Automorphic Representations, Shimura Varieties, and Motives. Ein Märchen*

R.P. Langlands

1. Introduction. It had been my intention to survey the problems posed by the study of zeta-functions of Shimura varieties. But I was too sanguine. This would be a mammoth task, and limitations of time and energy have considerably reduced the compass of this report. I consider only two problems, one on the conjugation of Shimura varieties, and one in the domain of continuous cohomology. At first glance, it appears incongruous to couple them, for one is arithmetic, and the other representation-theoretic, but they both arise in the study of the zeta-function at the infinite places.

The problem of conjugation is formulated in the sixth section as a conjecture, which was arrived at only after a long sequence of revisions. My earlier attempts were all submitted to Rapoport for approval, and found lacking. They were too imprecise, and were not even in principle amenable to proof by Shimura's methods of descent. The conjecture as it stands is the only statement I could discover that meets his criticism and is compatible with Shimura's conjecture.

The statement of the conjecture must be preceded by some constructions, which have implications that had escaped me. When combined with Deligne's conception of Shimura varieties as parameter varieties for families of motives they suggest the introduction of a group, here called the Taniyama group, which may be of importance for the study of motives of CM-type. It is defined in the fifth section, where its hypothetical properties are rehearsed.

With the introduction of motives and the Taniyama group, the report takes on a tone it was not originally intended to have. No longer is it simply a matter of formulating one or two specific conjectures, but we begin to weave a tissue of surmise and hypothesis, and curiosity drives us on.

Deligne's ideas are reviewed in the fourth section, but to understand them one must be familiar at least with the elements of the formalism of tannakian categories underlying the conjectural theory of motives, say, with the main results of Chapter II of [**40**].

The present Summer Institute is predicated on the belief that there is a close relation between automorphic representations and motives. The relation is usually couched in terms of L-functions,

* First appeared in Automorphic forms, representations, and L-functions, Proc. of Symp. in Pure Math. XXXIII (1977)

and no one has suggested a direct connection. It may be provided by the principle of functoriality and the formalism of tannakian categories. This possibility is discussed in the highly speculative second section.

However there is a small class of automorphic representations which are certainly not amenable to this formalism. I have called them anomalous, and in order to make their significance clearer I have discussed an example of Kurokawa at length in the third section. Although the anomalous representations form only a small part of the collection of automorphic representations, they are frequently encountered, especially in the study of continuous cohomology, and so we have come full circle. Our long divagation has not been in vain, for we have acquired concepts that enable us to appreciate the global significance of the local examples described in the seventh section, which deals with the second of our original two problems.

At all events, I have exceeded my commission and been seduced into describing things as they may be and, as seems to me at present, are likely to be. They could be otherwise. Nonetheless it is useful to have a conception of the whole to which one can refer during the daily, close work with technical difficulties, provided one does not become too attached to it, but takes pains to ensure that it continues to conform to the facts, and is prepared to abandon it when that is called for. The views of this report are in any case not peculiarly mine. I have simply fused my own observations and reflections with ideas of others and with commonly accepted tenets.

I have also wanted to draw attention to the specific problems, on which expert advice would be of great help, and I hope that the report is sufficiently loosely written that someone familiar with continuous cohomology but not with arithmetic can turn to the seventh section, overlooking the first few pages, and see what the study of Shimura varieties needs from that theory, or that someone familiar with Shimura varieties but not automorphic representations or motives will be able to find the definition of the Serre group in the fourth section and the Taniyama group in the fifth, and then turn to read the sixth section.

Finally a word about what is not discussed in this report. The investigations of Kazhdan [**26**] and Shih [**46**] on conjugation of Shimura varieties are not reviewed; their bearing on the problem formulated here is not yet clear. L-indistinguishability is not discussed. Little is known, and that is described in another lecture [**44**]. Problems caused by noncompactness are ignored. There is a tremendous amount of material on compactification and on the cohomology of noncompact quotients, but no one has yet tried to bring it to bear on the study of the zeta-functions. The omission I regret most is that of a discussion of the reduction of the Shimura varieties modulo a prime [**36**]. Here there is a great deal to be said, especially about the structure of the set of geometric points in the algebraic closure of a finite field, starting with the work of Ihara on curves. I hope to report on this topic on another occasion.

2. Automorphic representations. Our present knowledge does not justify an attempt to fix a language in which the relations of automorphic representations with motives are to be expressed. Nonetheless that of tannakian categories [**40**] appears promising and it might be worthwhile to take a few pages to draw attention to the problems to be solved before it can be applied in the study of automorphic representations.

We first recall the rough classification of irreducible admissible representations of $\mathrm{GL}(n, F)$, F being a local field, and of automorphic representations of $\mathrm{GL}(n, \mathbf{A}_F)$, F being a global field, reviewed in some of the other lectures ([**3**], [**5**], [**35**], [**48**], cf. also [**7**], [**31**]). In either case the representation π has a central character ω and $z \to |\omega(z)|$ may be extended uniquely from $Z(F)$, or $Z(A_F)$, to a positive character ν of $\mathrm{GL}(n, F)$, or of $\mathrm{GL}(n, \mathbf{A}_F)$.

If F is a local field then π is said to be cuspidal if for any K-finite vectors u and v, in the space of π and its dual, $\nu^{-1}(g)\langle\pi(g)u, v\rangle$ is square-integrable on the quotient $Z(F)\backslash\mathrm{GL}(n, F)$. To construct an arbitrary irreducible admissible representation one starts from a partition $\{n_1, \cdots, n_r\}$ of n and cuspidal representations $\pi_1, \cdots, \pi_r$ of $\mathrm{GL}(n_i, F)$. If ω_i is the central character of π_i, there is a real number s_i such that $|\omega_i(z)| \equiv |z|^{s_i}$ if z lies in the centre of $\mathrm{GL}(n_i, F)$, a group isomorphic to $F^\times$. Changing the order of the partition, one supposes that $s_1 \geqq \cdots \geqq s_r$. The partition defines a standard parabolic subgroup P of $\mathrm{GL}(n)$ and $\sigma = \otimes\pi_i$ a representation of $M(F)$, because the Levi factor M of P is isomorphic to $\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r)$. The representation σ yields in the usual way an induced representation I_σ of $G(F)$. I_σ may not be irreducible, but it has a unique irreducible quotient, which we denote $\pi_1 \boxplus \cdots \boxplus \pi_r$. Every representation is of this form and $\pi_1 \boxplus \cdots \boxplus \pi_r \simeq \pi'_1 \boxplus \cdots \boxplus \pi'_s$ if and only if $r = s$ and after renumbering $\pi_i \simeq \pi'_i$. Thus every representation can be represented uniquely as a formal sum, in the sense of this notation, of cuspidal representations. The representation $\pi_1 \boxplus \cdots \boxplus \pi_r$ is said to be tempered if all the s_i are 0. We can clearly define, in a formal manner, the sum of any finite number of representations.

We can in fact formally define an abelian category $\Pi(F)$ whose collection of objects is the union over n of the irreducible, admissible representations of $\mathrm{GL}(n, F)$. If $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$, $\pi' = \pi'_1 \boxplus \cdots \boxplus \pi'_s$, with π_i and π'_j cuspidal, we set

$$\mathrm{Hom}(\pi, \pi') = \bigoplus_{\{i,j \mid \pi_i \sim \pi'_j\}} \mathbf{C}.$$

The composition is obvious. The tempered representations form a subcategory $\Pi^\circ(F)$.

If F is a global field then π is said to be cuspidal if π is a constituent of the representation of $\mathrm{GL}(n, A_F)$ on the space of measurable cusp forms φ satisfying

(a) $\varphi(zg) = \omega(z)\varphi(g), z \in Z(A_F)$,

(b) $\int_{Z(A_F)G(F)\backslash G(A_F)} \nu^{-2}(g)|\varphi(g)|^2 dg < \infty$.

If $n_1 \cdots n_r$ is a partition of n and $\pi_1, \cdots, \pi_r$ cuspidal representations of $\mathrm{GL}(n_i, A_F)$ we may again change the order so that $s_1 \geqq \cdots \geqq s_r$ and then construct the induced representation I_σ. Every automorphic representation is a constituent of some I_σ. For an adequate classification, one needs more. The following statement may eventually result from the investigations of Jacquet, Shalika, and Piatetski-Shapiro, but has not yet been proved in general.

A. *If π is a constituent of I_σ and of $I_{\sigma'}$ then the partitions $\{n_1, \cdots, n_r\}$ and $\{n'_1, \cdots, n'_s\}$ have the same number of elements, and, after a renumbering, $n_i = n'_i$ and $\pi_i \sim \pi'_i$.*

If $\pi_i = \otimes_v \pi_i(v)$ then one constituent of I_σ is the representation $\pi = \otimes_v \pi(v)$ with local components $\pi(v) = \pi_1(v) \boxplus \cdots \boxplus \pi_r(v)$. This representation will be denoted $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$, but the notation is not justified until statement A is proved. The representations of this form will be called isobaric and can again be used to define an abelian category $\Pi(F)$.

We agree to call $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ tempered if each of the cuspidal representations π_i has a unitary central character, that is, if each of the s_i is 0. However the language is only justified if we can prove the following statement, the strongest form of the conjecture of Ramanujan to which the examples of Howe-Piatetski-Shapiro and Kurokawa allow us to cling.

B. *If $\pi = \otimes\pi(v)$ is a cuspidal representation with unitary central character then each of the factors $\pi(v)$ is tempered.*

The tempered representations form a subcategory $\Pi^\circ(F)$ of $\Pi(F)$.

It is clear that we have tried to define the categories $\Pi(F)$ and $\Pi^\circ(F)$ in such a way that if v is a place of F there are functors $\Pi(F) \rightarrow \{\Pi(F_v)$ and $\Pi^\circ(F) \rightarrow \Pi^\circ(F_v)$ taking π to its factor $\pi(v)$. However without a natural definition of the arrows, there is no unique way to define $\mathrm{Hom}(\pi, \pi') \rightarrow \mathrm{Hom}(\pi(v), \pi'(v))$.

If I_σ is not irreducible it will have other constituents in addition to $\pi_1 \boxplus \cdots \boxplus \pi_r$. These automorphic representations will be called anomalous. Although the principle of functoriality may apply to them, there is considerable doubt that they can be fitted into a tannakian framework. Observe that statement A and the strong form of multiplicity one imply that if π is any automorphic representation there is a unique isobaric representation π' such that $\pi(v) \sim \pi'(v)$ for almost all v.

If F is a global field the purpose of the tannakian formalism would be to provide us with a reductive group over $\mathbf{C}$ whose n-dimensional representations, or rather their equivalence classes, are to correspond bijectively to the isobaric automorphic representations of $\mathrm{GL}(n, A_F)$. This hypothetical group will have to be very large, a projective limit of finite-dimensional groups. We denote it by $G_{\Pi(F)}$.

The category $\mathrm{Rep}(G_{\Pi(F)})$ of the finite-dimensional representations over $\mathbf{C}$ of the algebraic group $G_{\Pi(F)}$ would certainly be abelian, but in addition it is a category in which tensor products can be defined. Moreover there is a functor to the category of finite-dimensional vector spaces over $\mathbf{C}$. If (φ, X), consisting of the space X and the representation φ of $G_{\Pi(F)}$ on it, belongs to $\mathrm{Rep}(G_{\Pi(F)})$ one simply ignores φ. The tensor product satisfies certain conditions of associativity, commutativity, and so on, and the functor, called a fibre functor, is compatible with tensor products and other operations of the two categories.

A theorem of [**40**], but not the principal one, asserts that, conversely, an abelian category with tensor products and a fibre functor is equivalent to the category of representations of a reductive group, provided certain natural axioms are satisfied. Thus it appears that if we are to be able to introduce $G_{\Pi(F)}$ we will have to associate to each pair consisting of a cuspidal representation π of $\mathrm{GL}(n, A)$ and a cuspidal representation π' of $\mathrm{GL}(n', A)$ an isobaric representation $\pi \boxtimes \pi'$ of $\mathrm{GL}(nn', A)$. In general, if $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ and $\pi' = \pi_1 \boxplus \cdots \boxplus \pi'_s$ we would set

$$\pi \boxtimes \pi' = \boxplus_{i,j} (\pi_i \boxtimes \pi'_j).$$

In addition, we will have to associate to each isobaric representation π of $\mathrm{GL}(n, A), n = 1, 2, \cdots$, a complex vector space $X(\pi)$ of dimension n, together with isomorphisms

$$X(\pi \boxplus \pi') \simeq X(\pi) \oplus X(\pi') \quad X(\pi \boxtimes \pi') \simeq X(\pi) \otimes X(\pi').$$

There are a large number of conditions to be satisfied, among them one which is perhaps worth mentioning explicitly. Suppose π and π' are cuspidal and $\pi \boxtimes \pi' = \boxplus_{i=1}^r \pi_i$ with π_i cuspidal. Then the set $\{\pi_i, \cdots, \pi_r\}$ contains the trivial representation of $\mathrm{GL}(1, A)$ if and only if π' is the contragredient of π, when it contains this representation exactly once.

At the moment I have no idea how to define the spaces $X(\pi)$; indeed, no solid reason for believing that the functor $\pi \to X(\pi)$ exists. Even though the attempt to introduce the groups $G_{\Pi(F)}$ may turn out to be vain the prize to be won is so great that one cannot refuse to hazard it. One would like to show, in addition, that if π and π' are tempered then $\pi \boxtimes \pi'$ is also, and thus be able to introduce a group $G_{\Pi^\circ(F)}$

classifying the tempered automorphic representations. If Ω_F^+ is the group of multiplicative type whose module of rational characters is the module of positive characters of the topological group $F^\times\backslash I_F$ then $G_{\Pi(F)}$ will be a direct product $G_{\Pi^\circ(F)} \times \Omega_F^+$.

One will also wish to introduce, by a similar process, groups $G_{\Pi(F)}$ and $G_{\Pi^\circ(F)}$, attached to a local field F and classifying the irreducible, admissible representations of $\mathrm{GL}(n, F), n = 1, 2, \cdots$, and the tempered representations of $\mathrm{GL}(n, F)$. The formalism is clearly intended to be such that if F_v is a completion of F there are homomorphisms $G_{\Pi(F_v)} \to G_{\Pi(F)}$ and $G_{\Pi^\circ(F_v)} \to G_{\Pi^\circ(F)}$ dual to $\Pi(F) \to \Pi(F_v)$ and $\Pi^\circ(F) \to \Pi^\circ(F_v)$.

If F is a local field the conjectured classification of the representations of $\mathrm{GL}(n, F)$ **[3]**, verified when F is archimedean, provides a concrete description of the category $\Pi(F)$ with its product ⊠. If F is archimedean and W_F is the Weil group of F then $\Pi(F)$ is equivalent to the category of continuous semisimple representations σ of W_F on complex vector spaces X. The tensor product is the usual one $(\sigma, X) \otimes (\sigma', X') = (\sigma \otimes \sigma', X \otimes X')$ and the fibre functor is $(\sigma, X) \to X$.

Thus $G_{\Pi(F)}$ is a kind of algebraic hull of the topological group W_F. In particular there is a homomorphism $W_F \to G_{\Pi(F)}(\mathbf{C})$ whose image is Zariski-dense. The subcategory corresponding to $\Pi^\circ(F)$ is obtained by taking only those (σ, X) for which the image of $\sigma(W_F)$ is relatively compact.

If F is nonarchimedean one should take not the Weil group but a direct product $W_F' = SL(2, \mathbf{C}) \times W_F$.

Conjecturally at least, σ is to be replaced by a continuous, semisimple representation of W_F' whose restriction to $SL(2, \mathbf{C})$ is complex analytic. To obtain a category equivalent to $\Pi^\circ(F)$ one should take only those σ for which $\sigma(W_F)$ is relatively compact. Observe that in order to obtain a semisimple category we have replaced the group WD_F employed by Borel and Tate **[47]** by the group W_F'. If $w \to |w|$ is the usual positive character of the Weil group, there is an obvious homomorphism of WD_F into W_F' which takes $w \in W_F \subseteq WD_F$ to

$$\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \times w.$$

Notice that according to these classifications there are homomorphisms of algebraic groups

$$G_{\Pi(F)} \to \mathrm{Gal}(\bar{F}/F) \quad \text{and} \quad G_{\Pi^\circ(F)} \to \mathrm{Gal}(\bar{F}/F),$$

the group on the right being a projective limit of finite groups. The principle of functoriality cannot be valid unless there are similar homomorphisms when F is a global field.

Let $\Omega_F, \Omega_F^+, \Omega_F^0$, be the groups of multiplicative type whose modules of rational characters are, respectively, the module of all characters of $F^\times$, or $F^\times \backslash \mathbf{I}_F$ if F is global, of all positive characters, or of all unitary characters. Then $\Omega_F = \Omega_F^+ \times \Omega_F^0$ and there will be homomorphisms

$$G_{\Pi(F)} \to \Omega_F, G_{\Pi^\circ(F)} \to \Omega_F^0.$$

If F is nonarchimedean and local we may also define $\Omega_{\mathrm{un}}, \Omega_{\mathrm{un}}^+$, and Ω_{un}^0, by replacing the modules of characters of various types by the modules of unramified characters of the same type. The groups $\Omega_{\mathrm{un}}, \Omega_{\mathrm{un}}^+$, and Ω_{un}^0 contain a distinguished point over $\mathbf{C}$, the Frobenius Φ, which is simply the image of a uniformizing parameter in $F^\times$. In any case the formalism will certainly allow us to introduce for any representation σ of the algebraic group $G_{\Pi(F)}$ over $\mathbf{C}$ an L-function $L(s,\sigma)$ and if σ corresponds to the representation π of $\mathrm{GL}(n, A_F)$ then $L(s,\sigma) = L(s,\pi)$.

But the reasons for wishing to introduce the groups $G_{\Pi(F)}$ and $G_{\Pi^0(F)}$ and the associated formalism are not simply, or even primarily, aesthetic. There are problems which will be difficult to formulate exactly without them. Suppose, for example, that $\pi = \otimes_v \pi_v$ is an isobaric representation of $\mathrm{GL}(n, A)$ and each of the factors π_v is tempered. For almost all v, π_v is unramified and associated to a conjugacy class $\{g_v\} = \{g(\pi_v)\}$ in $\mathrm{GL}(n, \mathbf{C})$. Since π_ν is supposed tempered this class meets the unitary group $U(n)$ and I may, as I prefer, regard it as a conjugacy class in $U(n)$. The general analytic analogue of the Tchebotarev theorem or the Sato-Tate conjecture would be a theorem or conjecture describing the asymptotic distribution of the classes $\{g_v\}$.

Suppose the formalism existed and π were associated to a representation σ of $G_{\Pi^\circ(F)}$. The image $\sigma(G_{\Pi^\circ(F)}(\mathbf{C}))$ would be a reductive subgroup $H(C)$ of $\mathrm{GL}(n, C)$ with a maximal compact subgroup K_H. For almost all v, σ_v would factor through $G_{\Pi^\circ(F_v)} \to \Omega_{\mathrm{un}}^0$ and $\sigma_v(\Phi_v)$ would be defined. Since its conjugacy class in $H(\mathbf{C})$ would meet K_H, it would define a conjugacy class in K_H, which we denote $\{\sigma_v(\Phi_v)\}$. Of course $\{\sigma_v(\Phi_v)\} \subseteq \{g_v\}$ and the asymptotic distribution of the classes $\{g_v\}$ can be inferred from that of the classes $\{\sigma_v(\Phi_v)\}$. There is a natural probability measure on the space of conjugacy classes in K_H. If X is a set of conjugacy classes and if the set $Y = \cup_{x \in X} x$ is measurable in K_H, one takes meas $X =$ meas Y. It is natural to suppose that it is this measure which defines the asymptotic distribution of the classes $\{\sigma_v(\Phi_v)\}$. To verify the supposition, it will be necessary to establish that if ρ is any representation of $G_{\Pi^\circ(F)}$ over $\mathbf{C}$ then the order of the pole of $L(s,\rho)$ at $s = 1$ is equal to the multiplicity with which the trivial representation of $G_{\Pi^\circ(F)}$ occurs in ρ. If the existence of $G_{\Pi^\circ(F)}$ were established, it would be easy enough to deduce this from the recent results of Jacquet and Shalika [**25**].

Within this formalism, the principle of functoriality asserts that if F is a local field and G a reductive group over F then any L-packet of representations of $G(F)$ is associated to a homomorphism $\varphi : G_{\Pi(F)} \to {}^L G$ of algebraic groups over $\mathbf{C}$ for which

$$\begin{tikzcd} G_{\Pi(F)} \arrow[rr, "\varphi"] \arrow[dr] & & {}^L G \arrow[dl] \\ & \mathrm{Gal}(\overline{F}/F) & \end{tikzcd}$$

is commutative. If $G_{\Pi(F)}$ is replaced by $G_{\Pi^\circ(F)}$, the L-packet should be tempered. If F is global, some caution will have to be exercised. If the L-packet Π consists of $\pi = \otimes \pi_v$ for which the π_v are always tempered, it should correspond to a $\varphi : G_{\Pi^\circ(F)} \to {}^L G$. Otherwise this may not be so, for a reason which will perhaps be clearer after an example of Kurokawa [**29**] is discussed in the next section. If there is a representation

$$\psi : {}^L G \to \mathrm{GL}(n) \times \mathrm{Gal}(\bar{F}/F)$$

and if the image $\psi_*(\Pi)$ of Π given by the principle of functoriality is not isobaric then Π can be associated to no φ.

One may nonetheless hope to prove, both locally and globally, that to each $\varphi : G_{\Pi(F)} \to {}^L G$ is associated an L-packet, provided φ commutes with the homomorphisms to the Galois group. If G is not quasi-split the local behaviour of φ with respect to parabolic subgroups will also have to be taken into account [**3**]. For archimedean fields one recovers the usual classification.

The few examples studied [**44**] suggest that questions about the multiplicity with which elements of Π occur in the space of automorphic forms will have to be answered in terms of φ.

The principal reason for wishing to define the group $G_{\Pi(F)}$ is that it provides the only way visible at present to express completely the relation between automorphic forms and the conjectural theory of motives [**40**]. The category of motives over F, a local or a global field, is **Q**-linear and tannakian, but it does not always possess a fibre functor over **Q** and seldom a single naturally defined one. Thus tannakian duality associates to it not a group over **Q** but a group-like object, a “gerbe” in the rustic terminology which has become so popular in recent years. Over $\mathbf{C}$ this object becomes a group $G_{\mathrm{Mot}(F)}$, and the relations between motives and automorphic representations will probably be adequately expressed by the existence of a homomorphism $\rho_F : G_{\Pi(F)} \to G_{\mathrm{Mot}(F)}$ defined over $\mathbf{C}$. The field F can be local or global. The local and global homomorphisms are to be compatible with each other and with the formation of L-functions. Both the image and the kernel of ρ_F will probably be rather large when F is a number field (cf. C.6.2 of [**43**]) but rather small when F is local.

