny$. If $\tau = \sigma^r$ belongs to $\mathfrak{G}$ set

$$c_\tau = y\sigma(y)\cdots\sigma^{r-1}(y)\sigma^{r-1}(x)^{-1}\cdots\sigma(x)^{-1}x^{-1}.$$

Since $Nx = Ny$, c_τ is well defined and

$$c_\sigma x \sigma(c_\tau) x^{-1} = c_{\sigma\tau}.$$

In other words $\tau \to c_\tau$ defines a cocycle of $\mathfrak{G}$ with values in $G_u^\sigma(E)$. Therefore there is an h satisfying

$$yx^{-1} = c_\sigma = h^{-1} x \sigma(h) x^{-1}$$

and

$$y = h^{-1} x \sigma(h).$$

Occasionally in later paragraphs Nx will simply stand for an element of $G(F)$ which is conjugate to $x\sigma(x)\cdots\sigma^{\ell-1}(x)$, but for now it is best to retain the convention that

$$Nx = x\sigma(x)\cdots\sigma^{\ell-1}(x).$$

Lemma 4.3 *Suppose*

$$u = \begin{pmatrix} a & av \\ 0 & a \end{pmatrix}$$

with $v \neq 0$. Then $u = Nx$ for some x in $G(E)$ if and only if $a \in NE^\times$. If $u = Nx$ and $h \in G(E)$ then $h^{-1}x\sigma(h)$ is upper-triangular if and only if h itself is.

If $u = Nx$ then $x \in G_u(E)$ and has the form

$$\begin{pmatrix} b & by \\ 0 & b \end{pmatrix}.$$

Consequently

$$Nx = Nb \begin{pmatrix} 1 & \operatorname{tr} y \\ 0 & 1 \end{pmatrix}.$$

The first assertion follows. To obtain the second we observe that if $h^{-1}x\sigma(h)$ is upper-triangular, then $h^{-1}uh$ is also.

Lemma 4.4 *Suppose*

$$u = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

with $a_1 = a_2$. Then $u = Nx$ if and only if a_1 and a_2 lie in $NE^\times$. If y is upper-triangular then Ny is of the form

$$\begin{pmatrix} a_1 & v \\ 0 & a_2 \end{pmatrix}$$

if and only if $y = h^{-1}x\sigma(h)$ with an upper-triangular h. If $h^{-1}x\sigma(h)$ is diagonal then h is of one of the two forms

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$

Since G_u is the group of diagonal matrices the first and last assertions are clear. Suppose y is upper-triangular and

$$Ny = \begin{pmatrix} a_1 & v \\ 0 & a_2 \end{pmatrix}.$$

Replacing y by $g^{-1}y\sigma(g)$ with g diagonal we may suppose that

$$y = x\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

perhaps with a different v. If

$$x = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

then

$$\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \; x \; \begin{pmatrix} 1 & \sigma(w) \\ 0 & 1 \end{pmatrix} = x \; \begin{pmatrix} 1 & \sigma(w) - b_1^{-1}b_2w \\ 0 & 1 \end{pmatrix}.$$

To complete the proof we need only verify the following supplementary lemma.

Lemma 4.5 *If $b \in E$ regard $w \to \sigma(w) - bw$ as a linear transformation of the vector space E over F. The determinant of this linear transformation is $(-1)^\ell(Nb-1)$.*

To compute the determinant we may extend scalars to E. Since

$$E \otimes_F E \simeq (E \oplus \cdots \oplus E)$$

and the linear transformation becomes

$$(x_1, \cdots, x_\ell) \to (x_2 - bx_1,\; x_3 - \sigma(b)\,x_2, \cdots, x_1 - \sigma^{\ell-1}(b)x_\ell),$$

we are reduced to calculating

$$\begin{vmatrix} -b & 1 & & & & \\ & -\sigma(b) & 1 & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & -\sigma^{\ell-2}(b) & 1 \\ 1 & & & & & -\sigma^{\ell-1}(b) \end{vmatrix}.$$

Elementary row and column operations yield the desired result.

Lemma 4.6 *Suppose $u \in G(F)$ has distinct eigenvalues which do not lie in F. Let $E' = F(u)$ be the centralizer of u in $M(F)$.*

(a) *If E' is isomorphic to E over F then $u = Nx$ has a solution.*

(b) *If E' is not isomorphic to E over F and $L = E' \otimes_F E$ then $u = Nx$ has a solution if and only if $u \in N_{L/E'}L^\times$.*

If T is the Cartan subgroup in which u lies and $u = Nx$ then $x \in T(E) = L^\times$. The second statement is therefore clear. If E' is isomorphic to E then $E' \otimes_F E$ is isomorphic to $E' \oplus E'$ with E' imbedded as $\{(y, y)\}$. Since σ acts as $(y_1, y_2) \to (y_2, y_1)$ every invariant is a norm.

Corollary 4.7 *If F is a local field and $u \in G(F)$ has distinct eigenvalues which do not lie in F then $u = Nx$ has a solution if and only if $\det u \in N_{E/F}E^\times$.*

This follows from the previous lemma and local class field theory.

Lemma 4.8 *Suppose $u \in Z(F) = F^\times$. If ℓ is odd then $u = Nx,\ x \in G(E)$, if and only if $u \in NE^\times$. If $\ell = 2$ then $u = Nx$ always has a solution.*

If $u = Nx$ then $u^2 = \det u \in NE^\times$. This makes the first statement clear. If $\ell = 2$ we may imbed E in $M(F)$; so the second statement follows from the proof of Lemma 4.6.

If F is a global field and v a place of it we set $E_v = E \otimes_F F_v$. $\mathfrak{G}$ acts on E_v. Either E_v is a field and $\mathfrak{G} = \mathfrak{G}(E_v/F_v)$ or E_v is isomorphic to a direct sum of ℓ copies of F_v and σ acts as

$$(x_1, \cdots, x_\ell) \to (x_2, \cdots, x_\ell, x_1).$$

Then

$$G(E_v) = G(F_v) \times \cdots \times G(F_v).$$

If $u = (u, \cdots, u)$ lies in $G(F_v) \subseteq G(E_v)$ and $x = (u, 1, \cdots, 1)$ then $u = NX$; so at a place which splits in E every element is a norm.

Lemma 4.9 *Suppose F is a global field and $u \in G(F)$. Then $u = Nx$ has a solution in $G(E)$ if and only if it has a solution in $G(E_v)$ for each place v.*

It is enough to show that the equation $u = Nx$ can be solved globally if it can be solved locally. We know that $a \in F^\times$ lies in $NE^\times$ if and only if it lies in $NE_v^\times$ for all v. If u is conjugate to an upper-triangular matrix the desired result follows from this and Lemmas 4.3, 4.4 and 4.8. Otherwise we apply Lemma 4.6.

Observe that if $u \in F^\times$ then the number of places v for which $u \notin NE_v^\times$ is finite and even.

We close this paragraph with a simple lemma which will be used frequently below. Suppose S is an abelian algebraic group over F, either a torus or the additive group G_a, and ω an invariant form of maximum degree on it. Let T be the group over F obtained from S over E by restriction of scalars and let ν be an invariant form of maximal degree on T. The two forms ω and ν and the exact sequences

$$1 \longrightarrow T^{1-\sigma} \longrightarrow T \xrightarrow{N} S \longrightarrow 1 \tag{4.1}$$

$$1 \longrightarrow S \longrightarrow T \xrightarrow{1-\sigma} T^{1-\sigma} \longrightarrow 1 \tag{4.2}$$

yield forms μ_1 and μ_2 on $T^{1-\sigma}$.

Lemma 4.10 *The forms μ_1 and μ_2 are equal, except perhaps for sign.*

The lemma need only be verified over the algebraic closure $\overline{F}$ of F. So we may assume S is either G_a or G_m and T is either $G_a \times \cdots \times G_a$ or $G_m \times \cdots \times G_m$. Suppose first that S is G_a. Then

$$T^{1-\sigma} = \left\{(x_1, \cdots, x_\ell) \;\middle|\; \sum x_i = 0\right\}.$$

We may suppose ω is dx and ν is $dx_1 \wedge \cdots \wedge dx_\ell$. The pullback of dx to T is $\sum dx_i$ and the restriction of dx_1 from T to S is dx. We may take $x_1, \cdots, x_{\ell-1}$ as coordinates on $T^{1-\sigma}$. Then

$$\mu_1 = dx_1 \wedge \cdots \wedge dx_{\ell-1}.$$

Pulling back μ_1 from $T^{1-\sigma}$ to T we obtain

$$d(x_1 - x_2) \wedge\, d(x_2 - x_3) \wedge \cdots \wedge\, d(x_{\ell-1} - x_\ell).$$

Multiplying by dx_1 we obtain

$$(-1)^{\ell-1} dx_1 \wedge \cdots \wedge dx_\ell$$

so $\mu_2 = (-1)^{\ell-1}\mu_1$. A similar computation can be made for G_m.

5. SPHERICAL FUNCTIONS

In this paragraph F is a non-archimedean local field and $O = O_F$ is the ring of integers in F. We want to study the algebra $\mathcal{H}$ of compactly supported functions on $G(F)$ spherical with respect to $G(O)$. A is the group of diagonal matrices and X^*, which is isomorphic to $\mathbf{Z}^2$, its lattice of rational characters. Set

$$X_* = \mathrm{Hom}(X^*, \mathbf{Z}).$$

If ϖ is a generator of the prime ideal of O then the map $\gamma \to \lambda(\gamma)$, where $\lambda(\gamma) \in X_*$ is defined by

$$|\lambda(\gamma)| = |\varpi|^{<\lambda, \lambda(\gamma)>}$$

establishes an isomorphism of $A(O)\backslash A(F)$ with X_*.

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

set

$$\Delta(\gamma) = \left| \frac{(a-b)^2}{ab} \right|^{1/2}$$

and, if $\Delta(\gamma) = 0$, let

$$F_f(\gamma) = \Delta(\gamma) \int_{A(F)\backslash G(F)} f(g^{-1}\gamma g) dg.$$

If $f \in \mathcal{H}$ then $F_f(\gamma)$ depends only on $\lambda(\gamma)$; so we write $F_f(\lambda)$. This function is invariant under permutation of the two coordinates of λ, and $f \to$meas $A(O) F_f(\lambda)$ defines an isomorphism of $\mathcal{H}$ with the subalgebra of the group ring of X_* over $\mathbf{C}$, formed by the invariant elements. We may look at this in a slightly different way. X_* may also be regarded as the lattice of rational characters of the diagonal matrices $A(\mathbf{C})$ in $GL(2, \mathbf{C})$, and every element $\sum a(\lambda)\lambda$ of the group ring defines a function

$$t \to \sum a(\lambda)\lambda(t)$$

on $A(\mathbf{C})$. The symmetric elements are precisely the functions obtained by restricting the elements of the representation ring of $GL(2, \mathbf{C})$ to $A(\mathbf{C})$. Thus $\mathcal{H}$ is isomorphic to an algebra of functions on $A(\mathbf{C})$. Let $f^\vee$ be the function corresponding to f.

There are a number of distributions, which will arise in the trace formula, whose value on f we shall have to be able to express in terms of $f^\vee$. We begin this paragraph by verifying the necessary formulae. Our method of verification will be simply to check that both sides are equal for $f = f_\lambda$, the characteristic function of a double coset $G(O)\gamma G(O)$ with $\lambda(\gamma) = \lambda$. It is easy to verify that $m(\lambda)$, the measure of $G(O)\gamma G(O)$, is meas $G(O)$ if $\lambda = (k, k)$ and is

$$q^{<\alpha,\lambda>} \left(1 + \frac{1}{q}\right) \mathrm{meas}\, G(O)$$

if $\lambda = (k', k),\ k' > k$. q is the number of elements in the residue field of O and α is the root for which $< \alpha, \lambda >> 0$, that is $< \alpha, \lambda >= k' - k$.

Lemma 5.1 *If $< \alpha, \lambda > \geq 0$, then $f_\lambda^\vee(t)$ is given by*

$$m(\lambda) \cdot \frac{q^{-\frac{<\alpha,\lambda>}{2}}}{1 + \frac{1}{q}} \left\{ \frac{1 - q^{-1}\alpha^{-1}(t)}{1 - \alpha^{-1}(t)} \lambda(t) + \frac{1 - q^{-1}\alpha(t)}{1 - \alpha(t)} \tilde{\lambda}(t) \right\}.$$

Here $\tilde{\lambda}$ is obtained from λ by permuting its two coordinates.

Taking $|\alpha(t)| < 1$ and expanding the denominators in a Laurent expansion we find that this expression is equal to

$$\begin{array}{ll} \operatorname{meas} G(O)\lambda(t) & < \alpha, \lambda >= 0 \\ \operatorname{meas} G(O) q^{\frac{<\alpha,\lambda>}{2}} \, (\lambda(t) + \tilde{\lambda}(t)) & < \alpha, \lambda >= 1 \\ \operatorname{meas} G(O) q^{\frac{<\alpha,\lambda>}{2}} \left\{ \sum_{j=0}^{<\alpha,\lambda>} \lambda(t)\alpha^{-j}(t) - \frac{1}{q} \sum_{j=1}^{<\alpha,\lambda>-1} \lambda(t)\alpha^{-j}(t) \right\} & < \alpha, \lambda > \geq 2. \end{array}$$

To verify the lemma we have only to calculate $F_{f_\lambda}(\mu)$ explicitly.

Let $\lambda(\gamma) = \mu$ and choose δ in $A(F)$ with $\lambda(\delta) = \lambda$. To make the calculation we use the building associated by Bruhat and Tits to $SL(2, F)$. This building is a tree $\mathfrak{X}$, the vertices of which are equivalence classes of lattices in F^2, two lattices being equivalent if one is a scalar multiple of the other. The vertices defined by lattices M_1, M_2 are joined by an edge if there are scalars α and β such that

$$\alpha M_1 \underset{\neq}{\supset} \beta M_2 \underset{\neq}{\supset} \varpi \alpha M_1.$$

If M_0 is the lattice of integral vectors let p_0 be the corresponding vertex. The action of $G(F)$ on lattices induces an action on $\mathfrak{X}$. Every vertex of $\mathfrak{X}$ lies on $q + 1$ edges. We associate to A an apartment $\mathfrak{A}$. This is a subtree whose vertices are the points $tp_0,\ t \in A(F)$, and whose edges are the edges joining two such points. The apartment $\mathfrak{A}$ is a line; every vertex lies on two edges. If p_1, p_2 are two points in $\mathfrak{X}$ there is a g in $G(F)$ and a t in $A(F)$ such that $p_1 = gtp_0$, $p_2 = gp_0$. If $\lambda(t) = (k', k)$ then $|k' - k|$ is uniquely determined, and is just the distance from p_2 to p_1.

We may also associate a simplicial complex $\mathfrak{X}'$ to $GL(2, F) = G(F)$. The points are lattices, two lattices M_1 and M_2 being joined by an edge if $M_1 \underset{\neq}{\supset} M_2 \underset{\neq}{\supset} \varpi M_1$ or $M_2 \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \varpi M_2$. We may define an apartment $\mathfrak{A}'$ and the type of an ordered pair (p_1', p_2'). It is a $\lambda = (k', k)$, the pair, not the ordered pair, (k', k) being uniquely determined, so that the type is in fact a double coset. There is an obvious map $p' \to p$ of $\mathfrak{X}'$ to $\mathfrak{X}$.

The type of $(\gamma p', p')$ depends only on the orbit under $A(F)$ to which p' belongs. If $\tau(p_1, p_2)$ denotes the type of (p_1, p_2) the integral

$$\int_{A(F)\backslash G(F)} f_\lambda(g^{-1}\gamma g) dg$$

is a sum over representatives of the orbits of $A(F)$ in X'

$$\sum\nolimits_{\tau(\gamma p', p')=\lambda} \operatorname{meas} G_{p'} \cap A(F)\backslash G_{p'}.$$

Here $G_{p'}$ is the stablizer of p'. We may choose the representatives p' so that the closest point to p in $\mathfrak{A}$ is p_0. If p'_0 is the vertex of $\mathfrak{X}'$ determined by the lattice of integral vectors and $p' = gp'_0$, let $d(p')$ be defined by

$$|\det g| = |\varpi|^{d(p')}.$$

Any point p lifts uniquely to a p' with $d(p') = \operatorname{dist}(p, p_0)$. We may also demand that the representatives p' be chosen so that $d(p') = \operatorname{dist}(p, p_0)$. Then $A(F) \cap G_{p'}$ will lie in $A(O)$. The number of choices for representatives satisfying the two conditions is $[A(O) : A(O) \cap G_{p'}]$. Since

$$\operatorname{meas} G_p = \operatorname{meas} G(O)$$

the integral is equal to

$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)} \sum\nolimits_{\tau(\gamma p', p)=\lambda} 1. \tag{5.1}$$

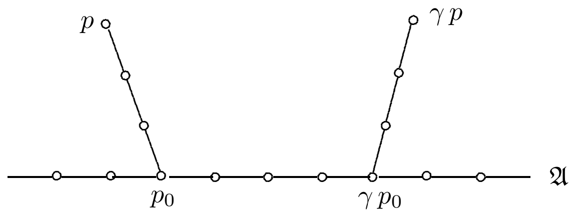
The sum is over all p' for which, in addition to the condition $\tau(\gamma p', p') = \lambda$ on the type of $\gamma p', p'$,

$$d(p') = \operatorname{dist}(p, p_0) = \operatorname{dist}(p, \mathfrak{A}).$$

This type of reduction will be used repeatedly, but without further comment, in the present paragraph.

We may suppose $<\alpha, \mu> \geq 0$ and $<\alpha, \lambda> \geq 0$. We have to show that $\Delta(\gamma)$ times the sum appearing in (5.1) is $q^{\frac{<\alpha,\lambda>}{2}}$ if $\lambda = \mu$, $q^{\frac{<\alpha,\lambda>}{2}} \left(1 - \frac{1}{q}\right)$ if $\lambda = \mu + n\alpha,\ n > 0$, and 0 otherwise.

There are two possibilities which have to be treated in different fashions. Suppose $\left|\frac{b}{a}\right| \neq 1$. Then we have the following picture



The distance between p_0 and γp_0 is $m' - m$ if $\mu = (m', m)$. If the distance of p from p_0 is k then the type of $\gamma p,\ p$ is $2k + m' - m$, provided $d(p, p_0) = d(p, \mathfrak{A})$, and the type of $(\gamma p', p')$ is $(m' + k, m - k)$. If $k = 0$ there is one choice for p and if $k > 0$ there are $q^k \left(1 - \frac{1}{q}\right)$. Since

$$\Delta(\gamma) = \left|\frac{b}{a}\right|^{1/2} \left|1 - \frac{a}{b}\right| = \left|\frac{b}{a}\right|^{1/2} = q^{\frac{m'-m}{2}}$$

and

$$q^{\frac{<\alpha,\lambda>}{2}} = q^{k+\frac{m'-m}{2}}$$

if $\lambda = (m'+k, m-k)$ the required equality follows.

Before treating the second possibility we establish another lemma.

Lemma 5.2 *Suppose* $\left|\frac{a}{b}\right| = 1$ *and* $\left|1-\frac{a}{b}\right| = q^{-r}$. *Then the points of* $\mathfrak{X}$ *fixed by* γ *are precisely those at a distance less than or equal to* r *from* $\mathfrak{A}$.

Since $G(F) = A(F)N(F)K$, with $K = G(O)$ and

$$N(F) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,|x \in F \right\}$$

any point of $\mathfrak{X}$ is of the form tnp_0, $t \in A(F),\ n \in N(F)$. Moreover γ fixes tnp_0 if and only if it fixes np_0; and $\mathrm{dist}(tnp_0, \mathfrak{A}) = \mathrm{dist}(np_0, \mathfrak{A})$.

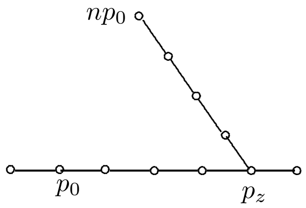
We may index the vertices of $\mathfrak{A}$ by $\mathbf{Z}$, the integer z corresponding to the vertex

$$p_z = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^z \end{pmatrix} p_0.$$

This vertex is fixed by $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ if and only if $\varpi^z x \in O$. If z is the smallest integer for which $\varpi^z x \in O$ then $np_0 = p_0$ for $z \leq 0$. Otherwise

$$\mathrm{dist}(np_0, \mathfrak{A}) = \mathrm{dist}(np_0, p_z) = \mathrm{dist}(np_0, np_z) = z.$$

Pictorially,

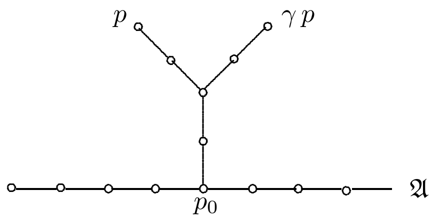


Certainly γ fixes np_0 if and only if $n^{-1}\gamma n$ or $\gamma^{-1}n^{-1}\gamma n$ belongs to K. Since

$$\gamma^{-1}n^{-1}\gamma n = \begin{pmatrix} 1 & x\left(1-\frac{b}{a}\right) \\ 0 & 1 \end{pmatrix}$$

the lemma follows.

To complete the proof of the first lemma, we have still to treat the case that $\left|\frac{b}{a}\right| = 1$. Let $\left|1 - \frac{a}{b}\right| = r$ so that $\Delta(\gamma) = q^{-r}$



If the distance of p from p_0 is $k+r$ with $k > 0$ then the type of $(\gamma p', p')$ is $(m'+k, m-k)$, with m' now equal to m. There are $q^{k+r}\left(1 - \frac{1}{q}\right)$ possible such points. If the distance of p from p_0 is less than or equal to r the type of $(\gamma p', p')$ is (m', m). There are

$$1 + \sum\nolimits_{j=1}^{r} q^j \left(1 - \frac{1}{q}\right) = q^r$$

such points. This gives the desired equality once again.

The group $A_0(\mathbf{C})$ of elements in $A(\mathbf{C})$ whose eigenvalues have absolute value 1 is compact. We introduce an inner product in the group ring of X_* by setting

$$< f_1, f_2 >= \int_{A_0(\mathbf{C})} f_1(t)\overline{f}_2(t).$$

The total measure of the group is taken to be one.

Lemma 5.3 *Suppose γ lies in $Z(F)$ and $\mu = \lambda(\gamma)$. If f belongs to $\mathcal{H}$ and*

$$\varphi_\gamma(t) = \frac{1+\frac{1}{q}}{2\,\mathrm{meas}\, G(O)}\ \frac{1-\alpha(t)}{1-q^{-1}\alpha(t)}\ \frac{1-\alpha^{-1}(t)}{1-q^{-1}\alpha^{-1}(t)}\lambda^\vee(t)$$

then

$$f(\gamma) =< f^\vee, \varphi_\gamma > .$$

We verify this for $f = f_\lambda$. If $\lambda = (k', k),\ k' \geq k$ and $\mu = (m, m)$ then both sides are 0 unless $k' + k = 2m$. If this condition is satisfied $f^\vee(t)\overline{\varphi}_\gamma(t)$ is constant with respect to elements

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

so that the integration may be taken with respect to

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

This gives

$$\frac{m(\lambda)}{\mathrm{meas}\, G(O)}\ \frac{q^{\frac{k-k'}{2}}}{2}$$

times

$$\frac{1}{2\pi i}\int_{|z|=1}\left\{\frac{1-z}{1-q^{-1}z}\,z^{k'-m}+\frac{1-z^{-1}}{1-q^{-1}z^{-1}}\,z^{k-m}\right\}\frac{dz}{z}.$$

Since $k'\geq m\geq k$ this integral is seen by inspection to be 0 unless $k'=m=k$ when it is 2. These are the required values.

Corollary 5.4 *If f_1 and f_2 belong to $\mathcal{H}$ then*

$$\int_{G(F)} f_1(g)\overline{f}_2(g)dg=\frac{1+\frac{1}{q}}{2\,\mathrm{meas}\,G(O)}\int_{A_0^\vee(\mathbf{C})} f_1^\vee(t)\overline{f}_2^\vee(t)\,\frac{1-\alpha(t)}{1-q^{-1}\alpha(t)}\cdot\frac{1-\alpha^{-1}(t)}{1-q^{-1}\alpha^{-1}(t)}.$$

Apply the previous formula for $f=f_1*f_2^*$ and $\gamma=1$ with $f_2^*(g)=\overline{f}_2(g^{-1})$.

Let

$$\nu(t)=\frac{1+\frac{1}{q}}{2}\cdot\frac{1-\alpha(t)}{1-q^{-1}\alpha(t)}\cdot\frac{1-\alpha^{-1}(t)}{1-q^{-1}\alpha^{-1}(t)}.$$

Then the family of function $f_\lambda^\vee$ is orthogonal with respect to $\nu(t)$ and

$$\int_{A_0(\mathbf{C})}|f_\lambda^\vee(t)|^2\nu(t)=m(\lambda)\mathrm{meas}\,G(O).$$

Let

$$n_0=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Lemma 5.5 *If $a\in F^\times,\ n=an_0$, and $\mu=\lambda\left(\begin{pmatrix}a&0\\0&a\end{pmatrix}\right)$, then*

$$\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg$$

is equal to

$$\frac{1}{\mathrm{meas}\,G_n(O)}\;\frac{< f^\vee,\mu >}{1-\frac{1}{q}}.$$

Since $\{p'\in\mathfrak{A}'|d(p')=\mathrm{dist}(p,p_0)\}$ is a set of representatives for the orbits of $G_n(F)$ in $\mathfrak{X}'$, the integral is equal to

$$\mathrm{meas}\,G(O)\cdot\sum_{\substack{p'\in\mathfrak{A}'\\ d(p')=\mathrm{dist}(p,p_0)\\ \tau(np',p')=\lambda}}\frac{1}{\mathrm{meas}\,G_n(F)\cap G_{p'}}$$

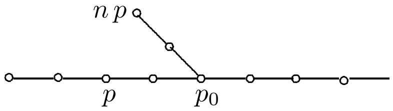
when $f=f_\lambda$. If $\mu=(m,m)$ this expression is 0 unless $\lambda=(m+k,m-k),\ k\geq 0$. If $k=0$ the sum is

$$\frac{1}{\mathrm{meas}\,G_n(O)}\sum_{z=0}^{\infty}\frac{1}{q^z}=\frac{1}{\mathrm{meas}\,G_n(O)}\cdot\frac{1}{1-\frac{1}{q}}.$$

If $k>0$ there is only one term in the sum and it equals

$$\frac{q^k}{\mathrm{meas}\,G_n(O)}.$$

Comparing with the explicit expansion of $f_\lambda^\vee$ we obtain the lemma. For these calculations we of course rely on the diagram



If γ is any semi-simple element in $G(F)$ with eigenvalues a and b we may set

$$\Delta(\gamma) = \left|\frac{(a-b)^2}{ab}\right|^{1/2}$$

and

$$F_f(\gamma) = \Delta(\gamma) \int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)dg$$

if T is the Cartan subgroup containing γ.

Lemma 5.6 *If f belongs to $\mathcal{H}$ and T splits over the unramified quadratic extension F' then*

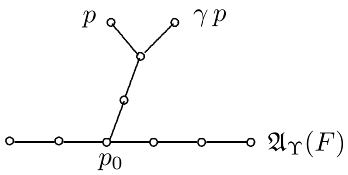
$$F_f(\gamma) = \left(1+\frac{1}{q}\right)\cdot\frac{\operatorname{meas} G_n(O)}{\operatorname{meas} T(O)} \int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg - 2\,\frac{\Delta(\gamma)}{q-1}\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} f(z).$$

Here $z \in F$ is determined by $|z| = |a| = |b|$, and

$$n = zn_0.$$

The group $T(O)$ consists of all matrices in $T(F)$ whose eigenvalues are units.

The Bruhat–Tits buildings $\mathfrak{X}$ and $\mathfrak{X}'$ over F may be regarded as subtrees of the buildings $\mathfrak{X}(F')$ and $\mathfrak{X}'(F')$ over F'. The torus T splits over F' and we may introduce the associated apartments $\mathfrak{A}_T(F')$ and $\mathfrak{A}'_T(F')$. They consist of all vertices fixed by $T(O')$ and the edges joining them. $\mathfrak{G}(F'/F)$ operates on these buildings and, because $H^1(\mathfrak{G}(F'/F), G(O'))$ is trivial, $\mathfrak{X}$ and $\mathfrak{X}'$ are formed by the fixed points of (F'/F). The intersection $\mathfrak{X} \cap \mathfrak{A}_T(F')$ consists of p_0 alone and $\mathfrak{X}' \cap \mathfrak{A}'_T(F')$ is formed by the points lying over p_0.



The integral defining $F_{f_\lambda}(\gamma)$ is equal to

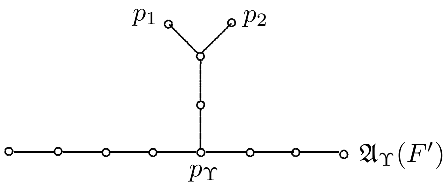
$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} \sum\nolimits_{\tau(\gamma p', p')=\lambda} 1$$

where p' runs over those points for which not only $\tau(\gamma p', p') = \lambda$ but also $d(p') = \operatorname{dist}(p, p_0)$. Since the closest point to p in $\mathfrak{A}_T(F')$ is p_0 and since the shortest path joining p to p_0 must lie completely in $\mathfrak{X}$ there are $q^{r+m}\left(1+\frac{1}{q}\right)$ such points if $\xi = \lambda(z) + (m, -m),\ m > 0$, and $\Delta(\gamma) = q^{-r}$, and there are

$$1 + (q+1)\sum\nolimits_{k=0}^{r-1} q^k = q^r \cdot \frac{q+1}{q-1} - \frac{2}{q-1}$$

if $\lambda = \lambda(z)$, but none otherwise. The lemma follows upon comparison with the calculations for the proof of Lemma 5.5.

Suppose the torus T splits over a ramified quadratic extension F'. It is no longer $\mathfrak{X}$ and $\mathfrak{X}'$ but their first barycentric subdivisions $\mathfrak{X}_1$ and $\mathfrak{X}'_1$ which are subcomplexes of $\mathfrak{X}(F')$ and $\mathfrak{X}(F')$. We may again introduce $\mathfrak{A}_T(F')$ and $\mathfrak{A}'_T(F')$ as well as the action of $\mathfrak{G}(F'/F)$. There is exactly one point p_T of $\mathfrak{A}_T(F')$ fixed by $\mathfrak{G}(F'/F)$ and it is a vertex. If p is a vertex of $\mathfrak{X}$ the closest point to it on $\mathfrak{A}_T(F')$ is p_T. There can be at most two points on $\mathfrak{X}$ at a minimal distance from $\mathfrak{A}_T(F')$ and these two points must be a distance 1 apart in $\mathfrak{X}$, for every second point on the path of shortest length joining them lies in $\mathfrak{X}$



There must be at least two such points p_1, p_2, for the set of them is fixed by $T(F)$, and $T(F)$ contains an element whose determinant has order 1, and which, as a consequence, fixes no point of $\mathfrak{X}$. Let δ be the distance of p_3 from p_T in $\mathfrak{X}(F')$.

Lemma 5.7 *Suppose* $|\det\gamma| = |\varpi|^{2m+1}$ *and set* $\mu = (m+1, m)$. *If*

$$\varphi_\gamma(t) = \frac{q^{\frac{-\delta-1}{2}}}{2\operatorname{meas} T(O)}\left\{\frac{\mu(t)}{1-q^{-1}\alpha(t)} + \frac{\widetilde{\mu}(t)}{1-q^{-1}\alpha^{-1}(t)}\right\}$$

then

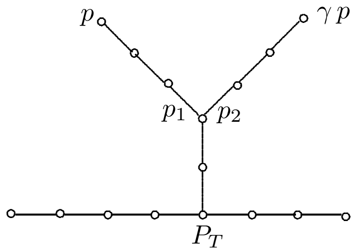
$$F_f(\gamma) = < f^\vee, \varphi_\gamma >$$

for $f^\vee$ *in* $\mathcal{H}$. *Here* $T(O)$ *is the stabilizer of* p'_T *in* $T(F)$.

Observe that by Lemma 5.2,

$$\Delta(\gamma) = q^{-\delta/2}$$

for γ must certainly interchange p_1 and p_2 and therefore p_3 is a fixed point of γ at maximal distance from $\mathfrak{A}_T(F')$. Arguing from a diagram



as usual we see that $F_{f_\lambda}(\gamma)$ is 0 unless $\lambda = (m+1+r, m-r),\ r \geq 0$ when it is

$$\Delta(\gamma)\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} q^r.$$

Moreover $< f_\lambda^\vee, \varphi_\gamma >$ is 0 unless $\lambda = (m+1+r, m-r)$ when it is

$$\frac{\operatorname{meas} G(O)}{2\operatorname{meas} T(O)}\, q^{r-\frac{\delta}{2}}$$

times

$$\frac{1}{2\pi i}\int_{|z|=1}\left\{\frac{z^r}{1-z^{-1}} + \frac{(1-q^{-1}z^{-1})}{(1-z^{-1})(1-q^{-1}z)}\,z^{r+1} + \frac{(1-q^{-1}z)}{(1-z)(1-q^{-1}z^{-1})}\,z^{-r-1} + \frac{z^{-r}}{1-z}\right\}\frac{dz}{z}.$$

This contour integral can be evaluated by shrinking the path a little and then integrating term by term. The first two terms have no poles inside the contour of integration and yield 0; the last two integrals are evaluated by moving the path to ∞, and each yields the residue 1 at $z = 1$. The lemma follows.

If $|\det\gamma| = |\varpi|^{2m}$ we may choose $z \in F$ so that $|z|_{F'} = |a|_{F'} = |b|_{F'}$.

Lemma 5.8 *If* $|\det\gamma| = |\varpi|^{2m}$ *then*

$$F_f(\gamma) = q^{\frac{-\delta-1}{2}}\,\frac{\operatorname{meas} G_n(O)}{\operatorname{meas} T(O)}\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg - \frac{\Delta(\gamma)}{q-1}\,\frac{\operatorname{meas}\mathrm{G(O)}}{\operatorname{meas}\mathrm{T(O)}} f(z)$$

with

$$n = zn_0.$$

If $\Delta(\gamma) = q^{-\alpha}$ then $\alpha \geq 0,\ 2\alpha - \delta - 1$ is even, and γ fixes all points in $\mathfrak{X}(F')$ at a distance less than or equal to 2α from $\mathfrak{A}_T(F')$. If $j - \delta - 1$ is even and non-negative there are

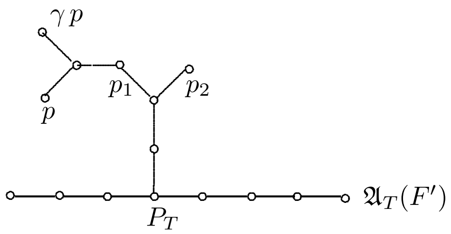
$$2q^{\frac{j-\delta-1}{2}}$$

points in $\mathfrak{X}$ whose distance from $\mathfrak{A}_T(F')$ is j. Certainly $F_{f_\lambda}(\gamma)$ is 0 unless $\lambda = (m+r, m-r),\ r \geq 0$. If $r > 0$ it is equal to

$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)}\, q^{r+\frac{-\delta-1}{2}}$$

and if $r = 0$ it equals

$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} \sum_{j=0}^{\frac{2\alpha-\delta-1}{2}} q^j = \frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} \; \frac{q^{\frac{-\delta+1}{2}} - q^{-\alpha}}{q-1}.$$



The factor 2 disappears because the orbits under $T(F)$ are twice as large as the orbits under $T(O)$. The lemma follows upon comparison with the proof of Lemma 5.5.

If

$$g = t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k$$

with t in $A(F)$ and k in $G(O)$, we set $\lambda(g) = 1$ if $x \in O$ and $\lambda(g) = |x|^{-2}$ otherwise. Then $\ell n\lambda(g)$ is $2\ell n|\varpi|$ times the distance of gp_0 from $\mathfrak{A}$. If $\Delta(\gamma) \neq 0$ set

$$A_1(\gamma, f) = \Delta(\gamma) \int_{A(F)\backslash G(F)} f(g^{-1}\gamma g)\ell n\lambda(g)dg.$$

If

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

lies in $A(\mathbf{C})$ and f belongs to $\mathcal{H}$ we write

$$f^\vee(t) = \sum_{j',j} a_f(j', j) t_1^{j'} t_2^{j}.$$

Lemma 5.9 *Let*

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and let $\lambda(\gamma) = (m', m)$ *or* (m, m'), $m' \geq m$. *If* $m' > m$ *then*

$$A_1(\gamma, f) = \frac{\ell n|\varpi|}{\operatorname{meas} A(O)} \sum_{\substack{j'+j=m'+m \\ |j'-j|>m'-m}} \left(1 - \frac{1}{q^s}\right) a_f(j', j)$$

with $2s = |j'-j| - (m'-m)$. *If* $m' = m$ *then* $A_1(\gamma, f)$ *is equal to the sum of three terms:*

$$\frac{\ell n|\varpi|}{\operatorname{meas} A(O)} \sum_{\substack{j'+j=m'+m \\ |j'-j|>0}} \left(1 - \frac{1}{q^s}\right) a_f(j', j)$$

and

$$\frac{\left(1 - \frac{1}{q}\right)\ell n\Delta(\gamma)}{\operatorname{meas} A(O)} \left\{2a_f(m', m) + \sum_{\substack{j'+j=m'+m \\ |j'-j|>0}} \frac{1}{q^{s+1}} a_f(j', j)\right\}$$

and, if $\Delta(\gamma) = q^{-\alpha}$ *and* $z \in F^{\times}$ *satisfies* $|z| = |\varpi|^m$*,*

$$2\ell n|\varpi|\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\sum\nolimits_{j=0}^{\alpha-1} j q^{j-2}\left(1-\frac{1}{q}\right) f(z).$$

It is enough to verify these formulae for $f = f_\lambda$. Suppose first that $m' > m$. The integral appearing in the definition of $A_1(\gamma, f)$ is equal to $\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}$ times the sum over all p' for which $\tau(\gamma p', p') = \lambda$ and $d(p') = \operatorname{dist}(p, p_0) = \operatorname{dist}(p, \mathfrak{A})$ of $2\ell n|\varpi| \operatorname{dist}(p, \mathfrak{A})$.

The sum is empty unless $\lambda = (m' + r, m - r)$, $r \geq 0$. However if this condition is satisfied it equals

$$2r\left(1-\frac{1}{q}\right) q^r \ell n|\omega|.$$

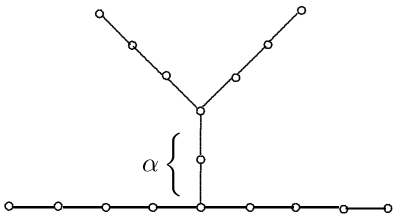
The sum appearing in the formula claimed for $A_1(\gamma, f)$ is also 0 unless λ has this form when, by the explicit expansion of $f^\vee(t)$, it equals

$$2\left\{\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)+\cdots+\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{r-2}}\right)+\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{r-1}}\right)\right\} q^r \operatorname{meas} G(O)$$

which is easily shown by induction to be

$$2rq^r \operatorname{meas} G(O).$$

If $m' = m$ we base our calculation on the diagram



$A_1(\gamma, f_\lambda)$ is certainly 0 unless $\lambda = (m + r, m - r)$, $r \geq 0$. If this condition is satisfied it equals

$$2\ell n|\varpi|(r+\alpha)q^r\left(1-\frac{1}{q}\right)\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}, \qquad r > 0,$$

or

$$2q^{-\alpha}\ell n|\varpi|\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\sum\nolimits_{j=0}^{\alpha} j q^j\left(1-\frac{1}{q}\right), \qquad r = 0.$$

The contribution

$$2\ell n|\varpi|\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\, rq^r\left(1-\frac{1}{q}\right)$$

is accounted for by the first of the three summands in the lemma. The contribution

$$2\ell n|\varpi|\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\alpha q^r\,\left(1-\frac{1}{q}\right)$$

by the second, and the remainder, which is 0 for $r>0$ and

$$2\ell n|\varpi|\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\sum_{j=0}^{\alpha-1} jq^{j-\alpha}\,\left(1-\frac{1}{q}\right)$$

for $r=0$, by the third.

The purpose of this paragraph is not simply to consider the algebra $\mathcal{H}$ by itself, but rather to compare it with the algebra $\mathcal{H}_E$ of spherical functions on $G(E)$, where E is an unramified extension of F of degree ℓ. The comparison can be motivated by the point of view exposed in [20].

We have already seen that $\mathcal{H}$ is isomorphic to the representation ring of $GL(2,\mathbf{C})$. With $\mathfrak{G}=\mathfrak{G}(E/F)$ we form the direct product

$$^LG = GL(2,\mathbf{C})\times\mathfrak{G}$$

which is the L-group of G. Let Φ be the Frobenius element in $\mathfrak{G}$. The representation ring of $GL(2,\mathbf{C})$ is isomorphic, by means of the map $g\to g\times\Phi$ from $GL(2,\mathbf{C})$ to $GL(2,\mathbf{C})\times\Phi\subseteq{}^LG$, to the algebra $\mathfrak{H}$ obtained by restricting to $GL(2,\mathbf{C})\times\Phi$ the representation ring of ^LG, which is the algebra of functions on ^LG formed by linear combinations of characters of finite-dimensional complex analytic representations of ^LG. It is the isomorphism of $\mathcal{H}$ with $\mathfrak{H}$ which is now important.

We may regard $G(E)$ as $G_E(F)$ where G_E is obtained from G by restriction of scalars. Its L-group is formed by setting

$$^LG_E^o = \prod\nolimits_{\mathfrak{G}} GL(2,\mathbf{C}),$$

on which we let G act by right translations on the coordinates, and then taking the semi-direct product

$$^LG_E = {}^LG_E^o\times\mathfrak{G}.$$

For simplicity index the coordinate $g\in{}^LG_E^o$ corresponding to Φ^j by j. Then

$$(h_1,\cdots,h_\ell)^{-1}\cdot(g_1,\cdots,g_\ell)\times\Phi\cdot(h_1,\cdots,h_\ell)$$

is equal to

$$(h_1^{-1}g_1h_2,h_2^{-1}g_2h_3,\cdots,h_\ell^{-1}g_\ell h_1)\times\Phi.$$

Taking $h_2=g_2h_3, h_3=g_3h_4,\cdots,h_\ell=g_1h_1$, and $h_1=h$ we obtain

$$(h^{-1}g_1g_2\cdots g_\ell,h,1,\cdots,1)\times\Phi.$$

Thus conjugacy classes in ${}^LG_E$ which project to Φ stand in a bijective correspondence with conjugacy classes in $GL(2,\mathbf{C})$. It follows easily that $\mathcal{H}_E$ is isomorphic to the algebra of functions $\mathfrak{H}_E$ obtained by restricting the representation ring of ${}^LG_E$ to ${}^LG_E^o \times \Phi$.

The map of ${}^LG$ to ${}^LG_E$ given by

$$g \times \tau \to (g, \cdots, g) \times \tau$$

yields a homomorphism $\mathfrak{H}_E \to \mathfrak{H}$ and hence a homomorphism $\mathcal{H}_E \to \mathcal{H}$. It is this homomorphism which must be studied. If ϕ in $\mathcal{H}_E$ has Fourier transform $\phi^\vee$, then maps to f, which is defined by

$$f^\vee(t) = \phi^\vee(t^\ell).$$

Fix $\sigma \in \mathfrak{G},\ \sigma \neq 1$. We have observed that if $\gamma \in G(F),\ \ \delta \in G(E)$, and $\gamma = N\delta$ then $G_\delta^\sigma(E)$ equals $G_\gamma^\sigma(F)$, where G_γ^σ is a twisted form of G_γ. We may therefore use the convention of [14] to transport Tamagawa measures from $G_\gamma(F)$ to $G_\gamma^\sigma(E)$.

Lemma 5.10 *Suppose ϕ in $\mathcal{H}_E$ maps to f in $\mathcal{H}$. If $\gamma = N\delta$ then*

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg = \xi(\gamma) \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg.$$

Here $\xi(\gamma)$ is 1 unless γ is central and δ is not σ-conjugate to a central element when it is -1. Moreover if γ in $G(F)$ is the norm of no element in $G(E)$ then

$$\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg = 0.$$

We check this when $\phi = \phi_\lambda$, the characteristic function of the double coset $G(O_E)tG(O_E) = K_E t K_E$, where $\lambda(t) = \lambda$. $\mathfrak{X}(E)$ and $\mathfrak{X}'(E)$ are the Bruhat–Tits buildings over E. To prove the lemma we are unfortunately, but probably inevitably, reduced to considering cases. Suppose first that δ is a scalar so that $G_\delta^\sigma(E) = G(F)$.

Then

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg$$

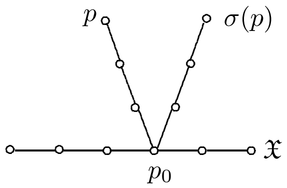
is equal to the sum over representatives $p' = gp_0'$ of the orbits of $G(F)$ in $\mathfrak{X}(E)$ for which the type of the pair $(\delta\sigma(p'), p')$ is λ of

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} G(F) \cap gG(O_E)g^{-1}}.$$

We choose representatives p' so that $d(p') = \operatorname{dist}(p, p_0)$ and so that $\operatorname{dist}(p, p_0) = \operatorname{dist}(p, \mathfrak{X})$. The reduction used repeatedly before shows that the integral is equal to

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} G(O)} \sum_{\substack{d(p')=\operatorname{dist}(p,p_0)=\operatorname{dist}(p,\mathfrak{X}) \\ \tau(\delta\sigma(p'),p')=\lambda}} 1.$$

If $\lambda(\delta) = (m, m)$, this is 0 unless $\lambda = (m+r, m-r),\ r \geq 0$. Since $\delta p = p$, the type $\tau(\delta\sigma(p'), p')$ is $\lambda = (m+r, m-r)$ if and only if $\operatorname{dist}(\sigma(p), p) = 2r$.



Since $\mathfrak{X}$ is the set of fixed points of σ in $\mathfrak{X}(E)$, the paths from p_0 to p and from p_0 to $\sigma(p)$ must start off in different directions. In other words the initial edge of the path from p_0 to p does not lie in X. This shows that there are

$$q^{\ell r}(1 - q^{1-\ell})$$

possibilities for the p' or, what is the same, the p occurring in the above sum if $r > 0$ and just 1 if $r = 0$.

To complete the verification in this case we have to evaluate

$$\int_{A_0(\mathbf{C})} \phi_\lambda(t^\ell)\overline{\varphi}_\gamma(t)$$

with φ_γ defined as in Lemma 5.3. Since $\lambda(\gamma) = \ell\lambda(\delta)$, the integral is certainly 0 unless $\lambda = (m+r, m-r)$. If this condition is satisfied it equals

$$\frac{m_E(\lambda)}{2 \operatorname{meas} G(O)}\ \frac{1+q^{-1}}{1+q^{-\ell}}\, q^{-r\ell}$$

times

$$\frac{1}{2\pi i}\int\limits_{|z|=1} \left\{ \frac{1-q^{-\ell}z^{-\ell}}{1-z^{-\ell}} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z}\, z^{\ell r} + \frac{1-q^{-\ell}z^{\ell}}{1-z^{\ell}} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z}\, z^{-\ell r} \right\} \frac{dz}{z}.$$

We have to show that this integral is

$$2\,\frac{1-q^{-\ell}}{1+q^{-1}}$$

if $r > 0$ and

$$2\,\frac{1+q^{-\ell}}{1+q^{-1}}$$

if $r = 0$.

Once we shrink the circle of integration a little, we may integrate term by term. The first term will have only one pole inside the new circle, that at 0, where the residue is 0 if $r > 0$ and $q^{1-\ell}$ if $r = 0$. The second term we write as

$$\frac{1}{1-z^\ell} \cdot \frac{1-z}{1-q^{-1}z} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}}\, \frac{1}{z} \left\{ \left(1 - \frac{z^\ell}{q^\ell}\right) z^{-\ell r} - \left(1 - \frac{1}{q^\ell}\right) \right\} + \frac{1-q^{-\ell}}{1-z^\ell} \cdot \frac{1-z}{1-q^{-1}z} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1}{z}.$$

The first summand is integratd by moving the path out. The residues are at q and ∞ and yield

$$\frac{1}{1-q^\ell}\cdot q-1\cdot\frac{1-q^{-1}}{1-q^{-2}}\,\left(1-\frac{1}{q^\ell}\right)+\begin{cases}0 & r>0\\ \frac{1}{q^{\ell-1}} & r=0\end{cases}.$$

The second is integrated by moving in; the residues are at 0 and $\frac{1}{q}$. They yield

$$1-\frac{1}{q^\ell}\cdot\frac{1}{1-q^{-\ell}}\cdot\frac{1-q^{-1}}{1-q^{-2}}\cdot\frac{q^{-1}-1}{q^{-1}}+\left(1-\frac{1}{q^\ell}\right)q.$$

If everything is put together the result follows.

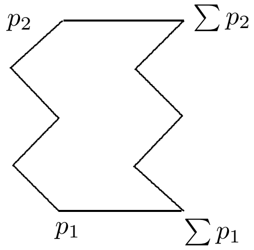
Now suppose that γ is central but δ is not σ-conjugate to a central element. Then as we observed in the previous paragraph $\ell=2$. Moreover since E is unramified

$$|\det\delta|=|\varpi|^{2m+1}$$

for some integer m. Let Σ be the map $p\to\delta\sigma(p)$ of $\mathfrak{X}(E)$ to itself. Then Σ has no fixed points, for $\Sigma: gp_0\to(\delta\sigma(g)g^{-1})gp_0$ and

$$|\det(\delta\sigma(g)g^{-1})|=|\varpi|^{2m+1}.$$

Suppose p_1 is a point for which $\mathrm{dist}(p_1,\Sigma p_1)$ is a minimum. Since Σ^2 is the identity, Σ defines an inversion of the path of shortest length joining p_1 to Σp_1. It follows immediately that $\mathrm{dist}(p_1,\Sigma p_1)=1$. I claim that if $\mathrm{dist}(p_2,\Sigma p_2)=1$ then $p_2\in\{p_1,\Sigma p_1\}$. If not take the path of shortest length joining p_2 to this set. Replacing p_2 by Σp_2 if necessary we may suppose the path runs from p_2 to p_1. Then Σ applied to the path joins Σp_2 to Σp_1, and we obtain a non-trivial cycle



This is a contradiction.

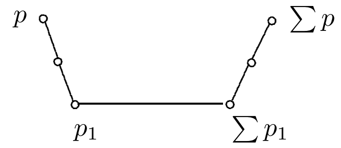
The integral

$$\int_{G_\delta^\sigma(E)\backslash G(E)}\phi(g^{-1}\delta\sigma(g))dg$$

is equal to

$$\frac{\mathrm{meas}\,G(O_E)}{\mathrm{meas}\,G_\delta^\sigma(E)\cap G_{p_1'}}\left\{\sum_{\substack{d(p')-d(p_1')=\mathrm{dist}(p,p_1)<\mathrm{dist}(p,\Sigma p_1)\\ \tau(p',\Sigma_{p'})=\lambda}}1\right\}.$$

Here p_1' is any fixed lifting of p_1 to $\mathfrak{X}(E)$. The sum is empty unless $\lambda = (m+1+r, m-r),\ r \geq 0$, when it is equal



to $q^{\ell r}$. Since $G_\delta^\sigma(E) \cap G_{p_1'}$ is a maximal compact subgroup of $G_\delta^\sigma(E)$, we can easily verify that (cf. p. 475 of [14])

$$\operatorname{meas} G_\delta^\sigma(E) \cap G_{p_1'} = \frac{\operatorname{meas} G(O)}{q-1}.$$

Since $\lambda(\gamma)$ must be $(2m+1, 2m+1)$, the integral

$$\int_{A_0(\mathbf{C})} \phi_\lambda(t^\ell)\overline{\varphi}_\gamma(t) \qquad (\ell = 2)$$

is 0 unless $\lambda = (m+1+r, m-r)$. If this condition is satisfied this inner product is

$$\frac{m_E(\lambda)}{2\operatorname{meas} G(O)} \cdot \frac{1+q^{-1}}{1+q^{-\ell}}\, q^{-(r+1/2)\ell} = \frac{\operatorname{meas} G(O_E)}{2\operatorname{meas}(G(O)} \cdot \left(1 + \frac{1}{q}\right) q^{-(r+1/2)\ell}$$

times

$$\frac{1}{2\pi i}\int_{|z|=1} \left\{\frac{1-q^{-\ell}z^{-\ell}}{1-z^{-\ell}} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z} z^{\ell r+\ell} + \frac{1-q^{-\ell}z^{\ell}}{1-z^{\ell}} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z} z^{-\ell r-\ell}\right\} \frac{dz}{z} \ (\ell = 2).$$

This integral has to be shown to equal

$$-2\frac{q-1}{q+1}.$$

This can be done much as before. Once the contour of integration is shrunk a little, the integral of the first term becomes 0. The second term is written as

$$\frac{1}{1-z^\ell} \cdot \frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z}\left\{\left(1-\frac{z^\ell}{q^\ell}\right) z^{-\ell(r+1)} - \left(1-\frac{1}{q^\ell}\right)\right\} + \frac{1-q^{-\ell}}{1-z^\ell}\frac{1-z^{-1}}{1-q^{-1}z^{-1}} \cdot \frac{1-z}{1-q^{-1}z}.$$

To integrate the first summand we move the path out. There is a residue at q which yields

$$\frac{1}{1-q^\ell} \cdot \frac{1-q^{-1}}{1-q^{-2}} \cdot q(q-1) \cdot \left(1 - \frac{1}{q^\ell}\right) = -\frac{q-1}{q+1} \qquad (\ell = 2).$$

For the second we move the path in; the residue at $\frac{1}{q}$ is

$$\left(\frac{1}{q} - 1\right) \cdot \frac{1-q^{-1}}{1-q^{-2}} = -\frac{q-1}{q+1}.$$

If γ is central but is not a norm then ℓ is odd and $\lambda(\gamma) = (m,m)$ with m prime to ℓ. It follows immediately that $\lambda(t^\ell)$ is always orthogonal to φ_γ, so that $f^\vee(\gamma) = 0$ if f is the image of ϕ.

We next suppose that

$$\gamma = an_0 = a \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\mu = \lambda(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix})0 = (m, m)$$

and γ is a norm if and only if ℓ divides m. It is clear that $< f^\vee, \mu >= 0$ if ℓ does not divide m. Suppose then $\gamma = N\delta$. We may write

$$\delta = b \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Then $G_\delta^\sigma(E) = G_\gamma(F)$. We may choose as a set of representatives for the orbits of $G_\gamma(F)$ in $\mathfrak{X}(E)$ the collection $\{p' = np'_z\}$, where $z \in \mathbf{Z}$, where p'_z is defined to be that element of $\mathfrak{A}'$ which projects to p_z in $\mathfrak{A}$ and satisfies $d(p'_z) = \mathrm{dist}(p_z, p_0)$, and where

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with x running over $E/F + \varpi^{-z} O_E$. Observe that

$$\frac{\mathrm{meas}\, G_{p'}}{\mathrm{meas}\, G_{p'} \cap G_\gamma(F)} = q^{-z}\ \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, G_\gamma(O)}.$$

Moreover $\tau(\Sigma p', p')$ is $(k + r, k - r),\ k = \frac{m}{\ell}$. with r equal to 0 if

$$\mathrm{order}(\sigma(x) - x - v) \geq -z$$

and equal to

$$-\mathrm{order}(\sigma(x) - x - v) - z$$

otherwise.

Thus

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi_\lambda(g^{-1}\delta\sigma(g))dg$$

is equal to 0 unless $\lambda = (k + r, k - r)$. Since trace$(\sigma(x) - x - v) = 1$, the order of $\sigma(x) - x - v$ is always less than or equal to 0. If we assume, as we may, that order $v = 0$, then $\mathrm{ord}(\sigma(x) - x - v)$ is $\mathrm{ord}(\sigma(x) - x)$ if this is negative and is 0 otherwise. If $\lambda = (k, k)$ then the integral equals

$$\frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, G_\gamma(O)}\ \sum\nolimits_{z=0}^{\infty} q^{-z} = \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, G_\gamma(O)} \cdot \frac{1}{1 - q^{-1}}.$$

If $\lambda = (k + r, k - r)$ with $r > 0$ then the integral is

$$\frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, G_\gamma(O)}\ \left\{ q^{(\ell-1)r} \cdot q^r + \sum\nolimits_{z=-r+1}^{\infty} q^{-z} q^{(\ell-1)r}\,(1 - q^{1-\ell}) \right\}$$

which equals

$$\frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, G_\gamma(O)}\ \frac{1 - q^{-\ell}}{1 - q^{-1}} \cdot q^{\ell r}.$$

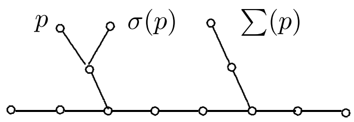
Since we can easily compute
$$\int_{A_0(\mathbf{C})} \phi_\lambda^\vee(t^\ell)\overline{\mu}(t)$$
by using the explicit expansion of $\phi_\lambda^\vee$, the required equality follows from Lemma 5.5.

We have still to treat the case that γ is regular and semi-simple. Let T be the Cartan subgroup containing γ. If $\gamma = N\delta$ then δ also belongs to T. If T is A then γ is a norm if and only if $\mu = \lambda(\gamma) = (m', m)$ with both m' and m divisible by ℓ. Since
$$\int_{A(F)\backslash G(F)} f(g^{-1}\gamma g)dg = \Delta(\gamma)^{-1}F_f(\gamma)$$
this integral is certainly 0 if m' and m are not both divisible by ℓ. However if ℓ divides m' and m and $\phi = \phi_\lambda$, with $\lambda = (k', k)$, the integral equals
$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\Delta(\gamma)^{-1}q^{\frac{\ell}{2}(k'-k)} \quad m' = \ell k', m = \ell k$$
and
$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)}\,\Delta(\gamma)^{-1}q^{\frac{\ell}{2}(k'-k)}(1-q^{-\ell}) \quad m' = \ell k' - \ell r \geq m = \ell k + \ell r, r > 0$$
but 0 otherwise.

The integral

$$\int_{A(F)\backslash G(E)} \phi_\lambda(g^{-1}\delta\sigma(g))dg \tag{5.2}$$

is equal to
$$\frac{\operatorname{meas} G(O)}{\operatorname{meas} A(O)} \sum_{\substack{d(p')=\mathrm{dist}(p,p_0)=d(p,\mathfrak{A})\\ \tau(\Sigma p',p')=\lambda}} 1.$$
If $m' \neq m$ then $\Delta(\gamma) = q^{\frac{m'-m}{2}}$ and the relevant diagram is



If $d(p, p_0) = r$ then the type of $(\Sigma p', p')$ is $(m' + r, m - r)$. For a given r, there are

$$q^{\ell r}(1-q^{-\ell}) \qquad r > 0$$

or

$$1 \qquad r = 0$$

possibilities for p or p'. The equality follows.

Suppose $m' = m$ and $\Delta(\gamma) = q^{-\alpha}$, $\alpha \geq 0$. Then γ fixes all points in $\mathfrak{X}(E)$ which are at a distance at most α from $\mathfrak{A}$, but no other points. We so choose δ that $N\delta$ lies in $G(F)$ and take $\gamma = N\delta$. If, as usual, $\Sigma : p \to \delta\sigma(p)$ then $\Sigma^\ell p = \gamma p$ so that Σ fixes p only if γ does. Suppose Σ fixes p and $\mathrm{dist}(p, \mathfrak{A}) < \alpha$. Then γ fixes all points which can be joined by p by an edge. If $p = gp_0$ these are the points $gkp_1, k \in K_E = G(O_E)$, p_1 being one of the points in the apartment $\mathfrak{A}$ adjacent to p_0. Thus $g^{-1}\gamma g \in G(O_E)$ and has trivial image in $G(\kappa_E)$, if κ_E is the residue field of O_E. Moreover

$$p_0 = g^{-1}p = g^{-1}\Sigma p = g^{-1}\delta\sigma(g)p_0;$$

so $g^{-1}\delta\sigma(g) \in K_E$. Since

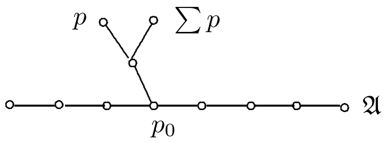
$$(g^{-1}\delta\sigma(g))\sigma(g^{-1}\delta\sigma(g)) \cdots \sigma^{\ell-1}(g^{-1}\delta\sigma(g)) = g^{-1}\gamma g,$$

we conclude that $g^{-1}\delta\sigma(g)$ defines a cocycle of $\mathfrak{G}$ in $G(\kappa_E)$. But all such cocycles are trivial; so we may suppose, upon replacing g by gk, that the image of $g^{-1}\delta\sigma(g)$ in $G(\kappa_E)$ is 1. Then

$$\Sigma(gkp_1) = g(g^{-1}\delta\sigma(g)\sigma(k))p_1 = g\sigma(k)p_1.$$

This is equal to gkp_1 if and only if $k^{-1}\sigma(k)p_1 = p_1$. It follows that the number of points in $\mathfrak{X}(E)$ which can be joined to p by an edge and are fixed by Σ is the same as the number of points in $\mathfrak{X}$ which can be joined to p_0 by an edge, namely $q + 1$.

The relevant diagram is now



The integral is certainly 0 unless $\lambda = \left(\frac{m}{\ell} + r, \frac{m}{\ell} - r\right)$, $r \geq 0$. If λ has this form and $r > 0$ the value of the integral is

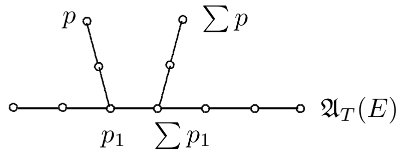
$$\frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)} \left\{ q^{\alpha-1}q^{\ell r}(1 - q^{1-\ell}) + q^\alpha \left(1 - \frac{1}{q}\right) q^{\ell r} \right\} = \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}\, q^{\alpha+\ell r}(1 - q^{-\ell r})$$

and if $r = 0$ it is

$$\frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}\, q^\alpha.$$

This again yields the correct result.

We suppose next that T is not split over F but that it splits over E. Then $\ell = 2$ and the equation $\gamma = N\delta$ can always be solved. If the eigenvalues of δ are a, b those of γ are $a\sigma(b)$, $b\sigma(a)$. Σ has exactly one fixed point in $\mathfrak{A}_T(E)$ and this point is a vertex or not according as the order of the eigenvalues of γ is even or odd. If it is odd, say $2m + 1$, then the diagram to be used is



The integral (5.2) (with A replaced by T), is 0, unless $\lambda = (m+1+r,\ m-r),\ r \geq 0$ when it is

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\, 2q^{\ell r} \tag{5.3}$$

for the only forbidden initial direction for the path from p_1 to p is the edge joining p_1 and Σp_1. Define $z \in F^\times = Z(F)$ by $|z| = |a\sigma(b)| = |b\sigma(a)|$ the set

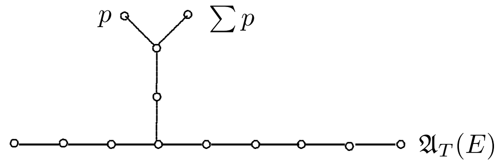
$$n = zn_0.$$

If we appeal, as we shall now constantly have occasion to do, to the calculations made for γ a scalar or a scalar times a unipotent we see that (5.3) equals $\Delta(\gamma)^{-1}$ times

$$\left(1+\frac{1}{q}\right)\frac{\operatorname{meas} G_n(O)}{\operatorname{meas} T(O)}\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg - \frac{2\Delta(\gamma)}{q-1}\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} f(z)$$

if f is the image of ϕ_λ. Observe in particular that the integral appearing here is 0 because the order of z is odd. In any case the desired equality follows from Lemma 5.6.

If the order of the eigenvalues is even, say $2m$, then the diagram to be brought into play is:



Thus (5.2) is 0 unless $\lambda = (m+r, m-r),\ r \geq 0$. If $\Delta(\gamma) = q^{-\alpha}$ and λ is of this form then it equals

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\left\{(1-q^{1-\ell})q^{\ell r}\left(1+(q+1)\sum\nolimits_{j=0}^{\alpha-1} q^j\right) + q^{\ell r}(q+1)q^{\alpha-\ell}\right\}$$

or

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\left\{q^{\alpha+\ell r}\cdot\frac{q+1}{q-1}\cdot(1-q^{-\ell}) - \frac{2q^{\ell r}(1-q^{1-\ell})}{q-1}\right\} \tag{5.5}$$

if $r > 0$, and

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\left\{1+(q+1)\sum\nolimits_{j=0}^{\alpha-1} q^j\right\} = \frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\left\{\frac{-2}{q-1} + q^\alpha\cdot\frac{q+1}{q-1}\right\} \tag{5.6}$$

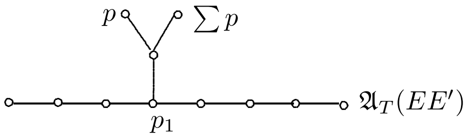
if $r = 0$. Our previous calculations show once again that this is equal to (5.4), so that we have only to appeal to Lemma 5.6.

Suppose that T does not split over E but that it does split over an unramified extension. Then ℓ is odd. If the order of the eigenvalues of γ is m then γ is a norm if and only if ℓ divides m. It is clear from Lemma 5.6 and the cases previously discussed that

$$\int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)dg = 0 \tag{5.7}$$

when f is the image of ϕ_λ, if ℓ does not divide m.

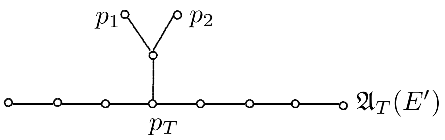
Suppose ℓ divides m. Let E' be the quadratic extension over which T splits and let $\Delta(\gamma) = q^{-\alpha}$. There is one point, denoted p_1, in $\mathfrak{A}_T(EE') \cap \mathfrak{X}(E)$



We can analyze the fixed points in Σ in $\mathfrak{X}(E)$ and evaluate (5.2) as before. It is 0 unless $\lambda = \left(\frac{m}{\ell} + r,\ \frac{m}{\ell} - r\right),\ r \geq 0$, when it is given by (5.5) and (5.6).

It remains to treat the case that T splits over a ramified quadratic extension E'. We shall appeal to Lemmas 5.7 and 5.8 as well as to some of our previous calculations. We know that γ is a norm if and only if $\det\gamma \in N_{E/F}E^\times$, that is, if and only if the order of $\det\gamma$ is divisible by ℓ.

The apartments $\mathfrak{A}_T(E')$ and $\mathfrak{A}_T(EE')$ are the same. Since this apartment is fixed by $\mathfrak{G}(EE'/F)$, the vertices in $\mathfrak{X}(E)$ closest to it lie in $\mathfrak{X}$. Let them be p_1, p_2 as before



We have all the information needed to calculate (5.2) (with T replacing A) at our disposal. If the order of $\det\gamma$ is odd, say $2m+1$, then Σ interchanges p_1 and p_2, and (5.2) is 0 unless $\lambda = (m+1+r, m-r),\ r \geq 0$, when it is

$$\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)} q^{\ell r}.$$

If $\ell = 2$ this is

$$\frac{-1}{q-1}\,\frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} f(z)$$

if $z \in F^\times$ and order $z = 2m+1$. Since the order of z is odd

$$\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg = 0$$

for $n = zn_0$; so we may conclude by an appeal to Lemma 5.8. If ℓ is odd we have to appeal to Lemma 5.7. This forces us to evaluate

$$\frac{< f^{\vee}, \varphi_{\gamma} >}{\Delta(\gamma)}. \tag{5.8}$$

If the order of $\det\gamma$ is $2m'+1$ the inner product is certainly 0 unless $\lambda = (k', k)$ with $\ell(k'+k) = 2m'+1$, and this implies in particular that it is always 0 unless ℓ divides $2m'+1$. If ℓ divides $2m'+1$, so that we can solve $\gamma = N\delta$, then $\ell(2m+1) = 2m'+1$ and $m' = \ell m + \frac{(\ell-1)}{2}$. When these necessary relations between k', k and m' are satisfied the expression (5.8) is equal to

$$\frac{q^{\frac{\ell r+\ell-1}{2}}}{2}\ \frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}$$

times

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\left\{\frac{1-q^{-\ell}z^{-\ell}}{1-z^{-\ell}}\ \frac{z^{\ell r+\frac{\ell-1}{2}}}{1-q^{-1}z^{-1}}+\frac{1-q^{\ell_z-\ell}}{1-z^{-\ell}}\ \frac{z^{\ell r+\frac{\ell+1}{2}}}{1-q^{-1}z}+\frac{1-q^{-\ell_z\ell}}{1-z^{\ell}}\ \frac{z^{-\ell r-\frac{\ell+1}{2}}}{1-q^{-1}z^{-1}}+\frac{1-q^{-\ell_z\ell}}{1-z^{\ell}}\cdot\frac{z^{-\ell r-\frac{(\ell-1)}{2}}}{1-q^{-1}z}\right\}\frac{dz}{z}.$$

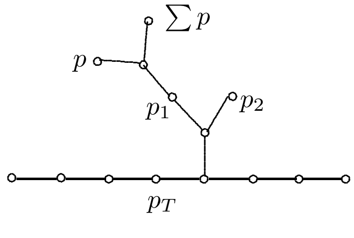
We again shrink the contour a little and then integrate term by term. The first two integrals are 0. For the last two we push the contours out to infinity. The only residues are at $z^{\ell} = 1$ and they are independent of $r \geq 0$. We may therefore evaluate the last two integrals by setting $r = 0$ and shrinking the contour to 0. The third integral then has residues at $\frac{1}{q}$ and 0, which yield altogether

$$\frac{1-q^{-2\ell}}{1-q^{-\ell}}\,q^{\frac{\ell+1}{2}} - q^{\frac{\ell+1}{2}} = q^{-\frac{\ell-1}{2}}.$$

The residue at 0 is easy to calculate because $\ell - \left(\frac{\ell+1}{2}\right) = \frac{\ell-1}{2}$ is positive, so that it is the same as the residue of

$$\frac{z^{-\frac{\ell+1}{2}}}{z-q^{-1}}.$$

The fourth has a residue only at 0 and there it is $q^{-\left(\frac{\ell-1}{2}\right)}$. The desired equality follows.



If the order of $\det\gamma$ is even, $2m'$, but not divisible by ℓ then Lemma 5.8 together with some of the previous calculations show that (5.7) is 0. Suppose $\gamma = N\delta$ and order $(\det\delta) = 2m$. Then Σ fixes p_1 and p_2 and the integral (5.2) is 0 unless $\lambda = (m+r, m-r),\ r \geq 0$. Let $\Delta(\gamma) = q^{-\frac{\delta-1}{2}-\alpha}$. Here α is necessarily integral. If $\lambda = (m+r, m-r)$ with $r > 0$ the integral (5.2) equals $\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}\cdot$ times

$$q^{\ell r}\left(1-\frac{1}{q^{\ell-1}}\right)\ \left(\sum\nolimits_{j=0}^{\alpha=1} q^j\right) + q^{\ell r}\cdot q^{\alpha} = q^{\ell r}\cdot q^{\alpha}\cdot \frac{1-q^{-\ell}}{1-q^{-1}} - \frac{q^{\ell r}}{q-1}\cdot(1-q^{1-\ell}).$$

If $r = 0$ it equals $\frac{\operatorname{meas} G(O_E)}{\operatorname{meas} T(O)}$ times

$$\sum\nolimits_{j=0}^{\alpha} q^j = \frac{q^{\alpha}}{1-q^{-1}} - \frac{1}{q-1}.$$

Our previous calculations show that these expressions equal

$$\frac{q^{\frac{-\delta-1}{2}}}{\Delta(\gamma)}\ \frac{\operatorname{meas} G_n(O)}{\operatorname{meas} T(O)}\ \int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg - \frac{1}{q-1}\ \frac{\operatorname{meas} G(O)}{\operatorname{meas} T(O)} f(z)$$

if $z \in F^{\times}$ and order $z = m'$. We have now merely to appeal to Lemma 5.8.

Lemma 5.10 is now completely proved but the tedious sequence of calculations is not quite finished. There is one more lemma to be proved, but its proof will be briefer.

The function $\lambda(g)$ was defined in the preamble to Lemma 5.9. If $\delta \in A(E)$ and $G^{\sigma}_{\delta}(E) \subseteq A(E)$ we set

$$A_1(\delta,\phi) = \Delta(\gamma)\ \int_{A(F)\backslash G(E)} \phi(g^{-1}\delta\sigma(g)\lambda(g))dg$$

with $\gamma = N\delta$.

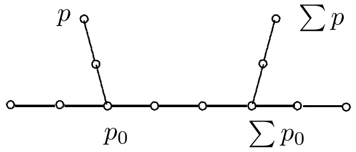
Lemma 5.11 *Suppose ϕ maps to f. Then*

$$\ell A_1(\gamma, f) = A_1(\delta,\phi).$$

Let

$$\delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and let $|a| = |\varpi|^{m'},\ |b| = |\varpi|^{m}$. Suppose first that $m' > m$. The relevant diagram is



If $\phi = \phi_\lambda$ then $A_1(\delta,\phi)$ is 0 unless $\lambda = (m'+r, m-r),\ r \geq 0$, when it is

$$2r\Delta(\gamma)\ell n|\varpi|^{\ell}\ \frac{\operatorname{meas} G(O_E)}{\operatorname{meas} A(O)} q^{\ell r}\ \left(1-\frac{1}{q^{\ell}}\right).$$

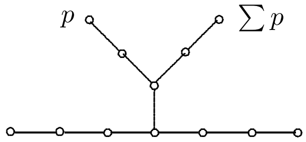
$A_1(\gamma, f)$ may be computed by combining the formula of Lemma 5.9 with the explicit expansion of $\phi^{\vee}_{\lambda}$. This yields

$$2\Delta(\gamma)\ell n|\varpi|\ \frac{\operatorname{meas} G(O_E)}{\operatorname{meas} A(O)} q^{\ell r}\ \sum\nolimits_{\substack{j+j'=m'+m \\ j'-j>m'-m}} \left(1-\frac{1}{q^{\ell s}}\right)\ \overline{a}_{\phi}(j,j')$$

if $\mathrm{meas}\, G(O_E)\overline{a}_\phi(j,j') = a_\phi(j,j')$, for
$$\Delta(\gamma) = q^{\ell\left(\frac{m'-m}{2}\right)}.$$

The above sum is
$$\left(1-\frac{1}{q^\ell}\right)\ \left(1-\frac{1}{q^\ell}\right) + \cdots + \left(1-\frac{1}{q^\ell}\right)\ \left(1-\frac{1}{q^{\ell(r-1)}}\right) + \left(1-\frac{1}{q^{\ell r}}\right) = r\left(1-\frac{1}{q^\ell}\right)$$
as required.

Now take $m' = m$. Let $\Delta(\gamma) = q^{-a}$



$A_1(\delta,\phi_\lambda)$ is 0 unless $\lambda = (m+r, m-r),\ r \geq 0$, when it is

$$(5.9)\quad 2q^{-\alpha}\ell n|\varpi|^\ell \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}\left\{rq^{\ell r}\left(1-\frac{1}{q^{\ell-1}}\right)+\sum_{j=1}^{\alpha-1}(j+r)q^{\ell r}\left(1-\frac{1}{q^{\ell-1}}\right) q^j\left(1-\frac{1}{q}\right)+(\alpha+r)q^{\ell r+\alpha}\left(1-\frac{1}{q}\right)\right\}$$

if $r > 0$ and

$$(5.10)\qquad 2q^{-\alpha}\ell n|\varpi|^\ell\ \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}\ \sum\nolimits_{j=0}^{\alpha} jq^j\left(1-\frac{1}{q}\right)$$

if $r = 0$. We sort this out and compare with the formula for $A_1(\gamma, f)$ given by Lemma 5.9
$$rq^{\ell r}\left\{\left(1-\frac{1}{q^{\ell-1}}\right) + \left(1-\frac{1}{q^{\ell-1}}\right)\sum\nolimits_{j=1}^{\alpha-1} q^j\left(1-\frac{1}{q}\right) + q^\alpha\left(1-\frac{1}{q}\right)\right\} = rq^{\ell r+\alpha}\left(1-\frac{1}{q^\ell}\right).$$
This yields the part corresponding to the first summand of the lemma.

The second summand of the lemma equals
$$\left(1-\frac{1}{q}\right)\ \alpha\ell n|\varpi|\ \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}$$
times
$$2q^{\ell r}\left\{\sum\nolimits_{s=0}^{r-1}\frac{1}{q^{\ell s}}\left(1-\frac{1}{q^\ell}\right)+\frac{1}{q^{\ell r}}\right\} = 2q^{\ell r}$$
and is therefore given by the term
$$\alpha q^{\ell r+\alpha}\left(1-\frac{1}{q}\right)$$
in the parentheses of (5.9) or by the last term of (5.10).

This leaves from (5.9)
$$2\ell n|\varpi|^\ell\ \frac{\mathrm{meas}\, G(O_E)}{\mathrm{meas}\, A(O)}\ q^{\ell r}\left(1-\frac{1}{q^{\ell-1}}\right)\sum\nolimits_{j=0}^{\alpha-1} jq^{j-\alpha}\left(1-\frac{1}{q}\right)$$

and from (5.10)

$$2\ell n|\varpi|^{\ell}\ \frac{\operatorname{meas} G(O_E)}{\operatorname{meas} A(O)}\ \sum\nolimits_{j=1}^{\alpha-1} jq^{j-\alpha}\ \left(1-\frac{1}{q}\right).$$

We know from the calculations made in the proof of Lemma 5.10 that these two expressions are equal to the last summand of Lemma 5.9.

We now have all the formulae for spherical functions that we need, but unfortunately for the wrong spherical functions. Suppose ξ is an unramified character of $N_{E/F}Z(E)$ and $\mathcal{H}'$ is the algebra of functions f' on $G(F)$ which are bi-invariant with respect to $G(O)$, of compact support modulo $N_{E/F}Z(E)$, and satisfy

$$f'(zg) = \xi^{-1}(z) f'(g) \qquad\qquad z \in N_{E/F}Z(E).$$

Multiplication is defined by

$$\int_{N_{E/F}Z(E)\backslash G(F)} f_1'(gh^{-1}) f_2'(h) dh.$$

The map $f \to f'$ with

$$f'(g) = \int_{N_{E/F}Z(E)} f(zg)\,\xi(z)dz$$

is a surjective homomorphism from $\mathcal{H}$ to $\mathcal{H}'$. There is a simple and obvious relation between the orbital integrals of f and f' as well as between $A_1(\gamma, f)$ and $A_1(\gamma, f')$. For example

$$A_1(\gamma, f') = \int_{N_{E/F}Z(E)} A_1(z\gamma, f)\,\xi(z)dz.$$

If ξ_E is the composite of ξ with the norm we may define $\mathcal{H}'_E$ in a similar manner. If $\phi' \in \mathcal{H}'_E$ then $\phi'(zg) = \xi_E^{-1}(z)\phi'(g)$ for $z \in Z(E)$. Moreover $\phi \to \phi'$ with

$$\phi'(g) = \int_{Z(E)} \phi(zg)\,\xi_E(z)dz.$$

There is also a commutative diagram

$$\begin{array}{ccc} \mathcal{H}' & \longrightarrow & \mathcal{H}'_E \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{H}_E \end{array}$$

If $\phi' \to f'$ then an analogue of Lemma 5.10 is valid.

$$\int_{Z(E)G_\delta^\sigma(E)\backslash G(E)} \phi'(g^{-1}\delta\sigma(g))dg = \int_{G_\gamma(F)\backslash G(F)} f'(g^{-1}\gamma g)dg.$$

To verify this we begin with

$$\int_{Z(E)G_\delta^\sigma(E)\backslash G(E)} \int_{Z(F)\backslash Z(E)} \phi(z^{-1}g^{-1}\delta\sigma(g)\sigma(z))dzdg = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg.$$

Replace δ by $\delta v,\ v \in Z(E)$ and hence γ by $\delta N v$. Both sides are then functions on $Z^{1-\sigma}(E)\backslash Z(E)$. Multiply by $\xi^{-1}(z)$ and integrate. The right side becomes

$$\int_{G_\gamma(F)\backslash G(F)} f'(g^{-1}\gamma g)dg.$$

Because of Lemma 5.10, the left side is

$$\int_{Z(E)G_\delta^\sigma(E)\backslash G(E)} \phi'(g^{-1}\delta\sigma(g))dg.$$

In order that the analogue of Lemma 5.11 be valid, we must set

$$A_1(\delta,\phi') = \Delta(\gamma) \int_{Z(E)A(F)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))\lambda(g)dg.$$

It is not difficult to see that $\mathcal{H}'$ is isomorphic to the algebra of functions on

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A(\mathbf{C}) \mid (\alpha\beta)^\ell = \xi(\varpi)^\ell \right\}$$

obtained by restriction from some $f^\vee,\ f \in \mathcal{H}$. This enables us to speak of $(f')^\vee$. Every homomorphism $\mathcal{H}' \to \mathbf{C}$ is of the form

$$f' \to (f')^\vee(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}) \qquad\qquad (\alpha\beta)^\ell = \xi(\varpi^\ell).$$

We may also speak of $(\phi')^\vee$.

6. ORBITAL INTEGRALS

The study of orbital integrals was initialed by Harish-Chandra in his papers on harmonic analysis on semi-simple Lie groups; the same integrals on p-adic groups were afterwards studied by Shalika. Some basic questions remain, however, unanswered. If they had been answered, much of this paragraph, which provides the information about orbital integrals, and twisted orbital integrals, to be used later, would be superfluous. But they are not and stop-gaps must be provided. No elegance will be attempted here; I shall simply knock together proofs out of the material nearest at hand.

Let F be a local field of characteristic 0. If f is a smooth function with compact support on $G(F)$, T is a Cartan subgroup of G over F, and γ is a regular element in $T(F)$ then we set

$$\Phi_f(\gamma, T) = \int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)dg.$$

The integral depends on the choice of measures on $G(F)$ and $T(F)$, measures which we always take to be defined by invariant forms ω_T and ω_G. When it is useful to be explicit we write $\Phi_f(\gamma, T; \omega_T, \omega_G)$. Since they complicate the formulae we do not use the local Tamagawa measures associated to forms ω as on p. 70 of [23] but simply the measures $|\omega|$, which could be termed the unnormalized Tamagawa measures.

It was observed on p. 77 of [23] that the map $\gamma \to \mathrm{Ch}(\gamma) = (\mathrm{trace}\,\gamma, \det\gamma)$ of G to the affine plane X is smooth except at the scalar matrices. If $a \in X$ is given then a two form μ on X which is regular and does not vanish in some neighborhood of a may be used to define an invariant form μ' on $G_\gamma\backslash G$ if γ is regular and $\mathrm{Ch}(\gamma)$ is close to a. Set

$$\Phi_f(\gamma, \mu) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)|d\mu'|.$$

If $\gamma \in T$ is regular there is a form $\omega_T(\mu)$ such that

$$\Phi_f(\gamma, \mu) = \Phi_f(T, \gamma; \omega_T(\mu), \omega_G).$$

$\omega_T(\mu)$ depends on γ.

We shall call a function $\gamma a, T \to \Phi(\gamma, T) = \Phi(\gamma, T; \omega_T, \omega_G)$ an HCS family if it satisfies the following conditions.

(i) If $\omega_T' = \alpha\omega_T$ and $\omega_G' = \beta\omega_G$ with $\alpha, \beta \in F^\times$ then

$$\Phi(\gamma, T; \omega_T', \omega_G') = \left|\frac{\beta}{\alpha}\right|\Phi(\gamma, T; \omega_T, \omega_G).$$

(ii) If $h \in G(F),\ T' = h^{-1}Th,\ \gamma' = h^{-1}\gamma h$, and if $\omega_{T'}$ is obtained from ω_T by transport of structure then

$$\Phi(\gamma', T'; \omega_{T'}, \omega_G) = \Phi(\gamma, T; \omega_T, \omega_G).$$

(iii) For each $T, \gamma \to \Phi(\gamma, T)$ is a smooth function on the set of regular elements in $T(F)$ and its support is relatively compact in $T(F)$.

(iv) Suppose $z \in Z(F)$ and $a = \mathrm{Ch}(z)$. Suppose μ is a two-form on X which is regular and non-zero in a neighborhood of a. There is a neighborhood U of a and for each T two smooth functions $\Phi'(\gamma, T; \mu)$ and $\Phi''(\gamma, T; \mu)$ on

$$T_U(F) = \{\gamma \in T(F) | \mathrm{Ch}(\gamma) \in U\}$$

such that

$$\Phi(\gamma, T; \omega_T(\mu), \omega_G) = \Phi'(\gamma, T, \mu) - \mathrm{meas}(T(F)\backslash G'(F))\Phi''(\gamma, T; \mu).$$

Here G' is the multiplicative group of the quaternion algebra over F. In the exceptional case that T is split, when G' may not exist, the function $\Phi''(\gamma, T; \mu)$ is not defined and we take

$$\mathrm{meas}(T(F)\backslash G'(F)) = 0.$$

Otherwise we regard T as a subgroup of G'. The measure on T is to be $|\omega_T(\mu)|$ and that on G' is given by the conventions on pp. 475–478 of [14]. If F is archimedean, X belongs to the center of the universal enveloping algebra and X_T is its image under the canonical isomorphism of Harish-Chandra [25] then the restriction of $X_T \Phi'(\gamma, T, \mu)$ to $Z(F)$ must be independent of T.

Lemma 6.1. *The collection $\{\Phi(\gamma, T\}$ is an* HCS *family if and only if there is a smooth function f with compact support such that*

$$\Phi(\gamma, T) = \Phi_f(\gamma, T)$$

for all T and γ. Then for $z \in F^\times = Z(F)$

$$\Phi'(z, T, \mu) = \Phi_f(n, \mu)$$

with

$$n = z \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and, if T is not split,

$$\Phi''(z, T, \mu) = f(z).$$

If F is archimedean, X belongs to the center of the universal enveloping algebra of the Lie algebra of T and X_T is its image under the canonical isomorphism of Harish-Chandra then

$$X_T \Phi'(z, T; \mu) = \Phi_{Xf}(n, \mu)$$

and

$$X_T \Phi''(z, T; \mu) = Xf(z).$$

If F is non-archimedean this is simply Lemma 6.2 of [23]. I observe however that in the formula for $a^T(\gamma)$ on p. 81 of [23] the function $\xi(z)$ should be replaced by

$$\frac{\xi(z)}{|\det\gamma|_p^{1/2}}.$$

In addition the discussion there is complicated by an infelicitous choice of measures.

That the family $\{\Phi_f(\gamma, T)\}$ satisfies conditions (i)–(iv) when F is archimedean is also well known but condition (iv) is usually formulated somewhat differently when T is not split. To reduce (iv) to the form usual for a T which is not split we remark first that if it is valid for one choice of μ then it is valid for all. Choose μ to be the standard translation invariant form $dx_1 dx_2$ on X. A simple calculation shows that $\omega_T(\mu)$ is, apart perhaps from sign, the form η_γ on p. 79 of [23]. Thus if

$$T = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right\}$$

and

$$\omega_T = \frac{da\,db}{a^2+b^2}$$

then

$$\Phi(\gamma, T; \omega_T(\mu), \omega_G) = 2|\beta|\Phi(\gamma, T; \omega_T, \omega_G)$$

for

$$\gamma = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Moreover if we take measures with respect to ω_T rather than $\omega_T(\mu)$ then $\mathrm{meas}(T(F)\backslash G'(F))$ must be replaced by

$$2|\beta|\mathrm{meas}(T(F)\backslash G'(F)).$$

The measure is now a constant.

Condition (iv) says simply that for any integer $n \geq 0$

$$2|\beta|\Phi(\gamma, T; \omega'_T, \omega_G) = \sum\nolimits_{k=0}^{n-1} \varphi_k(\alpha)\cdot\beta^k + \sum\nolimits_{k=1}^{n-1} \psi_k(\alpha)|\beta|^k + O(|\beta|^n)$$

near $\beta = 0$. The coefficients are smooth functions of α. Since the left side is in any case an even function of β, this relation says simply that its derivatives of even order with respect to β are continuous and that its derivatives of odd order are continuous except for a jump at $\beta = 0$ which is continuous in α. All this is well known [31] as are the additional properties of the family $\{\Phi_f(\gamma, T)\}$.

If F is $\mathbf{C}$ and

$$T(\mathbf{C}) = \left\{\begin{pmatrix} e^{z_1+z_2} & 0 \\ 0 & e^{z_1-z_2} \end{pmatrix}\,\middle|\, z_1, z_2 \in \mathbf{C}\right\}$$

then the image of the center of the universal enveloping under the canonical isomorphism is generated by $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_1}, \frac{\partial^2}{\partial z_2^2}, \frac{\partial^2}{\partial \overline{z}_2^2}$. Moreover there is a function $c(\alpha)$ on $Z(\mathbf{C})$ such that

$$f(\alpha) = c(\alpha)\frac{\partial^2}{\partial z_2 \partial \overline{z}_2}\Phi'(\alpha, T; \mu) \quad \alpha \in Z(\mathbf{C}).$$

We must still verify that if $\{\Phi(\gamma, T)\}$ is an HCS family then there is a smooth function f with compact support such that $\{\Phi(\gamma, T)\} = \{\Phi_f(\gamma, T)\}$. The field F may be supposed archimedean. If on each T the function $\gamma \to \Phi(\gamma, T)$ is 0 near $Z(F)$ we may proceed as in the proof of Lemma 6.2 of [23] to establish the existence of f. We must reduce the general problem to that case.

It is simpler to treat the real and complex fields separately. Suppose $F = \mathbf{R}$ and

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} | \alpha, \beta \in \mathbf{R}^\times \right\}$$
$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} | \alpha, \beta \in \mathbf{R}, \alpha^2 + \beta^2 \neq 0 \right\}.$$

If φ_A and φ_B are functions on A and B which satisfty $\varphi_A(\tilde{t}) = \varphi_A(t), \varphi_B(\tilde{t}) = \varphi_B(t)$, and $\varphi_A(z) = \varphi_B(z)$ for $z \in Z(\mathbf{R})$ there is a function ψ on X

(6.1) $$\varphi_A(t) = \psi(\mathrm{Ch}\, t) \qquad \varphi_B(t) = \psi(\mathrm{Ch}\, t).$$

If

$$t = \alpha \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}$$

then

$$\mathrm{Ch}\, t = (\alpha^2, 2\alpha \cosh u)$$

and if

$$t = \alpha \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then

$$\mathrm{Ch}\, t = (\alpha^2, 2\alpha\, \cos\theta).$$

Thus ψ is smooth on $\{(x_1, x_2) | x_1 \neq 0\}$ if and only if φ_A and φ_B are smooth and

$$\frac{d^{2n}}{d\theta^{2n}}\varphi_B(z) = (-1)^n \frac{d^{2n}}{du^{2n}}\varphi_A(z) \quad z \in Z(\mathbf{R}).$$

If

(6.2) $$\varphi_A(t) = \Phi(t, T; \omega_T(\mu), \omega_G) \quad A = T(\mathbf{R})$$

(6.3) $$\varphi_B(t) = \Phi(t, T; \omega_T(\mu), \omega_G) \quad B = T(\mathbf{R})$$

and $\{\Phi(\gamma, T)\}$ is an HCS family this is so if and only if $X_T \Phi''(\gamma, T; \mu)$ vanishes on $Z(\mathbf{R})$ for all X in the center of the universal enveloping algebra.