3. Anomalous representations. Since the anomalous representations cause some difficulty in the study of the zeta-functions of Shimura varieties, it will be useful to acquire some feeling for them before going on. The brief remarks of the previous section suggest that an automorphic representation π of the reductive group G, or rather the L-packet Π containing it, should be called anomalous if for some homomorphism $\psi : {}^LG \to \mathrm{GL}(n, \mathbf{C}) \times \mathrm{Gal}(\bar{F}/F)$ the principle of functoriality takes π or Π to an anomalous representation of $\mathrm{GL}(n, A)$. It may be that the counter-examples to the Ramanujan conjecture of Howe-Piatetski-Shapiro [**19**] are anomalous in this sense.

Since their paper is not available to me as I write, I have to test this suggestion on another example, discovered by Kurokawa and quite explicit. The group G is to be the projective symplectic group in four variables over $\mathbf{Q}$. The L-group is then the direct product of ${}^LG^\circ$, the symplectic group in four variables and the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Since all the groups with which we shall deal in this section will be split, we may ignore the factor $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.

Let G_1 be the product of $PGL(2)$ over $\mathbf{Q}$ with itself, so that ${}^LG_1^\circ = SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$ and let G_2 be $\mathrm{GL}(4)$ over $\mathbf{Q}$. Define $\varphi_1 : {}^LG_1^\circ \to {}^LG$ to be the homomorphism

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \times \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \to \begin{bmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & \alpha_2 & 0 & \beta_2 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & \gamma_2 & 0 & \delta_2 \end{bmatrix}$$

and let φ_2 be the standard imbedding of ${}^LG^\circ$ in ${}^LG_2^\circ$ which is $\mathrm{GL}(4, \mathbf{C})$.

In order to analyze Kurokawa's example one must formulate his statements representation-theoretically. I state the facts necessary to this purpose, but have to ask that the reader understand the discrete series sufficiently well to verify them for himself. There is nothing to prove. It is simply a question of writing down explicitly for the special case of concern to us here some of the results of [**17**] and [**18**], and some definitions from [**31**].

An automorphic representation $\pi = \otimes \pi_v$ of $G(\mathbf{A})$ is associated to a holomorphic form of weight k in the classical sense of [**39**] if and only if π_∞ is a member of the holomorphic discrete series and lies in an L-packet $\Pi_{\psi(k)}$, where the restriction of $\psi(k)$ to $\mathbf{C}^\times \subseteq W_{C/\mathbf{R}}$ has the form

$$z \to \begin{bmatrix} z^\lambda \bar{z}^{-\lambda} & & & \\ & z^\mu \bar{z}^{-\mu} & & \\ & & z^{-\lambda} \bar{z}^\lambda & \\ & & & z^{-\mu} \bar{z}^\mu \end{bmatrix}$$

with $\lambda = (2k-3)/2, \mu = \frac{1}{2}$. Since G is the projective group, k must be even. Notice that $z^\lambda \bar{z}^{-\lambda}$ is to be calculated as $z^{2\lambda}(z\bar{z})^{-\lambda} = \bar{z}^{-2\lambda}(z\bar{z})^\lambda$. We will eventually take $k = 10$.

On the other hand an automorphic representation $\pi = \otimes\pi_\nu$ of $PGL(2,\mathbf{A})$ is associated to a holomorphic form of weight $2k-2$ if and only if π_∞ belongs to the discrete series and to an L-packet $\Pi_{\varphi(k)}$, where

$$\varphi(k) : z \to \begin{pmatrix} z^\lambda \bar{z}^{-\lambda} & \\ & z^{-\lambda}\bar{z}^\lambda \end{pmatrix}$$

with λ as above. In particular φ_1 takes the L-packet $\Pi_{\varphi(k)} \otimes \Pi_{\varphi(2)}$, consisting in fact of a single representation, to $\Pi_{\psi(k)}$.

We want to apply the principle of functoriality to φ_1 and a very special automorphic representation π of $G_1(\mathbf{A})$. π must be a tensor product $\pi' \otimes \pi''$ of two representations of $PGL(2,\mathbf{A})$. π'_∞ and π''_∞ will both be members of the discrete series, the first in $\Pi_{\varphi(10)}$ and the second in $\Pi_{\varphi(2)}$. In fact π' will be the automorphic representation associated to the cusp form of weight 18, but π'' will be anomalous or, to be more exact, its pull-back $\tilde{\pi}''$ to $\mathrm{GL}(2,\mathbf{A})$ will be anomalous. To construct it we begin with the partition $\{1,1\}$ of 2 and the two characters

$$\eta : x \to |x|^{1/2}, \quad \nu : x \to |x|^{-1/2}$$

of $\mathrm{GL}(1,\mathbf{A})$, and then construct the induced representation of $\mathrm{GL}(2,\mathbf{A})$ as in the preceding section. Any constituent $\tilde{\pi}''$ of the induced representation factors through a representation $\tilde{\pi}''$ of $PGL(2,\mathbf{A})$. We so choose $\tilde{\pi}''$ that $\pi''_\infty \in \Pi_{\varphi(2)}$ while $\tilde{\pi}''_p = \eta_p \boxplus \nu_p$ for all p. The representation $\tilde{\pi}''$ is clearly anomalous.

The representation π is unramified. Thus the automorphic representation π° of $G(\mathbf{A})$ lies in the L-packet $\varphi_{1^*}(\{\pi\})$ defined by π, φ_1 and the principle of functoriality if π°_p is unramified for all p and $\pi^\circ_p \in \varphi_{1^*}(\{\pi_p\})$, and π°_∞ is in $\Pi_{\psi(10)}$. We take π° to be the representation defined by the cusp form χ_{10} of weight 10 [**39**]. Then π°_∞ lies in $\Pi_{\psi(10)}$. According to the definitions [**3**] the relation $\pi^\circ_p \in \varphi_{1^*}(\{\pi_p\})$ is a statement about eigenvalues of Hecke operators. These statements have been verified for small primes by Kurokawa [**29**]. The necessary equalities are too complicated to be merely coincidences, and we may assume with some confidence that $\pi^\circ \in \varphi_{1^*}(\{\pi\})$.

In any case the representation π° certainly is a counterexample to Ramanujan's conjecture. If $\varphi = \varphi_2 \circ \varphi_1$ then the principle of functoriality yields the same L-packet when applied to π and φ as it does when applied to π° and φ_2. Since G_2 is $\mathrm{GL}(4)$ the L-packet consists of a single representation. It is easily seen to be anomalous and to be equivalent almost everywhere to the isobaric representation $\pi' \boxplus \eta \boxplus \nu$. Thus π° itself is anomalous in the sense described at the beginning of the section.

4. Shimura varieties. In this section we review the definition of Shimura varieties, taken for now over $\mathbf{C}$, and their relations with motives. The point of view is Deligne's and most of what follows has been taken from his papers [**9**], [**10**], or learned in conversation with him. Of course the book of Saavedra Rivano [**40**] has again been a basic reference; many of the facts and definitions below will be found in it.

Recall that the data needed to define a Shimura variety are a connected reductive group G over **Q** and a homomorphism $h : \mathcal{R} \to G$ defined over $\mathbf{R}$. The symbol $\mathcal{R}$ is used to denote the group $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathrm{GL}(1)$. Thus we have a canonical isomorphism $\mathrm{GL}(1) \times \mathrm{GL}(1) \simeq \mathcal{R}$ over **C** and we may speak of the restriction of h to the first or the second factor. It is not h which matters but the set

$$\mathfrak{H} = \{\mathrm{Ad}\, g \circ h \mid g \in G(\mathbf{R})\}$$

and it will be best simply to let h denote an arbitrary element of $\mathfrak{H}$.

Recall that the pair (G, h) is subject to three conditions [**10**, §1.5]:

(a) *If w is the diagonal map $\mathrm{GL}(1) \to \mathrm{GL}(1) \times \mathrm{GL}(1)$ then the homomorphism $h \circ w$ is central.*

(b) *The Lie algebra $\mathfrak{G}$ of $G(\mathbf{C})$ is a direct sum $\mathfrak{G} = \mathfrak{p} + \mathfrak{k} + \bar{\mathfrak{p}}$ and if $(z_1, z_2) \in \mathcal{R}(\mathbf{C})$ then*

$$\begin{aligned} \mathrm{ad}\, h(z)(X) &= z_1^{-1} z_2 X, \qquad && X \in \mathfrak{p}, \\ &= X, && X \in \mathfrak{k}, \\ &= z_1 z_2^{-1} X, && X \in \bar{\mathfrak{p}}. \end{aligned}$$

In fact the summands $\mathfrak{p}, \mathfrak{k}$, and $\bar{\mathfrak{p}}$ vary with h, and when it is useful to make the dependence on h explicit we write $\mathfrak{p}_h, \mathfrak{k}_h$, and $\bar{\mathfrak{p}}_h$.

(c) *The adjoint action of $h(i, -i)$ on the adjoint group is a Cartan involution.*

Since $G(\mathbf{R})$ acts on the real manifold $\mathfrak{H}$ by conjugation every element of $\mathfrak{G}$ defines a complex vector field on $\mathfrak{H}$. Let X_h be the value of the vector field associated to X at $h \in \mathfrak{H}$. The complex structure on $\mathfrak{H}$ is so defined that the holomorphic tangent space at h is $\{X_h \mid X \in \bar{\mathfrak{p}}\}$ and the antiholomorphic tangent space is $\{X_h \mid X \in \mathfrak{p}\}$.

If $\mathbf{A}_f$ is the ring of adèles over **Q** whose component at infinity is 0 and K is an open compact subgroup of $G(\mathbf{A}_f)$ then $G(\mathbf{A}_f)/K$ is discrete and $X_K = \mathfrak{H} \times G(\mathbf{A}_f)/K$ is a complex manifold on which $G(\mathbf{Q})$ acts to the left. If K is sufficiently small then any $\gamma \in G(\mathbf{Q})$ with a fixed point lies in $Z(\mathbf{Q}) \cap K$ and thus fixes the whole manifold. We shall always assume that K is sufficiently small and then

$$\mathrm{Sh}_K(\mathbf{C}) = G(\mathbf{Q})\backslash\mathfrak{H} \times G(\mathbf{A}_f)/K$$

is a complex manifold, proved by Baily-Borel to be the set of complex points on an algebraic variety $\mathrm{Sh}_K = \mathrm{Sh}_K(G, h) = \mathrm{Sh}_K(G, \mathfrak{H})$ over **C**.

Deligne anticipates that Sh_K will often be a moduli space for a family of motives over $\mathbf{C}$. This is sometimes so, the motives then being those attached to abelian varieties, but can certainly not yet be proved in general. Nonetheless there is a good deal to be learned from a rehearsal of the considerations that suggest such an interpretation of Sh_K. In essence one observes that $\mathrm{Sh}_K(\mathbf{C})$ is the parameter space for a family of polarized Hodge structures; the difficulty is to show that these Hodge structures all arise from motives.

A real Hodge structure V is a finite-dimensional vector space $V_{\mathbf{R}}$ over $\mathbf{R}$ together with a decomposition of its complexification $V_{\mathbf{C}} = \oplus_{p,q\in\mathbf{Z}} V^{p,q}$, satisfying $V^{q,p} = \bar{V}^{p,q}$. The collection of real Hodge structures forms a tannakian category over $\mathbf{R}$ whose associated group is $\mathcal{R}$. Indeed to a real Hodge structure V, one associates the representation σ of $\mathcal{R}$ defined by

$$\sigma(z_1, z_2)v = z_1^{-p} z_2^{-q} v, \quad v \in V^{p,q}. \tag{4.1}$$

The relations $V^{q,p} = \bar{V}^{p,q}$ imply that σ is defined over $\mathbf{R}$. Conversely each representation of $\mathcal{R}$ that is defined over $\mathbf{R}$ yields a Hodge structure, the elements of $V^{p,q}$ being defined by (4.1). The real Hodge structure V is said to be of weight n if $V^{p,q} = 0$ whenever $p + q \neq n$. Certainly any Hodge structure is a direct sum $V = \oplus_n V^n$ with V^n of weight n.

We are however interested in the category of polarized rational Hodge structures. A rational Hodge structure V is formed by a finite-dimensional vector space $V_{\mathbf{Q}}$ over $\mathbf{Q}$, a direct sum decomposition $V_{\mathbf{Q}} = \oplus V^n_{\mathbf{Q}}$, and real Hodge structures of weight n on $V^n_{\mathbf{R}} = V^n_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. There is a distinguished object of weight -2, the Tate object $\mathbf{Q}(1)$, in the category of rational Hodge structures. The underlying rational vector space is $\mathbf{Q}(1)_{\mathbf{Q}} = 2\pi i\mathbf{Q} \subseteq \mathbf{C}$ and, by definition,

$$\mathbf{Q}(1)^{-1,-1} = \mathbf{Q}(1)_{\mathbf{C}}.$$

It seems to be customary to identify the underlying vector space of $\mathbf{Q}(n) = \mathbf{Q}(1)^{\otimes n}$ with $(2\pi i)^n\mathbf{Q}$ and $\mathbf{Q}(n)_{\mathbf{R}}$ with $(2\pi i)^n\mathbf{R} \subseteq \mathbf{C}$. The factors $2\pi i$ have been chosen for reasons which need not concern us. It is no trouble to carry them along.

If V is a rational Hodge structure and σ the associated representation of $\mathcal{R}$ let $\mathbf{C}$ be $\sigma(-i, i)$ acting on $V_{\mathbf{R}}$. If V is of weight n, a polarization of V is a bilinear form $P : V \times V \to \mathbf{Q}(-n)$ satisfying:

(a) *For all u and ν in $V_{\mathbf{C}}$ and all $r \in R(\mathbf{C})$*

$$P(\sigma(r)u, \sigma(r)\nu) = \sigma(r)P(u, \nu).$$

Thus the form is compatible with the Hodge structures.

(b) $P(v,u) = (-1)^n P(u,v)$.

(c) *The real-valued form* $(2\pi i)^n P(u, Cv)$ *on* $V_{\mathbf{R}}$ *is symmetric and positive-definite.*

A rational Hodge structure is said to be polarizable if each of its homogeneous components admits a polarization, a polarization of the full structure being defined by polarizations of the homogeneous components. The category $\mathcal{HOD}(\mathbf{Q})$ of polarizable Hodge structures is tannakian, with a natural fibre functor $\omega_{\mathrm{Hod}} : V \to V_{\mathbf{Q}}$ and an associated group G_{Hod}, reductive but overwhelmingly large. It does have factor groups of manageable size.

If V is a polarizable rational Hodge structure, one may take the tannakian category generated by V and $\mathbf{Q}(1)$ and the repeated formation of duals, sums, tensor products, and subobjects. The associated group is called the Mumford-Tate group of V and denoted by $\mathcal{MT}(V)$. It is finite-dimensional and reductive, and there is a surjection $G_{\mathrm{Hod}} \to \mathcal{MT}(V)$ defined over $\mathbf{Q}$. If σ is the representation of $\mathcal{R}$ attached to V then $\mathcal{MT}(V)$ is simply the smallest subgroup of the group of automorphisms of the rational vector space underlying V which contains $\sigma(\mathcal{R})$ and is defined over $\mathbf{Q}$ [**37**]. It is consequently connected.

The polarizable rational Hodge structures for which $\mathcal{MT}(V)$ is abelian play a particularly important role in the study of Shimura varieties. They are said to be of CM type. The second description of the groups $\mathcal{MT}(V)$ shows that the category of such Hodge structures is closed under sums and tensor products and thus is a tannakian category. The associated group has been studied at length in [**41**] and is often called the Serre group. At the risk of making a comparison with [**41**] difficult, for Serre himself employs a different notation, we shall denote the group by $\mathcal{S}$.

It is not difficult to describe $\mathcal{S}$. Let $\bar{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and let $\iota \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be complex conjugation. To construct $X^*(\mathcal{S})$, the module of rational characters of $\mathcal{S}$, we start with the module M of locally constant integral-valued functions on $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. The Galois group acts by right translation and

$$X^*(\mathcal{S}) = \{\lambda \in M \mid (\sigma - 1)(\iota + 1)\lambda = (\iota + 1)(\sigma - 1)\lambda = 0 \ \forall \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})\}.$$

In particular if $\lambda \in X^*(\mathcal{S})$,

$$(\sigma - 1)\lambda(1) + (\sigma - 1)\lambda(\iota) = 0$$

because the left side if $(\iota+1)(\sigma-1)\lambda(1)$. The lattice of rational characters of $\mathcal{R}$ is canonically isomorphic to $\mathbf{Z}\oplus\mathbf{Z}$ and the homomorphism $h_0 : \mathcal{R} \to \mathcal{S}$ dual to the homomorphism $X^*(\mathcal{S}) \to X^*(\mathcal{R})$ which sends

λ to $(\lambda(1), \lambda(\iota))$ is defined over $\mathbf{R}$. The composite $h \circ w : \mathrm{GL}(1) \to \mathcal{S}$ is dual to the homomorphism $X^*(\mathcal{S}) \to \mathbf{Z}$ taking λ to $\lambda(1) + \lambda(\iota)$ and is defined over $\mathbf{Q}$.

To verify that the group $\mathcal{S}$ just defined in terms of its module of characters is the Serre group defined in terms of Hodge structures is easy enough. The existence of the two homomorphisms h_0 and $h_0 \circ w$ implies that every representation of $\mathcal{S}$ defined over $\mathbf{Q}$ defines a rational Hodge structure. It is enough to show that these are polarizable when the representation is irreducible. To obtain the irreducible representations, one takes a $\lambda \in X^*(\mathcal{S})$ and defines the field F by

$$\mathrm{Gal}(\bar{\mathbf{Q}}/F) = \{\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \mid \sigma\lambda = \lambda\}.$$

The underlying space of the representation is the vector space over $\mathbf{Q}$ defined by F, and the representation $r = r_\lambda$ is that defined symbolically by $r_\lambda(s) : x \in F \to \lambda(s)x$. The weight of the associated Hodge structure is $-(\lambda(1) + \lambda(\iota)) = n$. There is an $\alpha \in F$ such that $\iota(\alpha) = (-1)^n\alpha$ and $(-1)^{\lambda(\iota)} i^n \alpha$ is totally positive. A possible polarization is

$$P(u, \nu) = (2\pi i)^{-n} Tr_{F/\mathbf{Q}} u\alpha\iota(\nu).$$

Conversely suppose one has a rational Hodge structure V whose Mumford-Tate group $\mathcal{MT}(V)$ is abelian. Since there is a homomorphism $\mathcal{R} \to \mathcal{MT}(V)$, the coweight $\mathrm{GL}(1) \to \mathcal{R}$ of the group $\mathcal{R}$ defined by $z \to (z, 1)$ also defines a coweight $\nu^\vee$ of $\mathcal{MT}(V)$. The lattice Y_* of coweights of $\mathcal{MT}(V)$ is a $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ module generated by $\nu^\vee$. We define an injective homomorphism of its lattice of rational characters V^* into $X^*(\mathcal{S})$ by sending $\nu \in Y^*$ to the element λ given by

$$\lambda(\sigma) = \langle \sigma\nu, \nu^\vee \rangle, \quad \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}).$$

The dual homomorphism $\mathcal{S} \to \mathcal{MT}(V)$ is surjective and V is defined by a rational representation of $\mathcal{S}$.

We return to the Shimura varieties $\mathrm{Sh}_K(\mathbf{C})$ and show how one attaches families of rational Hodge structures to $\mathrm{Sh}_K(\mathbf{C})$. Let G_0 be the largest quotient of G such that $(\sigma-1)(\iota+1)\nu^\vee = (\iota+1)(\sigma-1)\nu^\vee = 0$ for every coweight of the centre of G_0. Let ξ be a rational representation of G on V which factors through G_0.

To each $x = (h, g)$ in X_K we associate a triple $(V^x, \mathfrak{k}^x, \varphi^x)$. V^x is a rational Hodge structure whose underlying space is $V^x_{\mathbf{Q}} = V_{\mathbf{Q}}$, the Hodge structure being defined by the representation $\xi \circ h$ of $\mathcal{R}$. The third term φ^x is an isomorphism $V^x_{\mathbf{A}_f} \to V_{\mathbf{A}_f}$ and is given by $\nu \to \xi(g)^{-1}\nu$. It is only defined up to composition with an element of $\xi(K)$. The homogeneous components of $V^x_{\mathbf{Q}}$ are independent of x and may be written $V^n_{\mathbf{Q}}$. It follows from Lemma 2.8 of [**11**] that if $x \in X_K$ there is at least one collection P^x of bilinear forms $P^x_n : V^n_{\mathbf{Q}} \times V^n_{\mathbf{Q}} \to \mathbf{Q}(-n)_{\mathbf{Q}}$ which is invariant under the derived group of G and is a polarization of V^x. Let $\mathfrak{P}^x$ denote the collection of all such polarizations. If $g \in G(\mathbf{Q})$, $P^x \in \mathfrak{P}^x$, and $x' = \gamma x$ then the collection $P^{x'}$ given by $P^{x'}_n : (u,v) \to P^x_n(\xi(\gamma^{-1})u, \xi(\gamma^{-1})v)$ lies in $\mathfrak{P}^{x'}$.

We must also verify that the family $\{V^x\}$ over X_K is a family of rational Hodge structures in the sense of [**9**]. Otherwise it could not possibly be attached to a family of motives. There are two points to be verified. Let $V^x_{p,q}$ be the subspace of $V^x_{\mathbf{C}}$ of type p,q and set $V^x_p = \oplus_{p' \geqq p} V^x_{p',q'}$. The space $V^x_p \subseteq V_{\mathbf{C}}$ must be shown to vary holomorphically with x. In other words if $v(x)$ is any local section of V^x_p and Y any antiholomorphic vector field then $Yv(x)$ also takes values in V^x_p. The condition of transversality must also be established, to the effect that for the same $v(x)$ and any holomorphic vector field Y the values of $Yv(x)$ lie in V^x_{p-1}.

There is certainly no harm in supposing that $v(x)$ takes values in $V^x_{p,q}$. Then one has to show that if $x^\circ = (h^\circ, g^\circ_f)$ is fixed and Y is the vector field defined by $X \in \mathfrak{G}$ then $Yv(x^\circ)$ lies in V^x_p when $X \in \mathfrak{p}_{h^\circ}$ and in V^x_{p-1} when $X \in \bar{\mathfrak{p}}_{h^\circ}$. Let K_{h° be the stabilizer of h° in $G(\mathbf{R})$. We represent X_K as the quotient $G(\mathbf{A})/K_{h^\circ}K$ and lift $v(x)$ to a function on $G(\mathbf{A})$ which we write as

$$(g, g_f) \to \xi(g)u(g, g_f), \quad g \in G(\mathbf{R}), g_f \in G(\mathbf{A}_f).$$

The function u takes values in the constant space $V^{x^\circ}_{p,q}$. Moreover

$$Yv(x^\circ) = \xi(X)u(1, g^\circ_f) + Xu(1, g^\circ_f).$$

The second term lies in $V^{x^\circ}_{p,q}$ for all $X \in \mathfrak{G}$. If $X \in \mathfrak{p}$ then $\xi(X)V^{x^\circ}_{p,q} \subseteq V^{x^\circ}_{p+1,q-1}$ and if $X \in \bar{\mathfrak{p}}$ then $\xi(X)V^{x^\circ}_{p,q} \subseteq V^{x^\circ}_{p-1,q+1}$.