Since the map $t \to \operatorname{Ch} t$ is smooth away from Z a simple argument involving a partition of unity establishes that if $X_T \Phi''(\gamma, T, \mu)$ vanishes on $Z(\mathbf{R})$ for all X, so that the function ψ defined by (1) is smooth, then there is a smooth compactly supported f such that $\{\Phi(\gamma, T)\} = \{\Phi_f(\gamma, T)\}$.

This granted we argue as follows. Given an HCS family $\{\Phi(\gamma, T)\}$ there is ([29]) an f in the Schwartz space such that

$$\{\Phi(\gamma, T)\} = \{\Phi_f(\gamma, T)\}.$$

We may suppose that

$$\{x \in \mathbf{R}^\times | x = \det g \text{ for some } g \text{ with } f(g) \neq 0\}$$

is relatively compact in $\mathbf{R}^\times$. We write $f = f_1 + f_2$ where f_1 is compactly supported and f_2 vanishes near $Z(\mathbf{R})$. Replacing $\Phi(\gamma, T)$ by $\Phi(\gamma, T) - \Phi_{f_1}(\gamma, T)$, we obtain a family to which the argument above can be applied.

If F is $\mathbf{C}$ let

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} | \alpha, \beta \in \mathbf{C}^\times \right\}.$$

Then

$$\operatorname{Ch} : \alpha \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \to (\alpha^2, 2\alpha \cosh z).$$

Here $z = x + iy$ lies in $\mathbf{C}$. If φ is a smooth function on A satisfying $\varphi(\tilde{t}) = \varphi(t)$ we define ψ by

(6.4) $$\psi(\operatorname{Ch} t) = \varphi(t)$$

ψ is smooth if and only if the formal Taylor expansion of φ about $z = 0$ has the form

$$\sum\nolimits_{n=0}^{\infty} P_n(x^2 - y^2, xy; \alpha)$$

where $P_n(x^2 - y^2, xy; \alpha)$ is a polynomial of degree n in $x^2 - y^2$, xy whose coefficients are smooth functions of α. We may also write

$$P_n(x^2 - y^2, xy; \alpha) = Q_n(z^2, \overline{z}^2; \alpha).$$

It is easily seen that the expansion of y has this form if and only if

$$\frac{\partial^2}{\partial z \partial \overline{z}} \frac{\partial^{2m}}{\partial z^{2m}} \frac{\partial^{2n}}{\partial \overline{z}^{2n}} \varphi$$

vanishes on $Z(\mathbf{C})$ for every choice of m and n.

This allows us to proceed as before. We choose f in the Schwartz space of $G(\mathbf{C})$ so that $\{\Phi(\gamma, T)\} = \{\Phi_f(\gamma, T)\}$ ([29]), then write $f = f_1 + f_2$, where f_1 has compact support and f_2 vanishes near $Z(\mathbf{C})$, and replace $\Phi(\gamma, T)$ by $\Phi(\gamma, T) - \Phi_{f_1}(\gamma, T)$. If $T(\mathbf{C}) = A$ and

$$\varphi(t) = \Phi(t, T; \omega_T(\mu), \omega_G)$$

then the function ψ defined by (6.4) is smooth; so we may exploit the smoothness of $t \to \mathrm{Ch}\, t$ away from $Z(\mathbf{C})$ once again.

The purpose of this paragraph is however not the study of orbital integrals by themselves but the comparison of orbital integrals with twisted orbital integrals. Let E by a cyclic extension of prime degree ℓ and σ a fixed generator of $\mathfrak{G}(E/F)$. If ϕ is a smooth, compactly supported function in $G(E)$ and δ lies in $G(E)$ we consider

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg.$$

That these integrals converge will be manifest shortly.

It is clear that, sufficient care being taken with regard to measures, the integral depends only on $N\delta$. If $\gamma = N\delta$ lies in $G(F)$ then $G_\delta^\sigma(E) = G_\gamma^\sigma(F)$ and the principles of §15 of [14] may be used to carry measures from $G_\gamma(F)$ to $G_\delta^\sigma(E)$. Such a transfer is implicit in some of the formulae below.

We define a Shintani family $\{\Psi_\phi(\gamma, T)\}$ associated to ϕ. For this we have to fix for comparison a form ω_G^o on G over F as well as a form ω_G^E on G over E, and we define $\Psi_\phi(\gamma, T; \omega_T, \omega_G)$ at first only for this one choice. We extend the definition to other forms by Property (i) of an HCS family. If γ in $T(F)$ is regular we set

$$\Psi_\phi(\gamma, T; \omega_T, \omega_G^0) = 0$$

if $\gamma = N\delta$ has no solution. If it does we set

$$\Psi_\phi(\gamma, T; \omega_T,\ \omega_G^0) = \int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg.$$

Since $G_\delta^\sigma(E) = T(F)$, we may take the measure on it to be that defined by ω_T. The measure on $G(E)$ is that defined by ω_G^E.

If G' is the group over F obtained from G over E by restriction of scalars then $g \to \mathrm{Ch}(Ng)$ may be regarded as a morphism from G' to X over F. Indeed over $\overline{F}$

$$G' \simeq G \times \cdots \times G$$

and

$$N(g_1, \cdots, g_\ell) = (g_1 g_2 \cdots g_\ell, g_2 \cdots g_\ell g_1, \cdots, g_\ell g_1, \cdots g_{\ell-1}).$$

Hence

$$\mathrm{Ch}(Ng) = \mathrm{Ch}(g_1 \cdots g_\ell).$$

It is clear that this morphism is smooth off the locus $Ng \in Z$. Thus if $\delta \in G(E)$ and $N\delta \notin Z(F)$ we may associate to a two-form μ on X which is regular and non-zero in a neighborhood of $\mathrm{Ch}(N\delta)$ a measure on $G_\delta^\sigma(E)\backslash G(E)$ and hence

$$\Phi_\phi(\delta,\mu) = \int_{G_\delta^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta\sigma(g)).$$

We introduced earlier the form $\omega_T(\mu)$ on T. It is independent of ω_G. If φ is the restriction of Ch to T then at γ

$$\omega_T(\mu) = \frac{\varphi^*\mu}{\det((1-\mathrm{Ad}\,\gamma^{-1})_{\mathfrak{t}\backslash\mathfrak{g}})}$$

because under the map $(t,g) \to (g^{-1}tg)$ of $T \times T\backslash G$ to G the vector (X,Y) in $\mathfrak{t} \times \mathfrak{t}\backslash\mathfrak{g}$, a tangent vector at $(t,1)$, is sent to

$$X + (1-\mathrm{Ad}\,\gamma^{-1})Y.$$

If E is any finite extension of F and ψ the character on F used to define measures we may define ψ_E on E by

$$\psi_E(x) = \psi(\mathrm{trace}_{E/F}\, x).$$

If Y is a non-singular variety over E, $y_1,\cdots,y_m$ local coordinates on Y, and $a_1,\cdots,a_\ell$ a basis of E over F then we may introduce local coordinates y_{ij} on Y', the variety over F obtained from Y by restriction of scalars, by the partially symbolic equations

$$y_i = \sum\nolimits_j y_{ij}a_j.$$

If ω^E is a form of maximal degree on Y given locally by

$$\omega^E = e(y_1,\cdots,y_m)dy_1\wedge\cdots\wedge dy_m$$

we define ω' on Y' by

$$\omega' = N_{E/F}e(y_1,\cdots,y_m)(\det a_i^{\lambda_j})^m dy_{11}\wedge\cdots\wedge dy_{1\ell}\wedge dy_{21}\wedge\cdots\ .$$

Here $\{\lambda_j\}$ are the imbeddings of E into $\overline{F}$ and the norm is defined in an algebro-geometrical sense. ω' is not necessarily defined over F. But it is invariant up to sign under $\mathfrak{G}(\overline{F}/F)$ and hence the associated measure $|\omega'|$ on $Y'(F) = Y(E)$ is well defined. It is equal to that associated to ω^E.

These remarks apply in particular to our cyclic extension E and ω_G^E. The form ω_G' obtained from it, the form μ, and the morphism $\mathrm{Ch}(Ng)$ together define, for each δ in $T(E)$ with $\gamma = N\delta$ regular, a form $\omega_T'(\mu)$ on T satisfying

$$\Psi_\phi(\gamma,T,\omega_T'(\mu),\omega_G^0) = \Phi_\phi(\delta,\mu).$$

This elaborate introduction of $\omega'_T(\mu)$ is pretty much in vain because $\omega'_T(\mu)$ is independent of ω_G^E or ω'_G and equals $\omega_T(\mu)$, except perhaps for sign.

To see this begin by choosing a section of $G' \to T\backslash G'$ so that the Lie algebra $\mathfrak{g}'$ becomes $\mathfrak{t} \oplus \mathfrak{t}\backslash\mathfrak{g}'$. We may also write

$$\mathfrak{t}\backslash\mathfrak{g}' = \mathfrak{t}\backslash\mathfrak{t}' \oplus \mathfrak{m}$$

with $\mathfrak{m} \simeq \mathfrak{t}'\backslash\mathfrak{g}'$. The quotient $\mathfrak{t}\backslash\mathfrak{t}'$ may be identified with $t'^{1-\sigma}$. We write

$$\omega'_G = \omega_1 \wedge \omega_2 \wedge \omega_3$$

where ω_1 is a form on $\mathfrak{t}$, ω_2 a form on $\mathfrak{t}'^{1-\sigma}$, and ω_3 a form on $\mathfrak{m}$. If s is a section of

$$\begin{tikzcd} T' \arrow[r, "N"] & T \arrow[l, bend left] \end{tikzcd}$$

which takes γ to δ, then the map of $T \times T\backslash G'$ to G' given by $(t, g) \to g^{-1}s(t)\sigma(g)$ has the following effect on the tangent space $\mathfrak{t} \oplus \mathfrak{t}\backslash\mathfrak{g}' = \mathfrak{t} \oplus \mathfrak{t}'^{1-\sigma} \oplus \mathfrak{m}$ at $(\delta, 1)$. If $\mathfrak{m}$ is chosen to be invariant under the adjoint action of T' the vector (x, y, z) is sent to

$$(x, y, z - \mathrm{Ad}\delta^{-1}\sigma^{-1}(z)).$$

The section s is not rational, but analytic or formal, according to one's predilections.

The form $\omega_1 \wedge \omega_2$ is constructed from ω_1, ω_2 and the sequence

$$1 \longrightarrow T \longrightarrow T' \stackrel{1-\sigma}{\longrightarrow} T'^{1-\sigma} \longrightarrow 1.$$

By Lemma 4.10 we may also construct it starting from

$$1 \longrightarrow T'^{1-\sigma} \longrightarrow T' \stackrel{N}{\longrightarrow} T \longrightarrow 1.$$

Therefore pulling $\omega_1 \wedge \omega_2 \wedge \omega_3$ back to $T \times T\backslash G'$ we obtain

$$\det((1 - \mathrm{Ad}\gamma^{-1})|_{\mathfrak{t}\backslash\mathfrak{g}})\omega_1 \wedge \omega_2 \wedge \omega_3.$$

We conclude that if

$$\varphi^*(\mu) = \lambda\omega_1$$

at γ then

$$\det((1 - \mathrm{Ad}\gamma^{-1})_{\mathfrak{t}\backslash\mathfrak{g}})\omega'_T(\mu) = \lambda\omega_1$$

there as well. Our assertion follows.

These cumbersome remarks out of the way, we may state the principal lemma of the paragraph.

Lemma 6.2 *A Shintani family is an* HCS *family. An* HCS *family* $\{\Phi(\gamma,T)\}$ *is a Shintani family if and only if* $\Phi(\gamma,T)=0$ *whenever the equation* $\gamma=N\delta$ *has no solution. Moreover if* $\{\Phi(\gamma,T)\}=\{\Psi_\phi(\gamma,T)\}$ *then the function* $\Phi'(\gamma,T,\mu)$ *satisfies*

$$\Phi'(z,T,\mu)=\begin{cases}0, & z\notin NS(E),\\ \Phi_\phi(\delta,\mu), & z\begin{pmatrix}1&1\\0&1\end{pmatrix}=N\delta,\end{cases}$$

for $z\in F^\times=Z(F)$. *If* T *is not split, the value of* $\Phi''(z,T,\mu)$ *is 0 if the equation* $z=N\delta$ *has no solution. Otherwise it is*

$$\xi(\gamma)\int_{G^\sigma_\delta(E)\backslash G(E)}\phi(g^{-1}\delta\sigma(g))dg.$$

Here $\xi(\gamma)$ *is 1 if* δ *is* σ*-conjugate to a scalar and -1 if it is not.*

We begin by establishing the asserted properties of a Shintani family. They have only to be established when ϕ has small support about a given δ. If $N\delta$ is not central there is no problem for $g\to\mathrm{Ch}(Ng)$ is then smooth at δ. Suppose $N\delta$ is central.

It is convenient to treat two cases separately, that for which δ is σ-conjugate to a central element and that for which it is not. When treating the first, one may suppose that δ itself is central and then, translating if necessary, that it is 1. Choose an analytic section s of $G'(F)\to G(F)\backslash G'(F)$. The map of $G(F)\times G(F)\backslash G'(F)$ to $G'(F)$ given by $(g,w)\to s(w)^{-1}g\sigma(s(w))$ yields an analytic isomorphism in a neighborhood of the identity. If ϕ has support in such a neighborhood and δ lies in its intersection with $G(F)$ then

$$\Psi_\phi(N\delta,T)=\Phi_f(\delta,T)$$

if

$$f(\delta)=\int\phi(s(w)^{-1}\delta\sigma(s(w))),$$

the integral being taken over a small neighborhood of the trivial coset, $G(F)$ itself. It is therefore manifest that $\{\Phi_\phi(\gamma,T)\}$ is an HCS family. If $z\in Z(F)$ lies close to 1 then

$$f(z)=\int_{G(F)\backslash G'(F)}\phi(g^{-1}z\sigma(g))dg$$

and if

$$\varepsilon=z\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

then

$$\Psi_\phi(\varepsilon^\ell,\mu)=\Phi_f(\varepsilon,\mu)$$

and $N\varepsilon=\varepsilon^\ell$. The asserted formulae for $\Phi'(z,T,\mu)$ and for $\Phi''(z,T,\mu)$ follow.

Before we discuss the case that δ is not σ-conjugate to a scalar we comment on the manner in which one shows that an HCS family $\{\Phi(\gamma,T)\}$ for which $\Phi(\gamma,T)=0$ when $\gamma\notin NT(E)$ is a Shintani family. One can localize the problem and, once again, the only difficulty occurs for a family which is supported in a small neighborhood of a point in $Z(F)$. If this point lies in $NZ(E)$ we may suppose it is 1. If

$$\Phi'(\gamma, T) = \Phi(\gamma^\ell, T)$$

then $\{\Phi'(\gamma, T)\}$ is again an HCS family and hence there is an f such that

$$\Phi'(\gamma, T) = \Phi_f(\gamma, T).$$

We may suppose that the support of f consists of elements whose conjugacy class passes close to 1. Employing a partition of unity and then conjugating we may even suppose that f itself is upported in a small neighborhood of 1. Suppose a is a function on $G(F)\backslash G'(F)$ satisfying

$$\int_{G(F)\backslash G'(F)} \alpha(w)dw = 1.$$

If $w \in G(F)\backslash G'(F)$ and $h \in G(F)$ set

$$\phi(s(w)^{-1}h\sigma(s(w))) = \alpha(w)f(h).$$

Then

$$\Psi_\phi(\delta^\ell, T) = \Phi(\delta^\ell, T)$$

for δ close to 1. Since extraction of ℓth roots in a neighborhood of 1 is a well defined operation, we conclude that

$$\Psi_\phi(\gamma, T) = \Phi(\gamma, T)$$

for all γ.

Suppose now that $z = N\delta$ is central but that δ is not σ-conjugate to a central element. Then $\ell = 2$ and

$$G_\delta^\sigma(E) = \{y|\delta\sigma(y)\delta^{-1} = y\}$$

is the multiplicative group of a quanternion algebra. If $u \in G_\delta^\sigma(E)$ then

$$N(u\delta) = u\delta\sigma(u)\sigma(\delta) = u^2z.$$

It follows that if u is close to 1 then

$$G_{u\delta}^\sigma(E) \subseteq G_u(E)$$

if G_u is the centralizer of u. Since

$$G_{u\delta}^\sigma(E) = \{y|u\delta\sigma(y)\delta^{-1}u^{-1} = y\}$$

we conclude that

$$G^\sigma_{u\delta}(E) \subseteq G^\sigma_\delta(E).$$

This time we take s to be a section of $G'(F) \to G^\sigma_\delta(E)\backslash G'(F)$ and set

$$f(h) = \int \phi(s(w)^{-1}h\sigma(s(w))), \quad h \in G^\sigma_\delta(E),$$

so that

$$\Psi_\phi(zu^2, T) = \Phi_f(u, T).$$

The family $\{\Phi_f(\delta, T)\}$ is an HCS family not for $G(F)$ but for $G^\sigma_\delta(E) = G^\sigma_z(F)$. Since this group is the multiplicative group of a quanternion algebra the properties of HCS families for it are trivial to establish. The principal points to observe are that there is no longer a split Cartan subgroup, that $\Phi'(\gamma, T, \mu)$ does not occur, and that

$$\Phi(\gamma, T; \omega_T(\mu), \omega_{G'}) = \mathrm{meas}(T(F)\backslash G'(F))\Phi''(\gamma, T; \mu).$$

This said, one proceeds as before, and completes the proof of the lemma.

Lemma 6.2 allows us to associate to any smooth compactly supported ϕ on $G(E)$ a smooth compactly supported f on $G(F)$ for which

$$\{\Phi_f(\gamma, T)\} = \{\Psi_\phi(\gamma, T)\}.$$

The function f is not uniquely determined but its orbital integrals are, and this is enough for our purposes. The correspondence $\phi \to f$, which was introduced by Shintani, plays an important role in these notes. It is however essential to observe that if E is unramified and ϕ is spherical then f may be taken to be the image of ϕ under the homomorphism of the previous paragraph. In particular if ϕ is the characteristic function of $G(O_E)$ divided by its measure then f may be taken to be the characteristic function of $G(O)$ divided by its measure.

It should come as a surprise to no-one when I now confess that the map $\phi \to f$ has been defined for the wrong class of functions. If ξ is a given character of $NZ(E)$ we shall want ϕ to satisfy

$$\phi(zg) = \xi(Nz)^{-1}\phi(g) \quad z \in Z(E)$$

and f to satisfy

$$f(zg) = \xi(z)^{-1}f(g) \quad z \in NZ(E).$$

All we need do is start from the original ϕ and f and replace them with

$$\phi'(g) = \int_{Z(E)} \phi(zg)\xi(Nz)dz$$

and

$$f'(g) = \int_{NZ(E)} f(zg)\xi(z)dz.$$

The calculations at the end of the preceding paragraph show that if $\{\Phi_{\phi'}(\gamma, T)\}$ and $\{\phi_{f'}(\gamma, T)\}$ are defined in the obvious way then

$$\Phi_{\phi'}(\gamma, T) = \Phi_{f'}(\gamma, T).$$

This and the other relations between orbital integrals of f' and ϕ' which are deducible from Lemma 6.2 will play a central role in the comparison of Paragraph 11.

7. CHARACTERS AND LOCAL LIFTING

F is again a local field and E a cyclic extension of degree ℓ. σ is a fixed generator of $\mathfrak{G}$. If Π is an irreducible admissible representation of $G(E)$ then Π may or may not be equivalent to $\Pi^\sigma : g \to \Pi(\sigma(g))$. We shall be concerned only with those Π for which $\Pi^\sigma \simeq \Pi$. Then Π extends to a representation Π' of $G'(E) = G(E) \times \mathfrak{G}$. Π' is not unique, but any other extension is of the form $\omega \otimes \Pi'$, where ω is a character of $\mathfrak{G}$.

We may introduce the character of Π' along the lines of §7 of [14]. It is a distribution. We shall not be able to prove completely the following proposition until we have some of the results of Paragraph 11. If we had not thrown methodological purity to the winds, we would be bound to find a purely local proof for it.

Proposition 7.1. *The character of Π' exists as a locally integrable function.*

A good deal of this paragraph will be taken up with the proof of this proposition, although one case will be postponed until §11, appearing there as Lemma 11.2. In addition, we will begin the study of local base change, especially for the representations $\pi(\mu,\nu)$ and the special representations $(\sigma(\mu,\nu)$. They are, of course, both rather easy to handle. The last part of the paragraph is devoted to a computational proof of Lemma 7.17, which yields part of assertion (c) of §2.

Since the character of Π is a function and since σ is an arbitrary generator of $\mathfrak{G}$, it is enough to show that the character is a function on $G(E) \times \sigma$.

Let $\eta = (\mu,\nu)$ be the quasi-character of the group $A(E)$ of diagonal matrices and consider the representation $\rho(\eta) = \rho(\mu,\nu)$ introduced in Chapter 1 of [14]. If $\mu^\sigma = \mu, \nu^\sigma = \nu$ then $\rho(\eta)$ may be extended to a representation of $G'(E)$, which we still denote $\rho(\eta)$, by setting

$$\rho(\sigma,\eta)\varphi(g) = \varphi(\sigma^{-1}(g)) \qquad \varphi \in \mathcal{B}(\eta).$$

$\mathcal{B}(\eta)$ is introduced on p. 92 of [14].

For our purposes it is best to suppose that $\mu\nu = \xi_E$ on $E^\times$. If σ is smooth, satisfies $\phi(zg) = \xi_E^{-1}(z)\sigma(g),\ z \in E^\times = Z(E)$, and has compact support modulo $Z(E)$, we may set

$$\rho(\phi,\eta) = \int_{Z(E)\backslash G(E)} \phi(g)\rho(g,\eta)dg.$$

We may choose the Haar measure on K so that

$$\int_{Z(E)\backslash G(E)} h(g)dg = \int_K \int_{N(E)} \int_{Z(E)\backslash A(E)} h(tnk)dtdndk.$$

Then the kernel of $\rho(\phi,\eta)\rho(\sigma,\eta)$, which is a function on $K \times K$, is equal to

$$\int_{Z(E)\backslash A(E)} \int_{N(E)} \phi(k_1^{-1}tn\sigma(k_2))\eta(t)\left|\frac{\alpha}{\beta}\right|^{1/2} dtdn$$

if

$$t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

The trace of $\rho(\phi,\eta)\rho(\sigma,\eta)$ is obtained by integrating over the diagonal.

If $\gamma \in G(F)$ has distinct eigenvalues a, b we set

$$\Delta(\gamma) = \left|\frac{(a-b)^2}{ab}\right|_F^{1/2}.$$

It is easily seen that if $\delta \in A(E),\ \gamma = N\delta$, and $\Delta(\gamma) \neq 0$ then

$$\Delta(\gamma) \int\limits_{Z(E)A(F)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg =$$

$$\left|\frac{a}{b}\right|^{1/2} \int_K \int_{Z(E)A(F)\backslash A(E)} \int_{N(E)} \phi(k^{-1}t^{-1}\delta\sigma(t)n\sigma(k))dndtdk.$$

Thus if we denote the left hand side by $F_\phi(\delta)$,

$$\text{trace}\rho(\phi)\rho(\sigma) = \int_{Z(E)A^{1-\sigma}(E)\backslash A(E)} \eta(t)F_\phi(t)dt.$$

Since $F_\phi(t) = F_\phi(\tilde{t})$ if

$$\tilde{t} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

this may be written

$$\int_{Z(E)A^{1-\sigma}(E)\backslash A(E)} \frac{\eta(t)+\tilde{\eta}(t)}{2} F_\phi(t)dt.$$

We may extend the definition of F_ϕ to other tori, and there is an obvious, and easily verified, analogue of the Weyl integration formula

$$\int_{Z(E)\backslash G(E)} \phi(g)dg$$

is equal to

$$\frac{1}{2}\sum \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} \left\{ \int_{Z(E)T(F)\backslash G(E)} \phi(g^{-1}t\sigma(g))dg \right\} \Delta(Nt)^2 dt$$

or

$$\frac{1}{2}\sum \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} F_\phi(t)\Delta(Nt)dt.$$

The sum is over a set of representatives for the conjugacy classes of Cartan subgroups over F. We deduce the following lemma.

Lemma 7.2. *The character $\chi_{\rho(\eta)}$ of $\rho(\eta)$ is a function on $G(E) \times \sigma$. If $\gamma = N\delta$ is regular but not conjugate to an element of $A(F)$ then $\chi_{\rho(\eta)}(\delta \times \sigma) = 0$. If δ is σ-conjugate to t and $\gamma = N\delta$ is regular then*

$$\chi_{\rho(\eta)}(\delta \times \sigma) = \frac{\eta(t)+\tilde{\eta}(t)}{\Delta(\gamma)}.$$

If $\eta = \eta^\sigma$ then there exist μ', ν' such that $\mu(x) = \mu'(Nx), \nu(x) = \nu'(Nx)$. If $\phi \to f$ is defined as in the previous paragraph, the following corollary is clear.

Corollary 7.3. *If $\eta'(\mu', \nu')$ then*

$$\mathrm{trace}\rho(\phi, \eta)\rho(\sigma, \eta) = \mathrm{trace}\rho(f, \eta')$$

and

$$\chi_{\rho(\eta)}(\delta \times \sigma) = \chi_{\rho(\eta')}(\gamma)$$

if $\gamma = N\delta$.

If $\Pi = \pi(\mu, \nu)$ with $\mu(x) = \mu'(Nx), \nu(x) = \nu'(Nx)$ then $\Pi^\sigma \simeq \Pi$. We take $\Pi' = \pi'(\mu, \nu)$ to be the restriction of $\rho(\eta)$ to the subquotient of $\mathcal{B}(\eta)$ on which $\pi(\mu, \nu)$ acts. We see that if $\rho(\mu, \nu)$ is irreducible, then it is a lifting of $\rho(\mu', \nu')$ according to either of the criteria of §2. Notice that there are ℓ^2 choices for μ', ν'.

Lemma 7.4 *Suppose F is non-archimedean.*

(a) *If $\mu\nu^{-1}(x) \not\equiv |x|$ and $\mu\nu^{-1}(x) \not\equiv |x|^{-1}$ then*

$$\mathrm{trace}\,\pi'(\phi; \mu, \nu)\pi'(\sigma; \mu, nu) = \mathrm{trace}\,\pi(f; \mu', \nu')$$

and, if $\gamma = N\delta$ is regular,

$$\chi_{\pi'(\mu,\nu)}(\delta \times \sigma) = \chi_{\pi(\mu',\nu')}(\gamma).$$

(b) *If $\mu\nu^{-1}(x) \equiv |x|^{-1}$ and $\mu'\nu'^{-1}(x) \equiv |x|^{-1}$ the same equalities are valid.*

The only cases not covered by the lemma are those for which $\pi(\mu, \nu)$ is finite-dimensional while $\pi(\mu', \nu')$ is infinite-dimensional, when the equalities no longer hold. This is the reason that we have also had to introduce the criterion (i) of §2 for a local lifting. Observe that $\pi(\mu', \nu')$ then ceases to be unitary. The first part of the lemma is clear for, with the assumptions imposed there, $\pi(\mu, \nu) = \rho(\mu, \nu), \pi(\mu', \nu') = \rho(\mu', \nu')$.

If the conditions of the second hold, then

$$\begin{aligned}
\pi(g; \mu, \nu) &= \mu(\mathrm{det}g)|\mathrm{det}g|_E^{1/2}, \qquad g \in G(E),\\
\pi(\sigma; \mu, \nu) &= 1,\\
\pi(g; \mu', \nu') &= \mu'(\mathrm{det}g)|\mathrm{det}g|_F^{1/2}, \qquad g \in G(F).
\end{aligned}$$

The first of the desired equalities is clear; the other follows from the Weyl integration formulae. It is by the way implicit in the lemma that the characters appearing there are functions.

Lemma 7.5. *Suppose F is archimedean. If $\pi(\mu, \nu)$ and $\pi(\mu', \nu')$ are both infinite-dimensional or both finite-dimensional then the equalities of the previous lemma are again valid.*

This again follows from the corollary if both representations are infinite-dimensional. To check the remaining case we observe that F will be $\mathbf{R}$ and E will be $\mathbf{C}$. There is a finite-dimensional analytic representation ρ of $G(\mathbf{C})$ on a space V and a character χ of $\mathbf{C}^\times$ such that

$$\pi(g;\mu,\nu) \simeq \chi(\det g)\rho(\sigma(g)) \otimes \rho(g).$$

Since $\chi(z) = \chi(\sigma(z))$ there is no harm in supposing it is 1. If λ^{-1} is the highest weight of the contragredient to ρ and w a highest weight vector then

$$u^\sigma \otimes v \to (\sigma(w) \otimes w)(\rho(\sigma(g))\sigma(u) \otimes \rho(g)v)$$

maps V into $\mathcal{B}(\mu,\nu)$ if, as we may assume $\mu(x)|x|^{1/2} = \lambda(x\sigma(x))\nu(x)|x|^{-1/2} = \lambda(x\sigma(x))$. The absolute value is taken in the number-theoretical sense. Then $\pi'(\sigma;\mu,\nu)$ corresponds to $\sigma(u) \otimes v \to \sigma(v) \otimes u$.

If $\rho(g) = (\rho_{ij}(g))$ then a matrix form of $\pi'(g \times \sigma;\mu,\nu)$ is

$$\rho_{i',j}(\sigma(g)) \otimes \rho_{i,j'}(g).$$

Setting $i = j, i' = j'$ and summing we conclude that

$$\operatorname{trace}\pi'(g \times \sigma;\mu,\nu) = \operatorname{trace}\rho(g\sigma(g)).$$

Since $\pi(g,\mu',\nu')$ is either $\rho(g)$ or $\operatorname{sign}(\det g)\rho(g)$ the lemma follows.

The next lemma is an immediate consequence of Corollary 7.3 and Lemma 7.4.

Lemma 7.6. *Suppose F is non-archimedean and $\sigma(\mu,\nu)$ is a special representation. Let $\mu\nu^{-1}(x) = |x|_E$. Define $\sigma'(\sigma;\mu,\nu)$ to be the restriction of $\rho(\sigma;\mu,\nu)$ to the subspace of $\mathcal{B}(\mu,\nu)$ on which $\sigma(\mu,\nu)$ acts. If $\mu'\nu'^{-1}(x) = |x|_F$ then*

$$\operatorname{trace}\sigma'(\phi;\mu,\nu)\sigma'(\sigma;\mu,\nu) = \operatorname{trace}\sigma(f,\mu',\nu')$$

and if $\gamma = N\delta$ is regular

$$\chi_{\sigma'(\mu,\nu)}(\delta \times \sigma) = \chi_{\sigma(\mu',\nu')}(\gamma).$$

We see that $\sigma(\mu,\nu)$ is a lifting of $\sigma(\mu',\nu')$. There are now only ℓ choices for the pair (μ',ν').

If $\Pi = \sigma(\mu,\nu)$ then $\Pi^\sigma \simeq \Pi$ only if $\mu^\sigma = \mu, \nu^\sigma = \nu$. However if $\Pi = \pi(\mu,\nu)$ then Π^σ is also equivalent to Π if $\mu^\sigma = \nu, \nu^\sigma = \mu$ that is if $\tilde\eta = \eta^\sigma$. If $\eta^\sigma \neq \eta$ this can only happen for $\ell = [E:F] = 2$, as we now suppose. If $\eta^\sigma = \tilde\eta$ we can define an operator $R(\eta) : \mathcal{B}(\eta) \to \mathcal{B}(\eta^\sigma)$ as on p. 521 of [14]. Formally (and with a better choice of the ε-factor than in [14])

$$R(\eta)\varphi(g) = \varepsilon(0,\mu\nu^{-1},\psi_E)\frac{L(1,\mu\nu^{-1})}{L(0,\mu\nu^{-1})}\int_E \varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)dx.$$

ψ_E is a non-trivial additive character of E of the form $x \to \psi_F(\operatorname{trace} x)$ and dx is the Haar measure self-dual with respect to ψ_E. If ψ_F is replaced by $\psi_F(ax)$, $a \in F^\times$, then $\varepsilon(0, \mu\nu^{-1}, \psi_E)$ is multiplied by

$$|a|_E^{-1/2} \frac{\mu(a)}{\nu(a)} = |a|^{-1/2}.$$

Since dx is replaced by $|a|^{1/2} dx$, the expression as a whole is unchanged and $R(\eta)$ is well defined.

Lemma 7.7. (a) *If $\eta = \eta^\sigma = \tilde{\eta}$ then $R(\eta)$ is the identity.*

(b) *If $\eta^\sigma = \tilde{\eta}$ and $\rho(\sigma, \eta^\sigma) : \mathcal{B}(\eta^\sigma) \to \mathcal{B}(\eta)$ replaces $\varphi(g)$ by $\varphi(\sigma^{-1}(g))$ then $\rho(\sigma, \eta^\sigma) R(\eta)$, which takes $\mathcal{B}(\eta)$ to itself, is of order two.*

Since

$$\rho(\sigma, \eta^\sigma) R(\eta) \rho(\sigma, \eta^\sigma) R(\eta) = \rho(\sigma, \eta^\sigma) \rho(\sigma, \eta) R(\eta^\sigma) R(\eta)$$

and

$$\rho(\sigma, \eta^\sigma) \rho(\sigma, \eta) = 1$$

the assertion (b) is implied by the following lemma.

Lemma 7.8. *Let E be an arbitrary local field. Suppose $\eta = (\mu, \nu)$ and*

$$|\varpi_E| < |\mu(\varpi_E) \nu^{-1}(\varpi_E)| < |\varpi_E|^{-1}$$

then $R(\tilde{\eta})$ and $R(\eta)$ are defined and

$$R(\tilde{\eta}) R(\eta) = 1.$$

If ω is a quasi-character of $E^\times$, the map $\varphi \to \varphi'$ with $\varphi'(g) = \omega(\det g) \varphi(g)$ takes $\mathcal{B}(\mu, \nu)$ to $\mathcal{B}(\omega\mu, \omega\nu)$. It sends $R(\eta)\varphi$ to $R(\omega\eta)\varphi'$; so for the purposes of the lemma we may suppose $\nu = 1$. I also observe that if ψ_E is replaced by $x \to \psi_E(ax)$, $a \in E^\times$ then $\varepsilon(0, \mu\nu^{-1}, \psi_E)$ is multiplied by

$$|a|_E^{-1/2} \frac{\mu(a)}{\nu(a)}$$

and $\varepsilon(0, \nu\mu^{-1}, \psi_E)$ is multiplied by

$$|a|_E^{-1/2} \frac{\nu(a)}{\mu(a)}.$$

Thus $R(\tilde{\eta}) R(\eta)$ is not affected and if $\eta = \tilde{\eta}$ neither is $R(\eta)$.

First take E to be non-archimedean and μ to be unramified. Suppose $\mu(\varpi_E) = |\varpi_E|^s$, $\operatorname{Re} s > 0$. Let φ_0 be defined by

$$\varphi_0(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k) = \mu(\alpha) \left| \frac{\alpha}{\beta} \right|^{1/2}.$$

The factor $\varepsilon(0,\mu,\psi_E)$ is, almost by definition, equal to

$$|a|^{s+1/2}$$

if $a^{-1}O_E$ is the largest ideal on which ψ_E is trivial. Then

$$\int_{O_E} dx = |a|^{-1/2}.$$

The integrand of

$$\int_E \varphi_0(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})dx$$

is 1 if $x \in O_E$. Otherwise

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -\frac{1}{x} & -1 \end{pmatrix}$$

and the integrand equal $|x|^{-1-s}$. The integral equals

$$|a|^{-1/2}\left\{1+\sum\nolimits_{n=1}^{\infty} q^{-s}\left(1-\frac{1}{q}\right)\right\} = |a|^{-1/2}\frac{L(s,1_E)}{L(1+s,1_E)} = |a|^{-1/2}\frac{L(0,\mu)}{L(1,\mu)}.$$

Consequently

$$R(\eta)\varphi_0(1) = |a|^s\varphi_0(1) = |a|^s.$$

This relation may be analytically continued.

Any function φ is equal to

$$(\varphi - \varphi(1)\varphi_0) + \varphi(1)\varphi_0.$$

Thus to check that $R(\eta)\varphi(1)$ can be analytically continued for all φ, we need only check it when $\varphi(1) = 0$. The factor

$$\varepsilon(0,\mu,\psi_E)\frac{L(1,\mu)}{L(0,\mu)}$$

is certainly well defined if $|\mu(\varpi_E)| < |\varpi_E|^{-1}$. Moreover if $\varphi(1) = 0$ there is an N such that

$$\varphi\begin{pmatrix} -1 & 0 \\ -\frac{1}{x} & -1 \end{pmatrix} = 0$$

for $|x| > N$ and

$$\int \varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})dx = \int_{|x|\le N} \varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})dx.$$

The right side is well defined for any μ. We conclude that $R(\eta)$ is indeed defined for Re $s > -1$.

It is clear that $R(\eta)\varphi_0$ is a multiple of φ_0, for φ_0 is, up to a constant factor, the only function invariant under K_E. Therefore $R(\eta)\varphi_0 = |a|^s\varphi_0$. If $\mu = 1$, then $s = 0$ and $R(\eta)\varphi_0 = \varphi_0$. Since $R(\eta)$ then intertwines $\mathcal{B}(\eta)$ with itself and $\mathcal{B}(\eta)$ is irreducible, $R(\eta)=1$. This is the first part of the lemma. If Re $s<1$ then $R(\tilde{\eta})$ is also defined and $R(\tilde{\eta})R(\eta)$ which again intertwines $\mathcal{B}(\eta)$ with itself is a scalar. Since

$$R(\tilde{\eta})R(\eta)\varphi_0=\varphi_0$$

the scalar is 1.

Now suppose μ is ramified. The factor

$$\varepsilon(0,\mu,\psi_E)\frac{L(1,\mu)}{L(0,\mu)}=\varepsilon(0,\mu,\psi_E)$$

is well defined for all such μ. If

$$\varphi\begin{pmatrix}-1 & 0\\ -\frac{1}{x} & -1\end{pmatrix}=\varphi(-1)$$

for$|x|>N$, then

$$\int\varphi(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})dx=\int_{|x|\le N}\varphi(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix})dx.$$

The right side is meaningful for all values of $|\mu(\varpi_E)|$. For this μ we only wish to prove the second part of the lemma. For the sake of symmetry we put ν back in. We may also suppose that O_E is the largest ideal on which ψ_E is trivial.

If $|\varpi_E|<|\mu(\varpi_E)\nu^{-1}(\varpi_E)|<|\varpi_E|^{-1}$ then Propositions 3.2 and 3.4 of [14] give us two isomorphisms

$$A: W(\mu,\nu;\psi_E)\xrightarrow{\sim}\mathcal{B}(\mu,\nu)$$
$$B: W(\mu,\nu;\psi_E)=W(\nu,\mu;\psi_E)\xrightarrow{\sim}\mathcal{B}(\nu,\mu).$$

Suppose $W\in W(\mu,\nu;\psi_E)$ and $AW=\varphi,\ BW=\varphi'$. We shall show that

$$W(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})\sim|a|^{1/2}\mu\nu(-1)\{\varepsilon(0,\mu\nu^{-1},\psi_E)\mu(a)\varphi(1)+\varepsilon(0,\nu\mu^{-1},\psi_E)\nu(a)R(\eta)\varphi(a)\}\tag{7.1}$$

as $a\to 0$. Interchanging ν and μ we can also infer that

$$W(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix})\sim|a|^{1/2}\mu\nu(-1)\{\varepsilon(,\nu\mu^{-1},\psi_E)\nu(a)\varphi'(1)+\varepsilon(0,\mu\nu^{-1},\psi_E)\mu(a)R(\tilde{\eta})\varphi'(1)\}$$

as $a\to 0$. We conclude that

$$R(\eta)\varphi=\varphi'$$

and that

$$\varphi=R(\tilde{\eta})R(\eta)\varphi.$$

Hence

$$R(\tilde{\eta})R(\eta)=1.$$

To verify (1) we take $W = W_\Phi$ as on p. 94 of [14]. (There is a misprint there. The measure used to define $\theta(\mu_1, \mu_2; \Phi)$ should be $d^\times t$ rather than dt.) We may suppose that

$$\Phi(\alpha x, \alpha^{-1} y) = \mu^{-1}\nu(\alpha)\Phi(x, y)$$

if $|\alpha| = 1$. Then $\Phi(0,0) = 0$ and $\Phi(x, y)$ is 0 for x, y close to 0. If N is sufficiently large $W_\Phi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ is equal to

$$|a|^{1/2}\mu(a)\int_{\frac{1}{N}\le|t|\le N} \mu\nu^{-1}(t)\Phi(at, t^{-1})d^\times t + |a|^{1/2}\mu(a)\int_{|t|>N} \mu\nu^{-1}(t)\Phi(at, t^{-1})d^\times t$$

for all a. Fix N. When $|a|$ is small this expression equals

$$|a|^{1/2}\mu(a)\int_{E^\times} \nu\mu^{-1}(t)\Phi(0, t)d^\times t + |a|^{1/2}\nu(a)\int_{E^\times} \Phi(t, 0)d^\times t. \tag{7.2}$$

According to the definition of A, φ is equal to $f_{\Phi^\sim}$ with

$$f_{\Phi^\sim}(1) = \int \Phi^\sim(0, t)\mu\nu^{-1}(t)|t|d^\times t = \sum\nolimits_{n=-\infty}^{\infty} \Phi^\sim(0, \varpi_E^n)|\varpi_E|^n\mu\nu^{-1}(\varpi_E^n).$$

However, by the definition on p. 94 of [14]

$$\begin{aligned}\Phi^\sim(0, \varpi_E^n) &= \int \Phi(0, y)\psi_E(y\varpi_E^n)dy \\ &= \sum\nolimits_{m=-\infty}^{\infty}(1 - |\varpi_E|)|\varpi_E|^m\Phi(0, \varpi^m)\int_{|t|=1} \mu\nu^{-1}(t)\psi_E(t\varpi_E^{m+n})d^\times t.\end{aligned}$$

If r is the order of the conductor of $\mu\nu^{-1}$ the integral appearing here is 0 unless $m + n = -r$ when it equals

$$\varepsilon(0, \nu\mu^{-1}, \psi_E)\frac{\mu\nu^{-1}(\varpi^r)|\varpi|^r}{1 - |\varpi|}.$$

Thus

$$\Phi^\sim(0, \varpi_E^n) = \varepsilon(0, \nu\mu^{-1}, \psi_E)|\varpi_E|^{-n}\Phi(0, \varpi^{-n-r})\mu\nu^{-1}(\varpi^r)$$

and

$$\begin{aligned}f_{\Phi^\sim}(1) &= \varepsilon(0, \nu\mu^{-1}, \psi_E)\sum\nolimits_n \mu\nu^{-1}(\varpi^n)\Phi(0, \varpi^{-n}) \\ &= \varepsilon(0, \nu\mu^{-1}, \psi_E)\int_{E^\times} \nu\mu^{-1}(t)\Phi(0, t)d^\times t.\end{aligned}$$

Since, under the conditions imposed on ψ_E,

$$\varepsilon(0, \nu\mu^{-1}, \psi_E)\varepsilon(0, \mu\nu^{-1}, \psi_E) = \mu\nu^{-1}(-1)$$

we may substitute in the first term of (7.2) to obtain the first term of (7.1).

Lemma 3.2.1 of [14] gives

$$\mu\nu(-1)\varepsilon(0,\nu\mu^{-1},\psi_E)R(\eta)\varphi(1)$$

as

$$\lim_{N\to\infty}\int_{|x|\le N}\left\{\int W_\Phi(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix})\nu^{-1}(a)|a|^{-1/2}\psi_E(ax)da\right\}dx.$$

Interchanging the order of integration we see from (7.2) that this equals

$$\int_{E^\times}\Phi(t,0)dt.$$

In other words, the second term of (7.2) is equal to the second term of (7.1) as required.