If $\gamma \in G(\mathbf{Q})$ and $x' = \gamma x$ then $v \to \xi(\gamma)v$ provides an isomorphism between $(V^x, \mathfrak{P}^x, \varphi^x)$ and $(V^{x'}, \mathfrak{P}^{x'}, \varphi^{x'})$. Thus if $s \in S_K(\mathbf{C})$, any two elements of $\{(V^x, \mathfrak{P}^x, \varphi^x) \mid x \to s\}$ are canonically isomorphic, and we may take $(V^s, \mathfrak{P}^x, \varphi^s)$ to be any one of them, and redefine our family as a family of rational Hodge structures, with supplementary data, over the base $S_K(\mathbf{C})$. The locally constant sheaf $F_\xi(\mathbf{Q})$ of rational vector spaces underlying this family is the quotient of $V_{\mathbf{Q}} \times X_K$ by the action $\gamma : (v,x) \to (\xi(\gamma)v, \gamma x)$ of $G(\mathbf{Q})$. For this quotient to be well defined, the group K must be sufficiently small, for $G(\mathbf{Q}) \cap K_h K$ is then contained in the kernel of ξ for all h (cf. II.A.2 of [**41**]). It is here that the condition that ξ factors through G_0 intervenes.

When dealing with motives, one does not need to introduce the polarizations explicitly as part of the moduli problem, but in order to introduce a Hodge structure on the cohomology groups of the sheaves $F_\xi(\mathbf{Q})$ one must verify that on each connected component of $S_K(\mathbf{C})$ a locally constant section $s \to P^s$ can be defined. Let $G^0(\mathbf{R})$ be the connected component of $G(\mathbf{R})$ and choose $x^\circ = (h, g_f)$ in X_K. If

$$X_K^\circ = \{(\mathrm{Ad}\, g \circ h, g_f k) \mid g \in G^\circ(\mathbf{R}), k \in K\}$$

then the image of X_K° in $S_K(\mathbf{C})$ is open. If x and x' lie in X_K° and $x' = \gamma x$ then γ lies in

$$G(\mathbf{Q}) \cap G^\circ(\mathbf{R}) K_h g_f K g_f^{-1}. \tag{4.2}$$

All we need do is find a collection $P = \{P_n\}$ such that $P \in \mathfrak{P}^x$ for all $x \in X_K^\circ$ and $P_n(\xi(\gamma)u, \xi(\gamma)\nu)$ $= P_n(u, \nu)$ if γ lies in the group (4.2). Choose any P in $\mathfrak{P}^{x^\circ}$. Then $P \in \mathfrak{P}^x$ for all $x \in X_K^\circ$. There are certainly homomorphisms λ_n of G into the general linear group of V^n such that

$$P_n(\xi(\gamma)u, \xi(\gamma)v) = P_n(\lambda_n(\gamma)u, v).$$

The eigenvalues of $\lambda(\gamma)$ are positive if γ lies in $G^\circ(\mathbf{R})K_h$. Moreover λ is trivial on the derived group of G and factors through G_0. It therefore follows from the results of II.A.2 of [**41**] that each λ_n is the identity on all of (4.2).

Although we have defined the families $(V^s, \mathfrak{P}^s, \varphi^s)$ for any G, it is clear that $S_K(\mathbf{C})$ is not going to appear as a moduli space unless G is equal to G_0, and so for the rest of this section we assume this. The moduli problem is best formulated completely in the language of tannakian categories. We can drop the polarizations and retain only the pairs (V^s, φ^s), or (V^x, φ^x), but we now have to emphasize that (V^x, φ^x) is defined for every (finite-dimensional) representation ξ of G over **Q**, and so we write $(\xi, V(\xi))$ for the representation and $(V^x(\xi), \varphi^x(\xi))$ for the pair (V^x, φ^x).

On the category $\mathcal{REP}(G)$ of finite-dimensional representations of G we have the natural fibre functor $\omega_{\mathrm{Rep}(G)} : (\xi, V(\xi)) \to V(\xi)_{\mathbf{Q}}$ and $\eta^x : (\xi, V(\xi)) \to V^x(\xi)$ is a $\otimes$-functor from $\mathcal{REP}(G)$ to $\mathcal{HOD}(\mathbf{Q})$ which satisfies $\omega_{\mathrm{Hod}} \circ \eta^x = \omega_{\mathrm{Rep}(G)}$. Since $\mathcal{HOD}(\mathbf{Q})$ and $\mathcal{REP}(G_{\mathrm{Hod}})$ are the same categories, η^x defines a homomorphism [**40**, II.3.3.1] $\varphi^x : G_{\mathrm{Hod}} \to G$ and η^x may be defined by $(\xi, V(\xi)) \to (\xi \circ \varphi^x, V(\xi))$. When we emphasize η^x, φ^x appears as an isomorphism of two fibre functors

$$\varphi^x : \omega \overset{\mathbf{A}_f}{\mathrm{Hod}} \circ \eta^x \to \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G)}.$$

However, when we emphasize φ^x, as we shall, then these two fibre functors are the same, for they are both obtained from $\omega_{\mathrm{Hod}} \circ \eta^x = \omega_{\mathrm{Rep}(G)}$ by tensoring with $\mathbf{A}_f$, and φ^x may be interpreted as an isomorphism of $\omega_{\mathrm{Rep}(G)}^{\mathbf{A}_f}$. Such an isomorphism is given by a $g^{-1} \in G(\mathbf{A}_f)$ [**40**, II.3]. This is the g appearing in $x = (h, g)$. Only the coset $gK \subseteq G(\mathbf{A}_f)$ is well defined.

We have arrived, by a rather circuitous route, at the conclusion that X_K parametrizes pairs (φ, g), φ being a homomorphism from G_{Hod} to G defined over **Q**, and g in $G(\mathbf{A}_f)$ being specified only up to right multiplication by an element of K. In addition, φ is subject to the following constraint:

H. *The composition of φ with the canonical homomorphism $\mathcal{R} \to G_{\mathrm{Hod}}$ lies in $\mathfrak{H}$.*

If $\gamma \in G(\mathbf{Q})$ the pairs (φ, g) and $(\mathrm{ad}\,\gamma \circ \varphi, \gamma g)$ will be called equivalent. The variety $\mathrm{Sh}_K(\mathbf{C})$ parametrizes equivalence classes of these pairs.

One of the important tannakian categories is the category $\mathcal{MOT}(k)$ of motives over a field k. It cannot be constructed at present unless one assumes certain conjectures in algebraic geometry, referred to as the standard conjectures [**40**]. It is covariant in k, and rational cohomology together with its Hodge structure yields a $\otimes$-functor $h_{BH} : \mathcal{MOT}(\mathbf{C}) \to \mathcal{HOD}(\mathbf{Q})$. $\mathcal{MOT}(\mathbf{C})$ together with the fibre functor $\omega_{\mathrm{Mot}(\mathbf{C})}$ of rational cohomology also defines a group $G_{\mathrm{Mot}(\mathbf{C})}$ over **Q** and h_{BH} is dual to a homomorphism $h_{BH}^* : G_{\mathrm{Hod}} \to G_{\mathrm{Mot}(\mathbf{C})}$ defined over **Q**.

Implicit in Deligne's construction is the hope that any homomorphism $\varphi' : G_{\mathrm{Hod}} \to G$ satisfying H is a composite $\varphi' = \varphi \circ h_{BH}^*$. According to the Hodge conjecture, φ would be uniquely determined [**40**, VI.4.5] and $\mathrm{Sh}_K(\mathbf{C})$ would appear as the moduli space for pairs (φ, g), with g as before, but where φ is now a homomorphism from $G_{\mathrm{Mot}(\mathbf{C})}$ to G defined over **Q** and satisfying:

H$'$. *The composition of φ with the canonical homomorphism $\mathcal{R} \to G_{\mathrm{Mot}(\mathbf{C})}$ lies in $\mathfrak{H}$.*

This may be so but it will not be a panacea for all the problems with which the study of Shimura varieties is beset. So far as I can see, we do not yet have a moduli problem in the usual algebraic sense, and, in particular, no way of deciding over which field the moduli problem is defined. We can be more specific about this difficulty.

Suppose τ is an automorphism of **C**. Then τ^{-1} defines a $\otimes$-functor $\eta(\tau) : \mathcal{MOT}(\mathbf{C}) \to \mathcal{MOT}(\mathbf{C})$. Let $G_{\mathrm{Mot}(\mathbf{C})}^{\tau}$ be the group defined by $\mathcal{MOT}(\mathbf{C})$ and the fibre functor $\omega_{\mathrm{Mot}(\mathbf{C})} \circ \eta(\tau)$. The dual of $\eta(\tau)$ is then an isomorphism over **Q**:

$$\varphi(\tau) : G_{\mathrm{Mot}(\mathbf{C})} \to G_{\mathrm{Mot}(\mathbf{C})}^{\tau}.$$

The homomorphism φ has a dual, a $\otimes$-functor $\eta : \mathcal{REP}(G) \to \mathcal{MOT}(\mathbf{C})$ and the fibre functor $\omega_{\mathrm{Mot}(\mathbf{C})} \circ \eta(\tau) \circ \eta$ defines a group $G^{\tau,\varphi}$ over $\mathbf{Q}$. The $\otimes$-functor η then defines a dual

$$\varphi^\tau : G^\tau_{\mathrm{Mot}(\mathbf{C})} \to G^{\tau,\varphi}, \text{ and } \varphi' = \varphi^\tau \circ \varphi(\tau) : G_{\mathrm{Mot}(\mathbf{C})} \to G^{\tau,\varphi}.$$

Moreover the two fibre functors $\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})}$ and $\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})} \circ \eta(\tau)$ are canonically isomorphic. As a consequence, there is a canonical isomorphism $G(\mathbf{A}_f) \to G^{\tau,\varphi}(\mathbf{A}_f)$. Let g' be the image of g.

The pair (φ', g') seems once again to define a solution to our moduli problem. The difficulty is that $G^{\tau,\varphi}$ may not be the group G or, even if it is, the composition of φ' with the canonical homomorphism may not lie in $\mathfrak{H}$. One of the purposes of the next two sections is to discover what $G^{\tau,\varphi}$ is likely to be.

5. The Taniyama group. There is one type of Shimura variety which is very easy to study, that obtained when G is a torus T. Then the set $\mathfrak{H}$ reduces to a single point $\{h\}$. For each open compact subgroup U of $T(\mathbf{A}_f)$ the manifold $\mathrm{Sh}_U(\mathbf{C})$ consists of a finite set and $\mathrm{Sh}_U = \mathrm{Sh}_U(T,h)$ is zero-dimensional. In general a special point of $(G,\mathfrak{H})$ will be a pair (T,h) with $T \subseteq G$ and $h \in \mathfrak{H}$. If $U = K \cap T(\mathbf{A}_f)$ then $\mathrm{Sh}_U(\mathbf{C})$ is a subset of $\mathrm{Sh}_K(\mathbf{C})$, the points of which have traditionally been referred to as special points, and I shall continue this usage. But it is best to give priority to the pair (T,h) rather than to the points of $\mathrm{Sh}_K(\mathbf{C})$ it defines. There are a number of unsolved problems about Shimura varieties and their special points that I want to describe in the next section. To formulate them some Galois cocycles have to be defined. Deligne has shown me that my original construction gave, in particular, a specific extension of Gal$(\bar{\mathbf{Q}}/\mathbf{Q})$ by the Serre group, $\mathcal{S}$, an extension I venture to call the Taniyama group and denote by $\mathcal{T}$. Since the cocycles needed are, as Rapoport observed, often easily defined in terms of $\mathcal{T}$, I begin by constructing it.

The group $\mathcal{S}$ is an algebraic group over $\mathbf{Q}$, and $\mathcal{T}$ will also be defined over $\mathbf{Q}$. Thus we will have an exact sequence

$$1 \to \mathcal{S} \to \mathcal{T} \to \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to 1. \tag{5.1}$$

Recall that $X^*(\mathcal{S})$ is a module of functions on Gal$(\bar{\mathbf{Q}}/\mathbf{Q})$, and that the Galois action on $X^*(\mathcal{S})$ giving the structure of $\mathcal{S}$ as a group over $\mathbf{Q}$ is defined by right translation. We are still free to use left translation to define an algebraic action of Gal$(\bar{\mathbf{Q}}/\mathbf{Q})$ on $\mathcal{S}$, and it is this action which is implicit in (5.1). The extension will not split over $\mathbf{Q}$ but it will be provided with canonical splittings over each ℓ-adic field $\mathbf{Q}_\ell$, Gal$(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{T}(\mathbf{Q}_\ell)$, which will fit together to give Gal$(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{T}(\mathbf{A}_f)$.

Rather than attempting to work directly with $\mathcal{S}$, I choose a finite Galois extension L of $\mathbf{Q}$, let $\mathcal{S}^L$ be the quotient group of $\mathcal{S}$ whose lattice of rational characters consists of all functions in $X^*(\mathcal{S})$ invariant under $G(\bar{\mathbf{Q}}/L)$, and define extensions

$$1 \to \mathcal{S}^L \to \mathcal{T}^L \to \text{ Gal}(L^{ab}/\mathbf{Q}) \to 1, \tag{5.2}$$

afterwards lifting to $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, and then passing to the limit.

To motivate the construction we suppose that the extension is defined and that there is a section $\tau \to a(\tau)$ of $\mathcal{T}^L \to \text{ Gal}(L^{ab}/\mathbf{Q})$ with $a(\tau) \in \mathcal{T}^L(L)$. Let $a(\tau_1)a(\tau_2) = d_{\tau_1,\tau_2}a(\tau_1\tau_2)$ with $d_{\tau_1,\tau_2} \in \mathcal{S}^L(L)$, and with

$$\tau_1(d_{\tau_2,\tau_3})d_{\tau_1,\tau_2\tau_3} = d_{\tau_1,\tau_2}d_{\tau_1\tau_2,\tau_3}. \tag{5.3}$$

Observe that the elements of the Galois group play two different roles. They are first of all elements of a quotient group of $\mathcal{T}^L$, and secondly they are automorphisms of L^{ab} and thus act on $\mathcal{T}^L(L^{ab})$, since $\mathcal{T}^L$ is defined over $\mathbf{Q}$. In the first role they will be denoted by τ, perhaps with a subscript added, and in the second by ρ or σ.

We have $\rho(a(\tau)) = c_\rho(\tau)a(\tau)$ with $c_\rho(\tau) \in \mathcal{S}^L(L)$. Certainly

$$c_{\rho\sigma}(a(\tau)) = \rho(c_\sigma(\tau))c_\rho(\tau). \tag{5.4}$$

In addition

$$d_{\tau_1,\tau_2}c_\rho(\tau_1)\tau_1(c_\rho(\tau_2)) = \rho(d_{\tau_1,\tau_2})c_\rho(\tau_1\tau_2). \tag{5.5}$$

Conversely if we have collections $\{c_\rho(\tau)\}$ and $\{d_{\tau_1,\tau_2}\}$ satisfying (5.3), (5.4) and (5.5), we can construct $\mathcal{T}^L$ over $\mathbf{Q}$, together with the section a.

Any splitting $\text{Gal}(L^{ab}/\mathbf{Q}) \to \mathcal{T}(\mathbf{A}_f)$ will be of the form $\tau \to b(\tau)a(\tau)$ with $b(\tau) \in \mathcal{S}^L(\mathbf{A}_f(L))$. In order that it be a splitting, we must have

$$b(\tau_1)\tau_1(b(\tau_2))d_{\tau_1,\tau_2} = b(\tau_1\tau_2). \tag{5.6}$$

If the $b(\tau)a(\tau)$ are to lie in $\mathcal{T}^L(\mathbf{A}_f)$ we must have

$$\rho(b(\tau))c_\rho(\tau) = b(\tau). \tag{5.7}$$

Again any collection $\{b(\tau)\}$ satisfying (5.6) and (5.7) defines a splitting, and it is our task to construct $\{b(\tau)\}, \{c_\rho(\tau)\}$, and $\{d_{\tau_1,\tau_2}\}$.

The group $\mathcal{S}$ is a quotient of G_{Hod} and thus is provided with a canonical homomorphism $h : \mathcal{R} \to \mathcal{S}$. Over $\mathbf{C}$ the group $\mathcal{R}$ is canonically isomorphic to $\mathrm{GL}(1) \times \mathrm{GL}(1)$. Restricting h to the first factor we obtain a coweight μ of $\mathcal{S}$, the canonical coweight. If $\lambda \in X^*(\mathcal{S})$ then $\langle \lambda, \mu \rangle = \lambda(1)$. Since $\mathcal{S}^L$ is a quotient of $\mathcal{S}$, μ also defines a coweight of $\mathcal{S}^L$, which for convenience will also be denoted by μ. If ν is any coweight of $\mathcal{S}^L$ and x any invertible element of L or of $\mathbf{A}_f(L)$ then x^ν will be the element of $\mathcal{S}^L(L)$, or $\mathcal{S}^L(\mathbf{A}_f(L))$, satisfying $\lambda(x^\nu) = x^{\langle \lambda, \nu \rangle}$ for all $\lambda \in X^*(\mathcal{S}^L)$. Recall that we have two actions of $\mathrm{Gal}(L/\mathbf{Q})$, or $\mathrm{Gal}(L^{ab}/\mathbf{Q})$, on $X^*(\mathcal{S}^L)$ or on

$$X_*(\mathcal{S}^L) = \mathrm{Hom}(X^*(\mathcal{S}^L), \mathbf{Z}),$$

the lattice of coweights. That defined by right translation we write $\nu \to \sigma\nu$, and that defined by left translation we write $\nu \to \nu\tau$. Thus $\langle \lambda, \sigma\nu \rangle = \lambda(\sigma^{-1})$ while $\langle \lambda, \mu\tau \rangle = \lambda(\tau)$, because an inverse intervenes in the action by left translation, and $\mu\tau^{-1} = \tau\mu$. Moreover $\sigma(x^\mu) = \sigma(x)^{\sigma\mu}$ while $\tau(x^\mu) = x^{\mu\tau}$. These aspects of the notation have to be emphasized because at some points our convention of distinguishing between ρ, σ on the one hand, and τ on the other, fails us.

For the study of Shimura varieties, it is best to take $\bar{\mathbf{Q}}$ to be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and we shall do this. Thus we provide ourselves with an extension of the infinite valuation on $\mathbf{Q}$ to $\bar{\mathbf{Q}}$. There is a property of the Weil groups that will play a prominent role in our discussion. Let v be a valuation of $\bar{\mathbf{Q}}$ and hence of $\mathbf{Q}$, and let $\mathbf{Q}_v$ and $\bar{\mathbf{Q}}_v$ be the completions of $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ with respect to v. Eventually v will be defined by the inclusion $\bar{\mathbf{Q}} \subseteq \mathbf{C}$, and $\mathbf{Q}_v$ will be $\mathbf{R}$ and $\bar{\mathbf{Q}}_v$ will be $\mathbf{C}$. In any case, the data provide us with imbeddings

$$F_v^\times \hookrightarrow C_F \tag{5.8}$$

$$\mathrm{Gal}(F_v/\mathbf{Q}_v) \hookrightarrow \mathrm{Gal}(F/\mathbf{Q}), \tag{5.9}$$

if F is any finite Galois extension of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$. The local and global Weil groups $W_{F_v/\mathbf{Q}_v}$ and $W_{F/\mathbf{Q}}$ are defined as extensions

$$1 \to F_v^\times \to W_{F_v/\mathbf{Q}_v} \to \mathrm{Gal}(F_v/\mathbf{Q}_v) \to 1$$

and

$$1 \to C_F \to W_{F/\mathbf{Q}} \to \mathrm{Gal}(F/\mathbf{Q}) \to 1.$$

We may imbed the arrows (5.8) and (5.9) in a commutative diagram

$$\begin{array}{ccccccc}1 \to & F_v^\times \to & W_{F_v/\mathbf{Q}_v} \to & \mathrm{Gal}(F_v/\mathbf{Q}_v) \to 1\\ & \big\downarrow & \big\downarrow I_F & \big\downarrow \\ 1 \to & C_F \to & W_{F/\mathbf{Q}} \to & \mathrm{Gal}(F/\mathbf{Q}) \to 1\end{array}$$

Moreover we may so choose the central arrows that they are compatible with field extensions and upon passage to the limit yield I: $W_{\mathbf{Q}_v} \to W_{\mathbf{Q}}$. It is the image of $W_{\mathbf{Q}_v}$ in $W_{\mathbf{Q}}$ that will be fixed, and I_F may be changed to $w \to xI_F(w)x^{-1}$ where $x \in \mathbf{C}_F$ and $x\sigma(x)^{-1} \in F_v^\times$ for all $\sigma \in \mathrm{Gal}(F_v/\mathbf{Q}_v)$.

Now let v be the valuation given by $\bar{\mathbf{Q}} \subseteq \mathbf{C}$. Let $F_\infty^\times = \prod_{w|v} F_w^\times$. The natural map $F_\infty^\times \to \mathbf{C}_F$ is an imbedding, and we sometimes regard $F_\infty^\times$ as a subgroup of $\mathbf{C}_F$. If we take an element τ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, lift to $W_{\mathbf{Q}}$, and then project to $W_{L/\mathbf{Q}}$ we obtain an element $w = w(\tau)$ of $W_{L/\mathbf{Q}}$ which is well defined modulo the connected component, and in particular modulo the closure of $L_\infty^\times$.

We choose a set of representatives $w_\sigma, \sigma \in \mathrm{Gal}(L/\mathbf{Q})$, for the cosets of $\mathbf{C}_L$ in $W_{L/\mathbf{Q}}$ in such a way that the following conditions are satisfied:

(a) $w_1 = 1$.

(b) *If* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ *then* $w_\sigma \in W_{L_v/\mathbf{Q}_v}$.

(c) *If* $\rho \in \mathrm{Gal}(L/\mathbf{Q})$ *and* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ *then* $w_\rho w_\sigma = a_{\rho,\sigma} w_{\rho\sigma}$ *with* $a_{\rho,\sigma} \in L_\infty^\times$.