As a convenient, but in the long run unsatisfactory, expedient, we prove Lemma 7.8 for archimedean fields by appealing to the theory of Eisenstein series. E will be momentarily a global field, either **Q** or an imaginary quadratic field. If μ_∞,ν_∞ are arbitrary characters of $E_\infty^\times$, there are characters μ,ν of $E^\times\backslash I_E$ such that μ,ν restricted to $E_\infty^\times$ are μ_∞,ν_∞. Here I_E is the group of idèles. We introduce $M(\eta)=M(\mu,\nu;0)$ as on p. 513 of [14]

$$M(\eta)=\frac{L(1,\nu\mu^{-1})}{L(1,\mu\nu^{-1})}\otimes_v R(\eta_v).$$

Here

$$\frac{L(1,\nu\mu^{-1})}{L(1,\mu\nu^{-1})}=\lim_{s\to 0}\frac{L(1+s,\nu\mu^{-1})}{L(1-s,\mu\nu^{-1})}.$$

From the theory of Eisenstein series

$$M(\tilde\eta)M(\eta)=1.$$

Since the map $\eta\to\tilde\eta$ interchanges μ and ν and

$$\frac{L(1,\mu\nu^{-1})}{L(1,\nu\mu^{-1})}\,\frac{L(1,\nu\mu^{-1})}{L(1,\mu\nu^{-1})}=1$$

we conclude that

$$\otimes_v R(\tilde\eta_v)R(\eta_v)=1.$$

Applying our result for non-archimedean fields we conclude that

$$R(\tilde\eta_\infty)R(\eta_\infty)=1.$$

The first part of Lemma 7.1 must unfortunately still be proved directly. We revert to our earlier notation. We also suppose once again that $\nu=1$. Since we are dealing with the first part of the lemma, μ will also be 1. More generally let $\mu(x)=|x|^s$, Re$s>0$. We may suppose that

$$\psi_E(x)=e^{2\pi ix},\qquad E=\mathbf{R}$$

$$\psi_E(x)=e^{2\pi i\mathrm{Re}\,x},\quad E=\mathbf{C}.$$

Then dx is the usual Haar measure on E. Define φ_0 by

$$\varphi_0(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta/cr \end{pmatrix} k) = \mu(\alpha)\left|\frac{\alpha}{\beta}\right|^{1/2}.$$

Once again we need only show that

$$R(\eta)\varphi_0(1) = 1.$$

If $F = \mathbf{R}$, then, taking the definition of the L-functions and ε-factors into account [21],

$$R(\eta)\varphi_0(1) = \frac{\Gamma\left(\frac{1+s}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{s}{2}\right)} \int_{-\infty}^{\infty} (1+x^2)^{\frac{(s+1)}{2}} dx = 1$$

and if $F = \mathbf{C}$,

$$R(\eta)\varphi_0(1) = \frac{1}{\pi}\,\frac{\Gamma(1+s)}{\Gamma(s)} \int\int (1+x^2+y^2)^{-s-1} dxdy = 1.$$

We conclude from the second part of Lemma 7.7 that if $\eta^\sigma = \tilde{\eta}$ then $\Pi = \pi(\mu,\nu)$ may be extended to a representation Π' of $G'(E)$, a representation we shall sometimes denote $\tau(\eta)$, by setting

$$\Pi'(\sigma) = \tau(\sigma,\eta) = \rho(\sigma,\eta^\sigma)R(\eta).$$

Appealing to the first part of the lemma we see that this is consistent with our previous choice of Π' if $\eta^\sigma = \eta$. Observe also that if η is unramified and $\eta^\sigma = \tilde{\eta}$ then $\eta = \tilde{\eta}$. The following assertion is the part of Lemma 7.1 which will not be verified until §11, as Lemma 11.2.

(1) *The character of $\tau(\eta)$ is a locally integrable function.*

The following assertion is also part of that lemma.

(2) *If $\eta = (\mu, \mu^\sigma)$ and if $r = \mathrm{Ind}(W_{E/F}, W_{E/E}, \mu)$ then $\rho(\eta)$ is a lifting of $\pi(\tau)$.*

The only representations of $G(E)$ we have not yet considered are the absolutely cuspidal Π. Choose such a Π for which $\Pi^\sigma \simeq \Pi$ and extend Π in any way to Π'.

Lemma 7.9 *If Π is absolutely cuspidal then the character $\chi_{\Pi'}$ exists as a locally integrable function and is smooth on $\{g \in G(E) | N_g$ is regular and semi-simple$\}$.*

Moreover if Π is unitary

$$\frac{1}{2}\sum{}' \frac{1}{\mathrm{meas}\, NZ(E)\backslash T(F)} \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} |\chi_{\Pi'}(t \times \sigma)|^2 \Delta(Nt)^2 dt = \frac{1}{\ell}.$$

The sum is over a set of representatives of the conjugacy classes of non-split Cartan subgroups over F.

We shall imitate the proofs of Proposition 7.4 and Lemma 15.4 of [14]. In particular, it suffices to consider unitary Π. Then Π' is also unitary. Suppose f is a locally constant function on $G'(E) = G(E) \times \mathfrak{G}$ with compact support. Set

$$\Pi'(f) = \int_{G'(E)} f(g)\Pi'(g)dg.$$

Since Π' is a square-integrable representation of $G'(E)$ we may apply Lemma 7.4.1 of [14] to conclude that

$$\operatorname{trace}\Pi'(f) = d(\Pi') \int_{Z(E)\backslash G'(E)} \left\{ \int_{G'(E)} f(g)(\Pi'(g^{-1}hg)u, u)dh \right\} dg.$$

Here u is a unit vector in the space of Π' and $d(\Pi')$ the formal degree.

We introduce a subset $\widehat{G}'(E)$ of $G'(E)$ whose complement has measure 0 and a function $\xi(g)$ on it. If $g \in G(E)$ then $g \in \widehat{G}'(E)$ if and only if g has distinct eigenvalues a, b. We set

$$\Delta_E(g) = \left| \frac{(a-b)^2}{ab} \right|_E^{1/2}$$

and let $\xi(g) = \Delta_E(g)^{-1}$. If $\tau \in G,\ \tau \neq 1$, and $g \in G(E)$ then $g \times \tau \in \widehat{G}'(E)$ if and only if $g\tau(g)\cdots\tau^{\ell-1}(g)$ has distinct eigenvalues. If $g\tau(g)\cdots\tau^{\ell-1}(g)$ is conjugate to h in $G(F)$ we set $\xi(g) = \Delta(h)^{-1}$. We define $\xi(g)$ to be 0 on the complement of $\widehat{G}'(E)$ in $G'(E)$.

Lemma 7.10. *The function $\xi(g)$ is locally integrable on $G'(E)$.*

That it is locally integrable on $G(E)$ follows from Lemma 7.3 of [14]. It suffices then to show that it is locally-integrable on $G(E) \times \sigma$. Since any compactly supported locally constant function ϕ is dominated by a spherical function, it follows from the results on Paragraph 3 that

$$\sum \int_{T^{1-\sigma}(E)\backslash T(E)} |F_\phi(t)dt < \infty$$

if

$$F_\phi(\delta) = \Delta(N\delta) \int_{T(F)\backslash G(E)} \phi(g^{-\sigma}\delta g)dg.$$

The sum is over a set of representatives for the conjugacy classes of Cartan subgroups of G over F.

Take ϕ to be the characteristic function of a compact open set. By the Weyl integration formula, or rather the variant appropriate when $Z(E)\backslash G(E)$ is replaced by $G(E)$,

$$\int_{G(E)\times\sigma} \xi(g \times \sigma)\phi(g)dg$$

is equal to

$$\frac{1}{2} \sum \int_{T^{1-\sigma}(E)\backslash T(E)} F_\phi(t)dt$$

and is therefore finite.

Define T'_r as on p. 254 of [14], except that F is to be replaced by E. We have only to show that

$$\lim_{r\to\infty} \int_{T'_r \times \mathfrak{g}} (\Pi'(g^{-1}hg)u, u)dg$$

converges on $\widehat{G}'(E)$ and that the convergence is dominated by a constant times $\xi(h)$. It is easily seen that

$$\xi(\tau(h)) = \xi(h)$$

for all $\tau \in \mathfrak{g}$. It is therefore enough to verify the assertion for the sequence

$$\int_{T'_r} (\Pi'(g^{-1}hg)u, u)dg.$$

For h in $G(E)$, where Π' is Π, this has been done in [14]; so we replace h by $h \times \sigma$, with the new h in $G(E)$, and the first u by $v = \Pi'(\sigma)u$ and consider

(7.3) $$\varphi_r(h) = \int_{T'_r} (\Pi(g^{-1}h\sigma(g))v, u)dg.$$

Let $\widetilde{G}^\sigma(E)$ be the set of all h in $G(E)$ for which the eigenvalues of Nh do not lie in F. We need the following analogue of Lemma 7.4.2 of [14].

Lemma 7.11. *Let C'_1 be a compact subset of $\widetilde{G}^\sigma(E)$ and let C'_2 be a compact subset of $G(E)$. The image in $Z(E)\backslash G(E)$ of*

$$X = \{g \in G(E) | g^{-1}C'_1\sigma(g) \cap Z(E)C'_2 \neq \phi\}$$

is compact.

Let

$$C_1 = \{Nh | h \in C'_1\}$$
$$C_2 = \{Nh | h \in C'_2\}.$$

If $g \in X$ then

$$g^{-1}C_1 g \cap Z(E)C_2 \neq \phi.$$

Since C_1 and C_2 are compact we have only to apply Lemma 7.4.2.

We may choose C'_2 so that $(\Pi(g)v, u)$ is supported by $Z(E)C'_2$. Then for h in C'_1

$$\int_{T'_r} (\Pi(g^{-1}h\sigma(g))v, u)dg$$

becomes constant as soon as r is so large that T'_r contains $Z(E)\backslash X$. Moreover

$$\left| \int_{T'_r} (\Pi(g^{-1}h\sigma(g))v, u)dg \right| \leq \int_{Z(E)\backslash G(E)} |(\Pi(g^{-1}h\sigma(g))v, u)|dg$$

and the right side is equal to

$$\text{(7.4)} \qquad \operatorname{meas}(Z(E)\backslash Z(E)G_h^\sigma(E)) \int_{Z(E)G_h^\sigma\backslash G(E)} |(\Pi(g^{-1}h\sigma(g))v, u)|dg.$$

To estimate this we may replace the integrand by a positive spherical function ϕ, invariant under $Z(E)$, which dominates it, and h by any σ-conjugate element. Thus we may suppose h lies in $T(E)$, where T is one of the representatives for the conjugacy classes of Cartan subgroups of G over F. It follows from the lemmas of Paragraph 5 that (7.4) is bounded by $c(\phi)\xi(h \times \sigma)$ for all $h \in \widetilde{G}^\sigma(E)$.

It remain to consider the behaviour of the sequence (7.3) on the set of h for which Nh has distinct eigenvalues in F. If $k \in G(O_E)$ then

$$\varphi_r(k^{-1}h\sigma(k)) = \varphi_r(h)$$

so we need only consider h of the form

$$\text{(7.5)} \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

If h is constrained to lie within a compact subset C_3 of $G(E)$ then α, β are constrained to lie in a compact subset of $E^\times$ and x in a compact subset of E. If C_3 lies in $\widetilde{G}^\sigma(E)$ then in addition $\left|1 - N\left(\frac{\beta}{\alpha}\right)\right|$ remains bounded away from 0. We are going to show that there is a constant c such that

$$|\varphi_r(h)| \le c\left|1 - N\left(\frac{\beta}{\alpha}\right)\right|^{-1} = c\xi(h \times \sigma)$$

for all h in C_3 of the form (7.5) and that if $C_3 \subset \widetilde{G}^\sigma(E)$ then $\{\varphi_r(h)\}$ converges uniformly on C_3 to a locally constant function. Lemma 7.9 will follow.

As on p. 269 of [14], we are immediately led to the consideration of auxiliary sequences

$$\varphi_r^i(h) = \int \left(\Pi(\begin{pmatrix} \gamma & y \\ 0 & 1 \end{pmatrix}^{-1} h\sigma(\begin{pmatrix} \gamma & y \\ 0 & 1 \end{pmatrix})v_i, u_i)\right) |\gamma|_E^{-1} d^\times\gamma dy.$$

The integral is taken over those γ and y for which

$$\text{(7.6)} \qquad \begin{pmatrix} \gamma & y \\ 0 & 1 \end{pmatrix}$$

lies in T_r.

The product

$$\begin{pmatrix} \gamma & y \\ 0 & 1 \end{pmatrix}^{-1} h\sigma(\begin{pmatrix} \gamma & y \\ 0 & 1 \end{pmatrix})$$

is equal to

$$\begin{pmatrix} \gamma^{-1}\sigma(\gamma)\alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \sigma(\gamma)^{-1}(x + \sigma(y) - \alpha^{-1}\beta y) \\ 0 & 1 \end{pmatrix}.$$

We set

$$u_i' = \Pi(\begin{pmatrix} 1 & -\sigma(\gamma)^{-1}x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1}\sigma(\gamma)^{-1}\gamma & 0 \\ 0 & \beta^{-1} \end{pmatrix})u_i,$$

so that the integrand becomes

$$(\Pi \begin{pmatrix} 1 & \sigma(\gamma)^{-1}(\sigma(y) - \alpha^{-1}\beta y) \\ 0 & 1 \end{pmatrix} v_i, u_i')|\gamma|_E^{-1}. \tag{7.7}$$

We shall first integrate with respect to y. To do this we must find those values of γ and y for which the matrix (7.6) is in T_r. Let $|\gamma| = |\varpi_E|^m, |y| = |\varpi_E|^n$, and let the elementary divisors of (7.6) be $\varpi_E^j, \varpi_E^k,\ j \le k$. We list the possible values of j and k below, together with the condition that the matrix belongs to T_r.

(i) $m \ge n,\ n \ge 0$ then $j = 0,\ k = m,$ $\quad 0 \le m \le r,$

(ii) $m \ge 0,\ n \le 0$ then $j = n,\ k = m - n,$ $\quad 0 \le m - 2n \le r,$

(iii) $m \le 0,\ n \le m$ then $j = n,\ k = m - n,$ $\quad 0 \le m - 2n \le r,$

(iv) $m \le 0,\ n \ge m$ then $j = m,\ k = 0,$ $\quad -r \le m \le 0.$

Thus the matrix belong to T_r if and only if $-r \le m \le r$ and $m - r \le 2n$.

To evaluate the integral of (7.7) with respect to y, we take Π in the Kirillov form. Then u_i is a locally constant function on $E^\times$ with compact support; so is v_i' and it is bounded by a constant which does not depend on α, β, γ, or x. The inner product appearing in (7.7) is equal to

$$\int_{E^\times} \psi_E\left(\frac{a}{\sigma(\gamma)}(\sigma(y) - \alpha^{-1}\beta y)\right) v_i(a)\overline{u_i'(a)}d^\times a.$$

Let b' be the smallest integer greater than or equal to $\frac{m-r}{2}$. The integral with respect to y is equal to

$$|\varpi_E|^{b'} \int_{\{a||\rho(a) - \frac{\beta}{\alpha}a| \le |\varpi_E|^{-b}\}} v_i(a\sigma(\gamma))\overline{u_i'(a\sigma(\gamma))}d^\times a \quad \rho = \sigma^{-1}. \tag{7.8}$$

Here b is equal to b' if the largest ideal on which ψ_E is trivial is O_E. If this ideal is (ϖ_E^s) then $b' - b = s$.

Let ε be a small positive number. Since α and β are constrained to vary in a compact set there is an integer e such that

$$\left|1 - N\left(\frac{\beta}{\alpha}\right)\right| \ge \varepsilon \tag{7.9}$$

and

$$\left|\rho(a) - \frac{\beta}{\alpha}a\right| \le |\varpi_E|^{-b}$$

implies

$$|a| \le |\varpi_E|^{-e-b}.$$

Choose an integer d so that the support of each u_i is contained in

$$|\varpi_E|^{-d} \geq |a| \geq |\varpi_E|^d.$$

The integral (7.8) is certainly 0 unless $-e-b \leq d-m$ or $m-b \leq d+c$. Then

$$\frac{m+r}{2} \leq d+e+1.$$

Since $m+r \geq 0$ this gives a bound on the number of possibilities for m which is independent of r. Since the integral appearing in (7.8) is clearly bounded by

$$\{\sup |u_i(a)|\} \{\sup |v_i'(a)|\} \int_{|\varpi_E|^{-d} \geq |a| \geq |\varpi_E|^d} d^\times a$$

and

$$|\varpi_E|^b |\gamma|_E^{-1} = O(|\varpi_E|^{-\frac{m+r}{2}})$$

there is no difficulty bounding $|\varphi_r(h)|$ on the set of h in C_3 for which (7.9) is satisfied. To show that the limit exists and yields a locally constant function we replace γ by $\gamma\varpi_E^{-r}$ so that m now satisfies $0 \leq m \leq 2(d+c+1)$. Then $|\gamma|_E^{-1}|\varpi_E|^b$ is replaced by $|\gamma|_E^{-1/2}$ if m is even and by $|\gamma|_E^{-1/2}|\varpi_E|^{1/2}$ if m is odd. The integration with respect to γ becomes an integration over a compact set which does not depend on r. The integral of (7.8) appears in the integrand. Replacing a by $\frac{a}{\sigma(\gamma)}$ in it, we obtain

$$\int_{\{a \| \rho(a) - \frac{\beta}{\alpha}\frac{\gamma}{\sigma(\gamma)} a| \leq |\gamma|_E^{1/2} |\varpi_E|^{-s}\}} v_i(a)\overline{u_i'(a)} da$$

where s is 0 or $\frac{1}{2}$ according as m is even or odd. Mutliplying by $|\gamma|_E^{-1/2}|\varpi_E|^s$ and then integrating with respect to γ, we obtain a locally constant function of h.

We have still to estimate $\varphi_r(h)$ when $N\left(\frac{\beta}{\alpha}\right)$ is close to 1. We may write

$$\frac{\beta}{\alpha} = \lambda \frac{\zeta}{\sigma(\zeta)}$$

with $\lambda \in F^\times$. Let $|\zeta| = |\varpi_E|^c$. We may so choose ζ and λ that c remains uniformly bounded. We also choose λ close to 1. Change variables in the integral of (7.8) so that it becomes

$$\int_{|\rho(a) - \lambda a| \leq |\varpi_E|^{-b-c}} v_i(a\sigma(\zeta(\gamma))\overline{u_i'(a\sigma(\zeta\gamma))} d^\times a. \tag{7.10}$$

Write $a = a_1 + a_2$ with $a_1 \in F$, $\operatorname{trace} a_2 = 0$. Since λ varies in a neighborhood of 1 there is an integer $f \leq 0$ such that $|1-\lambda|_E = |\varpi_E|^e$, with some integer e, and $|\rho(a) - \lambda a| \leq |\varpi_E|^{-b-c}$ together imply

(i) $|a_1| \leq |\varpi_E|^{-b-c-e+f}$

(ii) $|a_2| \leq |\varpi_E|^{-b-c+f}$.

The integral (7.10) may be estimated by a constant times the measure of the intersection of the set defined by (i), (ii), and

(iii) $|\varpi_E|^{d-m-c} \le |a| \le |\varpi_E|^{-d-m-c}$.

If $|\varpi_F|_E = |\varpi_E|^g$, where g is 1 or ℓ, this measure is not affected if I replace m by $m - zg$, and r by $r + zg$, $z \in \mathbf{Z}$. Thus I may work with a finite set of m – but at the cost of letting r vary. What I want is that $d - m - c$ should be, for all practical purposes, constant. Then, for purposes of estimation, the multiplicative Haar measure may be replaced by an additive one. Moreover b now differs from $-\frac{m+r}{2}$, which does not change, by a bounded constant.

The set is clearly empty unless

$$-b - c - e + f \le d - m - c$$

or

$$m - b \le e - f + d.$$

Taking the relations between m, r, b', b and s into account, we conclude that

$$0 \le m + r \le 2(e - f + d + s + 1).$$

Because of (iii) the absolute value $|a_1|$ remains bounded, independent of r, and, because of (ii), the absolute value $|a_2|$ is now bounded by a constant times $|\varpi_E|^{\frac{m+r}{2}}$. Thus the measure of the set may be estimated by a constant times $|\varpi_E|^{\frac{(m+r)(\ell-1)}{2\ell}}$.

Since $|\gamma|_E^{-1}|\varpi_E|^b$, with the original b, is bounded by a constant times $|\varpi_E|^{-\frac{(m+r)}{2}}$, the absolute value of $\varphi_r(h)$ is bounded by a constant times

$$\sum_{0 \le k \le 2(e-f+d+s+1)} |\varpi_E|^{-\frac{k}{2\ell}} = O\left(|1-\lambda|_E^{\frac{1}{\ell}}\right)$$

Since

$$|1-\lambda|_E^{\frac{1}{\ell}} = |1-\lambda|_F$$

and

$$1 - N\left(\frac{\beta}{\alpha}\right) = 1 - \lambda^\ell$$

while, because λ is close to 1,

$$|1-\lambda^\ell|_F = |\ell(1-\lambda)_F$$

the proof of the first assertion of Lemma 7.9 is completed.

The proof of the second will be briefer. Let ζ be a primitive ℓth root of unity and let ω be the character of $G'(E)$ which is 1 on $G(E)$ and takes σ to ζ. The representations $\Pi'_i = \omega^i \otimes \Pi'$, $0 \le i < \ell$, are inequivalent. If u is a unit vector,

$$\phi(g) = d(\Pi')\overline{(\Pi'(g)u, u)}$$

and

$$\Pi_i'(\phi) = \int_{Z(E)\backslash G'(E)} \phi(g)\Pi_i'(g)dg$$

then

$$\operatorname{trace}\Pi_i'(\phi) = \begin{cases} 1, & i = 0, \\ 0, & 1 \le i < \ell. \end{cases}$$

Thus

$$1 = \sum\nolimits_{i=0}^{\ell-1} \zeta^{-i}\operatorname{trace}\Pi_i'(\phi) = \ell \int_{Z(E)\backslash G(E)\times\sigma} \phi(g)\chi_{\Pi'}(g)dg.$$

By the Weyl integration formula the right-hand side is equal to

$$\frac{1}{2}\sum\nolimits_T \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} \chi_{\Pi'}(t\times\sigma)\Big\{\int_{Z(E)T(F)\backslash G(E)} \phi(g^{-1}t\sigma(g))dg\Big\}\Delta(Nt)^2 dt.$$

The sum is over a set of representatives for the conjugacy classes of Cartan subgroups of G over F. If T is non-split the inner integral equals

$$\frac{1}{\operatorname{meas} Z(F)\backslash T(F)} \int_{Z(E)\backslash G(E)} \phi(g^{-1}t\sigma(g))dg.$$

Since

$$\int_{Z(E)\backslash G(E)} \phi(g^{-1}t\sigma(g))dg = \int_{Z(E)\backslash G(E)} \phi(g^{-1}\tau(t)\sigma(g))dg$$

for all $\tau \in \mathfrak{g}$ this expression equals

$$\frac{1}{\ell \operatorname{meas} Z(F)\backslash T(F)} \int_{Z(E)\backslash G'(E)} \phi(g^{-1}(t\times\sigma)g)dg$$

or

$$\frac{1}{\operatorname{meas} NZ(E)\backslash T(F)} \int_{Z(E)\backslash G'(E)} \phi(g^{-1}(t\times\sigma)g)dg$$

and the proof of the first part of the lemma has shown us that the integral appearing here is equal to $\overline{\chi_{\Pi'}(\sigma\times t)}$.

If T is split the inner integral is equal to

$$\int_{K_E}\int_{Z(E)T(F)\backslash T(E)}\int_{N(E)} \phi(k^{-1}n^{-1}s^{-1}t\sigma(nsk)\times\sigma)\left|\frac{\alpha}{\beta}\right|^{-1} dndsdk.$$

Here

$$s = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Setting

$$n_1 = n^{-1}t(\sigma n)\sigma(t)^{-1}$$

and changing variables in the usual way, we deduce from Proposition 2.22 of [14] that the integral is 0. The assertion of the lemma follows.

We shall also need a relation of orthogonality.

Lemma 7.12 *Suppose* Π_1, Π_2 *satisfy*

$$\Pi_i(z) = \xi(z) \qquad z \in Z(E)$$

and

$$\Pi_i^\sigma \simeq \Pi_i.$$

Extend Π_i *to* Π_i'*, a representation of* $G'(E) = G(E) \times \mathfrak{G}$*. Suppose moreover that* Π_2 *is absolutely cuspidal, that* Π_1 *is unitary, and that* $\chi_{\Pi_1'}$ *exists as a locally integrable function. Then*

$$\frac{1}{2}\sum{}' \frac{1}{\operatorname{meas} NZ(E)\backslash T(F)} \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} \chi_{\Pi_1'}(t \times \sigma)\overline{\chi_{\Pi_2'}(t \times \sigma)}\Delta(Nt)^2 dt$$

is equal to 0 if Π_1 *is not equivalent to* Π_2*.*

Just as above we take

$$\phi(g) = d(\Pi_2')\overline{(\Pi_2'(g)u, u)} \qquad g \in G'(E)$$

with a unit vector u. Since the proof of Proposition 5.21 of [14], and therefore the proposition itself, is valid in the present situation

$$(\omega^i \otimes \Pi_1')(\phi) = 0$$

for each i. Therefore

$$\frac{1}{\ell}\sum_{i=0}^{\ell-1} \zeta^{-i}\operatorname{trace}(\omega^i \otimes \Pi_1')(\phi) = 0.$$

The left-hand side is equal to

$$\int_{G(E)\times\sigma} \chi_{\Pi_1'}(g)\phi(g)dg.$$

Applying the Weyl integration formula and proceeding as before, we obtain the lemma.

We are not yet in a position to show that Π is the lifting of a π. However, there are some further lemmas toward that end which we can prove now. The map $t \to Nt$ imbeds $Z(E)T^{1-\sigma}(E)\backslash T(E)$ into $N_{E/F}Z(E)\backslash T(F)$ and is measure-preserving. Let ω be a non-trivial character of $N_{E/F}E^\times\backslash F^\times$.

Lemma 7.13 *If* π *is square-integrable then*

$$\frac{1}{2}\sum{}' \frac{1}{\operatorname{meas} NZ(E)\backslash T(F)} \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} |\chi_\pi(Nt)|^2\Delta(Nt)^2 dt = \begin{cases} 1 & \pi \simeq \omega\otimes\pi \\ \frac{1}{\ell} & \pi \not\simeq \omega\otimes\pi. \end{cases}$$

The sum is again over a set of representatives for the conjugacy classes of non-split Cartan subgroups over F*.*

If T is anisotropic then

$$\{Nt | t \in T(E)\} = \{s \in T(F) | \omega(\det s) = 1\}.$$

Thus, if $\pi_i = \omega^i \otimes \pi$,

$$\frac{1}{\ell}\sum_{i=0}^{\ell-1} \chi_{\pi_i}(s)$$

is 0 outside of this set and equal to $\chi_\pi(s)$ on it. Applying the orthogonality relations for characters of $G(F)$ (Proposition 15.4 of [14]) to this function, we obtain the lemma.

The same argument shows that if π_1 and π_2 are square-integrable and $\pi_1 \simeq \omega^i \otimes \pi_2$ for no i then

$$\frac{1}{2}\sum{}' \frac{1}{\operatorname{meas} NZ(E)\backslash T(F)} \int_{Z(E)T^{1-\sigma}(E)\backslash T(E)} \chi_{\pi_1}(Nt)\overline{\chi_{\pi_2}(Nt)}\Delta(Nt)^2 dt = 0.$$

Now we take a set of representatives T for the conjugacy classes of all Cartan subgroups of G over F. We want to introduce a collection $\mathfrak{S}$ of functions on

$$X = \bigcup\nolimits_T NT(E)$$

or rather on the regular elements therein. We introduce, for the sole purpose of defining this collection, an equivalence relation $\sim$ on the set of classes of irreducible admissible representations of $G(F)$. We write $\pi(\mu,\nu) \sim \pi(\mu',\nu')$ if for some i and $j, \mu' = \omega^i\mu, \nu' = \omega^j\nu$ and if $\pi(\mu,\nu), \pi(\mu',\nu')$ are both infinite-dimensional or both finite-dimensional. If π is square integrable we write $\pi \sim \pi'$ if $\pi' = \omega^i \otimes \pi$ for some i. It is clear that χ_π and $\chi_{\pi'}$ agree on X if $\pi \sim \pi'$. $\mathfrak{S}$ will be the collection of χ_π, with π varying over the equivalence classes.

If $\pi(\mu,\nu)$ is finite-dimensional let $\sigma(\mu,\nu)$ be the representation complementary to $\pi(\mu,\nu)$ in $\rho(\mu,\nu)$. The representation $\pi(\mu,\omega\nu)$ is infinite-dimensional and

(7.10)
$$\chi_{\pi(\mu,\nu)} + \chi_{\sigma(\mu,\nu)} = \chi_{\pi(\mu,\omega\nu)}$$

on X.

Lemma 7.14 *Every linear relation amongst the function in $\mathfrak{S}$ is a consequence of the relations* (7.10).

If this were not so there would be a relation which did not involve the χ_π, π finite-dimensional. The orthogonality relations then show that it involves no χ_π, π square-integrable. Therefore it involves only the $\chi_{\pi(\mu,\nu)}, \pi(\mu,\nu)$ infinite-dimensional, and the explicit expression for $\chi_{\pi(\mu,\nu)}$ in terms of μ,ν shows that the relation must be trivial.

There is one simple point which needs to be observed.

Lemma 7.15 *If $\pi(\mu,\nu)$ is finite-dimensional then $\pi(\mu,\omega\nu)$ is not unitary.*

First take $R = \mathbf{R}$. By Lemma 5.11 of [14]

$$\mu\nu^{-1}\omega^{-1} : t \to t^p$$

where p is a non-zero integer. Therefore

$$\overline{\mu}^{-1}\overline{\nu}\omega^{-1} : t \to t^{-p}$$

and $\pi(\mu,\omega\nu)$ can be unitary only if $\overline{\mu}^{-1}=\omega\nu, \omega\overline{\nu}^{-1}=\mu$. Then

$$\mu\overline{\mu}: t \to t^p$$

and p is even. The space of $\pi=\pi(\mu,\omega\nu)$ then contains a vector v invariant under $\mathrm{SO}(2,\mathbf{R})$. Standard formulae for spherical functions [31] show that the matrix coefficient $(\pi(g)u,u)$ must be unbounded. This is incompatible with unitarity.

Now take F to be non-archimedean. Then

$$\mu\nu^{-1}\omega^{-1}: x \to |x|^{\pm 1}\omega^{-1}(x)$$

and

$$\overline{\mu}^{-1}\overline{\nu}\omega^{-1}: x \to |x|^{\mp 1}\omega^{-1}(x).$$

Consequently $\pi(\mu,\omega\nu)$ can be unitary only if $\overline{\mu}^{-1}=\mu\nu, \omega\overline{\nu}^{-1}=\mu$. Then

$$\mu\overline{\mu}: x \to |x|^{\pm 1}\omega^{-1}(x).$$

This, fortunately, is a patent impossibility.

Lemma 7.16 *Choose for each representation Π of $G(E)$ invariant under $\mathfrak{G}$ an extension Π' to $G'(E)$. Then the restrictions of the characters $\chi_{|p'}$ to $G(E)\times\mathfrak{G}$ are linearly independent.*

Copying the proof of Lemma 7.1 of [14], one shows that the characters of the irreducible admissible representations of $G'(E)$ are linearly independent. If $\Pi'_i=\omega'\otimes\Pi'$ as before then

$$\frac{1}{\ell}\sum\nolimits_i \omega^{-i}(\sigma)\chi_{\Pi'_i}(g)$$

is 0 except for $g\in G(E)\times\sigma$, when it equals $\chi_{\Pi'}(g)$.

The most important fact about local lifting which remains to be proved is

(3) *If π is an absolutely cuspidal representation of $G(F)$ then π has a lifting in the sense of criterion*
(ii) *of* §2. *It is independent of σ.*

This will be proved in §11. See especially Proposition 11.5. I observe now that the results of this paragraph, including Lemma 7.17, which is to follow, imply, when taken together with the assertions (1), (2), and (3), the results (a)-(g) of §2, except for (e), which appears as Lemma 11.8.

It follows from (3) and Lemma 7.6 that every representation π has a lifting. Moreover it follows from Lemma 7.16 that it only has one lifting that satisfies (ii). Thus the unicity of the lifting could fail only if π had one lifting in the sense (i) and another in the sense (ii). By Corollary 7.3 and Lemmas 7.4 and 7.5 this could only happen if $\pi=\pi(\mu,\omega\nu)$ with $\mu\nu^{-1}(x)\equiv|x|^{\pm 1}$. Since π is a principal series representation and its lifting in the sense (i) is special, this lifting cannot also be a lifting in the sense (ii). It follows from Lemmas 7.4 and 7.9 that a lifting in the sense (ii) cannot be cuspidal, and from Lemma 7.14 that it cannot be a $\pi(\mu', \nu')$. So π has no lifting in the sense (ii), and the unicity is established.

By definition Π can be a lifting only if $\Pi^\tau \sim \Pi$ for all $\tau \in \mathfrak{g}$. If this condition is satisfied then, by Corollary 7.3 and Lemmas 7.4 and 7.5, Π is a lifting except perhaps when it is cuspidal. That it is a lifting when it is cuspidal follows from (3), Lemmas 7.9 and 7.12, and the completeness of the characters of square-integrable representations of $G(F)$, a consequence of Lemma 15.1 of [14]. The property (c) follows from Lemma 7.14 and 7.17; and (d) is a formality, as is (f). In so far as (g) is not an immediate consequence of the definitions and the unicity, it is a consequence of (3) and Lemma 7.6.

In our scheme for proving the results of these notes, the following lemma plays a critical role. It is a shame that our proof is so uninspired.

Lemma 7.17 *Suppose π is an irreducible, admissible representation of $G(F)$ and $\pi \simeq \omega \otimes \pi$. Then $\ell = 2$ and there is a quasi-character θ of $E^\times$ such that $\pi = \pi(\tau)$ with*

$$\tau = \mathrm{Ind}(W_{E/F}, W_{E/E}, \theta).$$

Moreover if $\pi = \pi(\tau)$ then $\pi \simeq \omega \otimes \pi$.

Suppose $\pi = \pi(\mu, \nu)$. Then $\omega \otimes \pi = \pi(\omega\mu, \mu\nu)$. Thus $\pi \simeq \omega \otimes \pi$ if and only if $\mu = \omega\nu, \nu = \omega\mu$; so $\omega^2 = 1$ and $\ell = 2$. If $\ell = 2$ and $\theta(x) = \mu(Nx)$, then $\pi(\tau) = \pi(\mu, \omega\mu)$ ([14], Theorem 4.6 together with the remarks on p.180). If F is non-archimedean and $\pi = \sigma(\mu, \nu)$ then $\pi \not\simeq \omega \otimes \pi$. (This follows readily from Proposition 3.6 of [14]). If F is $\mathbf{R}$ every square-integrable representation is a $\pi(\tau)$ for some τ, and it follows from Theorem 5.11 of [14] that $\pi(\tau \simeq \omega \otimes \pi(\tau)$.

Suppose finally that F is non-archimedean and π is absolutely cuspidal. We may as well suppose also that π is unitary. Let

$$G_+(F) = \{g \in G(F) | \omega(\det g) = 1\}.$$

We begin by remarking that if $\pi \simeq \omega \otimes \pi$ then the restriction of π to $G_+(F)$ is reducible. Indeed suppose the restriction were irreducible. There is an operator A on the space of π such that

$$A\pi(g)A^{-1} = \omega(\det g)\pi(g) \qquad g \in G(F).$$

The irreducibility and the admissibility of the restriction of π to $G_+(G)$ together imply that A is a scalar. We deduce a contradiction, viz.,

$$A\pi(g)A^{-1} = \pi(g) \qquad g \in G(F).$$

We also see immediately that ℓ must be 2, for if ℓ is odd ω is not trivial on $Z(F)$.

Take $\ell = 2$ and let π^+ be one of the irreducible components of the restriction of π to $G_+(F)$. Let π act on V. Define a $G(F)$-invariant map from the space of $r = \mathrm{Ind}(G(F), G_+(F), \pi^+)$ to V by

$$\varphi \to \sum\nolimits_{G_+(F)\backslash G(F)} \pi(g^{-1})\phi(g).$$

If π^+ extended to a representation π' of $G(F)$ then $\pi' \not\simeq \omega \otimes \pi'$,

$$r = \pi' \oplus (\omega \otimes \pi')$$

and $\pi \simeq \pi'$ or $\pi \simeq \omega \otimes \pi'$. This is impossible if $\pi \simeq \omega \otimes \pi$.

Choose h in $G(F)$ with $\omega(\det h) = -1$ and set $\pi^-(g) = \pi^+(h^{-1}gh)$. We conclude that if $\pi \simeq \omega \otimes \pi$ then $\pi^+ \not\simeq \pi^-$ and the restriction of π to $G(F)$ is $\pi^+ \oplus \pi^-$. A straightforward imitation of the proof of Proposition 7.4 of [14] shows that the characters of π^+ and π^- exist as locally-integrable functions on $G_+(F)$. For a t in $G_+(F)$ with distinct eigenvalues we define

$$\chi_\pi^+(t) = \chi_{\pi^+}(t) - \chi_{\pi^-}(t).$$

Let T be the Cartan subgroup to which t belongs. If T is split or the quadratic extension determined by T is not isomorphic to E there is an $s \in T(F)$ with $\omega(\det s) = -1$. Then

$$\chi_{\pi^-}(t) = \chi_{\pi^+}(s^{-1}ts) = \chi_{\pi^+}(t)$$

and

$$\chi_\pi^+(t) = 0.$$

I observe that the function χ_π^+ can be defined for any π for which $\pi \simeq \omega \otimes \pi$, provided that $\chi_{\pi^+}, \chi_{\pi^-}$ are known, for some reason or another, to exist as functions. It is only determined up to sign.

Choose a Cartan subgroup T so that the corresponding quadratic extension is isomorphic to E. The orthogonality relations for $G_+(F)$ yield the following lemma.

Lemma 7.18 *Suppose π is unitary and absolutely cuspidal and the function $\chi_{\pi'}^+$ is defined. Suppose also that the restrictions of π and π' to $Z(F)$ are the same. If π is not equivalent to π' then*

$$\int_{Z(F)\backslash T(F)} \chi_{\pi'}^+(t)\overline{\chi_\pi^+(t)}\Delta(t)^2 dt = 0.$$

If w lies in the normalizer of T in $G(F)$ but not in $T(F)$ then

$$\chi_\pi^+(wtw^{-1}) = \omega(\det w)\chi_\pi^+(t)$$

and the standard theory of crossed products shows that $\omega(\det w) = \omega(-1)$. Fix a regular t_0 in $T(F)$ with eigenvalues a_0, b_0. The ordering a_0, b_0 determines an order of the eigenvalues a, b of any t in $T(F)$. If θ is a quasi-character of E, which we identify with $T(F)$, we set

$$\chi_\theta(t) = \lambda(E/F,\psi)\omega\left(\frac{a-b}{a_0-b_0}\right)\frac{\theta(t)+\theta(wtw^{-1})}{\Delta(t)}.$$

Here ψ is a fixed non-trivial character of F and $\lambda(E/F,\psi)$ is defined as in [21]. We extend χ_θ to a locally integrable function on $G(F)$ by setting $\chi_\theta(g) = 0$ unless $g = h^{-1}th$ with t regular in $T(F)$ when we set

$$\chi_\theta(g) = \omega(\det h)\chi_\theta(t).$$

Lemma 7.17 is now a consequence of the completeness theorem for characters of $Z(F)\backslash T(F)$ and the following lemma.

Lemma 7.19 *Suppose*

$$\tau = \mathrm{Ind}(W_{E/F}, W_{E/E}, \theta)$$

and $\pi \simeq \pi(\tau)$. Then $\pi \simeq \omega \otimes \pi$, and χ_π^+ exists as a function and is equal to $\pm\chi_\theta$.

This is the lemma with the embarrassing proof. For $F = \mathbf{R}$ satisfactory proofs are available; but they are not elementary. A quick proof which is neither satisfactory nor elementary can be obtained along the following lines. It follows from the results of §5 of [14] that if $\pi = \pi(\tau)$ then $\pi \simeq \omega \otimes \pi$, and it follows from general results of Harish-Chandra that χ_π^+ is defined as a function. To compute it one has to find χ_{π^+} and χ_{π^-} on the regular elements of the non-split Cartan subgroup. For this it is enough to know the K-type of π^+ and π^- and that is given in §5 of [14]. No more need be said.

For non-archimedean fields it is possible to deduce the lemma from known formulae for the characters. Since it is harder to prove these formulae than to prove the lemma, and since no satisfactory proof of it, elementary or otherwise, is available, it is perhaps not entirely profitless to run through a verification by computation.

According to Theorem 4.6 of [14], in which the quasi-character θ is denoted ω and $\pi(\tau)$ is denoted $\pi(\omega)$, the representation $\pi(\tau)$ restricted to $G_+(F)$ is reducible. So $\pi(\tau) \simeq \omega\otimes\pi(\tau)$. We may take π^+ to be the representation $\pi(\theta,\psi)$ of that theorem. To show that χ_{π^+} and χ_{π^-} exist as functions, all we need do is show that the distribution $\chi_{\pi^+} - \chi_{\pi^-}$ is a function, for this is already known to be true for χ_π. I observe that the proof of the lemma which will now be given is also valid for completions of function fields.

As in [14] we realize $\pi^+ = \pi(\theta,\psi)$ on a space V_+ of functions on F_+. We need a representation of $\pi^+(g)$ as an integral operator when g is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $c \neq 0$. We may write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & b - \frac{ab}{c} \end{pmatrix}.$$

This allows us to effect the transformation $\varphi \to \pi^+(g)\varphi$ in three steps. The first is to replace the function φ by

$$u \to \omega\theta\left(\frac{ad-bc}{-c}\right) \psi\left(\frac{cdu}{ad-bc}\right) \varphi\left(\frac{c^2u}{ad-bc}\right).$$

In general the transformation

$$\pi^+(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

sends φ to φ' with

$$\varphi'(u) = \lambda(E/F, \psi)\theta(x)|x|_E^{1/2} \int_E \psi_E(x\bar{y})\theta^{-1}(y)|y|_E^{-1/2}\varphi(Ny)dy.$$

Here $Nx = u$,

$$\psi_E(w) = \psi(\mathrm{trace}_{E/F} w), \qquad w \in E,$$

and the bar denotes the non-trivial automorphism of E over F. The measure on E is also to be self-dual with respect to ψ_E.

The map $y \to Ny$ together with the measures on E and F self-dual with respect to ψ_E and ψ yields a measure on

$$\{y|Ny = u\}.$$

We set

$$J(u,v) = \lambda(E/F, \psi)\theta(u)|u|_F \int_{Ny=uv} \psi_E(y)\theta^{-1}(y)|y|_E^{-1/2}$$

and then

$$\varphi'(u) = \int_{F_+} J(u,v)\phi(v)dv.$$

Observe that

$$\int_{Ny=uNx} f(y) = \int_{Ny=u} f(xy).$$

Thus the second step takes us to the function

$$u \to \omega\theta\left(\frac{ad-bc}{-c}\right) \int_{F_+} J(u,v)\psi\left(\frac{cdv}{ad-bc}\right) \varphi\left(\frac{c^2v}{ad-bc}\right) dv$$

and the third to the function

$$u \to \omega\theta\left(\frac{ad-bc}{-c}\right) \psi\left(\frac{au}{c}\right) \int_{F_+} J(u,v)\psi\left(\frac{cdv}{ad-bc}\right) \varphi\left(\frac{c^2v}{ad-bc}\right) dv.$$

Changing variables we see that $\pi^+(g)$ is an integral operator with kernel

$$\left|\frac{ad-bc}{c^2}\right| \omega\theta\left(\frac{ad-bc}{-c}\right) \psi\left(\frac{au}{c}\right) \psi\left(\frac{dv}{c}\right) J\left(u, \frac{ad-bc}{c^2}v\right).$$

If f is a locally constant function on $G_+(F)$ with a support which is compact and does not meet the group of triangular matrices then

$$\pi^+(f) = \int_{G_+(F)} f(g)\pi^+(g)dg$$

is an operator of trace class and is defined by a kernel

$$F(u,v) = \int_{G_+(F)} \left|\frac{ad-bc}{c^2}\right| \omega\theta\left(\frac{ad-bc}{-c}\right) \psi\left(\frac{au+dv}{c}\right) J\left(u, \frac{ad-bc}{c^2}v\right) f(g)dg.$$

At the cost of multiplying π by a one-dimensional representation, I may suppose that θ is a character. Then there is an inner product on V_+ with respect to which the operators $\pi^+(g)$ are unitary. On $\mathcal{S}(F_+)$ this must be the inner product of Proposition 2.21.2 of [14]. If φ is orthogonal to $\mathcal{S}(F_+)$ then

$$(\pi^+(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\psi - \psi, \varphi) = 0$$

for all ψ and all x. As a consequence

$$\pi^+(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\varphi = \varphi$$

for all x. This we know to be impossible. We conclude that if E_N is the orthogonal projection on the space of functions supported by

$$\{x \in F_+ | \frac{1}{N} \leq |x| \leq N\}$$

then $E_N\varphi$ is equal to φ on this set and to 0 off it, and

$$\operatorname{trace} \pi^+(f) = \lim_{N\to\infty} \operatorname{trace} E_N \pi^+(f) E_N.$$

The kernel of $E_N\pi^+(f)E_N$ is $F(u,v)$ if $\frac{1}{N} \leq |u|,\ |v| \leq N$ and 0 otherwise. The trace is obtained by integrating the kernel along the diagonal.

The trace of $E_N\pi^+(f)E_N$ is obtained by taking the integral over $G_+(F)$ of the product of the expression $\lambda(E/F,\psi)f(g)$ with

$$\left|\frac{ad-bc}{c^2}\right| \omega\theta\left(\frac{ad-bc}{-c}\right) \int_{\frac{1}{N}\leq|u|\leq N} \psi\left(\frac{u(a+d)}{c}\right) \theta(u)|u|_F \left\{\int_{Ny=\frac{(ad-bc)u^2}{c^2}} \psi_E(y)\theta^{-1}(y)|y|_E^{-1/2}\right\} du.$$

If $\alpha = \det g$ and $\beta = \operatorname{trace} g$ this may be written

$$|\alpha| \frac{\theta(\alpha)}{\omega(-c)} \int_{\frac{1}{N|c|}\leq|u|\leq\frac{N}{|c|}} \psi(-\beta u)\theta(u)|u|_F \left\{\int_{Ny=\alpha u^2} \psi_E(y)\theta^{-1}(y)|y|_E^{-1/2}\right\} du.$$

It is understood that, in addition to the constraints explicitly given,

$$\omega(-cu) = 1.$$

The representation π^- is $\pi(\theta,\psi')$ where $\psi'(x)=\psi(ex)$ with $\omega(e)=-1$. It follows readily that

$$\chi_{\pi^+}(f)-\chi_{\pi^-}(f)$$

is equal to the limit as N approaches infinity of the integral over $G_+(F)$ of the product of $f(g)$ with

$$\lambda(E/F,\psi)|\alpha|\frac{\theta(\alpha)}{\omega(-c)}\int_{\frac{1}{N|c|}\le|u|\le\frac{N}{|c|}}\psi(-\beta u)\theta(u)|u|_F\left\{\int_{Ny=\alpha u^2}\psi_E(y)\theta^{-1}(y)|y|_E^{-1/2}\right\}du.$$

There is now no constraint on the value of $\omega(-cu)$.

Since we may confine the integration with respect to g to a region in which $|c|$ is bounded below by a positive constant and since f has compact support, the lower limit in the integration with respect to u may be taken to be 0. A change of variables in the inner integral yields

$$\lambda(E/F,\psi)\frac{|\alpha|\theta(\alpha)}{\omega(-c)}\int_{Ny=\alpha}\left\{\int_{|u|\le\frac{N}{|c|}}\psi(u(try-\beta))du\right\}\theta^{-1}(y)|y|_E^{-1/2}=\lambda_N(g).$$

We may suppose that N is approaching infinity through powers of $|\varpi|$, where ϖ is a uniformizing parameter. The inner integral is 0 if $|try-\beta|>\frac{|c|\ |\varpi|^{-n}}{N}$ and

$$\frac{N}{|c|}\,|\varpi|^{\frac{n}{2}}$$

otherwise. Here $\varpi^{-n}O_F$ is the largest ideal on which ψ is trivial.

Let $E=F(\delta)$ and let $y=a+b\delta$. If $r=\operatorname{trace}y$, $s=Ny$ then

$$\begin{vmatrix}\frac{\partial r}{\partial a} & \frac{\partial r}{\partial b}\\[2mm] \frac{\partial s}{\partial a} & \frac{\partial s}{\partial b}\end{vmatrix}=-b(\delta-\bar\delta)^2.$$

We may as well suppose that the largest ideal on which ψ is trivial is simply O_F. The self-dual measure on E is $|(\delta-\bar\delta)^2|_F^{1/2}dadb$ which equals

$$\frac{drds}{|(y-\bar y)^2|_F^{1/2}}.$$

If the support of f does not meet the set of matrices with equal eigenvalues, that is, the set where $\beta^2=4\alpha$, then for N large and $|c|$ bounded the relations

$$Ny=\alpha,\qquad |\operatorname{tr}y-\beta|\le\frac{|c|}{N},\quad f(g)\ne0,\qquad y\in E,$$

imply

$$|(y-\bar y)^2|_F=|(\operatorname{tr}y)^2-4Ny|_F=|(\operatorname{tr}y)^2-\beta^2+\beta^2-4\alpha|=|\beta^2-4\alpha|.$$

We conclude that $\lambda_N(g)$ remains bounded on the support of f. If the Cartan subgroup in which g lies is not conjugate to T then $\lambda_N(g)$ is 0 for large N. Otherwise it is

$$\frac{\lambda(E/F,\psi)\theta(\alpha)}{\omega(-c)\Delta(g)}\{\theta^{-1}(y)+\theta^{-1}(\bar{y})\} \tag{7.11}$$

if g is conjugate to y, because

$$|\alpha|^{1/2}=|y|_E^{-1/2}$$

and

$$\Delta(g)=\frac{|\beta^2-4\alpha|^{1/2}}{|\alpha|^{1/2}}.$$

Notice that

$$\theta(\alpha)\theta^{-1}(y)=\theta(\bar{y}) \qquad \theta(\alpha)\theta^{-1}(\bar{y})=\theta(y).$$

Suppose $g=t$ lies in $T(F)$. We have identified $T(F)$ with $E^\times$; but how, for the identification is not canonical? It does not matter for only the sign of $\chi_\theta(t)$ is affected. We may for example send $a+b\delta$ to

$$\begin{pmatrix} a & ub \\ b & a+bv \end{pmatrix}$$

if $\delta^2=u+v\delta$. If g corresponds to $y=a+b\delta$ the lower left-hand entry c of g is b and

$$-b=\frac{y-\bar{y}}{\bar{\delta}-\delta}.$$

Thus if we choose t_0 to correspond to $\bar{\delta}$ then (7.11) is equal to $\chi_\theta(t)$.

If θ does not factor through the norm map then π is absolutely cuspidal and $\chi_{\pi^+}, \chi_{\pi^-}$ are known to exist as functions; so the lemma is proved for such a θ. For a θ which factors through the norm we choose a θ_1 which agrees with it on $F^\times$ but does not so factor. To distinguish the two possibilities we write $\lambda_N(g,\theta)$ and $\lambda_N(g,\theta_1)$.

If we can show that there is a locally integrable function $\lambda(g)$ such that

$$\lim_{N\to\infty}\int_{G_+(F)} f(g)\lambda_N(g,\theta)dg=\int_{G_+(F)} f(g)\lambda(g)dg$$

when the support of f does not meet the group of triangular matrices we can conclude that $\lambda(g)=\chi_\theta(g)$ (provided the imbedding and t_0 are chosen as above) and that $\chi_{\pi^+}-\chi_{\pi^-}$ is a function outside the set of scalar matrices, and equals χ_θ there. Since we know that

$$\lim_{N\to\infty}\int_{G_+(F)} f(g)\lambda_N(g,\theta_1)dg=\int_{G_+(F)} f(g)\chi_{\theta_1}(g)dg$$

it is enough to establish the existence of a locally integrable η for which

$$\lim\int_{G_+(F)} f(g)(\lambda_N(g,\theta)-\lambda_N(g,\theta_1))dg=\int_{G_+(F)} f(g)\eta(g)dg.$$

For this it is sufficient to show that

(7.12) $$N\int_{Ny=\alpha}\chi_N(\mathrm{trace}\,y-\beta)|\theta^{-1}(y)-\theta_1^{-1}(y)|\;|y|_E^{-1/2}$$

remains bounded as α varies in a compact subset C of $F^\times$ and β in a compact subset of F. Here χ_N is the characteristic function of

$$\{u|\;|u|\le N\}.$$

Clearly there is a constant $\varepsilon\in 0$ such that

$$Ny\in C\qquad \theta^{-1}(y)-\theta_1^{-1}(y)\ne 0,$$

imply

$$|(y-\bar{y})^2|_F^{1/2}\ge\varepsilon.$$

It follows that the expression (7.12) is at most $\frac{2}{\varepsilon}$ for $\alpha\in C$.

We can now assert that the difference between the distribution $\chi_\pi^+=\chi_{\pi^+}-\chi_{\pi^-}$ and the distribution defined by the function χ_θ is concentrated on the scalar matrices. Therefore it is of the form

$$f\to a\int_{Z(F)}\xi(z)f(z)dz$$

if

$$\pi(z)=\xi(z),\qquad z\in Z(F).$$

Here a is a constant. Since the distribution $\chi_\pi=\chi_{\pi^+}+\chi_{\pi^-}$ exists as a function, we infer that a locally integrable function ζ satisfying

$$\mathrm{trace}\,\pi_+(f)=\frac{a}{2}\int_{Z(F)}\xi(z)f(z)dz+\int_{G_+(F)}f(g)\zeta(g)dg$$

exists. Let

$$K_n=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G(O)|a-1\equiv d-1\equiv b\equiv c\,(\mathrm{mod}\,\omega^n)\right)$$

and let $f_n(g)$ be 0 unless $g=zk,\ z\in Z(O),\ k\in K_n$ when $f_n(g)=\xi^{-1}(z)$. The function f_n is well defined for n large, and, since χ_π exists as a locally integrable function

$$\lim_{n\to\infty}\mathrm{trace}\,\pi(f_n)=0.$$

However

$$\mathrm{trace}\,\pi(f_n)\ge\mathrm{trace}\,\pi^+(f_n)\ge 0$$

and

$$\lim_{n\to\infty}\mathrm{trace}\,\pi^+(f_n)=\frac{a}{2}\int_{Z(O)}dz.$$

It follows that a is 0.

8. CONVOLUTION

Suppose F is a local field and E is a direct sum of ℓ copies of F on which the group $\mathfrak{G}$ of order ℓ acts by cyclic permutation. The correct notion of the lifting of an irreducible admissible representation of $G(F)$ to $G(E) = G(F) \times \cdots \times G(F)$ is patent: the representation π lifts to $\Pi = \pi \otimes \cdots \otimes \pi$. But there are some auxiliary constructions to be clarified.

The associate group ${}^L\overline{G}$ of $G \times \cdots \times G$ is a direct product

$$\mathrm{GL}(2, \mathbf{C}) \times \cdots \times \mathrm{GL}(2, \mathbf{C}) \times \mathfrak{G}(K/F).$$

There is an obvious homomorphism of ${}^LG$ to ${}^L\overline{G}$

$$\varphi : g \times \tau \to (g, \cdots, g) \times \tau.$$

The corresponding homomorphism of Hecke algebras takes a function ϕ of the form

$$\phi(g_1, \cdots, g_\ell) = f_1(g_1)f_2(g_2) \cdots f_\ell(g_\ell)$$

to the convolution

$$f = f_1 * \cdots * f_\ell.$$

For our purposes it is simplest to consider only functions, spherical or not, of the form

$$\phi : (g_1, \cdots, g_\ell) \to f_1(g_1) \cdots f_\ell(g_\ell)$$

and to define the map $\phi \to f$, which will play the same role as those introduced in Paragraphs 5 and 6, by the convolution product

$$f = f_1 * \cdots * f_\ell.$$

It is implicit that the factors of $\overline{G}$ have been ordered. The order is not important provided that $\sigma : (g_1, \cdots, g_\ell) \to (g_2, \cdots, g_\ell, g_1)$ is a generator of $\mathfrak{G}$.

If

$$\delta = (\delta_1, \cdots, \delta_\ell)$$

then

$$N\delta = (\delta_1 \cdots \delta_\ell, \delta_2 \cdots \delta_\ell \delta_1, \cdots, \delta_\ell \delta_1 \cdots \delta_{\ell-1})$$

is conjugate to

$$N\delta = (\delta_1 \cdots \delta_\ell, \delta_1 \cdots \delta_\ell, \cdots, \delta_1 \cdots \delta_\ell) = (\gamma, \cdots, \gamma)$$

which lies in $G(F)$, if $G(F)$ is identified with the set of fixed points of $\mathfrak{G}$ in $G(E)$.

The integral

$$\int_{G_\delta^\sigma(E)\backslash G(E)} \phi(h^{-1}\delta\sigma(h))dh$$

when written out in full becomes

$$\int_{G_\delta^\sigma(E)\backslash G(E)} f_1(h_1^{-1}\delta_1 h_2)\cdots f_{\ell-1}(h_{\ell-1}^{-1}\delta_{\ell-1}h_\ell)f_\ell(h_\ell^{-1}\delta_\ell h_1)dh.$$

We introduce new variables by

$$g_1 = h_1, g_2 = h_2^{-1}\delta_2\cdots\delta_\ell h_1, \cdots, g_{\ell-1} = h_{\ell-1}^{-1}\delta_{\ell-1}\delta_\ell h_1, g_\ell = h_\ell^{-1}\delta_\ell h_1.$$

Then $G_\delta^\sigma(E)$ becomes

$$\{(g,1,\cdots,1)|g\in G_\gamma(F)\}$$

and the integral itself becomes

$$\int_{G_\gamma(F)\backslash G(F)\times G(F)\times\cdots\times G(F)} f_1(g_1^{-1}\gamma g_1 g_2^{-1})f_2(g_2 g_3^{-1})\cdots f_\ell(g_\ell)dg_1\cdots dg_\ell$$

which is

$$\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg.$$

Suppose π is an irreducible admissible representation of $G(F)$ on V and let $\Pi = \pi\otimes\cdots\otimes\pi$. We extend Π to a representation Π' of $G(E)\times\mathfrak{G}$ by letting

$$\Pi'(\sigma) : v^1\otimes\cdots\otimes v^\ell \to v^2\otimes v^3\otimes\cdots\otimes v^\ell\otimes v^1.$$

We choose a basis $\{v_i\}$ for V so that

$$\operatorname{trace}\pi(f) = \sum \pi_{ii}(f)$$

if f is a compactly supported smooth function on $G(F)$. The matrix of

$$\Pi(\phi)\Pi'(\sigma) = (\pi(f_1)\otimes\cdots\otimes\pi(f_\ell))\Pi'(\sigma)$$

with respect to the basis $\{v_{i_1}\otimes\cdots\otimes v_{i_\ell}\}$ is

$$\pi_{i_1 j_2}(f_1)\pi_{i_2 j_3}(f_3)\cdots\pi_{i_\ell j_1}(f_\ell)$$

and its trace is

$$\sum \pi_{i_1 i_2}(f_1)\pi_{i_2,i_3}(f_3)\cdots\pi_{i_\ell i_1}(f_\ell) = \operatorname{trace}\pi(f_1 * \cdots * f_\ell) = \operatorname{trace}\pi(f).$$

Here f is the image of ϕ.

Since the character of π is a locally integrable function χ_π the trace of $\Pi(\sigma)\Pi'(\sigma)$ is equal to

$$\int_{\overline{G}(F)} f_1(g_1g_2^{-1})f_2(g_2g_3^{-1})\cdots f_\ell(g_\ell)dg_1\cdots dg_\ell.$$

If we change variables this integral becomes

$$\int_{\overline{G}(F)} \phi(g_1,\cdots,g_\ell)\chi_\pi(g_1\cdots g_\ell)dg.$$

Thus $\chi_{\Pi'}$ is a locally integrable function on $G(E)\times\sigma$ and $\chi_{\Pi'}(g\times\sigma)$ is $\chi_\pi(h)$ if h is Ng and has distinct eigenvalues.

It will be important to know the range of the map $\phi\to f$. It is clear that it is surjective if F is non-archimedean, that is, every smooth compactly supported f is the image of some smooth compactly supported ϕ. If F is archimedean we can apparently obtain all smooth f if we only demand that ϕ be highly differentiable and in addition allow finite linear combinations of the simple functions $\phi(g)=f_1(g_1)\cdots f_\ell(g_\ell)$ ([8], [19]). This is adequate, for the twisted trace formula will be valid for a function ϕ which is sufficiently differentiable. This will have to be the meaning attached to smooth in Paragraphs 10 and 11.

If ξ is a character of $F^\times$ or $Z(F)$ the observations of this paragraph are also valid for a function $\phi=(f_1,\cdots,f_\ell)$ with f_i satisfying

$$f_i(zg)=\xi^{-1}(z)f_i(g)\qquad z\in Z(F).$$

One has merely to define convolution suitably.

9. THE PRIMITIVE STATE OF OUR SUBJECT REVEALED

The derivation of the trace formula is such that it yields an expression for the trace, an invariant distribution, as a sum of terms of which some are not invariant and are not well understood. In many of the applications of the formula these terms have appeared with coefficient 0 and could be ignored. In the application we now have in mind they are not so easily suppressed. It is however possible to circumvent most of the difficulties they cause, but not all. Our ruse succeeds only if accompanied by some insight or hard work. The former failing we resort to the latter.

It is convenient to choose the forms defining the Tamagawa measures on N and $Z\backslash A$ to be

$$dn = dx, \qquad n = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix},$$
$$dt = \frac{b}{a} d\left(\frac{a}{b}\right), \qquad t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The maximal compact subgroup K of $G(F)$ will be chosen to be $G(O)$ if F is non-archimedean and to be the standard orthogonal or unitary group if F is archimedean. We choose the measure dk on K so that

$$\int_{Z(F)\backslash G(F)} h(g)dg = \int_{Z(F)\backslash A(F)} \int_{N(F)} \int_K h(ank)dadndk.$$

Let $\lambda(g)$ be the function on $A(F)\backslash G(F)$ obtained by writing $g = ank$, $a \in A(F)$, $n \in N(F)$, $k \in K$ and setting $\lambda(g) = \lambda(n)$, with $\lambda(n)$ defined as on p.519 of [14]. If $\gamma \in A(F)$ and $\Delta(\gamma) \neq 0$ set

$$A_1(\gamma, f) = \Delta(\gamma) \int_{A(F)\backslash G(F)} f(g^{-1}\gamma g)\ell n\lambda(g)dg.$$

We are interested in f which are smooth, satisfy

$$f(zg) = \xi^{-1}(z)f(g), \qquad z \in N_{E/F}Z(E),$$

and have compact support modulo $N_{E/F}Z(E)$. We shall write $\frac{A_1(\gamma,f)}{2}$ as the sum of $A_2(\gamma, f)$ and $A_3(\gamma, f)$ where $f \to A_2(\gamma, f)$ is an invariant distribution and where $A_3(\gamma, f)$ extends, for each f, to a continuous function on $A(F)$ whose support is compact modulo $N_{E/F}Z(E)$.

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

then

$$\frac{A_1(\gamma, f)}{2} = -\Delta(\gamma) \int_K \int_{|x|>1} f(k^{-1}\gamma \begin{pmatrix} 1 & \left(1 - \frac{b}{a}\right) x \\ 0 & 1 \end{pmatrix} k)\ell n|x|dxdk$$

which in turn equals

$$-\left|\frac{a}{b}\right|^{1/2} \int_K \int_{|x|>\left|1-\frac{b}{a}\right|} f(k^{-1}\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k) \left(\ell n|x| - \ell n\left|1 - \frac{b}{a}\right|\right) dxdk.$$

Suppose first that F is non-archimedean. If $\left|1-\frac{b}{a}\right|>1$ set $b(\gamma,f)$ and $c(\gamma,f)=0$. Otherwise set

$$\begin{aligned} b(\gamma,f) &= \left|\frac{a}{b}\right|^{1/2} f(a)\int_K\int_{|x|\le\left|1-\frac{b}{a}\right|}\left(\ell n|x|-\ell n\left|1-\frac{b}{a}\right|\right)dxdk \\ &= \left|\frac{a}{b}\right|^{1/2}\left|1-\frac{b}{a}\right| f(a)\int_K\int_{|x|\le 1}\ell n|x|dxdk \end{aligned}$$

and

$$c(\gamma,f)=-\left|\frac{a}{b}\right|^{1/2}|\varpi|\,\ell n\,|\varpi|\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)dg,$$

with

$$n=a\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

As usual, ϖ is a uniformizing parameter for F. Define $\omega(x,\gamma)$ by

$$\omega(x,\gamma)=\begin{cases}-\ell n|x| & |x|>\left|1-\frac{b}{a}\right| \\ -\ell n\left|1-\frac{b}{a}\right| & |x|\le\left|1-\frac{b}{a}\right|\le 1\end{cases}.$$

We define $A_2(\gamma,f)$ to be

$$\ell n\left|1-\frac{b}{a}\right|F(\gamma,f)+b(\gamma,f)+c(\gamma,f).$$

It clearly yields an invariant distribution. Then $A_3(\gamma,f)$ must be

$$\left|\frac{a}{b}\right|^{1/2}\int_K\int_F f(k^{-1}\gamma\begin{pmatrix}1&x\\0&1\end{pmatrix}k)\omega(x,\gamma)dxdk-b(\gamma,f)-c(\gamma,f).$$

If f is given and if we then choose γ so that $\left|1-\frac{b}{a}\right|$ is very small, the value of $A_3(\gamma,f)$ is

$$-\int_K\int_F f(k^{-1}a\begin{pmatrix}1&x\\0&1\end{pmatrix}k)\ell n|x|dxdk-\frac{|\varpi|\,\ell n\,|\varpi|}{1-|\varpi|}\int_K\int_F f(k^{-1}a\begin{pmatrix}1&x\\0&1\end{pmatrix}k)dxdk$$

so that $A_3(\gamma,f)$ clearly extends to all of $A(F)$ as a smooth function. The factor $1-|\varpi|$ appears in the denominator because we use the Tamagawa measures of [12].

We let $\beta(g)$ be the function on $G(F)$ defined by

$$\beta(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}c&0\\0&d\end{pmatrix}k)=\left|\frac{c}{d}\right|.$$

Departing from the notation on p. 520 of [14], we define the function $\theta(a,s,f)$ to be

$$\frac{1}{L(1+s,1_F)}\int_{G_n(F)\backslash G(F)} f(g^{-1}ng)\beta(g)^{-s}dg$$

with

$$n=a\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Then $\theta(a,s,f)$ is also equal to

$$\frac{1}{L(1+s,1_F)}\int_{Z(F)\backslash A(F)}\int_K f(k^{-1}t^{-1}ntk)\left|\frac{c}{d}\right|^{-1-s}dtdk$$

if

$$t=\begin{pmatrix} c & 0\\ 0 & d\end{pmatrix}$$

It is easy to check that the derivative of $\theta(a,s,f)$ at $s=0$ is $-A_3(a,f)$.

Suppose that f is the function f^0, where $f^0(g)=0$ unless $g=zk,\ z\in N_{E/F}Z(E),\ k\in K$, when it equals $\xi^{-1}(z)$. Of course f^0 exists only if ξ is unramified. All three terms in the definition of $A_3(\gamma,f^0)$ are 0 unless $\left|1-\frac{b}{a}\right|\le 1$. If this condition is satisfied the difference between the first two terms is

$$-f^0(a)\int_K\int_{|x|\le 1}\ell n|x|dxdk=-\frac{|\varpi|\,\ell n\,|\varpi|}{1-|\varpi|}f^0(a)\int_K\int_{|x|\le 1}dxdk.$$

Moreover

$$c(\gamma,f)=-\frac{|\varpi|\,\ell n\,|\varpi|}{1-|\varpi|}f^0(a)\int_K\int_{|x|\le 1}dxdk.$$

Thus $A_3(\gamma,f^0)$ is always 0.

If F is archimedean then

$$\begin{aligned}\frac{A_1(\gamma,f)}{2}&=-\frac{\Delta(\gamma)}{2}\int_K\int_F f(k^{-1}\gamma\begin{pmatrix}1 & \left(1-\frac{b}{a}\right)x\\ 0 & 1\end{pmatrix}k)\ell n(1+|x|^2)dxdk\\ &=-\frac{1}{2}\left|\frac{a}{b}\right|^{1/2}\int_K\int_F f(k^{-1}\gamma\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k)\left\{\ell n\left(\left|1-\frac{b}{a}\right|^2+|x|^2\right)-\ell n\left|1-\frac{b}{a}\right|^2\right\}dxdk.\end{aligned}$$

We may define

$$c(\gamma,f)=-\frac{L'(1,1_F)}{L(1,1_F)^2}\int_{G_n(F)\backslash G(F)}f(g^{-1}ng)dg.$$

This is just another form of the definition used before; we refrain here from writing out the L-functions explicitly ([21]).

We set

$$A_2(\gamma,f)=\ell n\left|1-\frac{b}{a}\right|F(\gamma,f)+c(\gamma,f)$$

and

$$A_3(\gamma,f)=-\frac{1}{2}\left|\frac{a}{b}\right|^{1/2}\int_K\int_F f(k^{-1}\gamma\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k)\ell n\left(\left|1-\frac{b}{a}\right|^2+|x|^2\right)dxdk-c(\gamma,f).$$

The desired properties are immediate. It is also clear that

$$\theta'(a,0,f)=-A_3(a,f)$$

once again.

Continuity of $A_3(\gamma,f)$ as a function on $A(F)$ is, however, not enough because we will want to apply a form of Poisson summation, for which we need to know at least that the Fourier-Mellin transform of $A_3(\gamma,f)$ is integrable. If we verify that the second derivatives of $A_3(\gamma,f)$ are measures when $F=\mathbf{R}$ and that the third derivatives are when $F=\mathbf{C}$, we will have adequate control on the Fourier-Mellin transform.*

A moment's thought and we are reduced to considering

$$\varphi(t)=\int_F h(x)\ell n(|t|^2+|x|^2)dx$$

at $t=0$. Here h is a smooth function with compact support on F. If $F=\mathbf{R}$ the first derivative of φ is

$$\varphi'(t)=\operatorname{sgn} t\int_F \frac{2h(tx)}{1+x^2}dx.$$

Since φ' is continuous except at 0 where it has a jump, the second derivative is a measure. If $F=\mathbf{C}$ a further reduction leads to

$$\varphi_1(t)=\int_0^1 h_1(x)\ell n(x^2+|t|^2)d(x^2).$$

A direct calculation shows that

$$\int_0^1 \ell n(x^2+|t|^2)d(x^2)=(1+|t|^2)\ell n(1+|t|^2)-|t|^2\ell n|t|^2-1$$

has third derivatives which are measures. We may therefore suppose that $h_1(0)=0$ and that $h_1(x)=O(x)$ as $x\searrow 0$. Computing the first, second, and third derivatives of $\ell n(x^2+|t|^2)$ with respect to the two components of t, one finds that after multiplication by x^2 they are $O\left(\frac{1}{|t|}\right)$ as $|t|\to 0$ and that the first and second remain bounded. It follows that the third derivatives of $\varphi_1(t)$ are measures.

This is all we need for the ordinary trace formula, but we must prepare ourselves for the twisted formula as well. Suppose E is either a cyclic extension of F of degree ℓ or the direct sum of ℓ copies of F and σ is a fixed non-trivial element in $\mathfrak{G}(E/F)$. If $\delta\in A(E)$, $\gamma=N\delta$, and $\Delta(\gamma)\neq 0$ we set

$$A_1(\delta,\phi)=\Delta(\gamma)\int_{Z(E)A(F)\backslash G(E)}\phi(g^{-1}\delta\sigma(g))\ell n\lambda(g)dg.$$

We are of course supposing that $\phi(zg)=\xi_E^{-1}(z)\phi(g),\ z\in Z(E)$. If E is a field then λ is defined as before except that E replaces F. If E is not a field and g has components $g_1,\cdots,g_\ell$ then $\lambda(g)=\Pi\lambda(g_i)$.

We are going to write $\frac{A_1(\delta,\phi)}{2}$ as the sum of $A_2(\delta,\phi)$ and $A_3(\delta,\phi)$. The latter will extend to a continuous function on $A^{1-\sigma}(E)\backslash A(E)$ whose support is compact modulo $Z(E)$. Moreover if ϕ and f are related as in Paragraphs 6 or 8 then

$$A_2(\delta,\phi)=\ell A_2(\gamma,f).$$

* I am grateful to J. Arthur for drawing to my attention that $A_3(\gamma,f)$ is not smooth.

Therefore if ϕ is a spherical function and f is related to it as in Paragraph 5, we will also have

$$A_3(\delta,\phi) = \ell A_3(\gamma, f).$$

We suppose first that F is non-archimedean. We set

$$F(\delta,\phi) = \Delta(\gamma)\int_{Z(E)A(F)\backslash G(E)} \phi(g^{-1}\delta\sigma(g))dg.$$

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

we set $b(\delta,\phi) = c(\delta,\phi) = 0$ unless $\left|1-\frac{b}{a}\right| \le 1$. If this condition is satisfied and

$$\delta = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

we set $b(\delta,\phi)$ equal to

$$\ell\left|\frac{a}{b}\right|^{1/2}\left\{\int_{Z(E)G(F)\backslash G(E)} \phi(g^{-1}c\sigma(g))dg\right\}\left\{\left|\frac{1-b}{a}\right|\int_K\int_{\{x\in F|\ |x|\le 1\}} \ell n|x|dxdk\right\}.$$

Moreover we choose z_0 with trace $z_0 = 1$ and set

$$n_0 = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix}$$

and

$$n = cn_0$$

and

$$c(\delta,\phi) = -\ell\left|\frac{a}{b}\right|^{1/2}|\varpi|\,\ell n\,|\varpi|\int_{G_n^\sigma(E)Z(E)\backslash G(E)} f(g^{-1}n\sigma(g))dg.$$

Finally

$$A_2(\delta,\phi) = \ell n\left|1-\frac{b}{a}\right|^{\ell} F(\delta,\phi) + b(\delta,\phi) + c(\delta,\phi)$$

and

$$A_3(\delta,\phi) = \frac{A_1(\delta,\phi)}{2} - A_2(\delta,\phi).$$

The only difficulty is to analyze the behavior of $A_3(\delta,\phi)$ as $\left|1-\frac{b}{a}\right|$ approaches 0. It is clear that $c(\delta,\phi)$ extends to a smooth function on $A^{1-\sigma}(E)\backslash A(E)$. Moreover $c(\delta,\phi)$ is independent of the choice of z_0. Since we are working with fields of characteristic 0 we may take $z_0 = \frac{1}{\ell} \in F$. We must consider

$$\frac{A_1(\delta,\phi)}{2} - b(\delta,\phi) - \ell n\left|1-\frac{b}{a}\right|^{\ell} F(\delta,\phi).$$

We are free to multiply δ by any element of $Z(E)A^{1-\sigma}(E)$. Since $\left|1-\frac{b}{a}\right|$ is taken small we may suppose that c and d belong to F.

We first treat the case that F is non-archimedean and E is a field. We must be careful to distinguish between absolute values in F and absolute values in E; so for the present discussion alone we denote absolute values in E by double bars. Observe that we may take $\frac{c}{d}$ close to 1 and write it as $1+u$ with $|u|$ small. Then

$$\left|1-\frac{b}{a}\right| = \left|1-\frac{d^\ell}{c^\ell}\right| = |\ell u|.$$

Denote $G(O_E)$ by K_E. We may rewrite the expression defining $\frac{A_1(\delta,\phi)}{2}$ as

$$-\Delta(\gamma)\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int_{\|x\|\geq 1}\phi(k^{-1}n^{-1}t^{-1}\delta\sigma(tnk))\ell n\|x\| dndtdk.$$

Here

$$n = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

If

$$t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

we rewrite this once again as

$$-\Delta(\gamma)\int_{K_E}\int\int_{\|x\|\geq\left\|\frac{\alpha}{\beta}\right\|}\phi(k^{-1}t^{-1}n^{-1}\delta\sigma(ntk))\left(\ell n\|x\| - \ell n\left\|\frac{\alpha}{\beta}\right\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1} dxdtdk.$$

The second integral is taken over $Z(E)A(F)\backslash A(E)$.

The integral with respect to $x \in E$ may be replaced by a double integral, for we may first integrate over F and then over $F\backslash E$. To be more precise we replace the variable x by $x+\frac{y}{\ell}$ and integrate first with respect to y. This forces us, if we use the usual normalizations of measures, to divide by $|\ell|$. The new x which appears is only determined modulo F and we choose it so that $\left\|x+\frac{y}{\ell}\right\| \geq \|x\|$ for all $y \in F$. Then $\frac{A_1(\delta,\phi)}{2}$ becomes

$$\frac{-\Delta(\gamma)}{|\ell|}\int_{K_E}\int\int\int\phi(k^{-1}t^{-1}n(-x)\delta n\left(\frac{uy}{\ell}\right)n(\sigma(x))\sigma(tk))\left(\ell n\left\|x+\frac{y}{\ell}\right\| - \ell n\left\|\frac{\alpha}{\beta}\right\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1}.$$

The inner integral is over the region $\left\|x+\frac{y}{\ell}\right\| \geq \left\|\frac{\alpha}{\beta}\right\|$.

The region of integration may be decomposed into two parts, defined by the inequalities $\left\|\frac{y}{\ell}\right\| \leq \|x\|$ and $\|x\| < \left\|\frac{y}{\ell}\right\|$. Since $\Delta(\gamma) = |\ell u|$, the integral over the second region yields

$$-\int\int\int\int\phi(k^{-1}t^{-1}n(-x)\delta n\left(\frac{y}{\ell}\right)n(\sigma(x))\sigma(tk))\left(\ell n\|y\| - \ell n\left\|\frac{\alpha}{\beta}\right\| - \ell n\|\ell u\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1}.$$

The inner integral is now over the region $\|y\| \geq \|\ell ux\|,\ \|y\| \geq \left\|\ell u\frac{\alpha}{\beta}\right\|$. Since $\|\ell u\| = |\ell u|^{\ell} = \left|1-\frac{b}{a}\right|^{\ell}$, this is the sum of

$$\ell n\left|1-\frac{b}{a}\right|^{\ell} F(\delta,\phi)$$

and

$$-\int\int\int\int \phi(k^{-1}t^{-1}n(-x)\delta n\left(\frac{y}{\ell}\right)n(\sigma(x))\sigma(tk))\left(\ell n\|y\| - \ell n\left\|\frac{\alpha}{\beta}\right\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1} \tag{9.1}$$

and

$$\int\int\int\int \phi(k^{-1}t^{-1}n(-x)\delta n\left(\frac{y}{\ell}\right)n(\sigma(x))\sigma(tk))\left(\ell n\|y\| - \ell n\left\|\frac{\alpha}{\beta}\right\| - \ell n\|\ell u\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1}, \tag{9.2}$$

and the integral being over $\|y\| \leq \|\ell ux\|,\ \|y\| \geq \left\|\frac{\alpha}{\beta}\right\|$, and

$$\int\int\int\int (\phi(k^{-1}t^{-1}n(-x)\delta n\left(\frac{y}{\ell}\right)n(\sigma(x))\sigma(tk))\left(\ell n\|y\| - \ell n\left\|\frac{\alpha}{\beta}\right\| - \ell n\|\ell u\|\right)\left\|\frac{\alpha}{\beta}\right\|^{-1}, \tag{9.3}$$

the integral now being taken over $\|y\| \leq \left\|\ell u\frac{\alpha}{\beta}\right\|$. In all of these integrals we may replace δ by c.

The integral (9.1) clearly extends to a smooth function on $A(E)$. Letting s in $A(F)$ be

$$\begin{pmatrix}\alpha_1 & 0\\ 0 & \beta_1\end{pmatrix}$$

and representing y as $\frac{\beta_1}{\alpha_1}$ we may change variables in (9.1) to obtain

$$(1-|\varpi|)\int\int\int \phi(k^{-1}t^{-1}n(-x)cn_0n(\sigma(x))\sigma(tk))\left\|\frac{\alpha}{\beta}\right\|^{-1}\ell n\left\|\frac{\alpha}{\beta}\right\| dxdtdk. \tag{9.4}$$

The integrals are over $K_E,\ Z(E)\backslash A(E)$, and $N(F)\backslash N(E)$.

In the first region $\left\|\frac{y}{\ell}\right\| \leq \|x\|$. Since ϕ is 0 where $\|x\|$ is large and $\|u\|$ is small we may replace y by 0 and δ by c in the integral over the first region, as well as in (9.2) and (9.3). The sum of the integral over the first region with (9.2) and (9.3) is equal to

$$\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int_{N(F)\backslash N(E)} \phi(k^{-1}t^{-1}n(-x)cn(\sigma(x))\sigma(tk))\psi(x,t)dxdtdk \tag{9.5}$$

where $\psi(x,t)$ is $\left\|\frac{\alpha}{\beta}\right\|^{-1}$ times

$$\int_{\substack{\|y\|\leq\|\ell ux\|\\ \|y\|>\left\|\ell u\frac{\alpha}{\beta}\right\|}} + \int_{\|y\|\leq\left\|\ell u\frac{\alpha}{\beta}\right\|}\left(\ell n\|y\| - \ell n\left\|\frac{\alpha}{\beta}\right\| - \ell n\|\ell u\|\right)$$

minus

$$|u|\int_{\substack{\|y\|\leq\|\ell x\|\\ \left\|x+\frac{y}{\ell}\right\|\geq\left\|\frac{\alpha}{\beta}\right\|}} \ell n\left\|x+\frac{y}{\ell}\right\| - \ell n\left\|\frac{\alpha}{\beta}\right\|.$$

We have to convince ourselves that the result in $b(\delta, \phi)$.

Although the principle to be invoked will be the same in both cases, it is simpler at this point to treat ramified and unramified E separately. If E is unramified then the tree $\mathfrak{X}$ is a subtree of $\mathfrak{X}(E)$ and every double coset in $Z(E)G(F)\backslash G(E)/K_E$ has a representative g for which

$$d(gp_0') = \mathrm{dist}(gp_0, p_0) = \mathrm{dist}(gp_0, \mathfrak{X}).$$

Two such representatives lie in the same double coset of $K\backslash G(E)/K_E$. Thus each double coset in $Z(E)G(F)\backslash G(E)/K_E$ is represented by a double coset in $K\backslash G(E)/K_E$. Moreover

$$\int_{Z(E)G(F)\backslash G(E)} \phi(g^{-1}c\sigma(g))dg$$

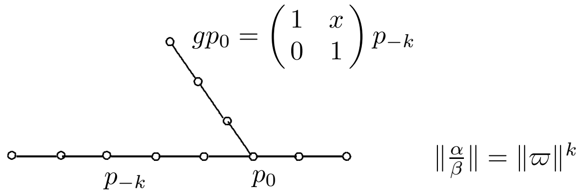
is equal to the sum over those p' in $\mathfrak{X}'(E)$ for which $d(p') = \mathrm{dist}(p, p_0) = \mathrm{dist}(p, \mathfrak{X})$ of

$$\frac{1}{\mathrm{meas}\, Z(O)\backslash K} \int_{gp_0'=p'} \phi(g^{-1}c\sigma(g))dg.$$

On the other hand every double coset has a representative

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

where $\|x\| = 1$ and $\|x + y\| \geq \|x\|$ for all $y \in F$. If $\|x\| > \left\|\frac{\alpha}{\beta}\right\|$ and $\|\beta\| = 1$ it is a representative of the type just described.



Two such g, say g_1 and g_2 lie in the same right coset of $G(E)/K_E$ if and only if $\|\alpha_1\| = \|\alpha_2\|$ and $x_1 \equiv x_2 (\mathrm{mod} \alpha_1 O_E)$. On the other hand, no matter what the absolute value of x is, if $\|x\| \leq \left\|\frac{\alpha}{\beta}\right\|$ then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

lies in $Z(E)G(F)K$.

We first examine that part of (9.5) for which $n(x)t$ lies in the trivial double coset. Then $\|\ell u x\| < \left\|\ell u \frac{\alpha}{\beta}\right\|$. Moreover $\|\ell x\| \geq \|y\|$ implies $\left\|x + \frac{y}{\ell}\right\| = \|x\|$; so $\left\|x + \frac{y}{\ell}\right\| \leq \left\|\frac{\alpha}{\beta}\right\|$. Thus

$$\begin{aligned} \psi(x,t) &= \left\|\frac{\alpha}{\beta}\right\|^{-1} \int_{\|y\| \leq \left\|\ell u \frac{\alpha}{\beta}\right\|} \ell n \left\|\frac{y\beta}{\ell u \alpha}\right\| dy \\ &= \ell \left\|\frac{\alpha}{\beta}\right\|^{-\frac{(\ell-1)}{\ell}} |\ell u| \int_{|y| \leq 1} \ell n |y| dy. \end{aligned}$$

We first integrate with respect to k. This allows us to replace $n(x)tk$ by k and yields, since $|\ell u| = \left|1-\frac{b}{a}\right|$, a product, labelled (9.6), of

$$\ell\left|1-\frac{b}{a}\right|\frac{\operatorname{meas} O_E}{\operatorname{meas} O}\;\frac{\operatorname{meas} Z(O_E)\backslash A(O_E)}{\operatorname{meas} Z(O)\backslash A(O)}$$

and

$$\left\{\int_{K_E}\phi(k^{-1}c\sigma(k))dk\right\}\left\{\int_{|y|\le 1}\ell n|y|dy\right\}.$$

There are however two Haar measures on K_E, dk and dg, the restriction of the Tamagawa measure of G, and

$$\operatorname{meas} O_E \operatorname{meas} Z(O_E)\backslash A(O_E)\int_{K_E}dk=\int_{Z(O_E)\backslash K_E}dg.$$

A similar observation applies to K. Thus (9.6) equals

$$\frac{\ell\left|1-\frac{b}{a}\right|}{\operatorname{meas} Z(O)\backslash K}\left\{\int_{K_E}\phi(g^{-1}c\sigma(g))dg\right\}\left\{\int_K\int_{|y|\le 1}\ell n|y|dydk\right\}$$

which is the contribution to $b(\delta,\phi)$ it is supposed to yield. If $\|x\| > \left\|\frac{\alpha}{\beta}\right\|$ then since $\|y\|\le\|\ell x\|$ implies $\left\|x+\frac{y}{\ell}\right\|=\|x\|$,

$$\begin{aligned}\psi(x,t)&=\left\|\frac{\alpha}{\beta}\right\|^{-1}\left\{\int_{\|y\|\le\|\ell u x\|}\ell n\left\|\frac{y\beta}{\ell u\alpha}\right\|-|\ell u|\int_{\|y\|\le\|x\|}\ell n\left\|\frac{x\beta}{\alpha}\right\|\right.\\&=\ell\left\|\frac{\alpha}{x\beta}\right\|^{-1}|\ell u|\int_{|y|\le 1}\ell n|y|dy.\end{aligned}$$

Once again we integrate with respect to K and then with respect to x and t, keeping $n(x)t$ in a fixed right coset of $G(E)/K_E$. Thus, for example, x varies over $x_0+\frac{\alpha_0}{\beta_0}O_E$ modulo F. The result is, as before,

$$\ell\left|1-\frac{b}{a}\right|\,\left\|\frac{\alpha_0}{x_0\beta_0}\right\|^{-\frac{1}{\ell}}\left\{\int_{g_0K_E}\phi(g^{-1}c\sigma(g))dg\right\}\left\{\int_K\int_{|y|\le 1}\ell n|y|dydk\right\}$$

if

$$g_0=n(x_0)t_0.$$

The expression is not changed if x_0 is replaced by $\frac{\alpha_1}{\beta_1}x_0$, and α_0,β_0 by $\alpha_1\alpha_0,\beta_1\beta_0$ with α_1,β_1 in $F^\times$. Thus we may always normalize so that $|x_0|=1$ and $\beta_0=1$. Then t is determined modulo $A(O_E)$. As we let x_0 and t_0 vary can we obtain all right cosets of K_E within a given double coset of $K\backslash G(E)/K_E$? No! - because x_0 is taken modulo F so that if the right coset represented by $n(x_0)t_0$ occurs then that represented by $n(x_0+y)t_0$, $y\in O$, $y\notin\frac{\alpha_0}{\beta_0}O_E$, does not. This means that a single right coset must stand for $\left\|\frac{\alpha_0}{\beta_0}\right\|^{-\frac{1}{\ell}}=\left\|\frac{\alpha_0}{x_0\beta_0}\right\|^{-\frac{1}{\ell}}$ altogether. Since this factor occurs in front of our integrals we may remove it, and then have a sum over all right cosets in the double coset of $K\backslash G(E)/K_E$ representing a double coset of $G(F)Z(E)\backslash G(E)/K_E$. We conclude that (9.5) is indeed $b(\delta,\phi)$.

If E is ramified then the set $\mathfrak{X}$ may still be regarded as contained in $\mathfrak{X}(E)$, but to obtain a subtree we have to subdivide each edge of $\mathfrak{X}$ into ℓ equal parts. The subdivision performed we may choose as representations p' for the orbits of $Z(E)G(F)$ in $\mathfrak{X}'(E)$ those points p' for which $d(p') = \operatorname{dist}(p, p_0)$ for which the closest point to p in $\mathfrak{X}$ is p_z in $A(E)$ with $0 \le z \le \frac{\ell}{2}$. If $z = 0$ let K_0 be $Z(O)K$. If $0 < z < \frac{\ell}{2}$ let

$$K_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(O)K | c \equiv 0 (\operatorname{mod} \varpi) \right\}.$$

If $z = \frac{\ell}{2}$ let it be the group generated by the previous group and

$$\varpi_E^{-1} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

where ϖ_E is a uniformizing parameter for E. Notice that if $z = \frac{\ell}{2}$ then $\ell = 2$. If p'_1 and p'_2 are two possible choices for representatives of an orbit then $z_1 = z_2$ and $p'_2 = kp'_1$ with $k \in K_0$. Thus each double coset $Z(E)G(F)hK_E$ is represented by a double coset $K_0 u K_E$. If ρ is 1 for $z = 0$, $\frac{q+1}{2}$ for $z = \frac{\ell}{2}$, and $q + 1$ for $0 < z < \frac{\ell}{2}$ then

$$\int_{Z(E)G(F)\backslash Z(E)G(F)hK_E} \phi(g^{-1} c\sigma(g)) dg$$

is equal to

$$\frac{\rho}{\operatorname{meas} K} \int_{K_E} \phi(g^{-1} u^{-1} c\sigma(ug)) dg$$

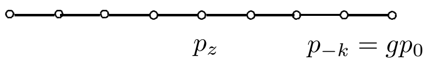
because

$$\frac{\operatorname{meas} Z(O)K}{\operatorname{meas} K} = \rho.$$

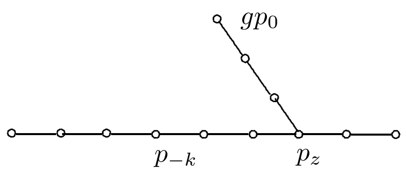
If $p' \to p$ and the closest point to p in X is p_z with $\frac{\ell}{2} < z \le \ell$ then p' lies in the orbit represented by p' where the closest point to $\bar{p}$ is $p_{\ell - z}$. Every double coset has a representative

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

where x is such that $\|x + y\| \ge \|x\|$ for all $y \in F$. Let $\|x\| = \|\varpi\|^{-\frac{z}{\ell}}$ and $\left\|\frac{\alpha}{\beta}\right\| = \|\varpi\|^{\frac{k}{\ell}}$. Here ϖ is a uniformizing parameter for F. Let j be the smallest integer greater than or equal to $\frac{k}{\ell}$. If $\|x\| \le \left\|\frac{\alpha}{\beta}\right\|$ then gp'_0 lies in the orbit of $Z(E)G(F)$ whose projection to $\mathfrak{X}(E)$ contains p_z with $z = \ell j - k$ if $\ell j - k \le \frac{\ell}{2}$ and $z = \ell - (\ell j - k)$ if $\ell j - k \le \frac{\ell}{2}$.