To arrange the final condition we may choose a collection of $\mathfrak{f}$ of representatives η for the cosets $\mathrm{Gal}(L/\mathbf{Q})/\mathrm{Gal}(L_v/\mathbf{Q}_v)$ and set $w_{\eta\sigma} = w_\eta w_\sigma$ if $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$. We suppose that $\mathfrak{f}$ contains 1. With this choice we also have:

(d) *If* $\{a_{\rho,\sigma}\}$ *is the cocycle defined by* $w_\rho w_\sigma = a_{\rho,\sigma} w_{\rho\sigma}$ *then* $a_{\eta,\rho} = 1$ *for* $\eta \in \mathfrak{f}$ *and* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q})$.

If $w \in W_{L/\mathbf{Q}}$, let $w_\sigma w = c_\sigma(w) w_\sigma, c_\sigma(w) \in C_L$. If $w = w(\tau)$ we set

$$b_0(\tau) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbf{Q})} c_\sigma(w)^{\sigma\mu}.$$

It lies in $\mathbf{C}_L \otimes X_*(\mathcal{S}^L)$, but is not well defined, because w is not. However we can show that it is well defined if taken modulo $L_\infty^\times \otimes X_*(\mathcal{S}^L)$, and that, in addition, it behaves properly under extensions of the field L.

The ambiguity in w now has no effect, for we are only free to replace w by uw where $u = \lim_n u_n$ and u_n lies in the image of $L_\infty^\times U$, where U is a subgroup of the group of units of L defined by a strong congruence condition. But [**41**, II.A.2],

$$\prod_{\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})} \sigma(u)^{\sigma\mu} = 1$$

for all $u \in U$. Consequently

$$\prod_\sigma c_\sigma(uw)^{\sigma\mu} = \left(\prod_\sigma \sigma(u)^{\sigma\mu}\right)\left(\prod_\sigma c_\sigma(w)^{\sigma\mu}\right)$$

is congruent modulo $L_\infty^\times \otimes X_*(\mathcal{S}^L)$ to $\prod_\sigma c_\sigma(w)^{\sigma\mu}$.

Suppose the representatives w_σ are replaced by $e_\sigma w_\sigma$, and, hence, $a_{\rho,\sigma}$ by

$$a'_{\rho,\sigma} = e_\rho \rho(e_\sigma) e_{\rho\sigma}^{-1} a_{\rho,\sigma}.$$

If $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ then $e_\sigma \in L_v^\times$ and $\rho(e_\sigma) \in L_\infty^\times$. Since $a_{\rho,\sigma}$ and $a'_{\rho,\sigma}$ must then both be in $L_\infty^\times$, we infer that $e_\rho \equiv e_{\rho\sigma} (\mathrm{mod}\, L_\infty^\times)$ when $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$. Moreover $c_\sigma(w)$ is replaced by $c'_\sigma(w) = e_{\sigma\tau} e_\sigma^{-1} c(w)$ and $b_0(\tau)$ by

$$b'_0(\tau) = \left\{\prod_\sigma e_{\sigma\tau}^{\sigma\mu} e_\sigma^{-\sigma\mu}\right\} b_0(\tau).$$

The factor may be written

$$\prod_\sigma e_\sigma^{\sigma(\tau^{-1}-1)\mu} \equiv \prod_{\tau\in\mathfrak{f}} \prod_{\sigma\in\mathrm{Gal}(L_v/\mathbf{Q}_v)} e_\eta^{\eta\sigma(\tau^{-1}-1)\mu}.$$

Since

$$\sum_{\sigma\in\, \mathrm{Gal}(L_v/\mathbf{Q}_v)} \sigma(1-\tau^{-1})\mu = (1+\iota)(1-\tau^{-1})\mu = 0,$$

this change has no effect on $b_0(\tau)$. In this argument we have denoted the image in $\mathrm{Gal}(L/\mathbf{Q})$ of $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ by the same symbol, a practice we shall continue to indulge in.

If we modify I then w is replaced by xwx^{-1} with $x \in C_L$ and $x\sigma(x)^{-1} \in L_v^\times$ for all $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q})$. Then $c_\sigma(xwx^{-1}) = \sigma(x)\sigma\tau(x^{-1})c_\sigma(w)$ and

$$\prod_\sigma \sigma(x)^{-\sigma\mu}\sigma\tau(x)^{\sigma\mu} = \prod_\sigma \sigma(x)^{\sigma(\tau^{-1}-1)\mu}.$$

Since $\sigma(x) \equiv x (\mathrm{mod}\, L_\infty^\times)$ if $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$, the same argument as before shows that $b_0(\tau)$ is unchanged.

Finally suppose that $L \subseteq L'$. Then $b'_0(\tau) \in C_L \otimes X_*(\mathcal{S}^{L'})$ and $b_0(\tau) \in \mathbf{C}_L \otimes X_*(\mathcal{S}^L)$ are both defined, and we must verify that $b'_0(\tau)$ is taken to $b_0(\tau)$ by the canonical mapping of the first group to the second. Either $L_v = \mathbf{R}$ or $L_v = \mathbf{C}$, and the two cases must be treated separately. Suppose first that $L_v = \mathbf{C}$ and hence that $\mathrm{Gal}(L'/L) \cap \mathrm{Gal}(L'_v/\mathbf{Q}_v) = \{1\}$. Since both $b_0(\tau)$ and $b'_0(\tau)$ are independent of the choices of coset representatives, we may choose those which make it easiest to verify that $b_0(\tau)$ is the image of $b_0'(\tau)$ is the image of $b_0'(\tau)$. Let $\mathfrak{e}$ be a set of representatives for the cosets $\mathrm{Gal}(L'/L)\backslash\mathrm{Gal}(L'/\mathbf{Q})/\mathrm{Gal}(L'_v/\mathbf{Q}_v)$ containing 1. Every elelment σ of $\mathrm{Gal}(L'/\mathbf{Q})$ may be written uniquely as a product $\sigma = \zeta\eta\rho, \zeta \in \mathrm{Gal}(L'/L), \eta \in \mathfrak{e}, \rho \in \mathrm{Gal}(L'_v/\mathbf{Q}_v)$. We may suppose that $w'_\sigma = w'_\rho w'_\eta w'_\rho$ with $w'_1 = 1$ and $w'_\rho \in w_{L'_v/\mathbf{Q}_v}$. Let w' be $w(\tau)$ with respect to L', and w be $w(\tau)$ with respect to L. Then under the canonical map $\pi : W_{L'/\mathbf{Q}} \to W_{L/\mathbf{Q}}$ the element w' maps to w. If $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$ it lifts to a unique element of $\mathrm{Gal}(L'/\mathbf{Q})$ of the form $\eta\rho$. We suppose that w_σ is the image of $w'_\eta w'_\rho$. Thus if

$$w'_\eta w'_\rho w' = d_{\eta,\rho}(w') w'_{\eta'} w'_{\rho'}$$

with $d_{\eta,\rho}(w') \in W_{L'/L}$ then $c_\sigma(w) = \pi(d_{\eta,\rho}(w'))$. On the other hand if $\sigma_1 = \zeta\eta\rho$ lies in $\mathrm{Gal}(L'/\mathbf{Q})$ then

$$w'_\zeta d_{\eta,\rho}(w') = c_{\sigma_1}(w') w'_\zeta.$$

Consequently, by the very definition of π,

$$c_\sigma(w) = \prod_{\sigma_1 \to \sigma} c_{\sigma_1}(w').$$

It follows immediately that $b_0(\tau)$ is the image of $b_0'(\tau)$.

If $L_v = \mathbf{R}$ then $X^*(\mathcal{S}^L) \simeq \mathbf{Z}$ and the Galois group acts trivially. Suppose we replace w_σ by $e_\sigma w_\sigma$ with $e_\sigma \in \mathbf{C}_L$. Then $c_\sigma(w)$ is replaced by $e_\sigma e_{\sigma\tau}^{-1}$. Since

$$\prod_\sigma (e_\sigma e_{\sigma\tau}^{-1})^{\sigma\mu} = \prod_\sigma (e_\sigma e_{\sigma\tau}^{-1})^{\mu} = 1,$$

this has no effect on $b_0(\tau)$, and when defining $b_0(\tau)$ we need not suppose that the collection $\{w_\sigma\}$ is subject to the constraints (a), (b), and (c). If we want to define $b_0'(\tau)$, we may still need to be careful about the choice of the coset representatives $w'_\sigma, \sigma \in \mathrm{Gal}(L'/\mathbf{Q})$. However, since we are only interested in the image of $b_0'(\tau)$ in $\mathcal{S}^L$, we may again ignore (a), (b), and (c). We choose a set $\mathfrak{e}$ of representatives for the cosets $\mathrm{Gal}(L'/L)\backslash\mathrm{Gal}(L'/\mathbf{Q})$, write $\sigma = \rho\eta, \rho \in \mathrm{Gal}(L'/L), \eta \in \mathfrak{e}$, and take $w'_\sigma = w'_\rho w'_\eta$. If $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$ is the image of η, we take $w_\sigma = \pi(w'_\eta)$. The argument can now proceed as before.

Let $\tilde{b}(\tau)$ be a lift of $b_0(\tau)$ to $I_L \otimes X_*(\mathcal{S}^L) = \mathcal{S}^L(\mathbf{A}(L))$ and let $b(\tau)$ be the projection of $\tilde{b}(\tau)$ on $\mathcal{S}^L(\mathbf{A}_f(L))$. The element $b(\tau)$ is well defined modulo $\mathcal{S}^L(L)$ and, as we shall see, this bit of ambiguity will cause us no difficulty. But we have to fix one choice. The first point to verify is that

$$d_{\tau_1,\tau_2} = b(\tau_1)\tau_1(b(\tau_2))b(\tau_1\tau_2)^{-1}$$

lies in $\mathcal{S}^L(L)$. When verifying this, we may choose the liftings $\tilde{b}(\tau_1), \tilde{b}(\tau_2)$, and $\tilde{b}(\tau_1\tau_2)$ in any way we like. We choose liftings $\tilde{c}_\sigma(w_1)$ and $\tilde{c}_\sigma(w_2)$ of $c_\sigma(w_1)$ and $c_\sigma(w_2)$ to I_L and take

$$\tilde{b}(\tau_1) = \prod_\sigma \tilde{c}_\sigma(w_1)^{\sigma\mu}, \quad \tilde{b}(\tau_2) = \prod_\sigma \tilde{c}_\sigma(w_2)^{\sigma\mu}.$$

Since $c_\sigma(w_1w_2) = c_\sigma(w_1)c_{\sigma\tau_1}(w_2)$, we may take $\tilde{c}_\sigma(w_1w_2)$ to be $\tilde{c}_\sigma(w_1)\tilde{c}_{\sigma\tau_1}(w_2)$. Because

$$\tau_1^{-1}(\tilde{b}(\tau_2)) = \prod_\sigma \tilde{c}_\sigma(w_2)^{\sigma\tau_1^{-1}\mu} = \prod_\sigma \tilde{c}_{\sigma\tau_1}(w_2)^{\sigma\mu},$$

the element d_{τ_1,τ_2} will then be 1.

Finally we have to establish that the elements $c_\rho(\tau)$ defined by equation (5.7) lie in $\mathcal{S}^L(L)$, for equations (5.4) and (5.5) will then follow immediately. It will suffice to show that for any $w \in W_{L/\mathbf{Q}}$ and any $\rho \in \mathrm{Gal}(L/\mathbf{Q})$

$$\left\{\prod_\sigma c_\sigma(w)^{\sigma\mu}\right\}\left\{\prod_\sigma \rho(c_\sigma(w))^{-\rho\sigma\mu}\right\} \tag{5.10}$$

lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$. Suppose $w = w_1w_2$ and w_1 projects to $\tau_1 \in \mathrm{Gal}(L/\mathbf{Q})$. Then $c_\sigma(w) = c_\sigma(w_1)c_{\sigma\tau_1}(w_2)$ and (5.10) is equal to

$$\left\{\prod_\sigma c_\sigma(w_1)^{\sigma\mu}\rho(c_\sigma(w_1))^{-\rho\sigma\mu}\right\}\tau_1\left\{\prod_\sigma c_\sigma(w_2)^{\sigma\mu}\rho(c_\sigma(w_2))^{-\rho\sigma\mu}\right\}.$$

Consequently we need only verify that (5.10) lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$ for w in $\mathbf{C}_L$ and for $w = w_\tau$. If w lies in $\mathbf{C}_L$ then $c_\sigma(w) = \sigma(w)$ and $\prod_\sigma \sigma(w)^{\sigma\mu} = \prod_\sigma \rho\sigma(w)^{\rho\sigma\mu}$. The expression (5.10) is therefore equal to 1. If $w = w_\tau$ then $c_\sigma(w) = a_{\sigma,\tau}$ and

$$\prod_\sigma a_{\sigma,\tau}^{\sigma\mu}\rho(a_{\sigma,\tau})^{-\rho\sigma\mu} = \prod_\sigma a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}. \tag{5.11}$$

However it follows from condition (c) that $a_{\rho,\sigma} \equiv a_{\rho,\sigma\iota} (\mathrm{mod}\ L_\infty^\times)$. Since $(1+\iota)\cdot(1-\tau^{-1})\mu = 0$, the right side of (5.11) lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$.

Since $b(\tau)$, although defined for $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, depends only on the image of τ in $\mathrm{Gal}(L^{ab}/\mathbf{Q})$, the groups $\mathcal{T}^L$ and $\mathcal{T}$ are now completely defined. The ambiguity in the $b(\tau)$ is easily seen to correspond to the ambiguity in the choice of the section $a(\tau)$.

If $E \subseteq \bar{\mathbf{Q}}$ is any finite extension of $\mathbf{Q}$, we let $\mathcal{T}_E$ be the inverse image of $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ in $\mathcal{T}$. If $E \subseteq L^{ab}$ we may also introduce $\mathcal{T}_E^L$. The group $\mathcal{T}_L^L$ has been introduced by Serre **[41]**, who uses it to formulate some ideas of Taniyama. He makes its arithmetic significance quite clear, but his definition is sufficiently different from that given here that an explanation of the reasons for their equivalence is in order.

If x is an idèle of L then $\psi(x) = \prod_{\mathrm{Gal}(L/\mathbf{Q})} \sigma(x)^{\sigma\mu}$ is an element of $\mathcal{S}^L(\mathbf{A}(L))$. By II.4 of **[41]** there is an open subgroup U of the group of idèles I_L such that $\psi(x) = 1$ if $x \in U \cap L^\times$. The standard map of I_L onto $\mathrm{Gal}(L^{ab}/L)$ restricts to U and if $\tau = \tau(x)$ is the image of x we may take $w = w(\tau)$ to be the image of x in $\mathbf{C}_L$. Then $\psi(x)$ is a lifting of $b_0(\tau)$ to $\mathcal{S}^L(\mathbf{A}(L))$. However $\psi(x)$ depends only on τ, and thus we may take $\tilde{b}(\tau) = \psi(x)$. Then $d_{\tau_1,\tau_2} = 1$ and $c_\rho(\tau) = 1$ if τ, τ_1, τ_2 lie in the image $\mathrm{Gal}(L^{ab}/F)$ of U. Here F is the finite extension of L defined by U. The elements $c_\rho(\tau)$ are in fact 1 for all $\tau \in \mathrm{Gal}(L^{ab}/L)$. If we choose a set of representatives $\mathfrak{e}$ for $\mathrm{Gal}(L^{ab}/F)\backslash\mathrm{Gal}(L^{ab}/L)$ and set $\tilde{b}(\tau\eta) = \tilde{b}(\tau)\tilde{b}(\eta), \tau \in \mathrm{Gal}(L^{ab}/F), \eta \in \mathfrak{e}$, then, in general, d_{τ_1,τ_2} will depend only on the images of τ_1, τ_2 in $\mathrm{Gal}(F/L)$, and $\mathcal{T}_L^L$ may be obtained by pulling back an extension

$$1 \to \mathcal{S}^L \to \mathcal{T}_U \to \mathrm{Gal}(F/L) \to 1$$

to $\mathrm{Gal}(L^{ab}/L)$. Here $\mathcal{T}_U$ is the quotient of $\mathcal{T}_L^L$ by the normal subgroup $\{a(\tau) \mid \tau \in \mathrm{Gal}(L^{ab}/F)\}$. It is the extension $\mathcal{T}_U$ that Serre defines directly. He denotes it by the symbol $S_\mathfrak{m}$.

The map ψ defines a homomorphism of $L^\times/L^\times \cap U \simeq L^\times U/L^\times$ into $\mathcal{S}^L(L)$ and to verify that $\mathcal{T}_U$ is the group studied by Serre, we have only to verify that it can be imbedded in a commutative diagram

$$\begin{array}{ccccccc}
1 \to & L^\times U/U \to & T_L/U \to & I_L/L^\times U \to 1 \\
& \downarrow & \downarrow & \downarrow \wr \\
1 \to & \mathcal{S}^L(\mathbf{Q}) \to & \mathcal{T}_U(\mathbf{Q}) \to & \mathrm{Gal}(F/L) \to 1
\end{array}$$

The right-hand arrow is $x \to \tau(x)^{-1}$. We do not have to pass to the quotient but may define the homomorphism

$$I_L/L^\times \to \mathcal{T}_L^L(L) \tag{5.12}$$

directly. The central arrow is then obtained by composing with the projection $\mathcal{T}_L^L(L) \to \mathcal{T}_U(L)$. The homomorphism (5.12) is

$$x \to \left\{\prod_\sigma \sigma(x)^{\sigma\mu}\right\} \tilde{b}(\tau)^{-1} a(\tau)^{-1}$$

if τ is the image of x in $\mathrm{Gal}(L^{ab}/L)$. The composition of our splitting $\mathrm{Gal}(L^{ab}/L) \to \mathcal{T}_L^L(\mathbf{Q}_\ell)$ with $\mathcal{T}_L^L(\mathbf{Q}_\ell) \to \mathcal{T}_U(\mathbf{Q}_\ell)$ is either Serre's ϵ_ℓ or its inverse, presumably its inverse, for we are so arranging matters that the eigenvalues of the Frobenius elements acting on the cohomology of algebraic varieties are greater than or equal to 1.

The cocycle $\rho \to c_\rho(\tau)$ certainly becomes trivial at every finite place, but is not necessarily trivial at the infinite place, v. Indeed under the isomorphism

$$H^1(\mathrm{Gal}(L_v(\mathbf{Q}_v), \mathcal{S}^L(L_v)) \simeq H^{-1}(\mathrm{Gal}(L_v/\mathbf{Q}_v), X_*(\mathcal{S}^L))$$

given by the Tate-Nakayama duality it corresponds to the element of the group on the right represented by $(1-\tau^{-1})\mu$. If, as has been our custom, we denote the image of τ in $\mathrm{Gal}(L/\mathbf{Q})$ again by τ we may suppose that $w(\tau) = w_\tau$, for the class of $\{c_\rho(\tau)\}$ depends only on this image. According to the discussion of formula (5.11)

$$\prod_\sigma a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu} = \prod_{\eta\in\mathfrak{f}} (a_{\rho,\eta\iota}a_{\rho,\eta}^{-1})^{\rho\eta(\tau^{-1}-1)\mu} = e_\rho(\tau)$$

lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L) = \mathcal{S}^L(L_\infty)$. By definition, the classes of $\{c_\rho(\tau)\}$ and $\{e_\rho(\tau)\}$ are inverse to one another in $H^1(\mathrm{Gal}(L/\mathbf{Q}), \mathcal{S}^L(\mathbf{A}(L)))$. Thus all we need do is calculate the projection of $e_\rho(\tau)$ on $\mathcal{S}^L(L_v)$ for $\rho \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$.

Observe first of all that if we agree to choose coset representatives satisfying (d), then $a_{\rho,\eta\iota}a_{\rho,\eta}^{-1}$ $= \rho(a_{\eta,\iota}^{-1})a_{\rho\eta,\iota} = a_{\rho\eta,\iota}$. Again if $\rho\eta = \eta_1\rho_1$ then

$$a_{\rho\eta,\iota} = a_{\eta_1\rho_1,\iota} = \eta_1(a_{\rho_1,\iota})a_{\eta_1,\rho_1\iota}a_{\eta_1,\rho_1}^{-1} = \eta_1(a_{\rho_1,\iota}).$$

Since $a_{\rho_1,\iota} \in L_v^\times$, the term on the right has a projection on $L_v^\times$ different from 1 only if $\eta_1 = 1$. Then η too equals 1, and so the projection of $e_\rho(\tau)$ on $\mathcal{S}^L(L_v)$ is

$$a_{\rho,\iota}^{\rho\iota(\tau^{-1}-1)\mu} = \prod_{\mathrm{Gal}(L_v/\mathbf{Q}_v)} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu},$$

in conformity with our assertion.

Suppose T is a torus over $\mathbf{Q}$, provided with a coweight μ such that

$$(1+\iota)(\tau-1)\mu = (\tau-1)(1+\iota)\mu = 0 \tag{5.13}$$

for all $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Then there exists a unique homomorphism $\psi : \mathcal{S} \to T$ such that the composition of ψ with the canonical coweight of $\mathcal{S}$ is μ. We can transport the cocycles $\rho \to c_\rho(\tau)$ from $\mathcal{S}$ to T, obtaining cocycles $\{c_\rho(\tau,\mu)\}$ as well as $b(\tau,\mu) \in T(\mathbf{A}_f(L))$. However if T and $h : \mathcal{R} \to T$ define a Shimura variety and μ is the restriction of h to the first factor of $\mathcal{R}$ the condition (5.13) will not necessarily be satisfied. Nonetheless, we may repeat the previous construction and define $b(\tau,\mu)$ and $\{c_\rho(\tau,\mu)\}$ for all τ such that $(1+\iota)(\tau^{-1}-1)\mu=0$. This generalization is necessary for the treatment of those Shimura varieties $\mathrm{Sh}(G,h)$ for which G is not equal to G_0. It should perhaps be observed that $b(\tau,\mu)$ is not insensitive to the ambiguity in the choice of $w=w(\tau)$, although $\{c_\rho(\tau,\mu)\}$ is. However, if Z is the centre of G the ambiguity all lies in $Z(\mathbf{A}_f)\cap K$, and may be ignored.

If $E\subseteq \mathbf{C}$ the motives over E whose associated Hodge structure is of CM type are themselves said to be of CM type. They form a tannakian category $CM(E)$ with a natural fibre functor $\omega_{\mathrm{CM}(E)}$, given by rational cohomology. Since one expects that the natural functor $CM(\bar{\mathbf{Q}})\to CM(\mathbf{C})$ is an equivalence, we may as well suppose that $E\subseteq\bar{\mathbf{Q}}$. According to the hopes expressed at the end of the previous section there should be an equivalence $\eta: CM(\bar{\mathbf{Q}})\to\mathcal{REP}(\mathcal{S})$ and an isomorphism $\omega_{\mathrm{Rep}(\mathcal{S})}\circ\eta\to\omega_{\mathrm{CM}(\bar{\mathbf{Q}})}$, which would enable us to identify $\mathcal{S}$ with the group $G_{\mathrm{CM}(\bar{\mathbf{Q}})}$, defined by the category $CM(\bar{\mathbf{Q}})$ and the functor $\omega_{\mathrm{CM}(\bar{\mathbf{Q}})}$. If $E\subseteq\bar{\mathbf{Q}}$, one hopes that in the same way it will be possible to identify $\mathcal{T}_E$ with $G_{\mathrm{CM}(E)}$.

There are properties which this identification whould have, and it is necessary to describe them explicitly. First of all, if $F\subseteq E$ the diagram

$$\begin{array}{ccccc}
\mathcal{S} \longrightarrow & & \mathcal{T}_E \longrightarrow & & \mathcal{T}_F \\
\uparrow & & \updownarrow & & \updownarrow \\
G_{\mathrm{CM}(\bar{\mathbf{Q}})} & \longrightarrow & G_{\mathrm{CM}(E)} & \longrightarrow & G_{\mathrm{CM}(F)}
\end{array}$$

should be commutative.

The other properties are more complicated to describe. Suppose $\tau\in\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and τ takes E to E'. Its inverse then naturally defines a $\otimes$-functor $\eta(\tau)$ from $CM(E')$ to $CM(E)$. Let $G^\tau_{\mathrm{CM}(E')}$ be the group defined by the category $CM(E')$ and the functor $\omega_{\mathrm{CM}(E)}\circ\eta(\tau)$. The dual of $\eta(\tau)$ is then an isomorphism $\varphi(\tau): G_{\mathrm{CM}(E)}\to G^\tau_{\mathrm{CM}(E')}$. In terms of representations $\eta(\tau)$ associates to every representation $(\xi', V(\xi'))$ of $G_{\mathrm{CM}(E')}$ a representation $(\xi, V(\xi))$ of $G_{\mathrm{CM}(E)}$. Since $\omega_{\mathrm{CM}(E')}$ and $\omega_{\mathrm{CM}(E)}\circ\eta(\tau)$ become isomorphic over $\bar{\mathbf{Q}}$ there is a family of homomorphisms, one for each $\xi', \psi(\xi'): V(\xi')_{\bar{\mathbf{Q}}}\to V(\xi)_{\bar{\mathbf{Q}}}$, compatible with sums and tensor products. If $\psi'(\xi)$ is another possible family then there is a $t\in\mathcal{T}_{E'}(\bar{\mathbf{Q}})$, such that $\psi'(\xi')=\psi(\xi')\xi'(t)$. Finally since the two functors $\omega^{\mathbf{A}_f}_{\mathrm{CM}(E')}$ and $\omega^{\mathbf{A}_f}_{\mathrm{CM}(E)}\circ\eta(\tau)$ are canonically isomorphic, arising as they do from the ℓ-adic cohomology, there is a canonical family of isomorphisms $\psi_{\mathbf{A}_f}(\xi'): V(\xi')_{\mathbf{A}_f}\to V(\xi)_{\mathbf{A}_f}$.

On the other hand, suppose $a(\tau)\in\mathcal{T}(\bar{\mathbf{Q}})$ maps to τ. Let $\sigma(a(\tau))=c_\sigma(\tau)a(\tau)$. We may use $a(\tau)$ to associate to every representation $(\xi', V(\xi'))$ of $\mathcal{T}_{E'}$ a representation $(\xi, V(\xi))$ of $\mathcal{T}_E$. The space $V(\xi)$ is obtained by twisting $V(\xi')$ by the cocycle $\{\xi'(c_\sigma(\tau)^{-1})\}$. The representation ξ is $t \to \xi'(a(\tau)ta(\tau)^{-1})$. This functor $(\xi', V(\xi')) \to (\xi, V(\xi))$ is to be (isomorphic to) that obtained from $\eta(\tau)$ by identifying $\mathcal{T}_E$ and $G_{\mathrm{CM}(E)}$ and $\mathcal{T}_{E'}$ and $G_{\mathrm{CM}(E')}$. Moreover one possible choice for $\psi(\xi')$ is to be the isomorphism $\psi_0(\xi') : V(\xi')_{\bar{\mathbf{Q}}} \to V(\xi)_{\bar{\mathbf{Q}}}$ implicit in the definition of $V(\xi)$. If $b(\tau)a(\tau)$ is the image of τ under the canonical splitting Gal$(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{T}(\mathbf{A}_f)$, then $\psi_{\mathbf{A}_f}(\xi')$ is to be $\psi_0(\xi') \circ \xi'(b(\tau))^{-1}$.

If $\tau \in$ Gal$(\bar{\mathbf{Q}}/E)$ then $E' = E$ and $\eta(\tau) : CM(E') \to CM(E)$ is the identity functor. Thus the functor $(\xi', V(\xi')) \to (\xi, V(\xi))$ from $\mathrm{Rep}(T_{E'}) = \mathrm{Rep}(T_E)$ to $\mathrm{Rep}(T_E)$ must be canonically isomorphic to the identity. A final property, which seems to be independent of the preceding ones, is that this isomorphism should be given by $\xi' \to \xi$ and $\psi_0(\xi') \circ \xi'(a(\tau)) : V(\xi') \to V(\xi)$. Observe that $a(\tau)$ is now in $\mathcal{T}_E$ and these transformations are defined over $\mathbf{Q}$.

I assume that all this is so, just to see where it leads, and especially to see what it suggests about the groups $G^{\tau,\varphi}$ introduced in the previous section. But there are some lemmas to be verified first. I conclude the present section by describing a property of the Taniyama group whose significance was pointed out to me by Casselman. It will be needed to show that the zeta-functions of motives, and especially abelian varieties, of CM type can be expressed as products of the L-functions associated to representations of the Weil group.

The point is that there is a natural homomorphism φ of the Weil group $W_{\mathbf{Q}}$ of $\mathbf{Q}$ into $\mathcal{T}(\mathbf{C})$ and thus for any finite extension F of $\mathbf{Q}$ a homomorphism $\varphi_F : W_F \to \mathcal{T}_F(\mathbf{C})$. To define it we work at a finite level, defining $W_{L/\mathbf{Q}} \to \mathcal{T}^L(\mathbf{C})$, and afterwards passing to the limit.

Fix for now a set of coset representatives w_σ which satisfies (a), (b), and (c). If $w \in W_{L/\mathbf{Q}}$ we define $b_0(w) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbf{Q})} c_\sigma(w)^{\sigma\mu}$. If τ is the image of w in Gal$(L^{ab}/\mathbf{Q})$ then $b_0(w) \equiv b_0(\tau) (\mathrm{mod}\ L_\infty^\times \otimes X^*(\mathcal{S}^L))$, and we may lift $b_0(w)$ to $b(w)$ in $\mathcal{S}^L(\mathbf{A}(L))$ in such a way that the projection of $b(w)$ in $\mathcal{S}^L(\mathbf{A}_f(L))$ is $b(\tau)$. However a simple calculation shows that if τ_1, τ_2 are the images of w_1, w_2 then $b(w_1)\tau_1(b(w_2))b(w_1w_2)^{-1} \in \mathcal{S}^L(L)$. Since its projection on $S^L(\mathbf{A}_f(L))$ is equal to d_{τ_1,τ_2}, it is itself equal to d_{τ_1,τ_2}. If $b_v(w)$ is the projection of $b(w)$ in $\mathcal{S}^L(L_\nu) = \mathcal{S}^L(\mathbf{C})$, we may define φ by $\varphi : w \to b_v(w)a(\tau)$. If the coset representatives w_σ are changed then φ is replaced by $\varphi' = \mathrm{ad}a \circ \varphi, a \in \mathcal{S}^L(\mathbf{C})$, but this is of no importance.

If G_{W_F} is the group over $\mathbf{C}$ defined by the tannakian category of continuous, finite-dimensional, complex, semisimple representations of W_F then φ_F is the composite of the imbedding $W_F \to G_{W_F}(\mathbf{C})$

and an algebraic homomorphism $\psi_F : G_{W_F} \to \mathcal{T}_F$. Moreover if the principle of functoriality is valid, there is a surjection $G_{\Pi(F)} \to G_{W_F}$ and we can expect to have a diagram

$$\begin{array}{ccc} G_{\Pi(F)} & \xrightarrow{\rho_F} & G_{\mathrm{Mot}(F)} \\ \big\downarrow & & \big\downarrow \\ G_{W_F} & \xrightarrow[\psi_F]{} & \mathcal{T}_F \end{array}$$

whose two composite arrows differ by $\mathrm{ad}s$, $s \in \mathcal{S}(\mathbf{C}) \subseteq \mathcal{T}_F(\mathbf{C})$.

6. Conjugation of Shimura varieties. The principal purpose of this section is to formulate a conjecture about the conjugation of Shimura varieties, a conjecture whose first justification is that it is a simple statement which implies what we need for the study of the zeta-functions at archimedean places and is compatible with all that we know. Some lemmas are necessary before it can be stated, and we shall see that these lemmas together with the hypothetical properties of the Taniyama group suggest an answer to the question that arose at the end of the fourth section. This answer in its turn throws new light on the conjecture, so that we can weave a consistent pattern of hypotheses, and our task will be ultimately to show that it has some real validity.

We need a construction, which we make in sufficient generality that it applies to all Shimura varieties and not just those associated to motives. Suppose the pair $(G, \mathfrak{H})$ defines a Shimura variety, T and $\bar{T}$ are two Cartan subgroups of G defined over $\mathbf{Q}$, and $h : \mathcal{R} \to T, \bar{h} : \mathcal{R} \to \bar{T}$ both lie in $\mathfrak{H}$. Let μ and $\bar{\mu}$ be the coweights of T and $\bar{T}$ obtained by restricting h and $\bar{h}$ to the first factor of $\mathcal{R}$, and choose a finite Galois extension L which splits T and $\bar{T}$. Let $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$; then the coweights $(1+\iota)(\tau^{-1}-1)\mu$ and $(1+\iota)(\tau^{-1}-1)\bar{\mu}$ are both central and they are equal. Choose $w = w(\tau)$ as before, and set

$$b_0(\tau, \mu) = \prod_\sigma c_\sigma(w)^{\sigma\mu}, \quad b_0(\tau, \bar{\mu}) = \prod_\sigma c_\sigma(w)^{\sigma\bar{\mu}}.$$

Let $\tilde{b}(\tau, \mu)$ and $\tilde{b}(\tau, \bar{\mu})$ be liftings to $T(\mathbf{A}(L))$ and $\bar{T}(A(L))$ and let $B(\tau) = B(\tau, \mu, \bar{\mu})$ be the projection of $\tilde{b}(\tau, \bar{\mu})^{-1}\tilde{b}(\tau, \mu)$ on $\mathbf{G}(\mathbf{A}_f(L))$. Although we may not be able to define the cocycle $\{c_\rho(\tau, \mu)\}$, we can define $\{c_\rho(\tau, \mu_{\mathrm{ad}})\}$ if μ_{ad} is the composition of μ with the projection to the adjoint group.

It may be as well to check that $B(\tau)$ is indeed independent of the choice of w and of the coset representatives w_σ, provided the usual conditions (a), (b), and (c) are satisfied. If w_σ is replaced by $e_\sigma w_\sigma$ then, apart from a factor in $L_\infty^\times \otimes X_*(T), b_0(\tau, \mu)$ is multiplied by $\prod_{\eta \in \mathfrak{f}} e_\eta^{\eta(1+\iota)(\tau^{-1}-1)\mu}$, and $b_0(\tau, \bar{\mu})$ by $\prod_{\eta \in \mathfrak{f}} e_\eta^{\eta(1+\iota)(\tau^{-1}-1)\bar{\mu}}$. Since these two terms are central and equal by assumption, the change has no effect on $B(\tau)$. If w is replaced by xwx^{-1} with $x\sigma(x^{-1}) \in L_v^\times$ for $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$, then $b_0(\tau, \mu)$ is modified by the product of an element in $L_\infty^\times \otimes X_*(T)$ and $\prod_\eta x^{\eta(1+\iota)(\tau^{-1}-1)\mu}\tau$. Since

$b_0(\tau,\bar\mu)$ undergoes a similar modification, $B(\tau)$ is not affected. One can also show easily that $B(\tau)$ is not changed when L is enlarged; the argument is once again basically the same as that used to treat $b(\tau)$.

$B(\tau)$ does depend on the choice of $\tilde b(\tau,\mu)$ and $\tilde b(\tau,\bar\mu)$. These choices made, we will use them consistently to define $c_\rho(\tau,\mu_{\rm ad}), c_\rho(\tau,\bar\mu_{\rm ad})$, and when

$$(1+\iota)(\tau^{-1}-1)\mu=(1+\iota)(\tau^{-1}-1)\bar\mu=0,$$

$c_\rho(\tau,\mu), c_\rho(\tau,\bar\mu)$. Thus $c_\rho(\tau,\mu_{\rm ad})$ is to be the projection of $\rho(\tilde b(\tau,\mu)^{-1})\tilde b(\tau,\mu)$ in $G_{\rm ad}(\mathbf{A}_f(L))$. With these conventions, the ambiguity in $B(\tau)$ will cause no harm in the construction to be given next.

Let $G^{\tau,\mu}$ and $G^{\tau,\bar\mu}$ be the groups obtained from G by twisting with the cocycles $\{c_\rho(\tau,\mu_{\rm ad})^{-1}\}$ and $\{c_\rho(\tau,\bar\mu_{\rm ad})^{-1}\}$. We are going to verify the following:

First Lemma of Comparison. (i) *If* $c_\rho=c_\rho(\tau,\mu_{\rm ad})$ *then*

$$\gamma_\rho=B(\tau){\rm ad}c_\rho^{-1}(\rho(B(\tau)^{-1}))$$

lies in $G^{\tau,\mu}(L)$.

(ii) *The cocycle* $\{\gamma_\rho\}$ *in* $G^{\tau,\mu}(L)$ *bounds.*

(iii) *If* $(1+\iota)(\tau^{-1}-1)\mu=(1+\iota)(\tau^{-1}-1)\bar\mu=0$ *then*

$$\gamma_\rho=c_\rho(\tau,\bar\mu)^{-1}c_\rho(\tau,\mu).$$

The third assertion is clear; it is the other two with which we must deal. The element γ_ρ can be obtained by projecting

$$\tilde b(\tau,\bar\mu)^{-1}\rho(\tilde b(\tau,\bar\mu))\rho(\tilde b(\tau,\mu)^{-1})\tilde b(\tau,\mu) \tag{6.1}$$

on $G(\mathbf{A}_f(L))$. This makes it perfectly clear that γ_ρ is not affected if $w=w(\tau)$ is replaced by $xw(\tau)$ with $x\in\mathbf{C}_L$. Thus we may assume that $w=w_\tau$, where τ is here also used to denote the image of τ in ${\rm Gal}(L/\mathbf{Q})$. Then we have to show that

$$\tilde b(\tau,\bar\mu)^{-1}\rho(\tilde b(\tau,\bar\mu))\equiv\tilde b(\tau,\mu)^{-1}\rho(\tilde b(\tau,\mu))\pmod{G(L_\infty)G(L)}.$$

It follows easily from (5.11) that if $\tilde a_{\rho,\eta}$ is a lift of $a_{\rho,\eta}$ to I_L, then both sides are congruent to

$$\prod_\eta \tilde a_{\rho,\eta}^{\rho\eta(1+\iota)(1-\tau^{-1})\mu}=\prod_\eta \tilde a_{\rho,\eta}^{\rho\eta(1+\iota)(1-\tau^{-1})\bar\mu}.$$

The desired equality follows.

To prove the second assertion we shall apply Hasse's principle, but for this we need a group G whose derived group G_{der} is simply connected. Let G_{sc} be the simply connected covering of G_{der}. The Cartan subgroup T defines T_{der} and T_{sc}. We have an imbedding $X_*(T_{\mathrm{sc}}) \to X_*(T)$ and $G_{\mathrm{der}} = G_{\mathrm{sc}}$ if and only if the quotient is torsion-free. If we can construct a diagram of Gal$(L/\mathbf{Q})$-modules

$$\begin{array}{ccccccccc} 0 \longrightarrow & X_*(T_{\mathrm{sc}}) & \longrightarrow & \mathbf{Q} & \longrightarrow & P & \longrightarrow & 0 \\ & \Big\| & & \Big\downarrow & & \Big\downarrow & & \\ 0 \longrightarrow & X_*(T_{\mathrm{sc}}) & \longrightarrow & X_*(T) & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which P is torsion-free and $P \to M$ is a surjection whose kernel is a free Gal$(L/\mathbf{Q})$-module, then we can use it to define a central extension G' of G with $X_*(T') = \mathbf{Q}$. We will have $G'_{\mathrm{der}} = G'_{\mathrm{sc}} = G_{\mathrm{sc}}$, and $G'(\mathbf{R}) \to G(\mathbf{R})$ will be surjective. To construct the diagram, we choose an exact sequence of Gal$(L/\mathbf{Q})$-modules

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0,$$

with P torsion-free and N free. Then $\mathbf{Q}$ is the set of all (x, ρ) in $X_*(T) \oplus P$ for which x and ρ have the same image in M.

We lift μ to $\mu' = (\mu, \nu)$ with $\nu \in P$. If $\bar{\mu} = \mathrm{ad}g \circ \mu$ and g is the image of g', we set $\bar{\mu}' = \mathrm{ad}g' \circ \mu'$. If the assertion is valid for $\mu', \bar{\mu}'$, and G' it is valid for $\mu, \bar{\mu}$, and G. Consequently we may suppose that G_{der} is simply-connected.

There are two types of ambiguity in $B(\tau)$. It can be changed to $tB(\tau)$ with $t \in \bar{T}(L)$. Then $\{\gamma_\rho\}$ is replaced by $\{(t\gamma_\rho \mathrm{ad}c_\rho^{-1}(\rho(t)^{-1})\}$, and its cohomology class is not affected. We can also change $B(\tau)$ to $B(\tau)t$ with $t \in T(L)$. Then $\{c_\rho\}$ is replaced by $\{c'_\rho\}$ with $c'_\rho = \rho(t_{\mathrm{ad}}^{-1})c_\rho t_{\mathrm{ad}}, t_{\mathrm{ad}}$ being the image of t in G_{ad}, and $\{\gamma_\rho\}$ is replaced by $\{\gamma'_\rho\}$, with

$$\gamma'_\rho = \gamma_\rho \mathrm{ad}c_\rho^{-1}(\rho(t^{-1}))t.$$

If $\gamma_\rho = \delta \mathrm{ad}c_\rho^{-1}(\rho(\delta^{-1}))$ then

$$\gamma'_\rho = (\delta t)\mathrm{ad}^{-1}c'_\rho(\rho(\delta t)^{-1}).$$

Consequently the ambiguity in $B(\tau)$ has no effect on the assertion (ii).

Rather than prove (ii) directly for a given choice of the pair $\mu, \bar{\mu}$, we want to prove it for a succession of pairs. For this one should first check that the validity of (ii) defines an equivalence relation. If μ and $\bar{\mu}$ are interchanged then $\{\gamma_\rho\}$ is replaced by $\{\gamma_\rho^{-1}\}$, and if $\gamma_\rho = \delta \mathrm{ad}c_\rho^{-1}(\rho(\delta^{-1}))$ then

$$\gamma_\rho^{-1} = \delta^{-1}\mathrm{ad}\bar{c}_\rho^{-1}(\rho(\delta)).$$

To show the transitivity, we introduce a new notation, denoting γ_ρ by $\gamma_\rho(\bar\mu,\mu)$, and c_ρ by $c_\rho(\mu)$. Suppose

$$\gamma_\rho(\mu_2,\mu_3) = t\mathrm{ad}c_\rho^{-1}(\mu_3)(\rho(t^{-1})),$$
$$\gamma_\rho(\mu_1,\mu_2) = s\mathrm{ad}c_\rho^{-1}(\mu_2)(\rho(s^{-1})).$$

Observing that

$$\gamma_\rho(\mu_2,\mu_3)\mathrm{ad}c_\rho^{-1}(\mu_3)(\rho(s^{-1}))\gamma_\rho(\mu_2,\mu_3)^{-1} = \mathrm{ad}c_\rho^{-1}(\mu_2)(\rho(s^{-1}))$$

and that $\gamma_\rho(\mu_1,\mu_3) = \gamma_\rho(\mu_1,\mu_2)\gamma_\rho(\mu_2,\mu_3)$, one deduces with little effort that

$$\gamma_\rho(\mu_1,\mu_3) = st\mathrm{ad}c_\rho(\mu_3)(\rho(st)^{-1}).$$

Transitivity established, we return to the original notation. If $\bar\mu$ is conjugate to μ under $G(\mathbf{Q})$, say $\bar\mu = \mathrm{ad}x \circ \mu$ then

$$\gamma_\rho = x\mathrm{ad}c_\rho^{-1}(x^{-1}) = x\mathrm{ad}c_\rho^{-1}(\rho(x^{-1})),$$

and certainly bounds. In general $\bar\mu$ and μ are not conjugate under $G(\mathbf{Q})$, but they are conjugate under $G(\mathbf{R})$. Since $G(\mathbf{Q})$ is dense in $G(\mathbf{R})$ we may take advantage of the transitivity and assume that they are conjugate under $G_{\mathrm{der}}(\mathbf{R})$.

It is now that the assumption that G_{der} is simply connected intervenes. If we are careful in our choice of $\tilde b(\tau,\mu)$ and $\tilde b(\tau,\bar\mu)$, defining them by liftings of $c_\sigma(w)$ to I_L, then $B(\tau)$ and the γ_ρ will lie in $G_{\mathrm{der}}^{\tau,\mu}$. Moreover the cocycle $\{\gamma_\rho\}$ in $G_{\mathrm{der}}^{\tau,\mu}(L)$ certainly bounds at every finite place. Since we are applying Hasse's principle, we need only verify that it bounds at infinity as well.