If $\|x\| > \left\|\frac{\alpha}{\beta}\right\|$ then, multiplying by an element of $A(F)$, we may suppose that $0 \le z < \ell$. Observe that p_z is the closest point to gp_0 in $\mathfrak{X}$.



We first examine that part of (9.5) for which $n(x)t$ lies in a double coset corresponding to an orbit which meet $\mathfrak{X}$, that is, for which $\|x\| \le \left\|\frac{\alpha}{\beta}\right\|$. Then

$$\psi(x,t) = \left|1-\frac{b}{a}\right| \left\|\frac{\alpha}{\beta}\right\|^{-1} \int_{\|y\|\le\left\|\frac{\alpha}{\beta}\right\|} \ell n\left\|y\frac{\beta}{a}\right\| dy.$$

In particular it is independent of x. We first integrate over K_E to obtain a factor

$$\int_{K_E} \phi(k^{-1}c\sigma(k))dk$$

and then with respect to x over $\frac{\alpha}{\beta}O_E$ modulo F. This yields a factor

$$\frac{\operatorname{meas} O_E}{\operatorname{meas} O}\left\|\frac{\alpha}{\beta}\right\| |\varpi|^{-j}.$$

Multiplying by $\psi(x,t)$ and ignoring the terms which do not depend on x we are left with an integrand

$$|\varpi|^{-j}\int_{\|y\|\le\left\|\frac{\alpha}{\beta}\right\|} \ell n\left\|y\frac{\beta}{\alpha}\right\| dy = \int_{\|y\|\le 1} \ell n\left\|y\frac{\varpi^j\beta}{\alpha}\right\| dy.$$

Since

$$\int_{|y|\le 1} \ell n|y|dy = \frac{|\varpi|\ell n|\varpi|}{1-|\varpi|}\int_{|y|\le 1} dy$$

the right-hand side equals

$$\ell\left\{1+(|\varpi|^{-1}-1)\left(j-\frac{k}{\ell}\right)\right\}\left\{\int_{|y|\le 1}\ell n|y|dy\right\}. \tag{9.7}$$

If we are interested in the double coset represented by the orbit whose projection on $\mathfrak{X}(E)$ contains p_z, $0 \le z \le \frac{\ell}{2}$ we must take $j-\frac{k}{\ell}$ to be $\frac{z}{\ell}$ or $1-\frac{z}{\ell}$. Integrating (9.7) over the relevant part of $Z(E)A(F)\backslash A(E)$ we obtain

$$\rho\ell\frac{\operatorname{meas} A(O_E)}{\operatorname{meas} A(O)}\int_{|y|\le 1}\ell n|y|dy$$

where ρ is 1 if $z=0$, $\frac{q+1}{2}$ if $z=\frac{\ell}{2}$, and $q+1$ if $0<z<\frac{\ell}{2}$. Gathering everything together we obtain ρ times (9.6), which is exactly what we need.

We next consider a double coset whose projection on $\mathfrak{X}(E)$ does not meet $\mathfrak{X}$. The product $n(x)t$ can lie in such a double coset only if $\|x\| > \left\|\frac{\alpha}{\beta}\right\|$, and then

$$\psi(x,t) = \left|1-\frac{b}{a}\right| \left\|\frac{\alpha}{\beta}\right\|^{-1} \int_{\|y\|\le\|x\|} \ell n\left\|\frac{y}{x}\right\| dy.$$

As before we first integrate with respect to k to obtain a factor

$$\int_{K_E} \phi(k^{-1}g^{-1}c\sigma(gk))dk$$

if g is some fixed representative of the right coset in which $n(x)t$ is constrained to lie. Take

$$g = \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 & 0 \\ 0 & \beta_0 \end{pmatrix}.$$

The integration over $x_0 + \frac{\alpha}{\beta} O_E$, on which $\|x\| = \|x_0\|$ is a constant, yields a new factor

$$\frac{\operatorname{meas} O_E}{\operatorname{meas} O} \left\| \frac{\alpha}{\beta} \right\| |\varpi|^{-j}$$

if $\left\|\frac{\alpha}{\beta}\right\| = \|\varpi\|^{\frac{k}{\ell}}$ and j is defined with respect to $\frac{k}{\ell}$ as above. If $\|x\| = \|\varpi\|^{-\frac{z}{\ell}}$ and i is the smallest integer greater than or equal to $-\frac{z}{\ell}$ the product of this with $\psi(x_0, t_0)$ is

$$\left| 1 - \frac{b}{a} \right| \frac{\operatorname{meas} O_E}{\operatorname{meas} O}$$

times

$$|\varpi|^{-j-i} \int_{\|y\| \le 1} \ell n \left\| \frac{y}{\varpi^i x} \right\| dy. \tag{9.8}$$

At this point we are free so to normalize x_0 and $0 \le z < \ell$ and $i = 0$ and $\beta_0 = 1$. Then (9.8) becomes, for z cannot be 0,

$$\ell |\varpi|^{-j} \left\{ \int_{|y| \le 1} \ell n |y| dy \right\} \left\{ 1 + (|\varpi|^{-1} - 1) \frac{z}{\ell} \right\}.$$

The final integration with respect to t simply introduces a factor

$$\frac{\operatorname{meas} Z(O_E) \backslash A(O_E)}{\operatorname{meas} Z(O) \backslash A(O)}.$$

We could now collect together the terms and find the contribution of the right coset $n(x_0)$ to K_E. However, we are interested in the total contribution from all the right cosets which, with x_0 and t_0 normalized, lie in a given double coset $K_0 u K_E$. As before, not all possible right cosets appear, for x_0 is taken modulo O. However, we may pretend that all occur if we suppress the factor $|\varpi|^{-j}$. We must also remember, since we are really interested in double cosets with respect to $Z(E)G(F)$, K_E, that we may obtain two double cosets in $K_0 \backslash G(E)/K_E$ which lie in the same double coset in $Z(E)G(F) \backslash G(E)/K_E$. To pass from one to the other we must replace z by $\ell - z$. Since

$$\left(1 + (|\varpi|^{-1} - 1) \frac{z}{\ell}\right) + \left(1 + (|\varpi|^{-1} - 1) \left(1 - \frac{z}{\ell}\right)\right) = q + 1$$

and

$$1 + (|\varpi|^{-1} - 1) \frac{\ell}{2\ell} = \frac{q+1}{2}$$

we can finish simply by gathering together the pieces.

Before analyzing the behavior of $A_3(\delta,\phi)$ when E is not a field, we introduce another expression for its value when $N\delta$ is a scalar. If c belongs to $E^\times$ and

$$n_0 = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{trace} z_0 = 1,$$

we introduce $\theta(c,s,\phi)$ as $L(1+\ell s, 1_F)^{-1}$ times

$$\int_{Z(E)\backslash A(E)} \int_{N(F)\backslash N(E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}cn_0\sigma(ntk)) \left\| \frac{a}{b} \right\|^{-1-s} dndtdk,$$

It is clearly independent of the choice of z_0; so we may take $z_0 = \frac{1}{\ell}$. The derivative of $\theta(c,s,\phi)$ at $s=0$ is equal to the sum of

$$\frac{-1}{L(1,1_F)} \int_{Z(E)\backslash A(E)} \int_{N(F)\backslash N(E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}cn_0\sigma(ntk)) \left\| \frac{a}{b} \right\|^{-1} \ell n \left\| \frac{a}{b} \right\| dndtdk,$$

which is the negative of (9.4), and

$$-\ell|\varpi|\ell n|\varpi| \int_{Z(E)\backslash A(E)} \int_{N(F)\backslash N(E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}cn_0\sigma(ntk)) \left\| \frac{a}{b} \right\|^{-1} dndtdk$$

which is $c(\delta,\phi)$. Thus

$$\theta'(c,0,f) = -A_3(c,f).$$

If F is non-archimedean and E is not a field then, as in Paragraph 8, ϕ is just a collection $f_1,\cdots,f_\ell$ of functions on $G(F)$ and if $\delta = (\delta_1,\cdots,\delta_\ell)$ then $A_1(\delta,\phi)$ is equal to

$$\sum\nolimits_{i=1}^{\ell} \Delta(\gamma) \int_{Z(E)A(F)\backslash G(E)} f_1(h_1^{-1}\delta_1 h_2)\cdots f_{\ell-1}(h_{\ell-1}^{-1}\delta_{\ell-1}h_\ell) f_\ell(h_\ell^{-1}\delta_\ell h_1) \ell n \lambda(h_i).$$

We choose an i, $1 \le i \le \ell$, and consider the corresponding term. We introduce new variables of integration by the equations

$$g_i = h_i, g_{i+1} = h_{i+1}^{-1}\delta_{i+1}\cdots\delta_\ell\delta_1\cdots\delta_{i-1}h_i\cdots, g_\ell = h_\ell^{-1}\delta_\ell\delta_1\cdots\delta_{i-1}h_i, \cdots, g_{i-1} = h_{i-1}^{-1}\delta_{i-1}h_i.$$

If $f^{(i)}$ is the convolution

$$f_i * f_{i+1}\cdots * f_\ell * f_1 * \cdots * f_{i-1}$$

the term in which we are interested is simply

$$A_1(\gamma, f^{(i)}).$$

Thus

$$A_1(\gamma,\phi) = \sum\nolimits_i A_1(\gamma, f^{(i)}).$$

A similar change of variables shows us that

$$F(\delta,\phi)=\sum\nolimits_i F(\gamma,f^{(i)})=\ell F(\gamma,f)$$

and

$$c(\delta,\phi)=\sum\nolimits_i c(\gamma,f^{(i)})=\ell c(\gamma,f)$$

if $f=f^{(1)}=f_1*\cdots*f_\ell$. We also see that

$$b(\delta,\phi)=\sum b(\gamma,f^{(i)})=\ell b(\gamma,f).$$

The required property of $A_3(\delta,\phi)$ follows therefore from the similar property of $A_3(\gamma,f)$.

We may again introduce $\theta(c,s,\phi)$ if $c\in Z(E)$. If $\beta(g)$ is the function on $G(E)$ defined by

$$\beta(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}k)=\left\|\frac{\alpha}{\beta}\right\|$$

then

$$\theta(c,s,\phi)=\frac{1}{L(1+\ell s,1_F)}\int_{G^\sigma_{n_0}(E)Z(E)\backslash G(E)}\phi(g^{-1}cn_0\sigma(g))\beta(g)^{-s}dg.$$

The derivative of $\theta(c,s,\phi)$ at $s=0$ is

$$\frac{-1}{L(1,F)}\int_{G^\sigma_{n_0}(E)Z(E)\backslash G(E)}\phi(g^{-1}cn_0\sigma(g))\ell n\beta(g)dg$$

plus

$$-\ell|\varpi|\ell n|\varpi|\int_{G^\sigma_{n_0}(E)Z(E)\backslash G(E)}\phi(g^{-1}cn_0\sigma(g))dg.$$

We may change variables as before to see that this equals

$$\ell\theta'(a,0,f)$$

if $a=N_{E/F}c$. Therefore

$$\theta'(c,0,\phi)=-A_3(c,\phi).$$

We must still discuss $A_2(\delta,\phi)$ and $A_3(\delta,\phi)$ for an archimedean field. We set

$$c(\delta,\phi)=-\ell\frac{L'(1,1_F)}{L(1,F)^2}\int_{G^\sigma_{n_0}(E)Z(E)\backslash G(E)}\phi(g^{-1}cn_0\sigma(g))dg$$

and

$$A_2(\delta,\phi)=\ell n\left|1-\frac{b}{a}\right|^\ell F(\delta,\phi)+c(\delta,\phi)$$

while

$$A_3(\delta,\phi) = \frac{A_1(\delta,\phi)}{2} - A_2(\delta,\phi).$$

If E is not a field we may proceed as in the non-archimedean case; so suppose E is a field. Then

$$\frac{A_1(\delta,\phi)}{2} = -\Delta(\gamma)\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int_E \phi(k^{-1}n^{-1}t^{-1}\delta\sigma(tnk))\ell n(1+|x|^2)dxdtdk.$$

Here

$$n = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Moreover the absolute value is that of an analyst and not of a number-theorist. To explain the disappearance of the 2 from the denominator we observe that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} = \begin{pmatrix} (1+x\bar{x})^{-1/2} & \\ & (1+x\bar{x})^{1/2} \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} k'$$

and

$$\lambda(n) = \|(1+x\bar{x})^{-1}\| = (1+|x|^2)^{-2}.$$

If

$$t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

we first write this expression as

$$-\Delta(\gamma)\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int_E \phi(k^{-1}t^{-1}n^{-1}\delta\sigma(ntk))\ell n\left(1+\left|\frac{\beta}{\alpha}x\right|^2\right)\left|\frac{\alpha}{\beta}\right|^{-2} dxdtdk.$$

Since we are again only interested in the behavior of this integral when γ is close to a scalar we may assume that δ lies in $A(F)$. Indeed we can always do this; for F is $\mathbf{R}$ and E is $\mathbf{C}$, so $A(E) = A^{1-\sigma}(E)A(F)$. But when γ is close to a scalar we may in addition suppose that $\frac{c}{d}$ is close to 1 if

$$\delta = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

The function $c(\delta,\phi)$ is clearly smooth on all of $A(E)$ and

$$\ell n\left|1-\frac{b}{a}\right|^{\ell} F(\delta,\phi) - \ell n\left|1-\frac{d}{c}\right|^{\ell} F(\delta,\phi)$$

is smooth as long as we keep $\frac{d}{c}$ close to 1, for then

$$\left|1-\frac{b}{a}\right| = \left|1-\frac{d}{c}\right|\,\left|1+\frac{d}{c}\right|$$

and

$$\ell n\left|1+\frac{d}{c}\right|$$

is smooth. Thus we have ony to investigate the behavior of

$$\frac{A_1(\delta,\phi)}{2} - \ell n\left|1-\frac{d}{c}\right|^{\ell} F(\delta,\phi).$$

Note that $\ell = 2$.

In the integral defining $\frac{A_1(\delta,\phi)}{2}$ we write x as $\frac{u}{2}+iv$ and integrate with respect to u and then with respect to v. We start from

$$n(-x)\delta n(\sigma(x)) = n(-iv)\delta n\left(\left(1-\frac{d}{c}\right)\frac{u}{2}\right)n(-iv)$$

and the change variables so that the $1-\frac{d}{c}$ disappears. This replaces $-\Delta(\gamma)$ by $-\left|\frac{a}{b}\right|^{1/2}$ and $\ell n\left(1+\left|\frac{\beta}{\alpha}x\right|^2\right)$ by

$$\ell n\left[\left|1-\frac{d}{c}\right|^2\left(1+\left|\frac{\beta}{\alpha}v\right|^2\right)+\frac{|\beta u|^2}{2\alpha}\right] - \ell n\left|1-\frac{d}{c}\right|^2.$$

If we subtract

$$\ell n\left|1-\frac{d}{c}\right|^2 F(\delta,\phi)$$

we are left with $-\left|\frac{a}{b}\right|^{1/2}$ times

$$\int\int\int \phi(k^{-1}t^{-1}n(-iv)\delta n\left(\frac{u}{2}\right)n(-iv)\sigma(tk))\ell n\left[\left|1-\frac{d}{c}\right|^2\left(1+\left|\frac{\beta}{a}v\right|^2\right)+\left|\frac{\beta u}{2\alpha}\right|^2\right]\left|\frac{\alpha}{\beta}\right|^{-2}.$$

The two outer integrals are over K_E and $Z(E)A(F)\backslash A(E)$. This is a continuous function of δ for $\frac{d}{c}$ close to 1.

As $\delta \to c$ in $E^\times$ the value of $A_3(\delta,\phi)$ approaches the sum of three terms:

$$-\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int \phi(k^{-1}t^{-1}n(-iv)cn\left(\frac{u}{2}\right)n(-iv)\sigma(tk))\left|\frac{\alpha}{\beta}\right|^{-2}\ell n\left|\frac{\beta u}{2\alpha}\right|^2;$$

and, since $1+\frac{d}{c}\to 2$,

$$-2\ell n 2\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int \phi(k^{-1}t^{-1}n(-iv)cn\left(\frac{u}{2}\right)n(-iv)\sigma(tk))\left|\frac{\alpha}{\beta}\right|^{-2};$$

and

$$\lim_{\delta\to c} -c(\delta,\phi).$$

The first two terms together yield

$$-\int_{K_E}\int_{Z(E)A(F)\backslash A(E)}\int \phi(k^{-1}t^{-1}n(-iv)cn\left(\frac{u}{2}\right)n(-iv)\sigma(tk))\left|\frac{\alpha}{\beta}\right|^{-1}\ell n\left|\frac{\beta u}{\alpha}\right|^2.$$

Writing $u=\frac{\beta_i}{\alpha_1}$ we see that this in turn equals

$$\frac{-1}{L(1,1_F)}\int_{K_E}\int_{Z(E)\backslash A(E)}\int_{N(F)\backslash N(E)}\phi(g^{-1}t^{-1}n^{-1}cn_0\sigma(ntk))\left|\frac{\alpha}{\beta}\right|^{-2}\ell n\left|\frac{\beta}{\alpha}\right|^2.$$

We conclude once again that

$$A_3(c,\phi) = -\theta'(c,0,\phi).$$

It is also easily shown that if $A_3(\delta,\phi)$ is regarded as a function on $A^{1-\sigma}(E)\backslash A(E)$ then its second derivatives are measures.

10. THE TRACE FORMULA

The results on global lifting as well as the remaining results on local lifting are obtained by combining the local analysis which we have carried out with a comparison of the trace formula over F and the twisted trace formula over E. This is also the method exploited by Saito and Shintani. The trace formula has been discussed extensively in recent years ([1], [9], [10], [14]) and we shall review it only briefly, stressing the modifications necessary for the present purposes. Our discussion of the twisted trace formula, of which the usual formula is a special case, will be only a little more extensive. Enough will be said that the reader familiar with the usual formula will be convinced of the validity of the twisted form, but the analytical aspects of the proof will be scamped. However, some calculations will be carried out in more detail for the twisted case, and it may occasionally be useful to glance ahead.

We recall some of the notation introduced in §2. Set

$$Z_E(\mathbf{A}) = Z(F)N_{E/F}Z(\mathbf{A}_E)$$

and let ξ be a unitary character of $Z_E(A)$ trivial on $Z(F)$. $L_s(\xi)$ is the space of measurable functions φ on $G(F)\backslash G(\mathbf{A})$ which satisfy

$$\text{(a)} \qquad \varphi(zg) = \xi(z)\varphi(g) \text{ for all } z \in Z_E(\mathbf{A})$$

$$\text{(b)} \qquad \int_{Z_E(\mathbf{A})G(F)\backslash G(\mathbf{A})} |\varphi(g)|^2 dg < \infty.$$

$G(\mathbf{A})$ acts on $L_s(\xi)$ by the right translations,

$$r(g)\varphi(h) = \varphi(hg).$$

The space $L_s(\xi)$ is the direct sum of three mutually orthogonal subspaces, $L_{sp}(\xi)$, $L^0_{se}(\xi)$, and $L^1_{se}(\xi)$, all defined in §2. The representation of $G(\mathbf{A})$ on $L_{sp}(\xi) + L^0_{se}(\xi)$ is denoted r.

Let f be a function on $G(\mathbf{A})$ defined by

$$f(g) = \prod\nolimits_v f_v(g_v),$$

where the f_v satisfy the conditions (i), (ii), and (iii) imposed in §2. Recall that

$$r(f)\varphi(h) = \int_{N_{E/F}Z(\mathbf{A}_E)\backslash G(\mathbf{A})} \varphi(hg)f(g)dg$$

if $\varphi \in L_{sp}(\xi) + L^0_{se}(\xi)$. It is of trace class. We start from the formula for its trace given on pages 516–517 of [14], taking account of the trivial modifications required by the substitution of $Z_E(\mathbf{A})$ or $N_{E/F}Z(\mathbf{A}_E)$ for $Z(\mathbf{A})$, and rewrite it in a form suited to our present needs. In particular, we shall express the trace as a sum of invariant distributions, along lines adumbrated in [23]. Unless the contrary is explicitly stated we shall use Tamagawa measures locally and globally (§6 of [23]). This will remove some of the normalizing constants of [14].

The first term of this sum, corresponding to (i) and (ii) of [14] together, is

$$\sum\nolimits_{\gamma} \varepsilon(\gamma)\mathrm{meas}(N_{E/F}Z(\mathbf{A}_E)G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)dg. \tag{10.1}$$

The sum is over conjugacy classes in $G(F)$ for which $G_\gamma(F)$ does not lie in a Borel subgroup taken modulo $N_{E/F}Z(F)$. $\varepsilon(\gamma)$ is $\frac{1}{2}$ or 1 according as the equation

$$\delta^{-1}\gamma\delta = z\gamma$$

can or cannot be solved for $\delta \in G(F)$ and $z \neq 1$ in $N_{E/F}Z(F)$. Observe that the extension is cyclic and that $N_{E/F}Z(E)$ is therefore

$$Z(F) \cap N_{E/F}Z(\mathbf{A}_E).$$

If A is the group of diagonal matrices the set D of all characters $\eta = (\mu,\nu)$ of $A(F)\backslash A(\mathbf{A})$ for which $\mu\nu = \xi$ on $Z_E(\mathbf{A})$ may be turned into a Riemann surface by introducing as parameter in the neighborhood $(\mu|\alpha|^{\frac{s}{2}}, \nu|\alpha|^{-\frac{s}{2}})$, $s \in \mathbf{C}$, of (μ,ν) the variable s. Differentiation with respect to s is well defined. We denote it by a prime. We may also introduce the measure $|ds|$ on the set D^0 of unitary characters in D.

We write $\rho(g,\eta)$ for the operator $\rho(g,\mu,\nu,0)$ introduced on p. 513 of [14] and set

$$\rho(f,\eta) = \int_{N_{E/F}Z(\mathbf{A}_E)\backslash G(\mathbf{A})} f(g)\rho(g,\eta)dg.$$

If η_v is the component of η at v we write $R(\eta_v)$ for the operator $R(\mu_v,\nu_v,0)$ introduced on p. 521 of [14], noting that the factor $\varepsilon(1-s,\mu_v^{-1}\nu_v,\psi_v)$ occurring in that definition should be $\varepsilon(s,\mu_v\nu_v^{-1},\psi_v)$, and set

$$M(\eta) = \frac{L(1,\nu\mu^{-1})}{L(1,\mu\nu^{-1})} \otimes_v R(\eta_v).$$

We also let $m(\eta)$ be the function

$$\frac{L(1,\nu\mu^{-1})}{L(1,\mu\nu^{-1})}.$$

The term (vi) of the trace formula of [14] may be written

$$-\frac{1}{4}\sum_{\nu=(\mu,\mu)} \mathrm{trace}\, M(\eta)\rho(f,\eta). \tag{10.2}$$

The term (vii) is

$$\frac{1}{4\pi}\int_{D^0} m^{-1}(\eta)m'(\eta)\mathrm{trace}\,\rho(f,\eta)|ds|. \tag{10.3}$$

It appears at first sight that a factor ℓ should appear in the numerators because the integral on line 4 of p. 540 of [14] is now over $G(F)N_{E/F}Z(\mathbf{A})\backslash G(\mathbf{A})$ rather than $G(F)Z(\mathbf{A})\backslash G(\mathbf{A})$. This is, however, compensated by a change in the measure on the dual D^0.

Let 1_F be the trivial character of the idèles of F and let λ_0 be the constant term of the Laurent expansion of $L(1+s, 1_F)$ at $s=0$. Let

$$n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The first term of (v) becomes

$$\sum_{a \in N_{E/F}Z(E)\backslash Z(F)} \ell\lambda_0 \prod\nolimits_v L(1, 1_{F_v})^{-1} \int_{G_n(F_v)\backslash G(F_v)} f(g^{-1}ang)dg. \tag{10.4}$$

Those who follow the discussion on p. 532 of [14] will see that the ℓ occurs in the numerator because we have replaced $Z(\mathbf{A})$ by $N_{E/F}Z(\mathbf{A}_E)$ and

$$[Z(\mathbf{A}) : Z(F)N_{E/F}Z(\mathbf{A}_E)] = \ell.$$

If

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

belongs to $A(\mathbf{A}_v)$ set

$$\Delta_v(\gamma) = \left| \frac{(a-b)^2}{ab} \right|_v^{1/2}$$

and let

$$F(\gamma, f_v) = \Delta_v(\gamma) \int_{A(\mathbf{Q}_v)\backslash G(\mathbf{Q}_v)} f(g^{-1}\gamma g)dg.$$

Let $\lambda(g)$ be the function on $A(F_v)\backslash G(F_v)$ obtained by writing $g = ank,\ , a \in A(F_v),\ n \in N(F_v),$
$k \in K_v$ and setting $\lambda(g) = \lambda(n)$ with $\lambda(n)$ defined as on p. 519 of [14]. If $\gamma \in A(F_v)$ set

$$A_1(\gamma, f_v) = \Delta_v(\gamma) \int_{A(F_v)\backslash G(F_v)} f(g^{-1}\gamma g)\ell n\lambda(g)dg.$$

Because of the product formula the term (iv) becomes

$$-\frac{1}{2}\ell\lambda_{-1} \sum\nolimits_v \sum_{\substack{\gamma \in N_{E/F}Z(E)\backslash A(F) \\ \gamma \notin Z(F)}} A_1(\gamma, f_v) \prod\nolimits_{w \neq v} F(\gamma, f_w).$$

λ_{-1}, the residue of $L(1+s, 1_F)$ at $s=0$, appears because we must pass to the normalized global Tamagawa measure. As before, an ℓ appears in the numerator because

$$[Z(\mathbf{A}) : Z(F)N_{E/F}Z(\mathbf{A}_E)] = \ell.$$

The results of the previous paragraph allow us to write the sum of (iv) and the second half of (v) as the sum of

$$-\ell\lambda_{-1}\sum\nolimits_v \sum\nolimits_{\substack{\gamma\in N_{E/F}Z(E)\backslash A(F)\\ \gamma\notin Z(F)}} A_2(\gamma,f_v)\prod\nolimits_{w\neq v}F(\gamma,f_w) \tag{10.5}$$

and

$$-\ell\lambda_{-1}\sum\nolimits_v \sum\nolimits_{\gamma\in N_{E/F}Z(E)\backslash A(F)} A_3(\gamma,f_v)\prod\nolimits_{w\neq v}F(\gamma,f_w). \tag{10.6}$$

We may apply the Poisson summation formula to (10.6) and the group $N_{E/F}Z(\mathbf{A}_E)\backslash A(\mathbf{A})$. However, departing a little from the usual convention, we apply it to a function, that of (10.6) which transforms under $N_{E/F}Z(\mathbf{A}_E)$ according to ξ^{-1}. Then the Fourier transform will be concentrated on D^0. We may compute the Fourier transform locally if we remember at the end to divide by λ_{-1}, for the global measure differs from the product of the local measures by this factor. The Fourier transform of $F(\gamma,f_v)$ is $\rho(f_v,\eta_v)$. Let $B_1(f_v,\eta_v)$ be the Fourier transform of $A_3(\gamma,f_v)$. Since

$$[Z(\mathbf{A}):Z_E(\mathbf{A})]=\ell$$

the dual measure on D^0 is

$$\frac{1}{2\pi\ell}|ds|.$$

Let

$$B(f_v,\eta_v)=\frac{1}{2}\,\mathrm{Trace}(R^{-1}(\eta_v)R'(\eta_v)\rho(f_v,\eta_v))-B_1(f_v,\eta_v).$$

Then (10.6) may be put together with (viii) of [14] to yield

$$\frac{1}{2\pi}\int_{D^0}\sum\nolimits_v B(f_v,\eta_v)\prod\nolimits_{w\neq v}\mathrm{trace}\,\rho(f_w,\eta_w)\ |ds|. \tag{10.7}$$

The trace in which we are interested is the sum of (10.1), (10.2), (10.3), (10.4), (10.5), and (10.7). Since (10.7) occures in a linear equality in which all other terms are invariant, it must be invariant. It is not hard to deduce from this that $f_v\to B(f_v,\eta_v)$ is also invariant. Since we do not need this fact we do not give its proof. The idea involved will come up later in a different context. Observe that $R(\eta_v)$ has been so defined that $B(f_v^0,\eta_v)=0$ for all η_v, if f_v^0 is the unit of the Hecke algebra. Thus f_v^0 is supported in $G(O_{F_v})N_{E_v/F_v}Z(E_v)$ and is invariant under $G(O_{F_v})$.

Let ξ_E be the character $z\to\xi(N_{E/F}z)$ of $Z(\mathbf{A}_E)$ and, as before, let $L_s(\xi_E)$ be the space of measurable functions φ on $G(E)\backslash G(\mathbf{A}_E)$ satisfying

(a) $\quad\varphi(zg)=\xi_E(z)\varphi(g)$ for all $z\in Z(\mathbf{A}_E)$

(b) $$\int_{Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)}|\varphi(g)|^2dg<\infty.$$

The representation r of $G(\mathbf{A}_E)$ on the sum of $L_{sp}(\xi_E)$ and $L^0_{se}(\xi_E)$ extends to a representation r of $G'(\mathbf{A}_E) = G(\mathbf{A}_E) \times \mathfrak{G}$ if we let $r(\tau),\ \tau \in \mathfrak{G}$, send φ to φ' with

$$\varphi'(h) = \varphi(\tau^{-1}(h)).$$

If

$$\phi(g) = \prod\nolimits_v \phi_v(g_v)$$

is a function on $G(\mathbf{A}_E)$, where the ϕ_v satisfy the conditions of §2, then we defined $r(\phi)$ by

$$r(\phi) = \int_{Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g) r(g) dg.$$

We can use the usual techniques to develop a formula for the trace of $r(\phi)r(\sigma)$.

The kernel of $r(\phi)r(\sigma)$ is

$$\sum\nolimits_{Z(E)\backslash G(E)} \phi(g^{-1}\gamma\sigma(h)).$$

Let P be the projection of $L_s(\xi_E)$ on $L^1_{se}(\xi_E)$. As on p. 538 of [14] we may find a formula for the kernel of $Pr(\phi)r(\sigma)$ in terms of Eisenstein series. Let D_E be the set of characters of $A(E)\backslash A(\mathbf{A}_E)$ which equal ξ_E on $Z(\mathbf{A}_E)$ and let D^0_E consist of the unitary characters in D_E. We may introduce the parameter s on D_E as before. If $\eta = (\mu, \nu)$ lies in D_E we introduce the space $\mathcal{B}(\eta) = \mathcal{B}(\mu, \nu)$, together with the representation $\rho(\eta)$ of $G(\mathbf{A})$ on it, as in Chap. 10 of [14]. Of course E is now to be substituted for F. As observed on p. 512 we may regard the space $\mathcal{B}(\eta)$ as depending only on the connected component of D_E in which η lies. In each of these connected components we choose an orthonormal basis $\{\varphi_i\}$ of $\mathcal{B}(\eta)$. Let

$$\varphi^\sigma_i(g) = \varphi_i(\sigma(g)).$$

If $E(g, \varphi, \eta)$ is the value of the Eisenstein series defined by φ at g and η then the kernel of $Pr(\phi)r(\sigma)$ is

$$\frac{1}{4\pi} \int_{D^0_E} \sum\nolimits_{i,j} \rho_{ij}(\phi, \eta) E(g, \varphi_i, \eta) \overline{E}(h, \varphi^\sigma_j, \eta^\sigma) |ds|.$$

It would be pointless to introduce the dependence of the basis on the connected component into the notation. Observe that

$$\rho(\phi, \eta) = \int_{Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g)\rho(g) dg.$$

We form the difference of the kernels and integrate along the diagonal. We begin by separating from the integrand some terms whose integral converges and can easily be put in the form we need. We take the sum

$$\sum \phi(g^{-1}\gamma\sigma(g))$$

over those elements γ, taken module $Z(E)$, which are not σ-conjugate to a triangular matrix in $G(E)$. We rewrite it as a sum over σ-conjugacy classes

$$\sum\nolimits_{\{\gamma\}} \varepsilon(\gamma) \sum\nolimits_{Z(E)F_\gamma^\sigma(E)\backslash G(E)} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g)).$$

Here $\varepsilon(\gamma)$ is $\frac{1}{2}$ or 1 according as the equation

$$\delta^{-1}\gamma\sigma(\delta) = z\gamma$$

can or cannot be solved for $\delta \in G(E)$ and z in $Z(E)$ but not in $Z(E)^{1-\sigma}$. Integrating we obtain

$$\sum\nolimits_{\{\gamma\}} \varepsilon(\gamma)\mathrm{meas}(Z(\mathbf{A}_E)G_\gamma^\sigma(E)\backslash Z(\mathbf{A}_E)G_\gamma^\sigma(\mathbf{A}_E)) \int_{Z(\mathbf{A}_E)G_\gamma^\sigma(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g^{-1}\gamma\sigma(g))dg$$

or

$$\sum\nolimits_{\{\gamma\}} \varepsilon(\gamma)\mathrm{meas}(Z(\mathbf{A})G_\gamma^\sigma(E)\backslash G_\gamma^\sigma(\mathbf{A}_E)) \int_{Z(\mathbf{A}_E)G_\gamma^\sigma(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g^{-1}\gamma\sigma(g))dg. \tag{10.8}$$

The convergence of the integral is a consequence of the basic properties of Siegel domains.

The next term we can break off has exactly the same form but the sum is over those γ for which $N\gamma$ is central.

$$\sum\nolimits_{\{\gamma\}} \varepsilon(\gamma)\mathrm{meas}(Z(\mathbf{A}_E)G_\gamma^\sigma(E)\backslash G_\gamma^\sigma(\mathbf{A}_E)) \int_{Z(\mathbf{A}_E)G_\gamma^\sigma(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} \phi(g^{-1}\gamma\sigma(g))dg. \tag{10.9}$$

For the γ appearing here, $\varepsilon(\gamma)$ is easily shown to be 1. Moreover all but a finite number of the terms in this sum are zero.

We turn now to the analogues of (16.2.1) and (16.2.2) of [14]. If B is the group of triangular matrices and N_1 the group of triangular matrices with equal eigenvalues the analogue of (16.2.1) is

$$\sum\nolimits_{B(F)N_1(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash N_1(E)\\ N\gamma\notin Z(F)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))$$

and that of (16.2.2) is

$$\frac{1}{2}\sum\nolimits_{A(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash A(E)\\ N\gamma\notin Z(F)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g)).$$

We introduce the function χ as on p. 529 of [14] and consider

$$\frac{1}{2}\sum\nolimits_{B(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash B(E)\\ N\gamma\notin N_1(E)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))(1-\chi(\delta g)-\chi(\omega(\gamma)\delta g).$$

Here $\omega(\gamma)$ is some element of $G(E)$ not in $B(E)$ for which

$$\omega(\gamma)\gamma\sigma(\omega(\gamma)^{-1}) \in B(E).$$

The integral of this sum over $Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)$ converges. It is equal to

$$\frac{1}{2}\int_{Z(\mathbf{A}_E)B(E)\backslash G(\mathbf{A}_E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash B(E)\\ N\gamma\notin N(E)}} \phi(g^{-1}\gamma\sigma(g))(1-\gamma(g)-\chi(\omega(\gamma)g))dg$$

which we may rewrite as

$$\frac{1}{2}\sum\nolimits_{\substack{\gamma\in A^{1-\sigma}(E)Z(E)\backslash A(E)\\ N\gamma\notin Z(F)}} \int_{Z(\mathbf{A}_E)A(F)\backslash G(\mathbf{A}_E)} \phi(g^{-1}\gamma\sigma(g))(1-\chi(g)-\chi(\omega g))dg \tag{10.10}$$

with

$$\omega=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

If we choose a measure onK_E, the standard maximal compact subgroup of $G(\mathbf{A}_E)$, so that

$$\int_{Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)} h(g)dg=\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)}\int_{N(\mathbf{A}_E)}\int_{K_E} h(ank)dadndk$$

then, as on pages 530–531, the integral (10.10) is equal to the sum of

$$\frac{\ell nc_1}{\ell}\sum\int_{Z(\mathbf{A}_E)A(\mathbf{A})\backslash G(\mathbf{A}_E)}\phi(g^{-1}\gamma\sigma(g))dg \tag{10.11}$$

and

$$-\frac{1}{2\ell}\sum\nolimits_\gamma\sum\nolimits_v\iint_{N(\mathbf{A}_E)}\int_{K_E}\phi(k^{-1}n^{-1}t^{-1}\gamma\sigma(tnk))\ell n\lambda(n_v)dtdndk. \tag{10.12}$$

The outer integral is taken over $Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)$.

The factor ℓ appears in the denominator because χ is defined with respect to absolute values on E. Moreover if E_v is not a field but a direct sum of fields, $\lambda(n_v)$ is the product of the values of λ at the components of n_v. We shall return to these expressions later.

We treat the analogue of (16.2.1) as on p. 532 of [14], separating off

$$\sum\nolimits_{B(F)N_1(E)\backslash G(E)}\sum\nolimits_{\substack{\gamma\in Z(E)\backslash N_1(E)\\ N\gamma\notin Z(F)}}\phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))(1-\chi(\delta g)).$$

The integral of this expression converges and is equal to

$$\int_{Z(\mathbf{A}_E)B(F)N(E)\backslash G(\mathbf{A}_E)}\sum\nolimits_{\substack{\gamma\in Z(E)\backslash N_1(E)\\ N\gamma\notin Z(F)}}\phi(g^{-1}\gamma\sigma(g))(1-\chi(g))dg.$$

If

$$n_0=\begin{pmatrix}1&z_0\\0&1\end{pmatrix}\qquad \operatorname{trace} z_0=1$$

the sum of the integrand is

$$\sum\nolimits_{N(F)Z(E)\backslash B(F)N_1(E)}\phi(g^{-1}\delta^{-1}n_0\sigma(\delta)\sigma(g))(1-\chi(g)).$$

Since $\chi(\delta g) = \chi(g)$ the integral itself is equal to

$$\int_{Z(\mathbf{A}_E)N(F)\backslash G(\mathbf{A}_E)\phi} (g^{-1}n_0\sigma(g))(1-\chi(g))dg.$$

If we write t in $A(\mathbf{A}_E)$ as

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

then this integral is the limit as s approaches 0 from above of

$$\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A})\backslash N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}n_0\sigma(ntk))(1-\chi(t))\left|\frac{a}{b}\right|^{-1-s} dndtdk.$$

If s is positive this integral is the difference of

$$\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A})\backslash N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}n_0\sigma(ntk))\left|\frac{a}{b}\right|^{-1-s} dndtdk \tag{10.13}$$

and

$$\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A})\backslash N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}n^{-1}n_0\sigma(ntk))\left|\frac{a}{b}\right|^{-1-s} \chi(t) dndtdk. \tag{10.14}$$

We suppose $\phi(g) = \prod \phi_v(g_v)$ and set $\theta(s, \phi_v)$ equal to $L(1+\ell s, 1_{F_v})^{-1}$ times

$$\int_{Z(E_v)\backslash A(E_v)} \int_{N(F_v)\backslash N(E_v)} \int_{K_{E_v}} \phi_v(k^{-1}t^{-1}n^{-1}n_0\sigma(ntk))\left|\frac{a}{b}\right|^{-1-s} dndtdk.$$

For almost all v, ϕ_v is ϕ_v^0, whose value at g_v is 0 unless $g_v = zk,\ z \in Z(E_v),\ k \in K_{E_v}$, when it is

$$\xi_{E_v}^{-1}(z)\mathrm{meas}^{-1}(Z(E_v) \cap K_{E_v}\backslash K_{E_v}).$$

If

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and if we take b to be 1 then

$$\phi_v^0(k^{-1}t^{-1}n^{-1}n_0\sigma(ntk)) = \phi_v^0(t^{-1}n^{-1}n_0\sigma(n)\sigma(t))$$

and

$$t^{-1}n^{-1}n_0\sigma(n)\sigma(t) = \begin{pmatrix} a^{-1}\sigma(a) & a^{-1}(1-x+\sigma(x)) \\ \\ 0 & 1 \end{pmatrix}.$$

For almost all v, this matrix can be in K_v only if $a = \alpha y$ where α^{-1} is integral in F_v, y is a unit in E_v, and αx is integral in E_v modulo F_v. If d_v is the product of the measures of the image in $Z(E_v)\backslash A(E_v)$ of

$$\left\{ \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; y \text{ a unit in } E_v \right\},$$

of the image in $N(F_v)\backslash N(E_v)$ of

$$\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\;\middle|\; x \text{ integral in } E_v\right\},$$

and of K_v, divided by the measure of $Z(E_v)\cap K_{E_v}\backslash K_{E_v}$, then

$$\theta(s,\phi_v^0)=d_v.$$

Since the product of the d_v converges and since each $\theta(s,\phi_v)$ is analytic for Res$>\frac{-1}{\ell}$ the product

$$\prod\nolimits_v \theta(s,\phi_v)=\theta(s,\phi)$$

is analytic for Res$>\frac{-1}{\ell}$ and its derivative at $s=0$ is

$$\sum\nolimits_v \theta'(0,\phi_v)\ \prod\nolimits_{w\neq v}\theta(0,\phi_v).$$

The expression (10.13) is equal to

$$L(1+\ell s,1_F)\,\theta(s,\phi).$$

It has a simple pole at $s=0$ and the constant term of its Laurent expansion is

$$\lambda_0\theta(0,\phi)+\frac{\lambda_{-1}}{\ell}\ \sum\nolimits_v \theta'(0,\phi_v)\ \prod\nolimits_{w\neq v}\theta(0,\phi_v). \tag{10.15}$$

This is one of the contributions to the twisted trace formula.

The pole of (10.13) at $s=0$ will have to be cancelled by a pole of (10.14) and is thus irrelevant. As on p. 534 of [14] we use the Poisson summation formula to treat (10.14). It equals the difference of

$$\int_{Z(\mathbf{A}_E)A(F)\backslash A(\mathbf{A}_E)}\iint_{K_E}\sum\nolimits_{\gamma\in N(E)}\phi(k^{-1}t^{-1}n^{-1}\gamma\sigma(ntk))\left|\frac{a}{b}\right|^{-1-s}\chi(t)dndtdk \tag{10.16}$$

and

$$\iiint_{K_E}\phi(k^{-1}t^{-1}n^{-1}\sigma(ntk))\left|\frac{a}{b}\right|^{-1-s}\chi(t)dndtdk. \tag{10.17}$$

The outer integrals are both over $Z(\mathbf{A}_E)A(F)\backslash A(\mathbf{A}_E)$ and the inner integrals in the two expressions are over the different spaces $N(\mathbf{A})N(E)\backslash N\ (\mathbf{A}_E)$ and $N(\mathbf{A})\backslash N(\mathbf{A}_E)$.

Let

$$N_0(\mathbf{A}_E)=\{n\in N(\mathbf{A}_E)\,|\,Nn=1\}.$$

If $g\in G(\mathbf{A}_E)$ and $t\in A(A)$ then

$$\int_{N(\mathbf{A})\backslash N(\mathbf{A}_E)}\phi(g^{-1}t^{-1}n^{-1}\sigma(ntg))dn=\left|\frac{a}{b}\right|_F^{\ell-1}\ \int_{N_0(\mathbf{A}_E)}\phi(g^{-1}n\sigma(g))dn.$$

Notice that an absolute value with respect to F intervenes in this formula; the other absolute values have been taken with respect to E. Also if

$$t' = \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$$

lies in $A(\mathbf{A}_E)$ then

$$\int_{Z(\mathbf{A})A(F)\backslash A(\mathbf{A})} \left|\frac{a}{b}\right|_F^{\ell-1} \left|\frac{a}{b}\right|^{-1-s} \chi(tt')dt = \frac{1}{1+\ell s} \left(\left|\frac{a'}{b'}\right| c_1^{-1}\right)^{\frac{1+\ell s}{\ell}}.$$

Here c_1 is the constant used to define χ. Thus (10.17) is equal to

$$\frac{1}{1+\ell s} \cdot c_1^{-\frac{1+\ell s}{\ell}} \cdot \iint_{N_0(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}n\sigma(t)\sigma(k))\left|\frac{a}{b}\right|^{\frac{1-\ell}{\ell}} dndtdk.$$

The outer integral is taken over $Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)$, and the entire integral is finite. This function is analytic at $s = 0$ and its value there approaches 0 as c_1 approaches infinity. Since our final step in the derivation of the trace formula is to let c_1 pass to infinity, it can be forgotten.