One begins with a calculation similar to the one made while studying the cocycle $\{c_\rho(\tau)\}$. If $\rho \in \mathrm{Gal}(L/\mathbf{Q})$, set

$$e_\rho(\tau,\mu) = \prod_{\eta\in\mathfrak{f}} a_{\rho\eta,\iota}^{\rho\eta(\tau^{-1}-1)\mu}$$

and define $e_\rho(\tau,\bar\mu)$ in a similar fashion. The projection of $\{\gamma_\rho\}$ on $G^{\tau,\mu}(L_\infty)$ is cohomologous to $e_\rho(\tau,\bar\mu)e_\rho(\tau,\mu)^{-1}$. If $\rho \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ the projection of $e_\rho(\tau,\mu)$ on $G^{\tau,\mu}(L_v)$ is

$$f_\rho(\tau,\mu) = \prod_{\sigma\in\mathrm{Gal}(L_v/\mathbf{Q}_v)} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}.$$

Thus all we need do is show that $f_\rho(\tau,\bar\mu)f_\rho(\tau,\mu)^{-1}$ bounds in $G^{\tau,\mu}(L_v)$. Recall that the cocycle defining $G^{\tau,\mu}(L_v)$ is

$$h_\rho = \prod_{\sigma\in\mathrm{Gal}(L_v/\mathbf{Q}_v)} a^{\rho\sigma(\tau^{-1}-1)\mu_{\mathrm{ad}}}.$$

For brevity, we write $\{\prod_\sigma a_{\rho,\sigma}^{\rho\sigma\nu}\} = \{\alpha_\rho(\nu)\}$. If $x \in G(\mathbf{R})$ we write $x(\mu) = \mathrm{ad}x \circ \mu$. If $x(\bar{\mu}) = \mu$ then

$$x f_\rho(\tau, \bar{\mu}) f_\rho(\tau, \mu)^{-1} \mathrm{ad} h_\rho(\rho(x^{-1})) = \alpha_\rho((x\tau^{-1}x^{-1} - \tau^{-1})\mu),$$

because $\rho(x) = x$ and

$$f_\rho(\tau, \mu)^{-1} \mathrm{ad} h_\rho(x^{-1}) = x^{-1} f_\rho(\tau, \mu)^{-1}.$$

If w lies in the normalizer of T_{der} in $G_{\mathrm{der}}(L_v)$ then

$$w \mathrm{ad} h_\rho(\rho(w^{-1})) = w\rho(w^{-1})\alpha_\rho((w-1)(\tau^{-1}-1)\mu). \tag{6.2}$$

However $\{w\rho(w^{-1})\}$ is a cohomology class in $T_{\mathrm{der}}(L_v)$ or $T_{\mathrm{der}}^{\tau,\mu}(L_v)$, the two groups being equal, and it is shown in **[45]** that it is equivalent to $\{\alpha_\rho((w-1)\mu)\}$. Thus the class of (6.2) in $G_{\mathrm{der}}^{\tau,\mu}(L_v)$ is $\{\alpha_\rho((w-1)\tau^{-1}\mu)\}$. Since we may take w such that its action on the weights of T is the same as that of $x\tau^{-1}x^{-1}\tau$, the verification is complete.

There is another fact that we should verify while this proof is fresh in our minds. Suppose there is an ω in the Weyl group such that

$$\omega\mu = \tau^{-1}\mu. \tag{6.3}$$

Then $(1+\iota)(\tau^{-1}-1)\mu = 0$ and $\{c_\rho^{-1}(\tau)\} = \{c_\rho^{-1}(\tau,\mu)\}$ is defined. It bounds in $G(L)$. Once again it is enough to verify this when G_{der} is simply connected, although this time the modifications made to arrive at a simply-connected derived group would be different. It is no longer important that $P \to M$ be surjective, or that its kernel be free, but there must be a $\nu \in P$ which maps to μ and is fixed by $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$.

The set of all τ which satisfy (6.3) for at least one ω form a group. Let $E = E(G,h) = E(G,\mathfrak{H})$ be its fixed field. Then $E \subseteq L$. In order to apply Hasse's principle we have to arrange that $\{c_\rho(\tau)\}$ lies in $G_{\mathrm{der}}(L)$ and that it obviously bounds in $G_{\mathrm{der}}(\mathbf{A}_f(L))$. Since $c_\rho(\tau)$ only depends on the image of τ in $\mathrm{Gal}(L/\mathbf{Q})$, we may calculate it when $w = w_\tau$. If $\mathfrak{S}$ is a set of coset representatives for $\mathrm{Gal}(L/\mathbf{Q})/\mathrm{Gal}(L/E)$, then

$$b_0(\tau) = \prod_{\nu\in\mathfrak{S}} \prod_{\sigma\in\mathrm{Gal}(L/E)} a_{\nu\sigma,\tau}^{\nu\sigma\mu}.$$

We write $a_{\nu\sigma,\tau} = \nu(a_{\sigma,\tau})a_{\nu,\sigma\tau}a_{\nu,\sigma}^{-1}$ and $\nu\sigma\mu = \nu\mu + \nu(\sigma-1)\mu$. Now for $\sigma \in \mathrm{Gal}(L/E), \nu(\sigma-1)\mu$ is a weight of the derived group and $\prod_\nu \prod_\sigma a_{\nu\sigma,\tau}^{\nu(\sigma-1)\mu}$ may be lifted to $T_{\mathrm{der}}(\mathbf{A}(L))$. On the other hand

$\prod_\nu \prod_\sigma a_{\nu,\sigma\tau}^{\nu\mu} a_{\nu,\sigma}^{-\nu\mu} = 1$. If $a = \prod_\sigma a_{\sigma,\tau}$ then a lies in C_E and lifts to a' in I_E. Moreover $\prod_\nu \nu(a)^{\nu\mu}$ lifts to $\prod_\nu \nu(a')^{\nu\mu}$ which lies in $T(\mathbf{A})$. If $\bar{a}_{\nu\sigma,\tau}$ is a lifting of $a_{\nu\sigma,\tau}$ to I_L then we so choose $c_\rho(\tau)$ that

$$c_\rho(\tau) \equiv \left\{ \prod_{\nu,\sigma} \rho(\bar{a}_{\nu\sigma,\tau})^{-\rho\nu(\sigma-1)\nu} \right\} \left\{ \prod_{\nu,\sigma} \bar{a}_{\nu\sigma,\tau}^{\nu(\sigma-1)\mu} \right\} \tag{6.4}$$

modulo $T(L_\infty)$.

It remains to verify that the $\{c_\rho^{-1}(\tau)\}$ so defined bounds at the infinite place. The projection of the right side of (6.4) on $C_L \otimes X_*(T_{\mathrm{der}})$ is

$$\prod_{\nu,\sigma} a_{\rho,\nu\sigma}^{\rho\nu\sigma(\tau^{-1}-1)\mu} = \prod_{\sigma\in\mathrm{Gal}(L/\mathbf{Q})} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}.$$

Thus, restricting $\{c_\rho^{-1}(\tau)\}$ to $\mathrm{Gal}(L/\mathbf{Q}_v)$ and projecting on $T_{\mathrm{der}}(L_v)$, we obtain a class cohomologous to $\{\alpha_\rho((\tau^{-1}-1)\mu)\} = \{\alpha_\rho((\omega-1)\mu)\}$. We have observed already that it is shown in [**45**] that the right side bounds in $G_{\mathrm{der}}(L_v)$.

Although there is one more consequence to be derived from (6.3), there are some things that must first be said about the general (T, h), or (T, μ).

We drop the assumption (6.3) for a while, and return to it later. We have associated to the pair (T, μ) and τ a twisted form $G^{\tau,\mu}$ of G. The twisting of T in G is trivial, and $T^\tau = T$ is a Cartan subgroup of $G^{\tau,\mu}$. Let μ^τ be $\tau^{-1}\mu$. There is a unique homomorphism $h^\tau : \mathcal{R} \to T^\tau$ whose restriction on the first factor is μ^τ and which is defined over $\mathbf{R}$.

The pair $(G^{\tau,\mu}, h^\tau)$ defines a Shimura variety.

The roots $\{\gamma\}$ of T in G are the same as the roots of T^τ in $G^{\tau,\mu}$. However the classification into compact and noncompact differs for the two pairs. The root γ of T in G is compact or noncompact according as $(-1)^{\langle\tau,\mu\rangle}$ is 1 or -1. In order for $(G^{\tau,\mu}, h^\tau)$ to define a Shmura variety, γ must be compact or noncompact as a root of T^τ according as $(-1)^{\langle\gamma,\tau^{-1}\mu\rangle}$ is 1 or -1. However the ideas used in the proof of Lemma A.8 of [**34**] show that the type of γ is changed on passing from T, G to $T^\tau, G^{\tau,\mu}$ if and only if $(-1)^{\langle\gamma,\tau^{-1}\mu-\mu\rangle} = -1$.

We are now almost ready to formulate a conjecture about the conjugation of Shimura varieties. After discussing the conjecture and its consequences, we shall show how it can be heuristically justified in terms of the Taniyama group and motives.

Recall that $\mathrm{Sh}_K(\mathbf{C}) = G(\mathbf{Q})\backslash\mathfrak{H} \times G(\mathbf{A}_f)/K$. Thus if $g \in G(\mathbf{A}_f)$ and $K_1 = g^{-1}Kg$, then right multiplication by g defines a morphism

$$\mathfrak{F}(g) : \mathrm{Sh}_K(\mathbf{C}) \longrightarrow \mathrm{Sh}_{K_1}(\mathbf{C}).$$

It is algebraic and is called a Hecke correspondence. If τ is an automorphism of $\mathbf{C}$, we set $\mathrm{Sh}^\tau_K(G,h) = \mathrm{Sh}^\tau_K = \mathrm{Sh}_K \otimes_{\tau^{-1}} \mathbf{C}$, the Shimura varieties being at the moment only defined over $\mathbf{C}$. Then $\mathfrak{F}^\tau(g) : \mathrm{Sh}^\tau_K \to \mathrm{Sh}^\tau_{K_1}$.

If $G = T$ is a torus and $K = U$ then Sh_U is 0-dimensional. Let τ also denote the element of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ defined by τ. If we set $U^\tau = U$ then

$$\mathrm{Sh}_U(\mathbf{C}) = T(\mathbf{A}_f)/U = T^\tau(\mathbf{A}_f)/U^\tau = \mathrm{Sh}_{U^\tau}.$$

This gives us an isomorphism $\mathrm{Sh}_U = \mathrm{Sh}_U(T,h) \to \mathrm{Sh}_{U^\tau} = \mathrm{Sh}_{U^\tau}(T^\tau, h^\tau)$. In addition there is the natural map τ from complex points of $\mathrm{Sh}^\tau_U(T,h)$ to complex points of $\mathrm{Sh}_U(T,h)$. Define $\varphi_\tau = \varphi_\tau(U,T,h)$ by the commutativity of the diagram

$$\begin{tikzcd}
 & \mathrm{Sh}_U(T,h) \arrow[d] \\
\mathrm{Sh}^\tau_U(T,h) \arrow[ur, "\tau"] \arrow[dr, "\varphi_\tau"'] & \\
 & Sh_{U^\tau}(T^\tau, h^\tau)
\end{tikzcd}$$

In general $G^{\tau,\mu}$ and G are different. But $G^{\tau,\mu}$ is defined by the cocycle $\{c_\rho^{-1}(\tau,\mu_{\mathrm{ad}})\}$ and in $G_{\mathrm{ad}}(\mathbf{A}_f(L))$

$$c_\rho^{-1}(\tau,\mu_{\mathrm{ad}}) = b(\tau,\mu_{\mathrm{ad}})^{-1}\rho(b(\tau,\mu_{\mathrm{ad}})).$$

Thus

$$g \longrightarrow g^\tau = \mathrm{ad} b(\tau,\mu_{\mathrm{ad}})^{-1}(g)$$

defines an isomorphism of $G(\mathbf{A}_f)$ with $G^{\tau,\mu}(\mathbf{A}_f)$.

Conjecture. There is a family of biregular maps $\varphi_\tau = \varphi_\tau(K,G,h), K \subseteq G(\mathbf{A}_f)$ defined over $\mathbf{C}$, taking $\mathrm{Sh}^\tau_K(G,h)$ to $\mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau)$, and rendering the following diagrams commutative:

(a)
$$\begin{tikzcd}
\mathrm{Sh}_U(T,h) \arrow[r, hook] & \mathrm{Sh}_K(G,h) \\
\mathrm{Sh}^\tau_U(T,h) \arrow[u, "\tau"] \arrow[r, hook] \arrow[d, "\varphi_\tau"] & \mathrm{Sh}^\tau_K(G,h) \arrow[u, "\tau"'] \arrow[d, "\varphi_\tau"'] \\
\mathrm{Sh}_{U^\tau}(T^\tau, h^\tau) \arrow[r, hook] & \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}h^\tau)
\end{tikzcd}$$

Here $U = T(\mathbf{A}_f) \cap K$ and the φ_τ in the left column is $\varphi_\tau(U, T, h)$.

(b)
$$\begin{tikzcd}
\mathrm{Sh}^\tau_K(G,H) \arrow[r, "\mathfrak{F}^\tau(g)"] \arrow[d, "\varphi_\tau"'] & \mathrm{Sh}^\tau_{K_1}(G,h) \arrow[d, "\varphi_\tau"] \\
\mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) \arrow[r, "\mathfrak{F}(g^\tau)"'] & \mathrm{Sh}_{K_1^\tau}(G^{\tau,\mu}, h^\tau)
\end{tikzcd}$$

The conjecture in this form refers to a specific T and a specific h factoring through T. If we choose another pair $(\bar{T}, \bar{h})$ we obtain another group $G^{\tau,\bar{\mu}}$ and another collection $\{\bar{\varphi}_\tau = \bar{\varphi}_\tau(K; G, \bar{h})\}$. The conjecture is inadequate as it stands, and must be supplemented by a statement relating φ_τ and $\bar{\varphi}_\tau$. Observe that there can be at most one family $\{\varphi_\tau\}$ satisfying the conditions (a) and (b).

We have already associated to the two pairs (T, h) and $(\bar{T}, \bar{h})$ a cocycle $\{\gamma_\rho\}$ which bounds in $G^{\tau,\mu}(L)$. Let $\gamma_\rho = u\mathrm{adc}_\rho^{-1}(\rho(u^{-1}))$. Then $g \to ugu^{-1}$ defines an isomorphism of $G^{\tau,\mu}(\mathbf{Q})$ with $G^{\tau,\bar{\mu}}(\mathbf{Q})$ or of $G^{\tau,\mu}(\mathbf{A}_f)$ with $G^{\tau,\bar{\mu}}(\mathbf{A}_f)$. Set ${}^uK^\tau = uK^\tau u^{-1}$.

Second Lemma of Comparison. *The homomorphism $\mathrm{ad}u \circ h^\tau$ is conjugate under $G^{\tau,\bar{\mu}}(\mathbf{R})$ to $\bar{h}^\tau$.*

If this statement is true when u is replaced by a v in $G(L_v) = G(\mathbf{C})$ which also trivializes $\{\gamma_\rho\}$ restricted to $\mathrm{Gal}(L_v/\mathbf{Q}_v)$ then it is true for u. An examination of the proof of the second property of $\{\gamma_\rho\}$ shows that we can take v to lie in $x^{-1}wT(L_v)$, x and w being as in that proof. Since $x^{-1}w(\tau^{-1}\mu) = \tau^{-1}\bar{\mu}$, we have $\mathrm{ad}\nu \circ h^\tau = \bar{h}^\tau$.

We infer that $\mathrm{ad}u$ carries $\mathfrak{H}$ to $\bar{\mathfrak{H}}^\tau$ and hence defines a bijection

$$G^{\tau,\mu}(\mathbf{Q})\backslash\mathfrak{H}^\tau \times G^{\tau,\mu}(\mathbf{A}_f)/K^\tau \longrightarrow G^{\tau,\bar{\mu}}(\mathbf{Q})\backslash\bar{\mathfrak{H}}^\tau \times G^{\tau,\bar{\mu}}(\mathbf{A}_f)/{}^uK^\tau$$

and an isomorphism

$$\psi : \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) \longrightarrow \mathrm{Sh}_{\mu_{K^\tau}}(G^{\tau,\bar{\mu}}, \bar{h}^\tau).$$

Since u and $B(\tau)$ both trivialize $\{\gamma_\rho\}$ in $G^{\tau,\mu}(\mathbf{A}_f(L))$ there is a y in $G^{\tau,\mu}(\mathbf{A}_f)$ with $u = B(\tau)y$. Let φ be the composite $\psi \circ \mathfrak{F}(y)$.

Supplement to the conjecture. The diagrams

$$\begin{tikzcd}
 & Sh^\tau_K(G,h) \arrow[dl, "\varphi_\tau"'] \arrow[dr, "\bar{\varphi}_\tau"] & \\
Sh_{K^\tau}(G^{\tau,\mu}, h^\tau) \arrow[rr, "\varphi"'] & & Sh_{K^\tau}(G^{\tau,\bar{\mu}}, \bar{h}^\tau)
\end{tikzcd}$$

are commutative.

The conjecture as it stands certainly implies that the conjugate of a Shimura variety is again a Shimura variety. Together with its supplement, it implies the usual form of Shimura's conjecture [**10**]. To verify this one applies the Weil criterion [**49**] for descent of the field of definition. For this we need families of isomorphisms $f_\rho : \mathrm{Sh}^\rho_K(G,h) \longrightarrow \mathrm{Sh}_K(G,h)$ defined for automorphisms ρ of $\mathbf{C}$ over $E(G,h)$ and satisfying $f_{\sigma\rho} = f_\rho f^\rho_\sigma$.

Choose a Cartan subgroup T and an h which factors through it. We know that when τ fixes $E(G,h)$ the cocycle $\{c^{-1}_\rho(\tau,\mu)\}$ is defined and bounds in $G(L)$. Let $c^{-1}_\rho(\tau,\mu) = \nu\rho(\nu^{-1})$. Then $g \to \nu g\nu^{-1}$ is an isomorphism of $G(\mathbf{Q})$ with $G^{\tau,\mu}(\mathbf{Q})$. Methods which we have already used show easily that

The composite $\mathrm{ad}\nu \circ h$ *is conjugate under* $G^{\tau,\mu}(\mathbf{R})$ *to* h^τ*.*

Consequently ad ν defines an isomorphism $\mathfrak{H} \to \mathfrak{H}^\tau$ and then, as before, we obtain from

$$G(\mathbf{Q})\backslash\mathfrak{H} \times G(\mathbf{A}_f)/K \longrightarrow G^{\tau,\mu}(\mathbf{Q})\backslash\mathfrak{H}^\tau \times G^{\tau,\mu}(\mathbf{A}_f)/{}^\nu K$$

an isomorphism

$$S_K(G,h) \longrightarrow S_{\nu_K}(G^{\tau,\mu}, h^\tau).$$

On the other hand $z\nu = b(\tau,\mu)^{-1}$ with $z \in G^{\tau,\mu}(\mathbf{A}_f)$. We define f_τ by the commutativity of

$$\begin{array}{ccc} \mathrm{Sh}^\tau_K(G,h) & \xrightarrow{f_\tau} & S_K(G,h) \\ \varphi_\tau \downarrow & & \downarrow \\ \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}h^\tau) & \xrightarrow[\mathfrak{F}(z)]{} & \mathrm{Sh}_{\nu_K}(G^{\tau,\mu},h^\tau) \end{array}$$

I omit the calculations, lengthy but routine, by which it is deduced from the conjecture and its supplement that f_τ does not depend on the choice of T and h and that the cocycle condition $f_{\sigma\rho} = f_\rho f^\rho_\sigma$ is satisfied.

Up to now the S_K have been taken as varieties over $\mathbf{C}$, but by the criterion for descent we may now define them over $E(G,h)$ in such a way that the f_τ are simply the identity maps. It has to be verified that the models thus obtained are canonical, but the construction is clearly such that only the case that G is a torus T need by considered. Let a be the transfer of $w = w_\tau$ to $\mathbf{C}_E$ and a' a lifting of a to I_E. The proof that in this case $c_\rho(\tau,\mu)$ is trivial shows in fact that we may take it to be 1 and $b = b(\tau,\mu)$ to be $\prod_{\mathrm{Gal}(L/\mathbf{Q})/\mathrm{Gal}(L/E)} \nu(a')^{\nu\mu}$. We take ν to be 1 and z to be $b^{-1} = b(\tau,\mu)^{-1}$. The composition

$$\mathrm{Sh}_U(T,h) \xrightarrow{\tau^{-1}} \mathrm{Sh}_U(T,h) \xrightarrow{\mathfrak{F}(b)} \mathrm{Sh}_U(T,h)$$

is then the identity for all τ fixing $E(T,h)$, and this is just the condition that $\mathrm{Sh}(T,h)$ be the canonical model.

Suppose $E(G,h) \subseteq \mathbf{R}$. If we take the canonical model for Sh_K then the complex conjugation defines an involution θ of the complex manifold $\mathrm{Sh}_K(\mathbf{C})$. It is necessary to have a concrete description of this involution in terms of the representation

$$\mathrm{Sh}_K(\mathbf{C}) = G(\mathbf{Q})\backslash\mathfrak{H} \times G(\mathbf{A}_f)/K,$$

and one purpose of the conjecture is to provide it.

Choose some special point (T,h). If $E(G,h) \subseteq \mathbf{R}$ then we may define $b(\iota,\mu)$ and $\{c_\rho(\iota,\mu)\}$. However the condition (c) on the coset representatives w_σ used to define $b(\iota,\mu)$ implies that $b(\iota,\mu) = 1$. We may also take $c_\rho(\iota,\mu)$ to be 1, and then ν may be taken to be 1 as well. It follows that h and h^ι are conjugate in $G(\mathbf{R})$. Since $K_{h^\iota} = K_h, \eta : \mathrm{ad}g \circ h^\iota \to \mathrm{ad}g \circ h$ is a well-defined map of $\mathfrak{H}$ to itself.

Consequence of the conjecture. The involution θ may be realized concretely as the mapping $(h,g) \to (\eta(h),g)$ of $G(\mathbf{Q})\backslash\mathfrak{H} \times G(\mathbf{A}_f)/K$ to itself.

Since we are comparing two continuous mappings which commute with the $\mathfrak{F}(g), g \in G(\mathbf{A}_f)$, it is enough to see that they coincide on the point in $S_U(T^\iota,h^\iota)$ represented by $(h^\iota,1)$. Since f_ι is the identity and $\nu = z = 1$, φ_ι takes this point to the point in $\mathrm{Sh}_{K^\iota}(G^{\iota,\mu},h^\iota) = \mathrm{Sh}_K(G,h)$ represented again by $(h^\iota,1)$. It follows immediately from condition (a) of the conjecture that ι applied to the point represented by $(h^\iota,1)$ is $(h,1)$.

For each T and μ let $\nu(\mu)$ be the fibre functor from $\mathcal{REP}(G)$ to $\mathcal{REP}(G^{\tau,\mu})$ which takes $(\xi',V(\xi'))$, to $(\xi,V(\xi))$ with $\xi = \xi'$, and with $V(\xi)$ being the space obtained from $V(\xi')$ by changing the Galois action to $\sigma : x \to \xi'(c_\sigma(\tau,\mu)^{-1})\sigma(x)$. If $(\bar{T},\bar{h})$ is another special point, and if $\nu(\mu,\bar{\mu})$ is defined by the diagram

$$\begin{tikzcd}
\mathcal{REP}(G^{\tau,\mu}) \arrow[dd, "\nu(\mu,\bar{\mu})"'] & \\
& \mathcal{REP}(G) \arrow[ul, "\nu(\mu)"'] \arrow[dl, "\nu(\bar{\mu})"] \\
\mathcal{REP}(G^{\tau,\bar{\mu}}) &
\end{tikzcd}$$

then, according to the first lemma of comparison, the two fibre functors $\omega_{\mathrm{Rep}(G^{\tau,\mu})}$ and $\omega_{\mathrm{Rep}(G^{\tau,\bar{\mu}})} \circ \nu(\mu,\bar{\mu})$ are isomorphic.