To treat (10.16) we choose a non-trivial character ψ of $F\backslash\mathbf{A}$. Write (10.16) as the integral over $Z(\mathbf{A}_E)A(F)\backslash A(\mathbf{A}_E)$ of

$$\sum\nolimits_{N_0(E)\backslash N(E)} \int_{K_E} \int_{N_0(\mathbf{A}_E)} \phi(k^{-1}t^{-1}\gamma n\sigma(t)\sigma(k))\left|\frac{a}{b}\right|^{-1-s} \chi(t)dndkdt$$

and take the Fourier transform $\Psi(\gamma, t)$, with respect to ψ, of the function

$$\int_{K_E} \int_{N_0(\mathbf{A}_E)} \phi(k^{-1}t^{-1}\gamma n\sigma(t)\sigma(k))dndk$$

on $N_0(\mathbf{A}_E)\backslash N(\mathbf{A}_E)$, which is isomorphic to $N(\mathbf{A})$ or $\mathbf{A}$. If $t \in A(\mathbf{A})$ then

$$\Psi(\gamma, tt') = \left|\frac{a}{b}\right|_F^{\ell-1} \Psi(t\gamma t^{-1}, t').$$

Since $\gamma \to t\gamma t^{-1}$ spreads apart lattice points when $\left|\frac{a}{b}\right|$ is large

$$\int_{Z(\mathbf{A}_E)A(F)\backslash A(\mathbf{A}_E)} \left\{\sum\nolimits_{\gamma\neq 0} \Psi(\gamma, t)\right\} \chi(t) \left|\frac{a}{b}\right|^{-1-s} dt$$

is a holomorphic function of s and its value at $s = 0$ approaches 0 as c_1 approaches ∞.

The remaining term is

$$\int_{Z(\mathbf{A}_E)A(F)\backslash A(\mathbf{A}_E)} \int_{K_E} \int_{N(\mathbf{A}_E)} \phi(k^{-1}t^{-1}n\sigma(t)\sigma(k))\chi(t)\left|\frac{a}{b}\right|^{-1-s} dndtdk$$

which equals

$$\frac{1}{\ell s} \frac{1}{c_1^s} \int_{Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)} \int_{K_E} \int_{N(\mathbf{A}_E)} \phi(k^{-1}t^{-1}n\sigma(t)\sigma(k))\left|\frac{a}{b}\right|^{-1} dndtdk.$$

The pole of this at $s = 0$ cancels with a pole we have met before, but we must keep its constant term with the opposite sign. This is

$$(10.19)\qquad \frac{\ell n c_1}{\ell} \int_{Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)} \int_{K_E} \int_{N(\mathbf{A}_E)} \phi(k^{-1}t^{-1}n\sigma(t)\sigma(k)) \left|\frac{a}{b}\right|^{-1} dndtdk.$$

The product formula together with a little measure-theoretic manipulation allows us to put (10.11) in a form that can be combined with (10.19) to yield

$$(10.20)\qquad \frac{\ell n c_1}{\ell} \sum \iint_{K_E} \int_{N(\mathbf{A}_E)} \phi(k^{-1}t^{-1}\gamma n\sigma(t)\sigma(k)) \left|\frac{a}{b}\right|^{-1} dndtdk.$$

The sum is over $A^{1-\sigma}(E)Z(E)\backslash A(E)$; the outer integral over $Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)$.

We treat what remains of the analogues of (16.2.1) and (16.2.2) as on pages 536–538. For the second we have the sum of

$$\frac{1}{2} \sum\nolimits_{B(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash B(E)\\ N\gamma\notin N_1(E)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))\chi(\delta g)$$

and

$$\frac{1}{2} \sum\nolimits_{B(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash B(E)\\ N\gamma\in N_1(E)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))\chi(\omega(\gamma)\delta g).$$

If $\gamma' = \delta^{-1}\gamma\sigma(\delta)$ with $\delta \in B(E)$ then we may chose $\omega(\gamma') = \delta^{-1}\omega(\gamma)\delta$. It follows easily that these two sums are equal and that together they yield

$$\sum\nolimits_{\substack{\gamma\in A^{1-\sigma}(E)Z(E)\backslash A(E)\\ N\gamma\notin Z(F)}} \sum\nolimits_{Z(E)A(F)\backslash G(E)} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))\chi(\delta g)$$

which may also be written

$$(10.21)\qquad \sum\nolimits_{\substack{\gamma_1\in A^{1-\sigma}(E)Z(E)\backslash A(E)\\ N\gamma_1\notin Z(F)}} \left\{\sum\sum \phi(g^{-1}\delta^{-1}\gamma_1\gamma_2\sigma(\delta)\sigma(g))\chi(\delta g)\right\}.$$

The inner sums are over δ in $A(F)N_1(E)\backslash G(E)$ and γ in $N(E)$. The expression in brackets is 0 for all but finitely many γ_1.

For given γ_1 and g, $\phi(g^{-1}\gamma_1\gamma_2\sigma(g))$ may be regarded as a function on $N(\mathbf{A}_E)$ or, what is the same, on $\mathbf{A}_E$. We choose a non-trivial additive character ψ_E of $E\backslash\mathbf{A}_E$ and set

$$\Psi(y,\gamma_1,g) = \int_{\mathbf{A}} \phi(g^{-1}\gamma_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \sigma(g))\psi(xy)dx.$$

We may apply Poisson summation to the innermost sum of (10.21). Now

$$\int_{Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)} \sum\nolimits_{\gamma_1} \sum\nolimits_{\delta} \sum\nolimits_{\substack{y\neq 0\\ y\in E}} \Psi(y,\gamma_1,\delta g)\chi(\delta g)dg$$

is equal to

$$\sum\nolimits_{\gamma_1} \int_{Z(\mathbf{A}_E)A(F)N(E)\backslash G(\mathbf{A}_E)} \sum\nolimits_{y\neq 0} \Psi(y,\gamma_1,g)\chi(g)dg.$$

Taking the structure of Siegel domains as well as the compact support of ϕ into account one sees that this integral is finite and that it approaches 0 as c_1 approaches infinity. This leaves

$$\sum\nolimits_{\{\gamma_1 \mid N\gamma_1\notin Z(F)\}} \sum\nolimits_{\delta} \Psi(0,\gamma_1,\delta g)\chi(\delta g) \tag{10.22}$$

to be considered.

The analogue of (16.2.1) still yields

$$\sum\nolimits_{B(F)N_1(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash N_1(E)\\ N\gamma\notin Z(F)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))\chi(\delta g).$$

If we observe that every element of $A(E)$ whose norm lies in $Z(F)$ is congruent module $A^{1-\sigma}(E)$ to an element of $Z(E)$, we see that we can apply Poisson summation to this expression to obtain a term which together with (10.22) yields

$$\sum\nolimits_{\gamma_1\in A^{1-\sigma}(E)Z(E)\backslash A(E)} \sum\nolimits_{\delta\in A(F)N_1(E)\backslash G(E)} \Psi(0,\gamma_1,\delta g)\chi(\delta g) \tag{10.23}$$

as well as two remainder terms:

$$-\sum\nolimits_{A(F)N_1(E)\backslash G(E)} \sum\nolimits_{\substack{\gamma\in Z(E)\backslash N_1(E)\\ N\gamma\in Z(F)}} \phi(g^{-1}\delta^{-1}\gamma\sigma(\delta)\sigma(g))\chi(\delta g);$$

and

$$\sum\nolimits_{A(F)N_1(E)\backslash G(E)} \sum\nolimits_{y\neq 0} \Psi(y,1,\delta g)\chi(\delta g).$$

The integrals over $Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)$ of both these functions converge and approach 0 as c_1 approaches ∞.

We now turn to the kernel of $Pr(\phi)r(\sigma)$ on the diagonal. We must separate from it a term which cancels (10.23) and calculate the integral of the reminder. Set $E_1(g,\varphi,\eta)$ equal to

$$\sum\nolimits_{\delta\in B(E)\backslash G(E)} \{\varphi(\delta g)+M(\eta)\varphi(\delta g\}\chi(\delta g)$$

and

$$E_2(g,\varphi,\eta)=E(g,\varphi,\eta)-E_1(g,\varphi,\eta).$$

In the sum it is implicit that φ lies in $\mathcal{B}(\eta)$, that is

$$\varphi(\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} g)=\mu(a)\nu(b)\varphi(g)$$

and that $M(\eta)$ takes $\mathcal{B}(\eta)$ to $\mathcal{B}(\tilde\eta)$ with $\tilde\eta=(\nu,\mu)$. If $1\le m,\ n\le 2$ set

$$H_{mn}(g,\eta;i,j)=\rho_{ij}(\phi,\eta)E_m(g,\varphi_i,\eta)\overline{E}_n(g,\varphi_j^\sigma,\eta^\sigma).$$

The kernel of $Pr(\phi)r(\sigma)$ is

$$\frac{1}{4\pi}\int_{D_E^0}\sum\nolimits_{m,n=1}^{2}\sum\nolimits_{i,j}H_{mn}(g,\eta;i,j)\,|ds|=\sum\nolimits_{m,n=1}^{2}\Phi_{mn}(g).$$

If m or n is 2 the integral of $\Phi_{mn}(g)$ over $Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)$ turns out to be finite and is equal to

$$\frac{1}{4\pi}\int_{D_E^0}\sum\nolimits_{i,j}\left\{\int_{Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)}H_{mn}(g,\eta;i,j)dg\right\}|ds|.$$

First take $m=n=2$. A formula for the inner product

$$\int_{Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)}E_2(g,\varphi_i,\eta)\overline{E}_2(g,\varphi_j^\sigma,\eta^\sigma)dg$$

is given on p. 135 of [22], but in a different notation and not in adelic form. It is easy enough to take these differences into account. Let

$$\alpha_t:\begin{pmatrix}a&0\\0&b\end{pmatrix}\to\left|\frac{a}{b}\right|^t.$$

We may as well suppose η is a unitary character. If $\eta\eta^{-\sigma}$ is trivial on

$$A^0=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}\;\middle|\;\left|\frac{a}{b}\right|=1\right\}$$

set

$$\eta\eta^{-\sigma}=\alpha_{s(\eta)}.$$

The inner product is the sum of two terms. The first is 0 if $\eta\eta^{-\sigma}$ is not trivial on A^0. Otherwise it is

$$\lim_{t\searrow 0}\frac{1}{s(\eta)+2t}\,\{c_1^{s(\eta)+2t}(\varphi_i,\varphi_j^\sigma)-c_1^{-s(\eta)-2t}(M(\eta\alpha_t)\varphi_i,M(\eta^\sigma\alpha_t)\varphi_j^\sigma)\}\tag{10.24}$$

if

$$(\varphi_i,\varphi_j^\sigma)=\int_{K_E}\varphi_i(k)\overline{\varphi}_j^\sigma(k)dk.$$

The second is 0 unless $\eta\tilde{\eta}^{-\sigma}$ is trivial on A^0, when it is

$$\lim_{t\searrow 0}\frac{1}{t(\eta)}\,\{c_1^{t(\eta)}(\varphi_i,M(\eta^\sigma\alpha_t)\varphi_j^\sigma)-c_1^{-t(\eta)}(M(\eta\alpha_t)\varphi_i,\varphi_j^\sigma)\}\tag{10.25}$$

if

$$\eta\tilde{\eta}^{-\sigma}=\alpha_{t(\eta)}.$$

Observe that $s(\eta)$ is constant on connected components of D_E^0 and that

$$t(\eta\alpha_t)=t(\eta)+2t.$$

The Riemann-Lebesgue lemma allows us to discard the integral of (10.24) over those connected components on which $s(\eta)$ is not 0. Those elements of D_E^0 for which $s(\eta)=0$ are all obtained from elements of D^0 by composing with the norm, and ℓ^2 different elements of D^0 give rise to each such η. If $s(\eta) = 0$ then (10.24) equals the sum of

$$2\ell nc_1(\varphi_i, \varphi_j^\sigma) \tag{10.26}$$

and

$$-\frac{1}{2}\,\{(M(\eta)\varphi_i, M'(\eta)\varphi_j^\sigma) + (M'(\eta)\varphi_i, M(\eta)\varphi_j^\sigma)\} = -(M^{-1}(\eta)M'(\eta)\varphi_i, \varphi_j^\sigma). \tag{10.27}$$

If $\eta = \eta^\sigma$ then $g \to \rho(g, \eta)$ may be extended to a representation of $G(\mathbf{A}_E) \times \mathfrak{G}$ for $\varphi \to \varphi^\sigma$ takes $\mathcal{B}(\eta)$ to itself. The trace of $\rho(\phi, \eta)\rho(\sigma, \eta)$ is, on the one hand,

$$\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}tn\sigma(k))\eta(t) \left|\frac{a}{b}\right|^{-1} dtdndk$$

or

$$\int \left\{ \iint_{N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}\gamma n\sigma(t)\sigma(k)) \left|\frac{a}{b}\right|^{-1} dtdndk \right\} \eta(\gamma) \left|\frac{a_0}{b_0}\right|^{-1} d\gamma,$$

the two missing domains of integration being $Z(\mathbf{A}_E)A^{1-\sigma}(\mathbf{A}_E)\backslash A(\mathbf{A}_E)$ and $Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)$, and γ now being

$$\begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix}.$$

On the other hand it is

$$\sum\nolimits_{i,j} \rho_{ij}(\phi, \eta)\,(\varphi_i, \varphi_j^\sigma).$$

We apply Poisson summation to see that $\frac{1}{2\pi}$ times the integral over those η for which $s(\eta) = 0$ of trace $\rho(\phi, \eta)\rho(\sigma, \eta)$ is the sum over γ in $A^{1-\sigma}(E)Z(E)\backslash A(E)$ of

$$\frac{1}{\ell}\left\{ \int_{Z(\mathbf{A}_E)A(\mathbf{A})\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A}_E)} \int_{K_E} \phi(k^{-1}t^{-1}\gamma n\sigma(t)\sigma(k)) \left|\frac{a}{b}\right|^{-1} dtdndk \right\}.$$

The factor ℓ appears in the denominator because the image of $A(\mathbf{A}_E)$ in $A(\mathbf{A})$ is of index ℓ module $A(F)$ and $\frac{1}{2\pi}|ds|$ is the dual of the Tamagawa measure on $Z(\mathbf{A})A(F)\backslash A(\mathbf{A})$ pulled back to characters of $Z(\mathbf{A}_E)A(E)\backslash A(\mathbf{A}_E)$. In any case the contribution of (10.26) cancels (10.20).

We define

$$m_E(\eta) = \frac{L_E(1, \nu\mu^{-1})}{L_E(1, \mu\nu^{-1})}$$

with a subscript to stress that the L-functions are defined with respect to E. We also introduce $R_E(\eta_v)$ so that

$$M(\eta) = m_E(\eta) \otimes_v R_E(\eta_v).$$

The contribution of (10.27) to the trace is the sum of

$$\frac{1}{4\pi}\int_{\{\eta\in D_E^0|s(\eta)=0\}} m_E^{-1}(\eta)m_E'(\eta)\mathrm{trace}\,\rho(\phi,\eta)\rho(\sigma,\eta) \tag{10.28}$$

and

$$\frac{1}{4\pi}\int\sum\nolimits_v \mathrm{trace}(\rho(\phi_v,\eta_v)\rho(\sigma,\eta_v)R_E^{-1}(\eta_v)R_E'(\eta_v))\prod\nolimits_{w\neq v}\mathrm{trace}\,\rho(\phi_w,\eta_w)\rho(\sigma,\eta_w)ds, \tag{10.29}$$

the integral being over $\{\eta\in D_E^0|s(\eta)=0\}$. The sum is over the places of F.

If $\eta^\sigma=\tilde\eta$ then $M(\eta^\sigma)$ is the adjoint $M^*(\eta)$ of $M(\eta)$ and, as on p. 543 of [14], the contribution of the integral of (10.25) to the trace formula is

$$-\frac{1}{4}\sum\nolimits_{\eta^\sigma=\tilde\eta}\mathrm{trace}\,(\rho(\phi,\eta)\rho(\sigma,\eta^\sigma)M(\eta)) \tag{10.30}$$

where

$$\rho(\sigma,\eta^\sigma):\mathcal{B}(\eta^\sigma)\to\mathcal{B}(\eta)$$

and

$$M(\eta):\mathcal{B}(\eta)\to\mathcal{B}(\eta^\sigma).$$

As on pages 543–544 of [14]

$$\int_{Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)} H_{mn}(g,\eta;i,j)dg$$

is 0 if $m\neq n$ and c_1 is sufficiently large.

To handle that part of the kernel given by $\Phi_{1,1}(g)$ we proceed as on p. 544 of [14]. If

$$F(g,\varphi,\eta)=\varphi(g)+M(\eta)\varphi(g)$$

where φ is here a function in $\mathcal{B}(\eta)$ then for c_1 sufficiently large

$$H_{1\,1}(g,\eta;i,j)=\sum\nolimits_{B(E)\backslash G(E)}\rho_{ij}(\phi,\eta)F(\delta g,\varphi_i,\eta)\overline{F}(\delta g,\varphi_j^\sigma,\eta^\sigma)\chi(\delta g).$$

The right side is the sum of four terms which we obtain by replacing $F(g,\varphi_i,\eta)$ by φ_i and $M(\eta)\varphi_i$, and $F(g,\varphi_j^\sigma,\eta^\sigma)$ by φ_j^σ and $M(\eta^\sigma)\varphi_j^\sigma$. Since $\tilde\eta\eta^{-\sigma}$ and $\eta\tilde\eta^{-\sigma}$ are not constant on the connected components, the cross terms $\varphi_i(g)\cdot\overline{M(\eta^\sigma)\varphi_j^\sigma(g)}$ and $M(\eta)\varphi_i(g)\cdot\overline{\varphi_j^\sigma}(g)$ contribute nothing to the trace, or at least only a term which approaches 0 as c_1 approaches ∞.

Thus that part of $\Phi_{1,1}(g)$ which we need to consider is the sum of

$$\sum\nolimits_\delta\frac{1}{4\pi}\int_{D_E^0}\sum\nolimits_{i,j}\rho_{ij}(\phi,\eta)\varphi_i(\delta g)\overline{\varphi_j^\sigma}(\delta g)|ds|\chi(\delta g)$$

and

$$\sum\nolimits_{\delta} \frac{1}{4\pi} \int_{D_E^0} \sum\nolimits_{i,j} \rho_{ij}(\phi,\eta)M(\eta)\varphi_i(\delta g)\overline{M(\eta^\sigma)\varphi_j^\sigma}(\delta g)|ds|\chi(dg).$$

The first integrand

$$\sum\nolimits_{i,j} \rho_{ij}(\phi,\eta)\varphi_i(g)\overline{\varphi_j^\sigma}(g)$$

is the kernel $\rho(\phi,\eta)\rho(\sigma,\eta^\sigma)$ restricted to the diagonal. The second is the kernel of

$$M(\eta)\rho(\phi,\eta)M^*(\eta)\rho(\sigma,\eta^\sigma) = \rho(\phi,\eta)\rho(\sigma,\eta^\sigma).$$

The kernel is also

$$\int_{Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)} \int_{N(\mathbf{A}_E)} \phi(g^{-1}nt\sigma(h))\eta(t)\left|\frac{a}{b}\right|^{-1} dtdn.$$

By Poisson summation our sum is

$$\sum\nolimits_{B(E)\backslash G(E)} \sum\nolimits_{Z(E)\backslash A(E)} \int_{N(\mathbf{A}_E)} \phi(g^{-1}\delta^{-1}\gamma n\sigma(\delta)\sigma(g))dn\chi(\delta g).$$

This is easily seen to equal (10.23); so the two cancel each other.

The twisted trace formula is given by the sum of (10.8), (10.9), (10.12), (10.15), (10.28), (10.29), and (10.30); but we must subject the expressions (10.12), (10.15), and (10.29) to further torture. We first remove

$$\lambda_0\theta(0,\phi) \tag{10.31}$$

from (10.15). If we observe that

$$\theta(s,\phi_v) = \theta(1,s,\phi_v)$$

we may appeal to the results of Paragraph 9 and write the sum of (10.12) and the remaining part of (10.15) as

$$\frac{-\lambda_{-1}}{\ell} \sum\nolimits_{\substack{\gamma\in A^{1-\sigma}(E)Z(E)/\mathbf{A}(E) \\ N\gamma\notin Z(F)}} \sum\nolimits_{v} A_2(\gamma,\phi_v) \prod\nolimits_{w\neq v} F(\gamma,\phi_w) \tag{10.32}$$

and

$$\frac{-\lambda_{-1}}{\ell} \sum\nolimits_{\gamma\in A^{1-\sigma}(E)Z(E)\backslash A(E)} \sum\nolimits_{v} A_3(\gamma,\phi_v) \prod\nolimits_{w\neq v} F(\gamma,\phi_w). \tag{10.33}$$

Poisson summation for the pair $A^{1-\sigma}(E)Z(E)\backslash A(E)$, $A^{1-\sigma}(\mathbf{A}_E)Z(\mathbf{A}_E)\backslash A(\mathbf{A}_E)$ may be applied to the latter sum. If η_v agrees with ξ_{E_v} on $Z(E_v)$ we set

$$B_1(\phi_v,\eta_v) = \int_{A^{1-\sigma}(E_v)Z(E_v)\backslash A(E_v)} A_3(t,\phi_v)\eta_v(t)dt.$$

Since λ_{-1} is just the discrepancy between the global Tamagawa measure and the product of the local Tamagawa measures (10.33) is equal to

$$-\frac{1}{2\pi}\int_{\{\eta\in D_E^0\,|\,s(\eta)=0\}}\sum\nolimits_v B_1(\phi_v,\eta_v)\prod\nolimits_{w\neq v}\mathrm{trace}(\rho(\phi_w,\eta_w)\rho(\sigma,\eta_w))|ds| \tag{10.34}$$

because, as observed in Paragraph 7,

$$\int_{A^{1-\sigma}(E_v)Z(E_v)\backslash A(E_v)}F(t,\phi_v)\eta_v(t)dt=\mathrm{trace}(\rho(\phi_v,\eta_v)\rho(\sigma,\eta_v)).$$

The ℓ has disappeared in (10.34) because the dual measure must be $\ell|ds|$.

If we set

$$B(\phi_v,\eta_v)=\frac{1}{2}\,\mathrm{trace}\,\rho(\phi,\eta_v)\rho(\sigma,\eta_v)R_E^{-1}(\eta_v)R_E'(\eta_v)-B_1(\phi_v,\eta_v)$$

then (10.29) and (10.34) may be combined to yield

$$\frac{1}{2\pi}\int_{\{\eta\in D_E^0\,|\,s(\eta)=0\}}\sum\nolimits_v B(\phi_v\eta_v)\prod\nolimits_{w\neq v}\mathrm{trace}(\rho(\phi_w,\eta_w)\rho(\sigma,\eta_w))|ds|. \tag{10.35}$$

11. THE COMPARISON

As pointed out in §2, the function of the trace formula is to establish the equality

$$\operatorname{trace} R(\phi)R(\sigma) = \operatorname{trace} r(f).$$

However we there defined the representation R only for ℓ odd, and we have now to complete the definition.

Let S be the set of η in D_E for which $\eta^\sigma \neq \eta$ but $\eta^\sigma = \tilde{\eta}$. If $\eta \in S$ and $\eta = (\mu, \nu)$ then $\mu^\sigma = \nu$ and $\nu^\sigma = \mu$ but $\mu^\sigma \neq \mu$ and $\nu^\sigma \neq \nu$. It follows, in particular, that $\ell = 2$ if S is not empty. If $\eta \in S$ we may extend $\rho(\eta)$ to a representation $\tau(\eta)$ of $G(\mathbf{A}_E) \times \mathfrak{G}$ by setting

$$\tau(\sigma) = \rho(\sigma, \eta^\sigma)M(\eta).$$

Indeed

$$\tau(g,\eta)\tau(\sigma,\eta) = \rho(g,\eta)\tau(\sigma,\eta) = \rho(\sigma,\eta^\sigma)M(\eta)\rho(g,\eta)$$

which, because $M(\eta)$ intertwines $\rho(\eta)$ and $\rho(\tilde{\eta}) = \rho(\eta^\sigma)$, is equal to

$$\rho(\sigma,\eta^\sigma)\rho(g,\eta^\sigma)M(\eta) = \rho(\sigma(g),\eta)\rho(\sigma,\eta^\sigma)M(\eta) = \tau(\sigma(g),\eta)\tau(\sigma,\eta).$$

Moreover, by the theory of Eisenstein series $M(\eta^\sigma)M(\eta) = M(\tilde{\eta})M(\eta) = 1$; so

$$\begin{aligned}\tau(\sigma,\eta)\tau(\sigma,\eta) &= \rho(\sigma,\eta^\sigma)M(\eta)\rho(\sigma,\eta^\sigma)M(\eta)\\ &= \rho(\sigma,\eta^\sigma)\rho(\sigma,\eta)M(\eta^\sigma)M(\eta)\\ &= 1.\end{aligned}$$

The representations $\tau(\eta)$ and $\tau(\tilde{\eta})$ are equivalent, for

$$M(\eta)\rho(g,\eta)M(\eta)^{-1} = \rho(g,\tilde{\eta})$$

and

$$M(\eta)\rho(\sigma,\eta^\sigma)M(\eta)M(\eta)^{-1} = M(\eta)\rho(\sigma,\eta^\sigma) = \rho(\sigma,\eta)M(\tilde{\eta}).$$

Since the involution $\eta \to \tilde{\eta}$ has no fixed points on S

$$\tau = \frac{1}{2}\bigoplus\nolimits_S \tau(\eta)$$

is actually a well-defined – up to equivalence – representation of $G(\mathbf{A}_E) \times \mathfrak{G}$. It is 0 if $\ell \neq 2$. Let R be the representation of $G(\mathbf{A}_E)\times\mathfrak{G}$ which is the direct sum of τ and ℓ copies of the representation r on $L_{sp}(\xi_E)\oplus L^o_{se}(\xi_E)$. We now let r denote solely the representation of $G(\mathbf{A})$ on $L_{sp}(\xi) + L^o_{se}(\xi)$.

Suppose $\phi = \Pi\phi_v$ is a function satisfying the conditions of the previous paragraph. Suppose moreover that if v splits in E then ϕ_v on $G(E_v) \simeq G(F_v) \times \cdots \times G(F_v)$ is itself a product of ℓ functions, one for each factor. Then we map $\phi_v \to f_v$, as in Paragraph 5 if v is unramified and ϕ_v is spherical, and as in Paragraph 6 or 8 otherwise.

Theorem 11.1 *The equality*

$$\operatorname{trace} R(\phi)R(\sigma) = \operatorname{trace} r(f)$$

is valid.

We will, as has been stressed, use the results of the previous paragraph to prove this equality. If our knowledge of local harmonic analysis were adequate we could prove it with no difficulty whatsoever; our ignorance however forces some rather inelegant gymnastics upon us. We begin by deriving a formula for

$$\operatorname{trace} R(\phi)R(\sigma) - \operatorname{trace} r(f).$$

We apply the trace formula, cancelling as much as possible.

We begin by observing that the contributions from (10.8) and (10.9) are cancelled by that from (10.1). First of all, if γ is one of the indices in (10.1), the corresponding term is 0 unless γ is a local norm everywhere, and hence a global norm. If $\gamma = N\delta$, then $\varepsilon(\gamma) = \varepsilon(\delta)$, for if

$$u^{-1}\gamma u = z\gamma, \qquad u \in G(F),$$

with $z = Nx,\ x \in Z(E),\ z \neq 1$ then

$$N(u^{-1}\delta u) = N(x\delta)$$

and

$$x\delta = v^{-1}u^{-1}\delta u\sigma(v) = v^{-1}u^{-1}\delta\sigma(u)\sigma(v).$$

Moreover

$$\operatorname{meas}(N_{E/F}Z(\mathbf{A}_E)G_\gamma(F)\backslash G_\gamma(\mathbf{A}A)) = \ell \operatorname{meas}(Z(\mathbf{A})G_\gamma(F)\backslash G_\gamma(\mathbf{A}))$$

and, by standard facts about Tamagawa numbers (formula 16.1.8 of [14]),

$$\begin{aligned}\operatorname{meas}(Z(\mathbf{A})G_\gamma(F)\backslash G_\gamma(\mathbf{A})) &= \operatorname{meas}(Z(\mathbf{A})G^\sigma_\gamma(F)\backslash G^\sigma_\delta(\mathbf{A}_E)) \\ &= \operatorname{meas}(Z(\mathbf{A}_E)G^\sigma_\gamma(F)\backslash Z(\mathbf{A}_E)G^\sigma_\delta(\mathbf{A}_E)).\end{aligned}$$

Since R is so defined that (10.8) and (10.9) have to be multiplied by ℓ, the cancellation follows from the definitions of Paragraphs 6 and 8, provided we recall from Paragraph 4 that if γ is central then the number of places at which δ is not σ-conjugate to a central element is even.

The term (10.4) is cancelled by (10.31), or rather ℓ times (10.31). To see this we have only to appeal to the definitions of Paragraphs 6 and 8, and to observe in particular that every term of (10.4) is 0 except the one indexed by $a \in N_{E/F}Z(E)$.

The terms (10.3) and (10.28) cancel each other. Observe first that there is a surjective map $\eta \to \eta_E$, with $\eta_E(t) = \eta(Nt)$, of D^0 to D^0_E, D^0 and D^0_E being the groups of unitary characters of $A(\mathbf{A})$ and $A(\mathbf{A}_E)$ introduced in the previous paragraph; and that, as we deduce from Paragraph 8,

$$\operatorname{trace} \rho(\phi, \eta_E)\rho(\sigma, \eta_E) = \operatorname{trace} \rho(f, \eta).$$

The expression (10.3) is equal to

$$\frac{1}{4\pi} \int_{D^0_E} \left(\sum\nolimits_{\eta \to \eta_E} m^{-1}(\eta)m'(\eta)\mathrm{trace}(\rho(\phi, \eta_E)\rho(\sigma, \eta_E))\right) |ds|.$$

Since

$$\sum\nolimits_{\eta \to \eta_E} m^{-1}(\eta)m'(\eta) = \ell m_E^{-1}(\eta_E)m'_E(\eta_E)$$

the two can be cancelled – provided of course that we do not forget to multiply (10.28) by ℓ.

The results of Paragraph 9 allow us to cancel (10.5) and (10.32). We should perhaps observe that the term of (10.5) indexed by γ and v is 0 unless γ is a norm everywhere except perhaps at v. But if γ is a norm at all but one place it is a norm everywhere, and hence a norm.

If we add the trace of $\tau(\phi)\tau(\sigma)$ to ℓ times (10.30) we obtain

$$-\frac{\ell}{4} \sum\nolimits_{\{\eta \in D^o_E \,|\, \eta = \eta^\sigma = \tilde{\eta}\}} M(\eta)\mathrm{trace}(\rho(\phi, \eta)\rho(\sigma, \eta)).$$

We have placed $M(\eta)$ outside the trace because it is now a scalar; it intertwines $\rho(\eta)$ with itself and $\rho(\eta)$ is irreducible ([14], Chapter I). If we subtract (10.2) from this we obtain

$$-\frac{1}{4} \sum\nolimits_{\eta} \left\{\sum\nolimits_{\substack{\eta' = \tilde{\eta}' \\ \eta' \to \eta}} M(\eta') - \ell M(\eta)\right\} \mathrm{trace}(\rho(\phi, \eta)\rho(\sigma, \eta)).$$

However, as we shall see in a moment, $M(\eta') = M(\eta) = -1$. Since there are ℓ different η' mapping to a given η, this expression is 0.

It will be enough to show that $M(\eta') = -1$, for $M(\eta)$ is the same object, defined with respect to a different field. First of all, since $\eta' = (\mu', \mu')$

$$m(\eta') = \lim_{t \to 0} m(\eta\alpha_t) = \lim_{t \to 0} \frac{L(1 - 2_t, 1_F)}{L(1 + 2t), 1_F)} = -1.$$

To conclude we have only to appeal to Lemma 7.7 which shows that each $R(\eta'_v)$ is the identity.

At this point only (10.7) and (10.35) are left. They yield the sum over v of

$$\frac{1}{2\pi} \int \left\{\ell B(\phi_v \eta_v) - \sum\nolimits_{\eta' \to \eta} B(f_v, \eta'_v)\right\} \prod\nolimits_{w \neq v} \mathrm{trace}(\rho(\phi, \eta_w)\rho(\sigma, \eta_w))|ds|, \tag{11.1}$$

the integral being taken over $\{\eta \in D_E^o | s(\eta) = 0\}$. Suppose v is unramified and ϕ_v is spherical. Then

$$B(\phi_v, \eta_v) = -B_1(\phi_v, \eta_v)$$

and

$$B(f_v, \eta_v') = -B_1(f_v, \eta_v).$$

If η_1' and η_2' both map to η and $\eta_1' = (\mu_1', \nu_1'),\ \eta_2' = (\mu_2', \nu_2')$, then $\frac{\mu_2'}{\mu_1'}$ and $\frac{\nu_2'}{\nu_1'}$ are both characters of $Z(F)N_{E/F}Z(\mathbf{A}_E)\backslash Z(\mathbf{A})$. Thus if v splits in E, η_v' is the same for all $\eta' \to \eta$. Denote it by η_v^0. Then

$$\sum\nolimits_{\eta' \to \eta} B_1(f_v, \eta_v') = \ell^2 \int_{Z(F_v)\backslash A(F_v)} A_3(t, f_v)\eta_v^0(t)dt.$$

Since

$$\ell A_3(Nt, f_v) = A_3(t, \phi_v)$$

the right side equals

$$\ell \int_{A^{1-\sigma}(E_v)A(E_v)\backslash A(E_v)} A_3(t, \phi_v)\eta_v(t)dt = \ell B_1(\phi_v, \eta_v).$$

If v remains prime in E, then

$$\sum\nolimits_{\eta' \to \eta} B_1(f_v, \eta_v') = \ell^2 \int_{N_{E/F}Z(E_v)\backslash N_{E/F}A(E_v)} A_3(t, f_v)\eta_v^0(t)dt$$

if η_v^o is the restriction of the η_v' to $N_{E/F}A(E_v)$. As before the right side equals $\ell B_1(\phi_v, \eta_v)$. We are led to suspect that

$$\ell B(\phi_v, \eta_v) = \sum\nolimits_{\eta' \to \eta} B(f_v, \eta_v')$$

for all v; so (11.1) should vanish. This however we have yet to prove.

We now know only that

$$\operatorname{trace} R(\phi)R(\sigma) - \operatorname{trace} r(f)$$

is equal to (11.1) above. We must show that this equality can hold only if both sides are 0.

The multiplicity one theorem is valid for the representation of $G(\mathbf{A}_E)$ on $L_{sp}(\xi_E) \oplus L_{se}^o(\xi_E)$ (Proposition 11.1.1 of [14]). If Π, acting on V_Π, is an irreducible constituent then so is $\Pi^\sigma : g \to \Pi(\sigma(g))$. If Π^σ is not equivalent to Π, that is, if $V_\Pi \neq V_{\Pi^\sigma}$ then the trace of $R(\phi)R(\sigma)$ on

$$V_\Pi \oplus V_{\Pi^\sigma} \oplus \cdots \oplus V_{\Pi^{\sigma^{\ell-1}}}$$

is 0. If $V_\Pi = V_{\Pi^\sigma}$ then $G(\mathbf{A}_E) \times \mathfrak{G}$ acts on V_Π. We denote the extended representation by Π'.

The representation Π is a tensor product $\otimes_v \Pi_v$ where Π_v is a representation of $G(E_v)$. If $\Pi^\sigma \simeq \Pi$ then $\Pi_v^\sigma \simeq \Pi_v$ for each v, so Π_v extends to a representation Π_v' of $G(E_v) \times \mathfrak{G}$. Π_v' is determined up to a character of $\mathfrak{G}$. We may suppose that $\Pi' = \otimes\Pi'_v$. Let V be a fixed finite set of places containing all infinite places and all places ramified in E. Suppose Π_v belongs to the unramified principal series for $v \notin V$; then we may also demand that for such v the operator $\Pi'_v(\sigma)$ fixes the K_{E_v} invariant vector. If we consider only ϕ for which ϕ_v is spherical outside of V, we have

$$\operatorname{trace}\Pi'_v(\phi_v)\Pi'_v(\sigma) = \operatorname{trace}\Pi_v(\phi_v) = f_v^\vee(t(\Pi_v))$$

for $v \notin V$. Here

$$t(\Pi_v) = \begin{pmatrix} a(\Pi_v) & 0 \\ 0 & b(\Pi_v) \end{pmatrix}$$

lies in $A(\mathbf{C})$ and

$$a(\Pi_v)b(\Pi_v) = \xi(\varpi_v), \qquad v \text{ split},$$

$$a(\Pi_v)^\ell b(\Pi_v)^\ell = \xi(\varpi_v^\ell), \qquad v \text{ not split},$$

if ϖ_v is a uniformizing parameter for F_v. Observe that it is really only the conjugacy class of $t(\Pi_v)$, that is, the pair $(a(\Pi_v),\, b(\Pi_v))$ which matters. Some of the equalities which are written below should be understood as equalities between conjugacy classes.

If we set

$$\alpha(\Pi) = \prod\nolimits_{v\in V} \operatorname{trace}\Pi'_v(\phi_v)\Pi'_v(\sigma)$$

then the trace of the operator $R(\phi)R(\sigma)$ on $L_{sp}(\xi_E) \oplus L^o_{se}(\xi_E)$ is

$$\sum \alpha(\Pi)\ \prod\nolimits_{v\notin V} f_v^\vee(t(\Pi_v)).$$

The sum is over those Π which are equivalent to Π^σ and for which Π_v belongs to the unramified principal series outside of V.

We need a similar expression for the trace of $\tau(\phi)\tau(\sigma)$. If $\eta \in S$ and $\eta = (\mu, \nu)$ then

$$m_E(\eta) = \frac{L(1, \nu\mu^{-1})}{L(1, \mu\nu^{-1})}.$$

In general this has to be evaluated as a limit. However both numerator and denominator are not finite and different from 0, for $\mu \neq \nu$. Thus the quotient is meaningful as it stands and equals

$$\frac{L(1, \nu\mu^{-1})}{L(1, \nu^\sigma\mu^{-\sigma})} = 1$$

because

$$L(s, \chi) = L(s, \chi^\sigma)$$

for all characters of $E^\times\backslash I_E$.

It follows that

$$M(\eta) = \otimes R_E(\eta_v).$$

If η_v is unramified then $R_E(\eta_v)$ fixes the K_{E_v}-invariant vectors. If ϕ_v is spherical outside of V then trace $\tau(\phi,\eta)\tau(\sigma,\eta)=0$ unless η_v is also unramified outside of v, when it equals

$$\Pi_v \mathrm{trace}\,\rho(\phi,\eta_v)\rho(\sigma,\eta_v^\sigma)R_E(\eta_v)=\alpha(\eta)\prod\nolimits_{v\notin V}(f_v^\vee(t(\eta_v)),$$

with

$$\alpha(\eta)=\prod\nolimits_{v\in V}\mathrm{trace}\,\rho(\phi,\eta_v)\rho(\sigma,\eta_v^\sigma)R_E(\eta_v)$$

and

$$t(\eta_v)=\begin{pmatrix}\mu_v'(\varpi_v) & \\ & \nu_v'(\varpi_v)\end{pmatrix}.$$

Here $\mu_v(x)=\mu_v'(Nx),\ \nu_v(x)=\nu_v'(Nx)$.

The trace of $R(\phi)R(\sigma)$ is, when ϕ_v is spherical outside of V, given by

$$\ell\ \sum\nolimits_\Pi \alpha(\Pi)\ \prod\nolimits_{v\notin V} f_v^\vee(t(\Pi_v))+\frac{1}{2}\ \sum\nolimits_\eta \alpha(\eta)\ \prod\nolimits_{v\notin V} f_v^\vee(t(\eta_v)). \tag{11.2}$$

The indices Π and η are constrained as above. We may treat the trace of $r(f)$ in a similar fashion to obtain

$$\sum\nolimits_\pi \alpha(\pi)\ \prod\nolimits_{v\notin V} F_v^\vee(t(\pi_v)) \tag{11.3}$$

where

$$t(\pi_v)=\begin{pmatrix}a(\pi_v) & 0\\ 0 & b(\pi_v)\end{pmatrix}$$

and

$$\begin{aligned} a(\pi_v)b(\pi_v)&=\xi(\varpi_v), && v \text{ split},\\ a(\pi_v)^\ell b(\pi_v)^\ell&=\xi(\varpi_v^\ell), && v \text{ not split}.\end{aligned}$$

We write the difference of (11.2) and (11.3) as

$$\sum\nolimits_k \alpha_k\ \prod\nolimits_{v\notin V} f_v^\vee(t_v^k) \tag{11.4}$$

with a family of distinct sequences $\{t_v^k | v\notin V\}$ and with none of the α_k equal to 0. Distinct must be understood to mean that either t_v^k and $t_v^{k'}$ are not conjugate for some v which splits in E or $(t_v^k)^\ell$ and $(t_v^{k'})^\ell$ are not conjugate for some v which does not split. We are trying to show that this sum is empty.

If we set

$$\beta(\eta)=\sum\nolimits_{\nu\in V}\left\{\ell\, B(\phi_v,\eta_v)-\sum\nolimits_{\eta'\to\eta} B(f_v,\eta_v')\right\}\left\{\prod\nolimits_{\substack{w\in V\\ w\neq v}}\mathrm{trace}\,\rho(\phi,\eta_w)\rho(\sigma,\eta_w)\right\}$$

and then (11.1) is equal to

$$\frac{1}{2\pi}\int\beta(\eta)\ \prod\nolimits_{v\notin V} f_v^\vee(t(\eta_v))|ds|. \tag{11.5}$$

The integral is taken over those $\eta \in D_E^0$ for which $\eta^\sigma = \eta$ and which are unramified outside V.

Fix a $v \notin V$. Suppose first that v splits in E. We choose $a,\ b$ in $\mathbf{C}$ with $|a| = |b| = 1$ and write any

$$t = \begin{pmatrix} a(t) & 0 \\ 0 & b(t) \end{pmatrix}$$

in $A(\mathbf{C})$ with $a(t)b(t) = \xi(\varpi_v)$ as

$$t = \begin{pmatrix} az & 0 \\ 0 & bz^{-1} \end{pmatrix}.$$

This allows us to regard any function in the Hecke algebra at v, $\mathcal{H}'_v$, onto which $\mathcal{H}'_{E_v}$ maps surjectively, as a finite Laurent series in z. These Laurent series will be invariant under $z \to \frac{b}{a}z^{-1}$. Moreover the Hecke algebra yields all such series.

We may assume that for all the

$$t_v^k = \begin{pmatrix} a_v^k & 0 \\ 0 & b_v^k \end{pmatrix}$$

occurring in (11.4), the inequality

$$|a_v^k| \geq |b_v^k|$$

obtains. It follows from Lemma 3.10 of [14] that

$$\left|\frac{a_v^k}{b_v^k}\right| \leq |\varpi_v|^{-1}.$$

If $\eta \in D_E^0$, η is unramified outside of V, and $v \notin V$, and

$$t(\eta_v) = \begin{pmatrix} a(\eta_v) & \\ & b(\eta_v) \end{pmatrix}$$

then

$$|a(\eta_v)| = |b(\eta_v)| = 1.$$

Let r_v^i, $i = 1, 2, \cdots$ be the distinct elements among the t_v^k for the given fixed v and set

$$c_i = \sum\nolimits_{t_v^k = r_v^i} \alpha_k \prod\nolimits_{\substack{w \notin V \\ w \neq v}} f_v^\vee(t_v^k).$$

We write (11.4) as

$$\sum\nolimits_i c_i f_v^\vee(r_v^i). \tag{11.6}$$

In a given connected component of D on which $\eta = \eta^\sigma$ and η is unramified outside V we may choose η^0 with

$$\eta_v^0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

If on each such component we choose an η^0 and set

$$d(s) = \sum \beta(\eta^0 \alpha_s) \ \prod\nolimits_{\substack{w \notin V \\ w \neq v}} f_w(t(\eta_w^0 \alpha_s))$$

then we may write (11.5) as

$$\frac{1}{2\pi} \int_{-i\infty}^{i\infty} d(s) f_v^\vee(\begin{pmatrix} a|\varpi_v|^s & 0 \\ 0 & b|\varpi_v|^{-s} \end{pmatrix})|ds|. \tag{11.7}$$

It will be recalled that

$$\alpha_s : \begin{pmatrix} a_1 & \\ & b_1 \end{pmatrix} \to \left|\frac{a_1}{b_1}\right|^s.$$

From the equality of (11.6) and (11.7) we want to deduce that all c_i are 0. It will follow that (11.7) is 0; so the theorem will be established, for given any ϕ we can always choose V so that ϕ_v is spherical outside of V as well as a v outside of V which splits in E. It is implicit in (11.6) and (11.7) that ϕ_w and f_w are fixed for $w \neq v$. However we are still free to vary ϕ_v and hence f_v.

Since the trace formula yields absolutely convergent sums and integrals and since, in addition, we can make $f_v^\vee = 1$,

$$\sum |c_i| = M_1 < \infty$$

and

$$\frac{1}{2\pi} \int_{-i\infty}^{i\infty} |d(s)| \ |ds| = M_2 < \infty.$$

Moreover

$$\frac{1}{2\pi} \sup_{-\infty < s < \infty} |d(is)| = M_3 < \infty.$$

We set

$$r_v^i = \begin{pmatrix} az_i & \\ & bz_i^{-1} \end{pmatrix}, \qquad |z_i| \geq 1.$$

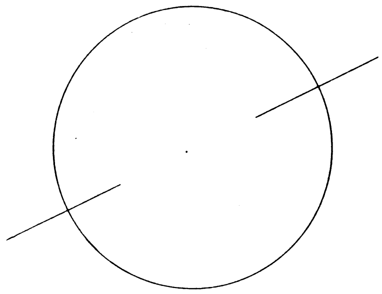
Since the Π_v, the $\rho(\eta_v)$, and the π_v which contribute to (11.6) are all unitary

$$f_v^\vee(r_v^i) = f_v^\vee(s_v^i)$$

with

$$s_v^i = \begin{pmatrix} \bar{a}^{-1}\bar{z}_i^{-1} & 0 \\ 0 & \bar{b}^{-1}\bar{z}_i^{-1} \end{pmatrix}.$$

That is, either $|z_i| = 1$ or $z_i = \frac{b}{a}\bar{z}_i$. Since $|\varpi_v|^{1/2} \leq |z_i| \leq |\varpi_v|^{-1/2}$, the z_i are constrained to lie in the compact set X depicted below.