On the other hand, we associated, at the end of the fourth section, to each pair (φ', g) a group $G^{\tau,\varphi}$ and a pair (φ', g'). The groups G and $G^{\tau,\varphi}$ are associated to the same tannakian category, and there is thus an equivalence of categories $\nu(\varphi) : \mathcal{REP}(G) \to \mathcal{REP}(G^{\tau,\varphi})$, determined up to isomorphism. If φ factors through the Serre group, then it can be factored through a Cartan subgroup $\bar{T}$ of G and defines a coweight $\bar{\mu}$ of T. The hypothetical properties of the Taniyama group imply that $G^{\tau,\varphi}$ may be taken to be $G^{\tau,\bar{\mu}}$ with $\nu(\varphi)$ being $\nu(\bar{\mu})$. Thus we have a diagram

$$\begin{tikzcd}
\mathcal{REP}(G^{\tau,\mu}) \arrow[dd, "\nu(\mu,\varphi)"'] & \\
& \mathcal{REP}(G) \arrow[ul, "\nu(\mu)"'] \arrow[dl, "\nu(\bar{\varphi})"] \\
\mathcal{REP}(G^{\tau,\varphi}) &
\end{tikzcd}$$

which is commutative up to isomorphism of functors. Moreover the two fibre functors $\omega_{\mathrm{Rep}(G^{\tau,\mu})} \circ \nu(\mu, \varphi)$ are isomorphic. If we choose an isomorphism between them, we obtain [**40**, II.3.3] an isomorphism over $\mathbf{Q}$, $G^{\tau,\varphi} \longrightarrow G^{\tau,\mu}$. Composing with φ' we obtain a homomorphism $\varphi'' : G_{\mathrm{Mot}(\mathbf{C})} \longrightarrow G^{\tau,\mu}$. Since there are so many special points, it is not unreasonable to hope that $\nu(\mu, \varphi)$ exists for all φ, and that $\omega_{\mathrm{Rep}(G^{\tau,\mu})}$ and $\omega_{\mathrm{Rep}(G^{\tau,\varphi})} \circ \nu(\mu, \varphi)$ are always isomorphic.

The second lemma of comparison in conjunction with the hypothetical properties of the Taniyama group implies that the composition of φ'' with the canonical homomorphism $\mathcal{R} \to G_{\mathrm{Mot}(\mathbf{C})}$ lies in $\mathfrak{H}^\tau$ with φ factors through the Serre group, and once again we may surmise or hope that this will be so in general.

If φ_τ is the biregular map appearing in the conjecture then the composition $\varphi_\tau \circ \tau^{-1}$ defines a map from the set of complex points on $\mathrm{Sh}_K(G, h)$ to the set of complex points on $\mathrm{Sh}_K^\tau(G^{\tau,\mu}, h^\tau)$. The idea is that $\varphi_\tau \circ \tau^{-1}$ will take (φ, g) to a pair (φ'', g''), by a process which can be defined within the moduli problem. We have just seen how to obtain φ'', at least at the hypothetical level at which we are working.

To obtain g'' we observe that we have two homomorphisms

$$\omega^{\mathbf{A}_f}_{\mathrm{Rep}(G^{\tau,\varphi})} \circ \nu(\mu, \varphi) \longrightarrow \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G^{\tau,\mu})}. \tag{6.5}$$

One is obtained from the chosen isomorphism over $\mathbf{Q}$ by extending scalars to $\mathbf{A}_f$. The other is obtained by a lengthy composition. The g from which we start provides an isomorphism $\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})} \circ \eta \to \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G)}$.

The canonical isomorphism $\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})} \circ \eta(\tau) \to \omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})}$ can be composed with η. Finally the definitions provide an isomorphism from $\omega_{\mathrm{Rep}(G^{\tau,\varphi})} \circ \nu(\varphi)$ to $\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})} \circ \eta(\tau) \circ \eta$. Putting these all together, we obtain an isomorphism

(6.6)
$$\omega^{\mathbf{A}_f}_{\mathrm{Rep}(G^{\tau,\varphi})} \circ \nu(\varphi) \longrightarrow \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G)}.$$

However we also have an isomorphism

(6.7)
$$\omega^{\mathbf{A}_f}_{\mathrm{Rep}(G)} \longrightarrow \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G^{\tau,\mu})} \circ \nu(\mu).$$

It is given by the isomorpphism $x \to \xi'(b(\tau,\mu)^{-1})x$ of $V(\xi')$ with $V(\xi)$. Composing (6.6) and (6.7) we obtain a second isomorphism between the two fibre functors figuring in (6.5). According to general principles, it can be obtained by composing the first with an element $(g'')^{-1}$ in $G^{\tau,\mu}(\mathbf{A}_f)$ [**40**, §II].

If one can establish, in some way or another, that the map $\psi_\tau : (\varphi, g) \to (\varphi'', g'')$ is really defined, then to prove the conjecture and its supplement one will only need to verify that the composite $\psi_\tau \circ \tau$ is complex analytic. However our purpose here has been to see how the wheels mesh, not to find the mainspring.

7. Continuous cohomology. If $G = G_0$ then, according to the principles of the fourth section, we should be able to attach to each point of $\mathrm{Sh}_K(\mathbf{C})$ an equivalence class of pairs (φ, g). Here φ is a homomorphism from $G_{\mathrm{Mot}(\mathbf{C})}$ to G defined over $\mathbf{Q}$ and if η is the associated θ-functor $\mathcal{REP}(G) \to \mathcal{MOT}(\mathbf{C})$, then g defines an isomorphism

$$\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})} \circ \eta \to \omega^{\mathbf{A}_f}_{\mathrm{Rep}(G)}.$$

In general we have mappings $\mathrm{Sh}_K(G,h) \to \mathrm{Sh}_{K_0}(G_0,h_0)$, and by pulling back we can associate to each point of $\mathrm{Sh}_K(\mathbf{C})$ a pair (φ, g) where g is again in $G(\mathbf{A}_f)$, but φ now takes $G_{\mathrm{Mot}(\mathbf{C})}$ to G_0.

If $(\xi, V(\xi))$ is a representation of G_0 over $\mathbf{Q}$ or, what is the same, a representation of G factoring through G_0, then to each point s of $\mathrm{Sh}_K(\mathbf{C})$ we may associate the motive $M(s,\xi)$ defined by $\xi \circ \phi$, together with the isomorphism

$$\omega^{\mathbf{A}_f}_{\mathrm{Mot}(\mathbf{C})}(M(s,\xi)) \simeq V(\xi)_{\mathbf{A}_f}$$

defined by g. The variety $\mathrm{Sh}_K(G,h)$ should be defined over $E = E(G,h)$ and so should this family of motives. Suppose now that the variety $\mathrm{Sh}_K(G,h)$ is proper. In the best of all possible worlds, one might be able to form the cohomology groups $M', 0 \leqq i \leqq 2\dim \mathrm{Sh}_K$, of the family $M(\cdot,\xi)$, which would again be motives, now over E, and thus correspond to representations σ^i of $G_{\mathrm{Mot}(E)}$.

If the formalism of the second section were established, one could compose σ^i with ρ_F to obtain representations $\rho^i = \sigma^i(G_{\Pi(E)})$. Then the basic problem would simply be to describe the image $\rho^i(G_{\Pi(E)})$.

But we do not have all this formalism, and one of the principal reasons for studying Shimura varieties is the hope that by grappling with the specific arithmetic problems they pose we will obtain an insight that will help us with its construction. Informed by the general principles and hypotheses we are attempting to establish, we can try to formulate questions that are, at least in part, tractable and which if answered will confirm or, if the answer is other than expected, perhaps refute these principles.

In the present context we can first observe that even if the M^i remained undefined, the zeta-function $Z(z, M^i)$ can be defined directly in terms of the data at our disposal. It is a product over the places of E, $\prod_v Z_v(z, M^i)$. At a nonarchimedean place it can be defined by the ℓ-adic representation of the Galois group on the ith cohomology group of the ℓ-adic sheaf $F_\xi(\mathbf{Q}_\ell)$ associated to ξ, as in the papers [**30**] and [**34**]. Since our principal concern now is with the factors for the archimedean places, we need not enter into details.

The field E is contained in $\mathbf{C}$, and the archimedean places are obtained by applying automorphisms of $\mathbf{C}$, or of $\bar{\mathbf{Q}}$, to E. We first define the factor $Z_v(z, M^i)$ for the place v given by $E \subseteq \mathbf{C}$. We have seen that we can associate to ξ a locally constant sheaf $F_\xi(\mathbf{Q})$ over $\mathrm{Sh}_K(\mathbf{C})$. Moreover, we have an analytic family of polarized Hodge structures on $F_\xi(\mathbf{C}) = F_\xi(\mathbf{Q}) \otimes \mathbf{C}$. By a construction of Deligne [**12**], [**50**] this defines a Hodge structure on the cohomology groups $H^i = H^i(\mathrm{Sh}_K(\mathbf{C}), F_\xi(\mathbf{C}))$.

In accordance with the ideas of Serre [**42**] the factor $Z_v(z, M^i)$ will be defined as $L(s, \rho^i)$ where ρ^i is a representation of the Weil group $W_{\mathbf{C}/E_v}$ on H^i. The Hodge structure on H^i defines a representation of $\mathcal{R}(\mathbf{R})$. Since $\mathbf{C}^\times = \mathcal{R}(\mathbf{R})$, this can be used to define ρ^i on $\mathbf{C}^\times \subseteq W_{\mathbf{C}/E_v}$. If E_v is equal to $\mathbf{C}$, this defines ρ^i completely. If $E_v = \mathbf{R}$ then to define ρ^i completely we also have to define $\rho^i(w)$, if w is the element of the Weil group which projects to ι and has square -1.

Since Sh_K is defined over E, ι also defines an involution on $\mathrm{Sh}_K(\mathbf{C})$ which we denoted by θ. What we need is a map of order two

$$\psi : \theta^* F \longrightarrow F, \quad F = F_\xi(\mathbf{C}),$$

such that on each fibre

$$\theta^* F_s^{p,q} = F_{\theta(s)}^{p,q} \longrightarrow F_s^{q,p}.$$

The associated map ι^* on cohomology takes $H^{p,q}$ to $H^{q,p}$ and we set $\rho^i(w)$ equal to $(-1)^p \iota^*$.

To define ψ we have to assume that the consequence of the conjecture which was described in the previous section is valid. It can be proved directly in several cases. Then θ can be obtained by taking the map $(h, g) \to (\eta(h), g)$ and passing to the quotient. Given (h, g), the fibres at the image of (h, g) and $(\eta(h), g)$ may both be identified with $V(\xi)_{\mathbf{C}}$ and ψ is simply the identity map.

If we replace the imbedding $E \subseteq \mathbf{C}$ by $\tau^{-1} : E \to \mathbf{C}$, τ being an automorphism, then the complex manifold $\mathrm{Sh}_K(\mathbf{C})$ is replaced by $\mathrm{Sh}_K^{\tau}(\mathbf{C})$, which we may identify by means of φ_τ with $\mathrm{Sh}_{K^\tau}(\mathbf{C})$, the manifold associated to $\mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau)$. Thus the factor of the zeta-function defined by the place associated to $\tau^{-1} : E = E(G, h) \to \mathbf{C}$ can be calculated by replacing G by $G^{\tau,\mu}$, h by h^τ, and E by $\tau^{-1}(E) = E(G^{\tau,\mu}, h)$ with the place defined by its inclusion in $\mathbf{C}$. The space $V(\xi)$ has also to be twisted by the cocycle $\{\xi(c_\sigma(\tau, \mu)^{-1})\}$.

The function $Z(z, M^i)$ defined, the immediate problem is to show that it can be expressed as a product of L-functions associated to automorphic representations

$$Z(z, M^i) = \prod_j L(z - a_j, \pi_j, r_j). \tag{7.1}$$

Here $a_j \in \mathbf{C}$ is a translation, π_j is an automorphic representation of some group H_j, and r_j is a representation of the L-group ${}^L H_j$. The first step is to decide which H_j, which π_j, and which r_j intervene in the product.

The first step is to use the theory of continuous cohomology to compute the cohomology groups of the sheaves $F_\xi(\mathbf{C})$ together with their Hodge structure, and thus to compute $Z_v(z, M^i)$, v being again defined by $E \subseteq \mathbf{C}$. Using this together with an analysis of the L-packets of automorphic representations of G [**44**], one searches for an identity (7.1) which is at least valid when both sides are replaced by their factors at v. An example is discussed in detail in [**34**]. The identity found, it must be verified for the local factors at the other places v'. If v' is an archimedean place, then the theory of continuous cohomology will allow us to compute $Z_{v'}(z, M^i)$ in terms of the automorphic representations of a $G^{\tau,\mu}$, a group which differs from G by an inner twisting. To make the comparison it will be necessary to have established the principle of functoriality for the pair G and $G^{\tau,\mu}$, and to have understood in detail how it manifests itself. This is bound up with the study of L-packets and is primarily an analytic problem, for we expect that the trace formula will give us a good purchase on it [**23**].

At the finite places the identity (7.1) is difficult to treat as it stands, and for reasons familiar from topology one replaces the left side by

$$Z(z) = \prod_i Z(z, M^i)^{(-1)^i} \tag{7.2}$$

modifying the right side accordingly. The right side is then analyzed by the trace formula, at least if there is no ramification or, at worst, a mild sort [**8**], [**14**], [**30**]. I do not see at the moment any general way of dealing with a truly nonabelian situation, although a rather curious method has been discovered by Deligne for treating the group $\mathrm{GL}(2)$ [**13**].

If there is no ramification, the factor of (7.2) at a finite place can be analyzed by the fixed point formulae of ℓ-adic cohomology. Apart from combinatorial difficulties [**28**] the critical factor is to have a reasonably explicit description of the set of geometric points on $\mathrm{Sh}_K(G,h)$ in $\bar{\kappa}_{\mathfrak{p}}$, the algebraic closure of the residue field at a prime $\mathfrak{p}$ of E, together with the action of the Frobenius on it [**32**]. This is an idea first applied by Ihara [**20**], who has since intensively studied the structure of this set for Shimura curves [**21**], [**22**].

Not much has been done when there is ramification. The first thing is to analyze in reasonably simple cases the manner in which the variety reduces badly. Some interesting discoveries have been made for curves [**8**], [**14**], [**15**], but higher dimensional varieties behave in a more complicated manner. However tools are available for studying their reduction, and it is time to begin.

None of these steps will be easy to carry out. The study of L-packets is in an embryonic stage, and even the combinatorial problems will demand considerable ingenuity in their solution [**27**]. There is still a great deal to be learned from the study of specific examples.

The theory of continuous cohomology is itself in its infancy, and my purpose in this section is to draw attention to some problems which arise in the study of Shimura varieties and which the mature theory should resolve.

I begin by introducing a representation r of the L-group ${}^LG$ which will play a fundamental role in the discussion. The group ${}^LG$ is a semidirect product ${}^LG^\circ \times \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. If T is a Cartan subgroup of G over $\mathbf{Q}$ then there is an isomorphism of $X_*(T)$, the lattice of coweights of T, with $X^*({}^LT^\circ)$, the lattice of weights of ${}^LT^\circ$, defined up to an element of the Weyl group. In particular if (T,h) is a special point then μ defines an orbit Θ in $X^*({}^LT^\circ)$, and Θ is independent of (T,h). Let r° be the representation of ${}^LG^\circ$ whose set of extreme weights is Θ. The group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on the weights of ${}^LT^\circ$ and preserves the set of dominant weights. The group $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ fixes the set Θ. Thus $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ fixes the dominant element $\mu^\vee$ in Θ, and we may extend r° to ${}^LG^\circ \times \mathrm{Gal}(\bar{\mathbf{Q}}/E)$ in such a way that $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ acts trivially on the weight space of $\mu^\vee$. The extended representation will also be called r°, and we define r to be the induced representation

$$r = \mathrm{Ind}({}^LG, {}^LG^\circ \times \mathrm{Gal}(\bar{\mathbf{Q}}/E), r^\circ).$$

Let d be the dimension of $\mathrm{Sh}_K(G,h)$. One expects that the function (7.2) will be equal to a product of functions

$$L(z-d/2,\pi,\rho). \tag{7.3}$$

Here π is an automorphic representation of one of the groups H attached to G in [**33**]. There is, in general, an imbedding $\varphi : {}^LH \hookrightarrow {}^LG$ and ρ is a subrepresentation of $r\circ\varphi$.

If w is any place of $\mathbf{Q}$, let r_w be the restriction of r to the L-group ${}^LG_w$, which equals ${}^LG^\circ \times \mathrm{Gal}(\bar{\mathbf{Q}}_w/\mathbf{Q}_w)$. Implicit in this notation is an imbedding $\bar{\mathbf{Q}} \subseteq \bar{\mathbf{Q}}_w$. Then the double cosets in $\mathrm{Gal}(\bar{\mathbf{Q}}/E)\backslash\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})/\mathrm{Gal}(\bar{\mathbf{Q}}_w/\mathbf{Q}_w)$ parametrize the places of E dividing w, and $r_w = \oplus_{v|w} r_v$, with

$$r_v = \mathrm{Ind}({}^LG_w, {}^LG^\circ \times\ \mathrm{Gal}(\bar{\mathbf{Q}}_w/E_v), r_v^\circ)$$

and $r_v^\circ(\tau) = r^\circ(\sigma_v\tau\sigma_v^{-1})$ if σ_v is some element in the coset defining v. The representation ρ will also be a direct sum $\rho = \oplus_{v|w}\rho_v$, and the function (7.3) will be a product $\prod_w\prod_{v|w} L(z-d/2,\pi_w,\rho_v)$. The factor corresponding to the place v is $L(z-d/2,\pi_w,\rho_v)$.

Since we shall only be interested in the place v given by $E\subseteq\mathbf{C}$, we shall write r and ρ instead of r_v. Moreover we shall write an automorphic representation of $G(\mathbf{A})$ (or of $H(\mathbf{A})$) as $\pi\otimes\pi_f$, π being a representation of $G(\mathbf{R})$ and π_f of $G(\mathbf{A}_f)$.

Any irreducible representation π of $G(\mathbf{R})$ lies in some L-packet Π_φ where φ is a homomorphism from $W_{\mathbf{C}/\mathbf{R}}$ to ${}^LG$. If α is the character of $W_{\mathbf{C}/\mathbf{R}}$ obtained by composing $W_{\mathbf{C}/\mathbf{R}}\to\mathbf{R}$ with the absolute value, let $\psi_1(\pi) = \alpha^{-d/2}\otimes(r\circ\varphi)$. Then $L(s-d/2,\pi,r) = L(s,\psi_1(\pi))$.

On the other hand, suppose $\pi\otimes\pi_f$ is an automorphic representation of $G(\mathbf{A})$. Let it act on the subspace $U\otimes U_f$ of the space of automorphic forms. If $h\in\mathfrak{H}$ then, according to the principles of continuous cohomology [**4**], its contribution to the cohomology of $\mathrm{Sh}_K(\mathbf{C})$ with values in $F_\xi(\mathbf{C})$ in dimension i is

$$\mathrm{Hom}_{K_h}(\Lambda^i\mathfrak{g}/\mathfrak{k}\otimes\tilde{V}, U)\otimes U_f^K. \tag{7.4}$$

Here $\mathfrak{k}$ is the Lie algebra of K_h and $\tilde{V}$ the dual of $V(\xi)$. The space U_f^K is the space of vectors in U_f fixed by K.

The action of $z\in\mathbf{C}^\times = \mathcal{R}(\mathbf{R})$ defining the Hodge structure sends $\varphi\otimes u$ to $\varphi'\otimes u$ with

$$\varphi'(X\otimes\tilde{\nu}) = \varphi(\eta(z)X\otimes\tilde{\xi}(h(z^{-1}))\tilde{\nu}).$$

Here $X \to \eta(z)X$ is the action which multiplies the exterior product of p holomorphic and q antiholomorphic vectors by $z^{-p}\bar{z}^{-q}$. Thus $\eta(z) = \mathrm{ad}\ h(z^{-1}) \circ \eta(\bar{z})$ and

$$\varphi'(X \otimes \tilde{\nu}) = \pi(h(z^{-1}))\varphi(\eta(\bar{z})X \otimes \tilde{\nu}).$$

If $E \subseteq \mathbf{R}$ we may extend this action of $\mathbf{C}^\times$ to an action of $W_{\mathbf{C}/\mathbf{R}} = W_{\mathbf{C}/E_v}$. Let $n \in G(\mathbf{R})$ be such that $\mathrm{ad}\ n \circ h = h^\iota$. Then the element of w which projects to ι and has square -1 sends $\varphi \otimes u$ to $\varphi' \otimes u$ with

$$\varphi'(X \otimes \tilde{\nu}) = \pi(n)\varphi(\mathrm{ad}\ n^{-1}(X) \otimes \tilde{\xi}(n^{-1})\tilde{\nu}).$$

In either case the representation of $W_{\mathbf{C}/E_v}$ on (7.4) factors as $\psi^i(\pi) \otimes 1$, where $\psi^i(\pi)$ acts on $\mathrm{Hom}_{K_h}(\Lambda^i \mathfrak{G}/\mathfrak{k} \otimes \tilde{V}, U)$. Let $\psi_2(\pi)$ be the element in the representation ring of $W_{\mathbf{C}/\mathbf{R}}$ defined by

$$\psi_2(\pi) = \oplus(-1)^i \,\mathrm{Ind}\,(W_{\mathbf{C}/\mathbf{R}}, W_{\mathbf{C}/E_v}, \psi^i(\pi)).$$

Let $m(\pi_f, K)$ be the dimension of U_f^K. If for all π we had

$$\psi_2(\pi) = m(\pi)\psi_1(\pi) \tag{7.5}$$

we could expect a relation

$$Z(z) = \prod_\pi L(z - d/2, \pi, r)^{m(\pi_\infty)m(\pi_f, K)}. \tag{7.6}$$

Here, as a single exception, we have taken π to be a representation of $G(\mathbf{A})$, its component at infinity being denoted π_∞, and r to be the representation of ${}^LG$. However (7.5) is not always valid, and it is the true form of the relation between $\psi_1(\pi)$ and $\psi_2(\pi)$ that we must discover, for it is the clue to the correct expression of (7.2) as a product of L-functions associated to automorphic representations.

Let $\Pi(\xi) = \{\pi_1, \cdots, \pi_r\}$ be the set of discrete series representations with the same central and infinitesimal characters as $\tilde{\xi}$. Then $\Pi(\xi) = \Pi$ is an L-packet Π_φ and the representations $\psi_1(\pi_i)$ are all equal. We denote them by $\psi_1(\Pi)$. The continuous cohomology of the representations π_i is completely understood **[4]**, and it is a simple exercise to prove the following lemma.

Lemma $\oplus_{j=1}^{r}\psi_2(\pi_j) = (-1)^d\psi_1(\Pi)$.

Thus, in this case, the relation (7.5) fails when the L-packet has more than one element. In order to correct (7.6) one has to replace r by a subrepresentation. However r is in general irreducible as a representation of ${}^LG$, and so we have to introduce the groups ${}^LH$ of [**33**], and begin the study of L-indistinguishability.

If we accept L-indistinguishability, but expect no other difficulties with the correction of (7.6), then we have to be prepared to prove that every irreducible component of $\psi_2(\pi)$ is a component of $\psi_1(\pi)$. But we will again be deceived. There is another difficulty.

It appears already in the simplest of the examples considered by Casselman [**6**] and Milne [**36**], although they had no occasion to draw attention to it. Suppose G is the group associated to a quaternion algebra over **Q** which is split at infinity but not at p. Let $\pi_f = \pi_p \otimes \pi^p$, and suppose $\pi \otimes \pi_f$ is trivial on the centre, π_p is one-dimensional and trivial on the maximal compact subgroup K_p of $G(\mathbf{Q}_p)$, and π is either one-dimensional or the first element of the discrete series. ξ is taken to be trivial. If $K = K^pK_p$ then $L(s-\frac{1}{2}, \pi\otimes\pi_f, r)$ should appear in the zeta-function $Z(s, \mathrm{Sh}_K)$ with the exponent $\pm m(\pi^p, K^p)$. Here $m(\pi^p, K^p)$ is the multiplicity with which the trivial representation of K^p occurs in π_f^p, and the sign is positive if π is one-dimensional and negative if it is the first element of the discrete series. As Casselman and Milne show in their lectures, this is so locally almost everywhere.

One can probably show without great difficulty that the local statement is correct at p as well when π belongs to the discrete series, for π then contributes to the cohomology in dimension one and $\pi_p = \pi(\sigma_p)$ where σ_p is a special representation of the thickened Weil group. In particular

$$L(z-\frac{1}{2}, \pi_p, r) = \frac{1}{1-\epsilon/p^z},\ |\epsilon| = 1.$$

However if π is one-dimensional then π contributes to the cohomology in dimensions zero and two and the corresponding local contribution to the zeta-function should be

$$\left\{\frac{1}{(1-\epsilon/p^z)(1-\epsilon p/p^z)}\right\}^{m(\pi^p,K^p)}.$$

The factor inside the brackets is not $L(z-\frac{1}{2}, \pi_p, r)$.

The difficulty is resolved if we realize that when π, and hence $\pi\otimes\pi_f$, is one-dimensional we should not be using $L(z-\frac{1}{2}, \pi\otimes\pi_f, r)$ at all but rather $L(z-\frac{1}{2}, \pi'\otimes\pi'_f, r)$ where $\pi'\otimes\pi'_f$ is the one-dimensional representation of $G'(\mathbf{A}) = \mathrm{GL}(2, \mathbf{A})$ defined by the same character of the idèle-class group as $\pi \otimes \pi_f$. Since $G'(\mathbf{Q}_v) \sim G(\mathbf{Q}_v)$ and $\pi'_v \sim \pi_v$ for almost all places v, the error of using $\pi \otimes \pi_f$ instead of $\pi' \otimes \pi'_f$ is not detected when one only considers the local zeta-function almost everywhere.

The significance of the considerations of the second and third sections begins to appear. The representation $\pi \otimes \pi_f$ and the representation $\pi'' \otimes \pi''_f$ of $G'(\mathbf{A})$ associated to it by the principle of functoriality are anomalous, because π''_v is one-dimensional for almost all places v while π''_p is infinite-dimensional. The isobaric representation equivalent to $\pi'' \otimes \pi''_f$ almost everywhere is $\pi' \otimes \pi'_f$. It was implicit in the discussion of the second section that anomalous representations would have nothing to do with motives, and so it should come as no surprise now that we must discard $\pi \otimes \pi_f$ and replace it by $\pi' \otimes \pi'_f$.

In this example π itself was not changed for $G(\mathbf{R}) \sim G'(\mathbf{R})$ and $\pi \sim \pi'$. However in general we must expect that π itself will have to be modified. Thus the proper factor will not be $L(z - d/2, \pi \otimes \pi_f, r)$ but $L(z - d/2, \pi' \otimes \pi'_f, r)$, where $\pi' \otimes \pi'_f$ is an automorphic representation of a group G' obtained from G by an inner twisting. Again π'_v will have to be equivalent to π_v almost everywhere.

Since at the moment we are primarily interested in the infinite place, we simply ask whether it is possible to find a candidate for π' or, rather, for an L-packet $\{\pi'\} = \Pi'$. There are apparently two conditions to be satisfied, the first arising from the compatibility of functional equations.

(a) *Let $\pi \in \Pi_\varphi$ and let $\{\pi'\} = \Pi_{\varphi'}$. For any additive character ψ of $\mathbf{R}$ and any representation σ of ${}^LG = {}^LG'$,*

$$\epsilon'(z, \sigma \circ \varphi, \psi) = \epsilon(z, \sigma \circ \varphi, \psi) \frac{L(1 - z, \sigma \circ \varphi)}{L(z, \sigma \circ \varphi)}$$

is equal to

$$\epsilon'(z, \sigma \circ \varphi', \psi) = \epsilon(z, \sigma \circ \varphi', \psi) \frac{L(1 - z, \sigma \circ \varphi')}{L(z, \sigma \circ \varphi')}.$$

(b) *It is possible to find a summand $\psi_0(\pi')$ of $\psi_1(\pi')$ which is such that $\psi_2(\pi) = \alpha\psi_0(\pi'), \alpha \in \mathbf{Z}$.*

These conditions are only tentative, and may have to be modified in the course of time, but they will serve for the explanation of our problem.

The first condition involves only φ and φ' and we begin by constructing some pairs that satisfy it. Fix an element w of $W_{\mathbf{C}/\mathbf{R}}$ that projects to ι and satisfies $w^2 = -1$. We may suppose that $\varphi(w) = a \times \iota$ with a in the normalizer of ${}^LT^\circ$ in ${}^LG^\circ$. Then $\varphi(w)$ also normalizes ${}^LT^\circ$. Let $\varphi(\iota)$ denote the transformation of $X^*({}^LT^\circ)$ or of $X_*({}^LT^\circ)$ defined by $\varphi(w)$. We may also suppose that φ takes $\mathbf{C}^\times$ to ${}^LT^\circ$ and that $\varphi(z) = z^\Lambda \bar{z}^{\varphi(\iota)\Lambda}$ with $\Lambda \in X_*({}^LT^\circ) \otimes \mathbf{C}$ and $\Lambda - \varphi(\iota)\Lambda \in X_*({}^LT^\circ)$.

The representation φ' will be defined in a similar way. Thus $\varphi'(w) = a' \times \iota$ with a' in the normalizer of ${}^LT^\circ$, and $\varphi'(z) = z^\Lambda \bar{z}^{\varphi'(\iota)\Lambda}$. Notice that Λ is to be the same for φ' as for φ. However a, which is given, is replaced by a', which we must now define.

We suppose that $\varphi(\iota)$ sends every root to its negative, and choose λ in $X_*({}^LT^\circ)$ such that $\lambda^\vee(a) = e^{2\pi i\langle\lambda,\lambda^\vee\rangle}$ for any weight $\lambda^\vee$ of ${}^LT^\circ$ which is orthogonal to all roots. We shall take a' to lie in ${}^LT^\circ$ and to be such that $\lambda^v(a') = e^{2\pi i\langle\lambda',\lambda^\vee\rangle}$ when $\lambda^\vee$ is orthogonal to all roots. Here λ' is still to be defined. If we also denote by a the operator on $X_*({}^LT^\circ) \otimes \mathbf{C}$ defined by a and if we let q be one-half the sum of the positive roots then λ' is to be given by the equation

$$\lambda' = \frac{1+a}{2}\lambda - \frac{(1-a)(1+\varphi'(\iota))}{8}\Lambda + \frac{q}{2}.$$

Observe that the action of $\varphi'(\iota)$ is the same as that of ι.

We are assuming that φ is a given, well-defined homomorphism, and hence [**31**] that

$$\lambda + \varphi(\iota)\lambda \equiv \frac{\Lambda - \varphi(\iota)\Lambda}{2} - q(\operatorname{mod} X_*({}^LT^\circ)).$$

In order to show that φ' is also well defined we must verify that

$$\lambda' + \varphi'(\iota)\lambda' \equiv \frac{\Lambda - \varphi(\iota)\Lambda}{2}(\operatorname{mod} X_*({}^LT^\circ)). \tag{7.7}$$

We begin with the equations $a\varphi'(\iota) = \varphi'(\iota)a = \varphi(\iota)$ and $a(1+\varphi(\iota)) = 1+\varphi(\iota)$, remarking also that the square of both $\varphi(\iota)$ and $\varphi'(\iota)$ is the identity. We infer that the left side of (7.7) is equal to

$$\frac{(1+\varphi'(\iota))(1+\varphi(\iota))\lambda}{2} - \frac{(1-a)(1+\varphi'(\iota))}{4}\Lambda + q,$$

because $(1+\varphi'(\iota))q/2 = q$. The sum is in turn congruent to

$$\frac{1-\varphi(\iota)}{2}\Lambda - \frac{(1-a)(1+\varphi'(\iota))}{4}\Lambda = \frac{1-\varphi'(\iota)}{2}\Lambda.$$

Consequently the homomorphism φ' can be constructed whenever φ is defined and $\varphi(\iota)$ sends every root to its negative. The following lemma is valid in this generality.

Lemma. *For any representation σ of the Weil form of ^LG and any nontrivial character ψ of $\mathbf{R}$,*

$$\epsilon'(\tau, \sigma \circ \varphi, \psi) = \epsilon'(\tau, \sigma \circ \varphi', \psi).$$

The proof is a computation based on the proof of Lemma 3.2 of [**31**] and on Chapters 5 and 6 of [**24**], but it is rather lengthy, and not worth including here.

Observe that we could have started with φ', defined by an a' in $^LT^\circ$, and reversing the process, passed to λ and a. More generally if ^LM and $^LM'$ are two parabolic subgroups of ^LG, and $\varphi : W_{\mathbf{C}/\mathbf{R}} \to {}^LM$ has an image which lies in no proper parabolic subgroup of ^LM, then we can use the process to pass to a φ'' whose image lies in the minimal parabolic of ^LM, and thus of ^LG or $^LM'$, and afterwards reverse it to pass from φ'' to a $\varphi' : W_{\mathbf{C}/\mathbf{R}} \to {}^LM'$ whose image lies in no proper parabolic subgroup of $^LM'$.

My intention now is simply to show, by means of a few examples, how for a given π in some Π_φ one can choose one of the φ' just described so that the condition (b) is satisfied for the elements π' of $\Pi_{\varphi'}$. Of course the problem is to decide if such a choice is always possible. Without more examples or a general theorem, we cannot be at all confident that this is so.

If all the continuous cohomology of $\xi \otimes \pi$ is zero there is no difficulty satisfying (b). We take $\varphi' = \varphi$ and $\alpha = 0$. The simplest nontrivial example is obtained by taking ξ trivial and π trivial. Let $\pi \in \Pi_\varphi$. Then $\varphi(z) = z^q \bar{z}^{\varphi(\iota)q}$, with q equal again to one-half the sum of the positive roots. If ^LM is the parabolic subgroup of ^LG corresponding to the minimal parabolic of G over $\mathbf{R}$, then the image of φ lies in ^LM and $\varphi(\iota)$ takes every root of $^LT^\circ$ in ^LM to its negative. Define φ' as above, with $a' \in {}^LT^\circ$. G' can be taken to be the quasi-split form of G over $\mathbf{R}$. The continuous cohomology of π is all in even dimensions and all of type (p, p) for some p. To compute it one observes that it is the same as the cohomology of the compact dual, which can be computed by using Schubert cells. One verifies without difficulty that for $\pi' \in \Pi_{\varphi'}$ the representation $\psi_1(\pi')$, which depends in reality only on φ', is equivalent to $\psi_2(\pi)$.

If G is not quasi-split over $\mathbf{R}$ then φ' is different from φ. If G is not quasi-split over $\mathbf{R}$ then it is certainly not quasi-slit over $\mathbf{Q}$, and the trivial representation of $G(\mathbf{A})$ is anomalous. Once again we see that the passage from π to π' is the local expression of the passage from an anomalous representation to one which is not anomalous.

Other interesting examples are the representations $\pi = J_{i,j}$ of $PSU(n,1)$ discussed in Chapter XI of the notes of Borel-Wallach [**4**]. Take ξ trivial. In this case $\psi_2(\pi)$ is $(-1)^{i+j}$ times a representation induced from $\mathbf{C}^\times$, the representation of $\mathbf{C}^\times$ used having the weights

$$z^{-i}\bar{z}^{-j}, z^{-i-1}\bar{z}^{-j-1}, \cdots, z^{-(n-j)}\bar{z}^{-(n-i)}. \tag{7.8}$$

Here $0 \leqq i+j \leqq n-1$ and $0 \leqq i, j$. Borel and Wallach lapse into vagueness at one point, and it may be that the roles of i and j should be reversed, but that is of little consequence.

The group ${}^LG^\circ$ is SL$(n+1, \mathbf{C})$ and ${}^LT^\circ$ may be taken to be the group of diagonal matrices. The representation r° is the standard representation of SL$(n+1,\mathbf{C})$. It is easy enough to deduce from [**4**] that if $\pi \in \Pi_\varphi$ then

$$\varphi(z) = z^\Lambda \bar{z}^{\varphi(\iota)\Lambda}, \quad z \in \mathbf{C}^\times,$$

with Λ being equal to $(n/2-i, n/2, n/2-1, \cdots, n/2-i+1, n/2-i-1, \cdots, -n/2+j+1, -n/2+j-1, \cdots, -n/2, -n/2+j)$. The numbers occurring here are $n/2, n/2-1, \cdots, -n/2$, but the order is somewhat unusual. The transformation $\varphi(\iota)$ is given by

$$(x_1, \cdots, x_{n+1}) \longrightarrow (-x_{n+1}, -x_2, \cdots, -x_n, -x_1).$$

We are of course using the obvious representation of the elements of $X_*({}^LT^\circ) \otimes \mathbf{C}$ as sequences of $n+1$ complex numbers whose sum is 0.

Suppose, to be definite, that $i \leqq j$. We will choose φ' to be such that the transformation $\varphi'(\iota)$ takes $(x_1, \cdots, x_{n+1})$ to $(-x_{n+1}, -x_n, \cdots, -x_{n-i+1}, -x_{n-j}, \cdots, -x_{i+2}, -x_{n-j+1}, \cdots, -x_{n-i}, -x_{i+1}, \cdots, -x_1)$. The indices within the gaps decrease or increase regularly by one. If $\pi' \in \Pi_{\varphi'}$ then the representation $\psi_1(\pi')$ is induced from a representation of $\mathbf{C}^\times$ with weights

$$\begin{gathered} z^{-i}\bar{z}^{-j}, 1, z^{-1}\bar{z}^{-1}, \cdots, z^{-i+1}\bar{z}^{-j+1}; \\ z^{-i-1}\bar{z}^{-j-1}, \cdots, z^{-(n-j-1)}\bar{z}^{-(n-i-1)}; \\ z^{-(n-j+1)}\bar{z}^{-(j-1)}, \cdots, z^{-(n-i)}\bar{z}^{-i}; \\ z^{-(n-i+1)}\bar{z}^{-(n-i+1)}, \cdots, z^{-n}\bar{z}^{-n}, z^{-(n-j)}\bar{z}^{-(n-i)}. \end{gathered} \tag{7.9}$$

Happily the set (7.8) is a subset of (7.9) and the condition (b) is satisfied.

It should be observed that the representation $\psi_0(\pi')$ that is chosen to satisfy (b) will have to be, except for some degenerate values of i and j, a proper subrepresentation of $\psi_1(\pi')$. This phenomenon will, I hope, be taken into account by L-indistinguishability. For example if ϵ is the element of ${}^LT^\circ$ with diagonal entries

$$1,\ \overbrace{-1,-1,\cdots,-1}^{i}\ 1,\cdots,1,\ \overbrace{-1,\cdots,-1,}^{j}\ 1$$

then ϵ commutes with $\varphi'(W_{\mathbf{C}/\mathbf{R}})$ and $\psi_0(\pi')$ may be taken to be the restriction of $\psi_1(\pi')$ to the $+1$ eigenspace of $r(\epsilon)$.

References

1. J. Arthur, *Eisenstein series and the trace formula*, these PROCEEDINGS, part 1, pp. 253-274.
2. W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442-528.
3. A. Borel, *Automorphic L-functions*, PROCEEDINGS, part 2, pp. 27-61.
4. A. Borel and N. Wallach, *Seminar on the cohomology of discrete subgroups of semi-simple groups*, Institute for Advanced Study (1976/77).
5. P. Cartier, *Representations of $\mathfrak{p}$-adic groups: A survey*, these PROCEEDINGS, part 1, pp. 111-155.
6. W. Casselman, *The Hasse-Weil ξ-function of some moduli varieties of dimension greater than one*, these PROCEEDINGS, part 2, pp. 141-163.
7. ________ , $GL(n)$, Algebraic Number Fields (L-Functions and Galois Properties), A. Fröhlich, ed., Academic Press, New York, 1977.
8. I.V. Cherednik, *Algebraic curves uniformized by discrete arithmetical subgroups*, Uspehi Mat. Nauk **30** (1975), 183-184. (Russian)
9. P. Deligne, *Travaux de Griffiths*, Sém. Bourbaki (1970.
10. ________ , *Travaux de Shimura*, Sém. Bourbaki (1971).
11. ________ , *La conjecture de Weil pour les surfaces $K3$*, Invent. Math. **15** (1972), 206-226.
12. ________ , *Théorie de Hodge*, III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5-78.
13. ________ , Letter to Piatetski-Shapiro.
14. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular Functions of One Variable. II, Lecture Notes in Math., vol. 349, Springer, New York, 1972, pp. 143-316.
15. V.G. Drinfeld, *Coverings of p-adic symmetric spaces*, Functional Anal. Appl. **10** (1976), 29-40. (Russian)
16. J. Giraud, *Cohomologie non-abélienne*, Springer-Verlag, Berlin and New York, 1971.
17. Harish-Chandra, *Representations of semi-simple Lie groups*. VI, Amer. Math. J. **78** (1956), 564-628.
18. ________ , *Discrete series for semi-simple Lie groups*. I, Acta Math. **113** (1965), 241-318; II, **116** (1966), 1-111

19. R. Howe and I. Piatetski-Shapiro, *A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups*, these PROCEEDINGS, part 1, pp. 315-322.

20. Y. Ihara, *Hecke polynomials as congruence ξ-functions in elliptic modular case*, Ann. of Math. (2) **85** (1967), 267-295.

21. ________ , *On congruence monodromy problems*, vols. 1,2, Lecture notes from Tokyo Univ., 1968, 1969.

22. ________ , *Some fundamental groups in the arithmetic of algebraic curves over finite fields*, Proc. Nat. Acad. Sci. **72** (1975).

23. S. Gelbart and H. Jacquet, *Forms of $GL(2)$ from the analytic point of view*, these PROCEEDINGS, part 1, pp. 213-251.

24. H. Jacquet and R.P. Langlands, *Automorphic forms on $GL(2)$*, Lecture Notes in Math., vol. 114, Springer, New York, 1970.

25. H. Jacquet and J. Shalika, *A non-vanishing theorem for zeta-functions of $GL(n)$*, Invent. Math. **38** (1976).

26. D. Kazhdan, *On arithmetic varieties*, Lie Groups and Their Representations, Halsted, New York, 1975.

27. R. Kottwitz, Thesis, Harvard Univ., 1977.

28. ________ , *Combinatorics and Shimura varieties* mod p, these PROCEEDINGS, part 2, pp. 185-192.

29. N. Kurokawa, *Examples of eigenvalues of Hecke operators in Siegel modular forms*, 1977 (preprint).

30. R.P. Langlands, *Modular forms and ℓ-adic representations*, Modular Forms of One Variable. II, Lecture Notes in Math., vol. 349, Springer, New York, 1973, pp. 361-500.

31. ________ , *On the classification of irreducible representations of real algebraic groups*, Institute for Advanced Study, Princeton, N.J., 1973.

32. ________ , *Some contemporary problems with origins in the Jugendtraum*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976, pp. 401-418.

33. ________ , *Stable conjugacy: definitions and lemmas*, Canad. J. Math. (to appear).

34. ________ , *On the zeta-functions of some simple Shimura varieties*, Canad. J. Math. (to appear).

35. ________ , *On the notion of automorphic representation*, these PROCEEDINGS, part 1, pp. 203-207.

36. J. Milne, *Points on Shimura varieties* mod$p$, these PROCEEDINGS, part 2, pp. 165-184.

37. D. Mumford, *Families of abelian varieties*, Proc. Sympos. Pure Math., vol. 9., Amer. Math. Soc., Providence, R.I., 1966, pp. 347-351.

38. I. Piatetski-Shapiro, *Multiplicity one theorems*, these PROCEEDINGS, part 1, pp. 206-212.

39. H. Resnikoff and R.L. Saldaña, *Some properties of Fourier coefficients of Eisenstein series of degree two*, Crelle **265** (1974), 90-109.

40. N. Saavedra Rivano, *Catégories tannakiennes*, Lecture Notes in Math., vol. 265, Springer, New York, 1972.

41. J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, Benjamin, New York, 1968.

42. ________ , *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures), Sém. Delange-Pisot-Poitou, 1970.*

43. ________ , *Répresentations ℓ-adiques*, Algebraic Number Theory, Tokyo, 1977.

44. D. Shelstad, *Notes on L-indistinguishability*, these PROCEEDINGS, part 2, pp. 193-203.

45. ________ , *Orbital integrals and a family of groups attached to a real reductive group*, 1977 (preprint).

56. K.-Y. Shih, *Conjugations of arithmetic automorphic function fields*, 1977 (preprint).

47. J. Tate, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3-26.

48. N. Wallach, *Representations of reductive Lie groups*, these PROCEEDINGS, part 1, pp. 71-86.

49. A. Weil, *The field of definition of a variety*, Amer. J. Math. **78** (1956), 509-524.

50. S. Zucker, *Hodge theory with degenerating coefficients*. I, 1977 (preprint).

The Institute for Advanced Study