A finite Laurent series

$$\varphi(z) = \sum \lambda_j z^j$$

is yielded by the Hecke algebra if and only if $a^j\lambda_{-j} = b^j\lambda_j$. If $\lambda_j^* = \bar{\lambda}_{-j}$ this condition is equivalent to $a^j\lambda^*_{-j} = b^j\lambda^*_j$; so φ is yielded by the Hecke algebra if and only if

$$\varphi^*(z) = \sum \lambda_j^* z^j$$

is. Since

$$\lambda_j^* = \left(\frac{a}{b}\right)^j \bar{\lambda}_j$$

the equality

$$\varphi^*(z) = \overline{\varphi(z)}$$

is valid on X. We appeal to the Stone-Weierstrass theorem to conclude that any continuous function φ on X satisfying

$$\varphi(z) = \varphi\left(\frac{b}{a}\,z^{-1}\right) \tag{11.8}$$

can be uniformly approximated by the functions associated to elements of the Hecke algebra.

Both (11.6) and (11.7) then extend to continuous linear functionals in the space of continuous functions satisfying (11.8). It follows from the Riesz representation theorem that they are both zero, for one is given by an atomic measure and the other by a measure absolutely continuous with respect to the Lebesgue measure on the circle.

The theorem gives the equality easiest to state, but we shall work with a sharper form. Observe first that we could have applied a similar argument if v were not split. The only difference is that the Laurent series coming into play would only involve power of z^ℓ. But we would have to notice that it is then only the ℓth power $(t_v^k)^\ell$ of

t_v^k which is relevant. It is clear that by repeatedly applying our argument we can show that if U is any finite set of places disjoint from V then

$$\sum \alpha_k \prod\nolimits_{v \notin U \cup V} f_v^\vee(t_v^k) = 0. \tag{11.9}$$

Here we choose $r_v,\ v \in U$, and take the sum over those k for which

$$\begin{aligned} t_v^k &= r_v, & &v \text{ split},\\ (t_v^k)^\ell &= r_v^\ell, & &v \text{ not split}. \end{aligned}$$

The equality is to be read as an equality of conjugacy classes. It simply means that the two matrices have the same eigenvalues.

We show next that each α_k is 0. Suppose for example that $\alpha_0 \neq 0$. Choose an N such that

$$\sum\nolimits_{k \geq N} |\alpha_k| \leq \frac{|\alpha_0|}{2}.$$

Then choose U disjoint from V so that if $1 \leq k < N$ then for some $v \in U$ either i) v is split and $t_v^k \neq t_v^o$, or ii) v is not split and $(t_v^k)^\ell \neq (t_v^o)^\ell$. Applying (11.9) with $r_v = t_v^o,\ v \in U$, and with all $f_v^\vee$ equal to 1 we deduce a contradiction.

Before going on we review the facts now at our disposal. Let V be a finite set of places containing all infinite places and all finite places ramified in E. Suppose that for each $v \notin V$ we are given

$$r_v = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$$

where $a_v b_v = \xi(\varpi_v)$ if v is split and $(a_v b_v)^\ell = \xi(\varpi_v^\ell)$ if v is not split. Set

$$A_1 = \sum \prod\nolimits_{v \in V} \operatorname{trace}(\Pi_v(\phi_v)\Pi_v'(\sigma)).$$

The sum is taken over all Π occurring in the representation of $G(E)$ on $L_{sp}(\xi_E) \oplus L_{se}^o(\xi_E)$ for which Π_v is unramified outside of V and for which

$$\operatorname{trace} \Pi_v(\phi_v) = f_v^\vee(r_v)$$

for all $v \notin V$ and all spherical ϕ_v. Observe that by the strong form of the multiplicity one theorem (Lemma 3.1), the sum is either empty or contains a single term.

We set

$$A_2 = \sum \prod\nolimits_{v \in V} \operatorname{trace} \tau(\phi_v, \eta_v)\tau(\sigma, \eta_v).$$

Since $\tau(\eta) \sim \tau(\tilde{\eta})$ we take the sum over unordered pairs $(\eta, \tilde{\eta})$ for which i) $\eta^\sigma = \tilde{\eta}$, ii) $\eta \neq \tilde{\eta}$, iii) $\eta = (\mu, \nu)$ and $\mu\nu = \xi_E$, iv) η_v is unramified for $v \notin V$, and v) if $\phi_v,\ v \notin V$, is spherical then

$$\operatorname{trace} \tau(\phi_v, \eta_v) = \operatorname{trace} \rho(\phi_v, \eta_v) = f_v^\vee(r_v).$$

According to Lemma 12.3 of [14], $\eta = \eta'$ or $\tilde{\eta} = \eta'$ if for almost all v either $\eta_v = \eta'_v$ or $\tilde{\eta}_v = \eta'_v$. Thus the sum defining A_2 is either empty or contains a single term. By examining the poles of the L-functions $L(s, \chi \otimes \Pi)$ and $L(s, \chi \otimes \rho(\eta))$ one sees readily that one of the two sums, either that defining A_1 or that defining A_2, must always be empty. Set

$$A = \ell A_1 + A_2.$$

Finally set

$$B = \sum \prod\nolimits_{v \in V} \operatorname{trace} \pi_v(f_v).$$

The sum is taken over all π occurring in the representation r for which π_v is unramified outside of V and for which

$$\operatorname{trace} \pi_v(f_v) = f_v^\vee(r_v)$$

if f_v is the image of some spherical ϕ_v. We know that

$$A = B,$$

and it is this equality with which we shall work.

We begin by studying the representation $\tau(\eta)$, and hence suppose for the moment that E is quadratic over F. Given η with $\eta^\sigma = \tilde{\eta},\ \eta \neq \tilde{\eta}$ choose V and $\{r_v\}$ so that A_2 is

$$\prod\nolimits_{v \notin V} \operatorname{trace} \tau(\phi, \eta_v)\tau(\sigma, \eta_v).$$

If $\eta = (\mu, \mu^\sigma)$ and

$$\rho = \operatorname{Ind}(W_{E/F}, W_{E/E}, \mu),$$

then $\pi = \pi(\rho)$ (§12 of [14]) defines a term entering the sum B. I claim there is only this one term.

If π' also contributes to B then it must be cuspidal. To show that it must be π I apply a theorem of Jacquet-Shalika ([15]), according to which it is enough to show that the function $L(s, \pi' \times \tilde{\pi})$ employed by them has a pole at $s = 1$. Here $\tilde{\pi}$ is the contragredient of π. According to them it suffices for this purpose to show that

$$L(s, \pi'_v \times \tilde{\pi}_v) = L(s, \pi_v \times \tilde{\pi}_v)$$

for almost all v. We take v outside of V. If v splits in E then $\pi_v = \pi'_v$ and the equality is certainly valid. Otherwise

$$L(s, \pi'_v \times \tilde{\pi}_v) = \det{}^{-1}(1 - |\varpi_v|^s t(\pi'_v) \otimes \tilde{\rho}(\Phi_v))$$

if $\tilde{\rho}$ is the contragredient of ρ and Φ_v the Frobenius at v. Since ρ is induced the right side is equal to

$$\det{}^{-1}(1 - |\varpi_v|^{2s}\mu(\Phi_v^2)t(\pi_v')^2).$$

Since the analogous formula is valid for $L(s, \pi_v \times \tilde{\pi}_v)$, the asserted local equality is clear.

We conclude that

$$\prod\nolimits_{v \in V} \operatorname{trace} \tau(\phi_v, \eta_v)\tau(\sigma, \eta_v) = \prod\nolimits_{v \in V} \operatorname{trace} \pi(f_v) \tag{11.10}$$

f $\pi = \pi(\rho)$. We want to deduce the equality

$$\operatorname{trace} \tau(\phi_v, \eta_v)\tau(\sigma, \eta_v) = \operatorname{trace} \pi(f_v) \tag{11.11}$$

for all ϕ_v. We know from Paragraphs 7 and 8 that this equality is valid if v splits, or if $\eta_v = (\mu_v, \nu_v)$ is unramified for then $\mu_v = \nu_v$.

Given F and a non-archimedean v we may choose another quadratic extension E' so that $E'_v = E_v$ and so that every infinite place of F splits in E'. Given any character μ_v of E'_v we may extend it to a character μ of $E'^{\times}\backslash I_E$, which is unramified outside of v. Take $\eta = (\mu, \mu^\sigma)$ and apply the equality (11.10) to E', η. Since we can always choose ϕ so that

$$\operatorname{trace} \tau(\phi_w, \eta_w)\tau(\sigma, \eta_w) \neq 0 \qquad\qquad w \in V,\ w \neq v$$

we deduce (11.11). To prove (11.11) for $F_v = \mathbf{R}$ we take E to be an imaginary quadratic field and F to be $\mathbf{Q}$. Any character of E_∞ extends to a character of $E^\times\backslash I_E$, and we can proceed as before, since we now know that (11.11) is valid at all non-archimedean places. The next lemma is an immediate consequence of the relation (11.11).

Lemma 11.2 *Suppose F is a local field, E a quadratic extension, and $\eta = (\mu, \mu^\sigma)$. Then the character of $\tau(\eta)$ exists as a function and if*

$$\rho = \operatorname{Ind}(W_{E/F}, W_{E/E}, \mu)$$

then

$$\chi_{\tau(\eta)}(g \times \sigma) = \chi_{\pi(\tau)}(h)$$

if h in $G(F)$ is conjugate to Ng and h has distinct eigenvalues, and the representation $\pi(\mu, \mu^\sigma)$ is a lifting of $\pi(\rho)$.

Actually we have only proved the lemma when μ is a unitary character, but the general case reduces immediately to this. Observe that with this lemma, the proof of Proposition 5.1 is complete.

The first assertion of the next lemma is already proved. The others, in which the degree of E over F is an arbitrary prime, will also be deduced from the equality $A = B$.

Lemma 11.3 (a) *If E is a quadratic extension of the global field F and ρ is the representation induced from an idèle class character μ of E then $\pi(\mu, \mu^\sigma)$ is a lifting of $\pi(\rho)$.*

(b) If π is a cuspidal automorphic representation and π is not a $\pi(\rho)$ with ρ dihedral and induced from an idèle class character of the given E then there is a cuspidal automorphic representation Π of $G(\mathbf{A}_E)$ which is a quasi-lifting of π.

(c) f Π is a cuspidal automorphic representation of $G(\mathbf{A}_E)$ and $\Pi^\sigma \sim \Pi$ then Π is a quasi-lifting of some π.

To begin the proof, suppose Π is finite-dimensional and choose $V, r_v,\ v \notin V$ so that A_1 is equal to

$$\prod\nolimits_{v\in V} \mathrm{trace}(\Pi_v(\phi_v)\Pi'_v(\sigma)).$$

If $\Pi(g) = \chi(\det g)$ then $\chi^\sigma = \chi$ and there exists a χ' with $\chi(x) = \chi'(Nx)$. $\Pi'(\sigma)$ is the identity; so we may take each $\Pi'_v(\sigma)$ to be the identity. This means that the extension of Π_v to $G(E_v)\times\mathfrak{G}$ agrees with that of Paragraphs 7 and 8. If ω is again a non-trivial character of $F^\times N_{E/F} I_E\backslash I_E$ then the representations $\pi(g) = \omega^i\chi'(\det g),\ 0 \le i < \ell$, each contribute a term to B. By Lemmas 7.4 and 7.5 and the results of Paragraph 8 the sum of these terms is equal to A. If B' denotes the sum over those π entering into B which are not of the form $g \to \omega^i\chi(\det g)$ of

$$\prod\nolimits_v \mathrm{trace}\,\pi_v(f_v)$$

then B' equals 0. We must show that this implies the sum defining B' is empty.

If we knew that the sum contained only a finite number of terms, this would be an easy application of Lemma 7.13. But we do not, and have to work a little harder. We have a finite set of places $V = (v_1, \cdots, v_r)$, and a sequence $\{(\pi^k_{v_1}, \cdots, \pi^k_{v_r}) \mid k \ge 0\}$, which may terminate or be empty, in which $\pi^k_{v_i}$ is an irreducible, admissible, infinite-dimensional, unitary representation of $G(F_{v_i})$. For each i

$$\pi_{v_i}(zg) = \xi_{v_i}(z)\pi_{v_i}(g) \qquad z \in N_{E_{v_i}/F_{v_i}} E^\times_{v_i}.$$

Moreover for every collection $(f_{v_1}, \cdots, f_{v_r})$ where f_{v_i} is the image of some ϕ_{v_i} on $G(E_{v_i})$ the series

$$\sum\nolimits_k \prod\nolimits_{i=1}^r \mathrm{trace}\,\pi^k_{v_i}(f_{v_i}) \tag{11.12}$$

is absolutely convergent and its sum is 0. We show by induction on r that this implies the sequence is empty.

Take a square-intergrable representation π^0 of $G(F_{v_r})$ satisfying

$$\pi^0(zg) = \xi_{v_r}(z)\pi^0(g), \qquad z \in N_{E_{v_r}\backslash F_{v_r}} E^\times_{v_r}, \tag{11.13}$$

and let $f^0_{v_r}$ be such that

$$\mathrm{trace}\,\pi(f^0_{v_r}) = 0$$

for infinite-dimensional π unless $\pi \simeq \omega_v^i \otimes \pi^0$ for some i, ω_v being the character of $F_v^\times$ associated to the extension E_v. Then the trace is to be $\frac{1}{\ell}$ if $\pi^0 \not\simeq \omega \otimes \pi^0$ and 1 if $\pi^0 \simeq \omega \otimes \pi^0$. Notice that $\pi(f_{v_r}^0)$ is defined only if π too satisfies (11.13). The function $f_{v_r}^0$ is defined by

$$\int_{A(F_{v_r})\backslash G(F_{v_r})} f_{v_r}^0(g^{-1}\gamma g)dg = 0 \tag{11.14}$$

for regular γ in the group $A(F_{v_r})$ of diagonal matrices, and

$$\int T(F_{v_r})\backslash G(F_{v_r}) f_{v_r}^0(g^{-1}\gamma g)dg = \begin{cases} (\operatorname{meas} Z(F_{v_r})\backslash T(F_{v_r}))^{-1}\overline{\chi_{\pi^\circ}(\gamma)}, & \gamma \in NT(E), \\ 0, & \gamma \notin NT(E) \end{cases} \tag{11.15}$$

Here of course γ must in addition be regular, T is a non-split Cartan subgroup, and χ_{π° is the character of π^0.

Substituting $f_{v_r}^0$ for f_{v_r} in (11.12) and applying the induction assumption, we see that $\pi_{v_r}^k$ is never square-integrable. As a consequence (11.12) is not affected by the values of the orbital integrals of f_{v_r} on the non-split Cartan subgroups.

Choose a character $\eta^0 = (\mu^0, \nu^0)$ of $A(F_{v_r})$ such that $\mu^0\nu^0 = \xi_{v_r}$ on $NE_{v_r}^\times$. For simplicity choose η^0 so that if, for some s,

$$\mu^0(x) = \nu^0(x)|x|^s, \qquad \text{for } x \in NE_{v_r}^\times,$$

then $\mu^0 = \nu^0$. This can always be arranged by replacing μ^0 by $x \to \mu^0(x)|x|^{-\frac{s}{2}}$ and ν^0 by $x \to \nu^0(x)|x|^{\frac{s}{2}}\omega_{v_r}^j(x)$. Let

$$A^0(F_{v_r}) = \left\{ t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in NA(E_{v_r}) \;\Big|\; |\alpha| = |\beta| \right\}.$$

If φ is a smooth function on $NA(E_{v_r})$ compactly supported modulo $NZ(E_{v_r})$ and satisfying

$$\varphi(zt) = \eta^0(z)^{-1}\varphi(t), \qquad z \in A^0(F_{v_r}),$$

there is an f_{v_r} such that

$$F_{f_{v_r}}(t) = \begin{cases} \varphi(t) + \varphi(\tilde{t}), & t \in NA(E_{v_r}), \\ 0, & t \in A(F_{v_r}),\ t \notin NA(E_{v_r}). \end{cases}$$

We set

$$\varphi^\vee(s) = \int_{NZ(E_{v_r})\backslash NA(A_{v_r})} \varphi(t)\eta^0(t)\left|\frac{\alpha}{\beta}\right|^s dt.$$

If $\pi = \pi(\mu, \nu)$ is infinite-dimensional and $\mu\nu = \xi_{v_r}$ on $NE_{v_r}^\times$ then

$$\operatorname{trace} \pi(f_{v_r}) = 0$$

unless there is an s such that $\mu(x) = \mu^0(x)|x|^s,\ \nu(x) = \nu^0(x)|x|^{-s}$ for $x \in NE_{v_r}^\times$ and then

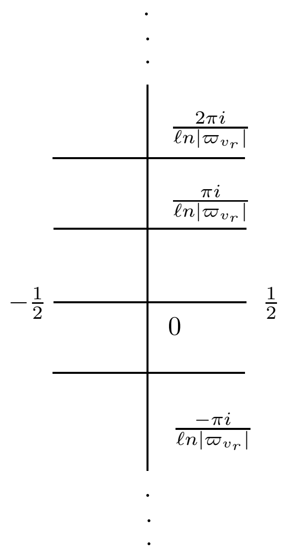
$$\operatorname{trace}\pi(f_{v_r}) = \begin{cases} \varphi^\vee(s), & \tilde\eta^o \neq \eta^o, \\ \varphi^\vee(s) + \varphi^\vee(-s), & \tilde\eta^o = \eta^o. \end{cases}$$

Since the collection of functions $\varphi(t)$ is closed under convolution and, if $\eta^0 = \tilde\eta^0$, also under $\varphi \to \tilde\varphi$ with $\tilde\varphi(t) = \varphi(\tilde t)$, the collection $\varphi^\vee(s)$ or $\varphi^\vee(s) + \varphi^\vee(-s)$ is closed under multiplication.

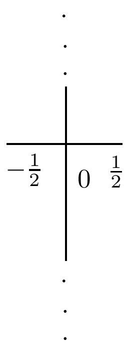
Suppose $\pi = \pi(\mu,\nu)$ is unitary. Then either $\mu = \bar\mu^{-1},\ \nu = \bar\nu^{-1}$ and then s may be taken purely imaginary or $\nu = \omega\mu$ and $\bar\mu^{-1} = \omega\mu,\ \bar\omega^{-1}\bar\mu^{-1} = \mu$. Then $\omega = (\mu\bar\mu)^{-1} : x \to |x|^u$ with u positive. This implies in particular that $\eta^0 = \tilde\eta^0$.

Thus if $\eta^0 \neq \tilde\eta^0$ it is only the values of $\varphi^\vee(s)$ for purely imaginary s which matter. Applying the Stone-Weierstrass Theorem we see that if v is non-archimedean any continuous function on the imaginary axis which is periodic of period $\frac{2\pi i}{\ell n|\varpi_{v_r}|}$ or $\frac{2\pi i}{\ell n|\varpi_{v_r}^\ell|}$, the latter only if v_r is unramified and does not split, may be uniformly approximated by the functions $\varphi^\vee(s)$ and that if v_r is archimedean then any continuous function on the imaginary axis which approaches 0 at infinity may be uniformly approximated by these functions.

If $\eta^0 = \tilde\eta^0,\ \mu(x) = \mu^0(x)|x|^s,\ \nu(x) = \mu^0(x)|x|^{-s}$, and $\mu\nu^{-1}(x) = |x|^{-u}$ with u real then s is real if v_r is archimedean and of the form $\frac{a\pi i}{\ell n|\varpi_{v_r}|} + b,\ a \in \mathbf{Z}$ or $\frac{\mathbf{A}}{\ell},\ b \in \mathbf{R}$ if v_r is non-archimedean. As we observed before an examination of the asymptotic behavior of the spherical functions shows that $\pi(\mu,\nu)$ cannot be unitary unless $-\frac{1}{2} \le s \le \frac{1}{2}$. The Stone-Weierstrass Theorem shows that the $\varphi^\vee(s) + \varphi^\vee(-s)$ uniformly approximate continuous symmetric functions on the set



if v_r is non-archimedean, and continuous symmetric functions on the set



which go to zero at infinity if v_r is archimedean. In the first diagram ϖ_{v_r} is to be replaced by $\varpi_{v_r}^{\ell}$ if the extension is unramified.

Suppose $\pi_{v_r}^0$ in (11.12) is $\pi(\mu,\nu)$ with $\mu(x)=\mu^0(x)|x|^{s_0},\ \nu(x)=\nu^0(x)|x|^{-s_0}$ for $x\in NE_{v_r}^\times$. Choose φ_1 so that

$$1=\begin{cases}\varphi_1^\vee(s_0), & \eta^0\neq\tilde\eta^0,\\[2ex] \varphi_1^\vee(s_0)+\varphi_1^\vee(-s_0), & \eta^0=\tilde\eta^0.\end{cases}$$

Let $s_0, s_1\cdots$ be the collection of s for which there is a k such that

$$\pi_{v_r}^k=\pi(\mu',\nu')$$

with $\mu'(x)=\mu^0(x)|x|^s,\ \nu'(x)=\nu^0(x)|x|^{-s}$ for $x\in NE_{v_r}^\times$. We suppose that the pairs $\{\mu',\nu'\}$ of characters of $NE_{v_r}^\times$ which arise from distinct s_j are distinct. Let $\mu_j(x)=\mu^0(x)|x|^{s_j},\ \nu_j(x)=\nu^0(x)|x|^{-s_j}$ and set

$$\alpha_j=\sum\ \prod\nolimits_{i=1}^{r-1}\operatorname{trace}\pi_{v_i}^k(f_i).$$

The sum is over those $\pi_{v_r}^k$ which have the same lifting as $\pi(\mu_j,\nu_j)$. Then

$$\sum\nolimits_j\alpha_j\varphi_1^\vee(s_j)$$

or

$$\sum\nolimits_j\alpha_j(\varphi_1^\vee(s_j)+\varphi_1^\vee(-s_j))$$

is absolutely convergent. If we choose any φ_2

$$\sum\nolimits_j\alpha_j\varphi_1^\vee(s_j)\varphi_2^\vee(s_j)$$
$$\sum\nolimits_j\alpha_j(\varphi_1^\vee(s_j)+\varphi_1^\vee(-s_j))(\varphi_2^\vee(s_j)+\varphi_2^\vee(-s_j))$$

is equal to 0. The argument used to prove the quality of Theorem 11.1 allows us to conclude that $\alpha_0=0$. From this and the induction hypothesis we immediately derive a contradiction.

We can infer not only that if the Π defining A is finite-dimensional then all the π contribution to B are finite-dimensional but also that if the sum defining A is empty then so is the sum defining B. It is clear that the sum A is empty whenever the sum B is. Parts (b) and (c) of Lemma 11.3 follow immediately from these facts.

As our last piece of serious work we verify the assertion (F) of §2.

Proposition 11.4 *A quasi-lifting is a lifting.*

Once again we exploit the quality $A = B$. Choose a Π occurring in the space of cusp forms and then a V and a collection $\{r_v\}$ so that

$$A = \ell \prod\nolimits_{v \in V} \mathrm{trace}(\Pi_v(\phi_v)\Pi'_v(\sigma)).$$

Let

$$B = \sum\nolimits_k \prod\nolimits_{v \in V} \mathrm{trace}\, \pi_v^k(f_v).$$

The proof of Proposition 11.4 proceeds as follows:

1) We show that if for some $v \in V$ the representation Π_v is the lifting of a π_v then for all k it is the lifting of π_v^k.

2) We let V' be the set of $v \in V$ for which Π_v is not a lifting. We show that if V' is not empty then it contains more than one element.

3) From (2) we deduce the following proposition, which in conjunction with (1) in turn implies Proposition 11.4.

Proposition 11.5 *Suppose F is a local field and E a cyclic extension of prime degree ℓ. Fix a generator σ of $\mathfrak{G}(E/F)$. Every absolutely cuspidal representation π of $G(F)$ has a lifting in the sense of criterion* (ii) *of* §2. *Moreover every representation Π of $G(E)$ for which $\Pi^\sigma \sim \Pi$ is a lifting.*

We begin with (1). Observe that Π_v is not finite-dimensional. If

$$\alpha = \prod\nolimits_{\substack{w \in V \\ w \neq v}} \mathrm{trace}(\Pi_w(\phi_w)\Pi'_w(\sigma))$$

there is an integer i such that

$$A = \zeta^i \alpha\, \mathrm{trace}\, \pi_v(f_v).$$

The power ζ occurs because the Π'_v occurring in the definition of A may not be the Π'_v which satisfies the local lifting condition. The equality $A = B$ becomes

$$\ell \zeta^i \alpha\, \mathrm{trace}\, \pi_v(f_v) = \sum\nolimits_k \beta_k \mathrm{trace}\, \pi_v^k(f_v) \tag{11.16}$$

with

$$\beta_k = \prod\nolimits_{\substack{w \in V \\ v \neq w}} \mathrm{trace}\, \pi_w^k(f_w).$$

Let π'_v be square-integrable and choose f_v so that for infinite-dimensional π''_v with $\pi''_v(z) = \xi(z),\ z \in NE_v^\times$,

$$\text{trace}\,\pi''_v(f_v) = \begin{cases} 0, & \pi''_v \not\simeq \omega_v^j \otimes \pi'_v, \\ 1, & \pi''_v \simeq \omega_v^j \otimes \pi'_v, \pi'_v \simeq \omega_v \otimes \pi'_v, \\ \frac{1}{\ell}, & \pi''_v \simeq \omega_v^j \otimes \pi'_v, \pi'_v \not\simeq \omega_v \otimes \pi'_v. \end{cases} \tag{11.17}$$

Here ω_v is, as usual, a character of $F_v^\times$ associated to the extension E_v. If π_v is not of the form $\omega_v^j \otimes \pi'_v$ then substitution in (11.16) yields

$$0 = \sum\nolimits_{\pi_v^k \simeq \omega_v^j \otimes \pi'_v} \beta_k,$$

all possible j being allowed. The arguments used in the proof of Lemma 11.3 show that the sum is empty. If however π_v is equivalent to some $\omega_v^j \otimes \pi'_v$, then

$$\ell\zeta^i\alpha = \sum\nolimits_{\pi_v^k \simeq \omega_v^j \otimes \pi_v} \beta_k. \tag{11.18}$$

In conjunction with (11.16) this equality yields

$$0 = \sum\nolimits_{\pi_v^k \not\simeq \omega_v^j \otimes \pi_v^k} \beta_k.$$

The sum on the right must once again be empty.

We have shown that if Π_v is the lifting of a square-integrable π_v then it is the lifting of each π_v^k. Suppose it is the lifting of a π_v which is not square-integrable. Then we have shown that no π_v^k is square-integrable. We may introduce the functions $\varphi^v(s)$ as before and show in the same way that every π_v^k has the same lifting as π_v.

Now suppose that there is a single v in V for which Π_v is not a lifting. It is necessarily non-archimedean. The equality $A = B$ becomes

$$\text{trace}\,\Pi_v(\phi_v)\Pi'_v(\sigma) = \sum\nolimits_k \text{trace}\,\pi_v^k(f_v). \tag{11.19}$$

By Lemma 7.9 there is a function χ_v on the union of $NT(E_v)$, where T runs over a set of representatives for the conjugacy classes of Cartan subgroups of G over F_v, such that

$$\text{trace}\,\Pi_v(\phi)\Pi'_v(\sigma) = \frac{1}{2}\sum \int_{NZ(E_v)\backslash NT(E_v)} \chi_v(t)F_{f_v}(t)\Delta(t)dt.$$

Moreover

$$\frac{1}{2}\sum\nolimits' \frac{1}{\text{meas}\,NZ(E_v)\backslash T(F_v)} \int_{NZ(E_v)\backslash NT(E_v)} |\chi_v(t)|^2\Delta(t)^2 dt = \frac{1}{\ell}.$$

By the completeness of the characters of the square-integrable representations of $G(F_v)$, which is a consequence of Theorem 15.1 of [14], there is a square-integrable π_v such that

$$\frac{1}{2}\sum\nolimits' \frac{1}{\text{meas}\,NZ(E_v)\backslash T(F_v)} \int_{NZ(E_v)\backslash NT(E_v)} \chi_v(t)\overline{\chi_{\pi_v}(t)}\Delta(t)^2 dt = \alpha_v \neq 0.$$

It follows from Lemmas 7.6, 7.12, and 11.2 that π_v is absolutely cuspidal and not $\pi(\rho_v)$ for any dihedral ρ_v associated to ρ_v. By Lemma 7.17, $\pi_v \not\simeq \omega_v \otimes \pi$ and then, by Lemma 7.13,

$$\frac{1}{2} \sum{}' \frac{1}{\operatorname{meas} NZ(E_v)\backslash T(F_v)} \int_{NZ(E_v)\backslash NT(E_v)} |\chi_{\pi_v}(t)|^2 \Delta(t)^2 dt = \frac{1}{\ell}.$$

Here ω is a non-trivial character of $F^\times NI_E\backslash I_F$ and ω_v is its component at v. We conclude that

$$|\alpha_v| \le \frac{1}{\ell}$$

with equality only if $\chi_v = \ell\alpha_v\chi_{\pi_v}$ on $NT(E_v)$ whenever T is not split. Choose f_v so that it satisfies (11.17). Taking first $\pi_v' = \pi_v$, we deduce from (11.19) that

$$\ell \prod\nolimits_{v\in V'} \alpha_v = \frac{1}{\ell} \left\{\sum 1\right\}. \tag{11.20}$$

The sum is over those k such that $\pi_v^k \simeq \omega_v^j \otimes \pi_v$ for some j. If π^k contributes to the sum in brackets, so does $\omega \otimes \pi^k$ and $\omega \otimes \pi^k \not\simeq \pi^k$. The sum is therefore a multiple of ℓ. We conclude that $\alpha_w = \frac{1}{\ell}$ and that

$$\chi_v = \chi_{\pi_v} \tag{11.21}$$

on the norms in non-split Cartan subgroups. Moreover the sum on the right of (11.20) contains exactly ℓ terms.

Renumbering if necessary we assume that

$$\pi_v^k \simeq \omega^k \otimes \pi_v, \qquad v \in V',\ 0 \le k < \ell.$$

Choosing the π_v' defining f_v to be inequivalent to each $\omega_v^j \otimes \pi_v$, we conclude from (11.19), (11.21), and the orthogonality relations for characters of square-integrable representations of $G(F_v)$ that if $k \ge \ell$ then π_v^k is not square-integrable. We want to show that there are only ℓ terms on the right of (11.19). Suppose not, so that k takes on the value ℓ.

Choose $\eta^0 = (\mu^0, \nu^0)$ and φ as before, replacing v_r by v and demanding that $\pi_v^\ell = \pi(\mu, \nu)$ with $\mu(x) = \mu^0(x)|x|^s,\ \nu(x) = \nu^0(x)|x|^{-s}$ for $x \in NE_v^\times$. As before we choose f_v so that

$$F_{f_v}(t) = \begin{cases} \varphi(t) + \varphi(\tilde{t}), & t \in NA(E_v), \\ 0, & t \in A(F_v),\ t \notin NA(E_v). \end{cases}$$

We then substitute in (11.19). The terms for $k \ge \ell$ yield a sum

$$\sum\nolimits_j \alpha_j \varphi^\vee(s_j), \qquad \eta^0 \ne \tilde{\eta}^0, \tag{11.22a}$$

or

$$\sum\nolimits_j \alpha_j (\varphi^\vee(s_j) + \varphi^\vee(-s_j)), \qquad \eta^0 = \tilde{\eta}^0. \tag{11.22b}$$

The sum is finite but not empty, and the α_j are positive integers. It is equal to a difference

$$\frac{1}{2}\sum\int_{NZ(E_v)\backslash NT(E_v)}\Delta(t)\chi_v(t)F_{f_v}(t)dt$$

minus

$$\frac{1}{2}\sum\int_{NZ(E_v)\backslash NT(E_v)}\Delta(t)\chi_{\pi_v}(t)F_{f_v}(t)dt.$$

The first part is contributed by the left-hand side of (11.19); the second by the first ℓ terms on the right. Because of (11.21) the contributions from the non-split Cartan subgroups to this difference cancel.

The proofs of Lemma 7.9, and of Proposition 7.4 of [14] show that

$$\frac{\Delta(t)}{2}\{\chi_v(t)-\chi_{\pi_v}(t)\}$$

is bounded on $NA(E_v)$ and that it has support which is compact modulo $NZ(E_v)$. If we choose η^0 and φ as above and set

$$\psi(s)=\int_{NZ(E_v)\backslash NA(E_v)}\frac{\Delta(t)}{2}\{\chi_v(t)-\chi_{\pi_v}(t)\}\eta^0(t)^{-1}\left|\frac{\alpha}{\beta}\right|^{-s}dt$$

then $\psi(-s)=\psi(s)$ if $\eta^0=\tilde{\eta}^0$ and

$$\frac{1}{2}\int_{NZ(E_v)\backslash NA(E_v)}\Delta(t)\{\chi_v(t)-\chi_{\pi_v}(t)\}F_{f_v}(t)dt$$

is equal to

(11.23a)
$$\frac{b}{2\pi\,\mathrm{meas}\,NZ(E_v)\backslash A^0(F_v)}\int_0^{\frac{2\pi i}{b}}\psi(s)\varphi^\vee(s)|ds|$$

if $\eta^0\neq\tilde{\eta}^0$ and to

(11.23b)
$$\frac{b}{2\pi\,\mathrm{meas}\,NZ(E_v)\backslash A^0(F_v)}\int_0^{\frac{2\pi i}{b}}\psi(s)\{\varphi^\vee(s)+\varphi^\vee(-s)\}|ds|$$

if $\eta^0=\tilde{\eta}^0$. Here b is $\ell n|\varpi_v|$ if v is ramified and $\ell n|\varpi_v^\ell|$ if it is not. Both (11.22) and (11.23) are linear functionals of $\varphi^\vee(s)$ given by measures. One is atomic, one is continuous, and they are equal; and so, by the Riesz representation theorem, they are both zero. This is a contradiction.

We conclude that there are only ℓ representations π which contribute to the sum B, namely $\pi^0,\cdots,\pi^{\ell-1}$, with $\pi^j=\omega^j\otimes\pi^0$. It now follows from (11.19) that (11.21) is valid on all norms, and hence that Π_v is a lifting of π_v.

We next prove Proposition 11.5. The proposition has already been proved for F archimedean, and for π and Π not absolutely cuspidal. We may therefore suppose π and Π are absolutely cuspidal. There is then a trivial reduction to unitary π and Π, which we omit. It is moreover enough to show that every π has a lifting, for we can then conclude from the completeness of the characters of square-integrable representations of $G(F)$, which follows from Theorem 15.1 of [14], and the orthogonality relations of Lemma 7.12 that if Π is not a lifting then $\chi_{\Pi'}(t \times \sigma) = 0$ when Nt lies in a non-split Cartan subgroup. This contradicts Lemma 7.9.

If a non-archimedean local field and a cyclic extension of it of order ℓ are given there is a totally real global field F, a place v of it, and a cyclic extension E, totally real and again of degree ℓ, such that the pair F_v, E_v is isomorphic to the given local field with the given cyclic extension. Suppose π_v is a unitary absolutely cuspidal representation of $G(F_v)$. To prove the proposition we have to show that π_v has a lifting. By Step (2), we have only to show that there is a cuspidal automorphic representation π of $G(\mathbf{A})$, whose local component at v is π_v and whose local components at the non-archimedean places other than v are unramified. This will be done with the help of the trace formula.

There is a character ζ_v of $Z(F_v) = F_v^\times$ such that

$$\pi_v(z) = \zeta_v(z), \qquad\qquad z \in Z(F_v).$$

There is also a character ζ of $F^\times \backslash I_F$, unramified outside of v, whose component at v is ζ_v. Let $v_1, \cdots, v_r$ be the infinite places of F. Let $\zeta_{v_i}(-1) = (-1)^{m_i}$. If $n_i > 0$ and $n_i - m_i \equiv 1 (\operatorname{mod} 2)$ there is a pair μ_{v_i}, ν_{v_i} of characters of $F_{v_i}^\times$ such that

$$\mu_{v_i}\nu_{v_i}^{-1} : t \to t^{n_i} \operatorname{sgn} t$$

$$\mu_{v_i}\nu_{v_i} = \zeta_{v_i}.$$

The representation $\pi_{v_i} = \sigma(\mu_{v_i}, \nu_{v_i})$ introduced in Theorem 5.11 of [14] is square-integrable.

There is a smooth function f_{v_i} on $G(F_{v_i})$ compactly supported modulo $Z(F_{v_i})$ such that:

(i) if γ in $A(F_{v_i})$ is regular then

$$\int_{A(F_{v_i})\backslash G(F_{v_i})} f_{v_i}(g^{-1}\gamma g)dg = 0;$$

(ii) if T is a non-split Cartan subgroup over F_{v_i} and γ in $T(F_{v_i})$ is regular then

$$\int_{Z(F_{v_i})\backslash G(F_{v_i})} f_{v_i}(g^{-1}\gamma g)dg = \overline{\chi_{\pi_{v_i}}(\gamma)};$$

(iii) if $z \in Z(F_{v_i})$ then

$$f_{v_i}(zg) = \zeta_{v_i}^{-1}(z) f_{v_i}(g).$$

We may replace v_i by v, π_{v_i} by π_v and then define f_v in a similar manner. If w is a non-archimedean place and $w \neq v$ define f_w by $f_w(g) = 0$ if $g \notin Z(F_w)K_w$ while

$$f_w(zk) = \frac{\zeta_w^{-1}(z)}{\operatorname{meas}(Z(O_w)\backslash K_w)}$$

The trace of $\Phi = \prod f_w$, the product being taken over all places, on the space $L_{sp}(\zeta) \oplus L_{se}^o(\zeta)$ is given by the trace formula on pages 516–517 of [14]. Of the terms given there only (i) and (ii) do not vanish. If the term in (ii) defined by γ is non-zero then $F(\gamma)$ is a totally imaginary quadratic extension of F. Denote the automorphism of this field over F by a bar. Then $\frac{\bar\gamma}{\gamma}$ is a root of unity, for $\frac{\bar\gamma}{\gamma}$ must have absolute value 1 at all places. Moreover we are only interested in γ modulo $Z(F)$ and if the term in (ii) defined by γ does not vanish then, replacing γ by $\frac{\gamma}{\delta}$, $\delta \in Z(F)$, if necessary, we may assume that γ is itself a unit except perhaps at the places in V, if V is a finite set of non-archimedean places containing v and set of generators for the ideal class group of F. Since there are only a finite number of possibilities for the root of unity, there is a finite set of integers $\{k_1, \cdots, k_s\}$ such that the non-zero terms of (ii) are given by γ for which, for at least one i, γ^{k_i} lies in F and is a unit away from V. Applying the unit theorem for the set $\{v_1, \cdots, v_r\} \cup V$ we see that there is a finite set of γ, taken modulo $Z(F)$, which can yield a non-zero contribution to (ii). This set may be chosen to be independent of $n_1, \cdots, n_r$.

If γ_i is the image of γ in an imbedding $F(\gamma) \to \mathbf{C}$ extending v_i and if $(\gamma_i\bar\gamma_i)^{1/2}$ is the positive square root, the contribution of a given γ to (ii) is

$$\frac{1}{2}\,\mathrm{meas}(Z(\mathbf{A})B(F)\backslash B(\mathbf{A}))\ \prod_{i=1}^{r}\left\{\frac{-\zeta_{v_i}((\gamma_i\bar\gamma_i)^{-1/2})}{(\gamma_i\bar\gamma_i)^{\frac{n-1}{2}}\,\mathrm{meas}(Z(F_{v_i})\backslash B(F_{v_i}))}\quad\frac{\gamma_i^{n_i}-\bar\gamma_i^{n_i}}{\gamma_i-\bar\gamma_i}\right\}$$

times the product over the non-archimedean places of

$$\int_{B(F_w)\backslash G(F_w)} f_w(g^{-1}\gamma g)dg.$$

The conclusion to be drawn is that the contribution of (ii) is uniformly bounded.

On the other hand the well-known formulae described in Paragraph 6 show that the term (i) is equal to

$$\mathrm{meas}(Z(F)G(F)\backslash G(\mathbf{A}))\ \prod_{i=1}^{r}\ \frac{n_i}{\mathrm{meas}\,Z(F_{v_i})\backslash G'(F_{v_i})}$$

times

$$\prod_w f_w(1).$$

Here G' is the multiplication group of the quaternion algebra over F_{v_i} and w runs over the non-archimedean places. It is clear that $f_w(1) \neq 0$ if $w \neq v$. Since we may take

$$f_v(g) = d(\pi_v)\overline{(\pi_v(g)u, u)}$$

with a unit vector u we also have $f_v(1) = d(\pi_v) \neq 0$. We infer that a suitable choice of $n_1, \cdots, n_r$ will make (i) arbitrarily large and the trace non-zero. We conclude that for such a choice of $n_1, \cdots, n_r$ there is a constituent π' of the representation on $L_{sp}(\zeta)$ such that if $\pi' = \otimes_w \pi'_w$ then $\pi'_{v_i} = \pi_{v_i}$, $1 \leq i \leq r$, $\pi'_v = \pi_v$, and π_w is unramified if w is non-archimedean but different from v.

There is one more conclusion to be drawn from the equality $A = B$.

Lemma 11.6 (a) *Suppose E is a quadratic extension of the global field F and $\Pi = \pi(\mu, \mu^\sigma)$ with $\mu^\sigma \neq \mu$. Then Π is the lifting of a unique π.*

(b) *Suppose E is cyclic of prime degree ℓ and Π is a cuspidal automorphic representation of $G(\mathbf{A}_E)$ with $\Pi^\sigma \simeq \Pi$. Then Π is the lifting of ℓ cuspidal automorphic representations π.*

Let N be the number of π which lift to Π. The equality $A = B$ now reduces to $N = 1$ in case (a) and to $N = \ell$ in case (b).

The following lemma is important for a complete understanding of the notion of lifting. It is trivial if ℓ is odd, but does not appear to be so if ℓ is even. Indeed the proof is lengthy enough that it seemed best to omit it from these notes and to include it in [18], in which it more easily finds a place.

Lemma 11.7 *Suppose ω is a non-trivial character of $F^\times NI_E\backslash I_F$ and π is a constituent of $L_{sp}(\zeta)$, for some quasi-character ζ of $F^\times\backslash I_F$. Then $\pi \simeq \omega \otimes \pi$ if and only if $\ell = 2$ and there is a character θ of $E^\times\backslash I_E$ such that $\pi = \pi(\tau)$ with*

$$\tau = \mathrm{Ind}(W_{E/F}, W_{E/E}, \theta).$$

There is now no problem in verifying the properties (A)-(G) of global liftings. If π is not cuspidal then it is a constituent of $\rho(\mu, \nu)$, for some pair of idèle-class characters. Its lifting is then a constituent of $\rho(\mu', \nu')$, with $\mu' = \mu \circ N_{E/F},\ \nu' = \nu \circ N_{E/F}$, and, by [25], is also automorphic. If π is cuspidal, then by Lemmas 11.3 and 11.4 it has a lifting. The global unicity is a consequence of the local unicity.

If Π is isobaric and not cuspidal then $\Pi = \pi(\mu, \nu)$ and $\Pi^\sigma \sim \Pi$ if and only if $\mu^\sigma \sim \mu$ and $\nu^\sigma \sim \nu$ or $\mu^\sigma \sim \nu,\ \nu^\sigma \sim \mu$. Thus (B) too follows from Lemmas 11.3 and 11.4. We observe also that, since the notion of a quasi-lifting is independent of σ, the notion of a global lifting is independent of σ. It then follows from the proof of Proposition 11.5 together with Corollary 7.3 and Lemmas 7.4 and 7.5 that the notion of a local lifting is also independent of σ.

Those parts of (C) which are not manifest follow from Lemmas 11.3, 11.6, 11.7 and Lemma 12.3 of [14]. (F) has been proved, and (D) and (E) follow from the corresponding properties of local liftings.

We have still to verify property (e) of local liftings.

Lemma 11.8 *Suppose F_v is a local field, ρ_v an irreducible two-dimensional representation of the Weil group of F_v, and E_v a cyclic extension of F_v of prime degree ℓ. If ρ_v is dihedral or tetrahedral then $\pi(\rho_v)$ exists and the lifting of $\pi(\rho_v)$ is $\pi(P_v)$ if P_v is the restriction of ρ_v to the Weil group of E_v.*

The existence of $\pi(\rho_v)$ follows from the results of §3 and of §12 of [14]. Indeed we may choose global F, E and ρ so that F_v, E_v, and ρ_v are obtained by localization at the place v, which we take to be non-archimedean, the lemma being clear otherwise. Since it is clear from property (G) of global liftings, given in §3, that $\pi(P)$ is the lifting of $\pi(\rho)$, we infer that $\pi(P_v)$ is the lifting of $\pi(\rho_v)$.

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