Modular Forms and ℓ-adic Representations*

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1. Introduction. This report is another attempt on the part of its author to come to terms with the circumstance that L-functions can be introduced not only in the context of automorphic forms, with which he has had some experience, but also in the context of diophantine geometry. That this circumstance can be the source of deep problems was, I believe, first perceived by E. Artin. He was, to be sure, concerned with forms on $GL(1)$ and with varieties of dimension 0. This remains the only case in which results of any profundity have been obtained. These have been hard won. Their mathematical germ is the theory of cyclotomic fields; itself easy only in comparison to the general theory.

One assertion of the theory, a simple statement of existence, is that an L-functions for $GL\big(1, \mathbf{A}(\mathbf{Q})\big)$ associated to a Grössencharakter χ trivial on the connected component of $GL(1, \mathbf{R})$, that is, a Dirichlet L-function, is an Artin L-function associated to a character χ' of the Galois group of a cyclotomic field. In order that such an assertion really have concrete reciprocity laws, such as the quadratic, as consequences, it must be supplemented by a procedure for obtaining χ from χ'. This is usually given locally, not merely at almost every place, but at every place.

There are similarities, as well as dissimilarities, between the role played by cyclotomic equations in the study of $GL(1)/\mathbf{Q}$ and that played by elliptic modular varieties in the study of $GL(2)/\mathbf{Q}$. According to the Eichler–Shimura theory, the non-trivial part of the zeta-function of each of the varieties $M_K \otimes \mathbf{Q}$ studied in a previous lecture can be expressed as a product of L-functions associated to automorphic forms on $GL\big(2, \mathbf{A}(\mathbf{Q})\big)$. Both Eichler and Shimura, as well as Igusa and Deligne, have contented themselves with results valid for, in one sense or another, almost all primes. This may represent the most important step. Nonetheless, for reasons explained at Nice and elsewhere, I want a result valid at all primes.

The zeta-functions and the automorphic forms can be brought together because of the close relation of both to certain cohomology groups. These groups are introduced, and evaluated in terms of automorphic forms, in §2. In §3 they are interpreted as groups in the étale theory. Combining the results of §2 and §3, we formulate in §4 the problem with which these lectures are concerned.

To explain this problem, which is only partially solved in the later paragraphs, we formulate it as the solution to an existence question. Suppose $\pi' = \otimes \pi'_v$ is a representation of $GL\big(2, \mathbf{A}(\mathbf{Q})\big)$ occurring in the space of automorphic forms. Let $W(\mathbf{C}/\mathbf{R})$ be the Weil group of $\mathbf{C}$ over $\mathbf{R}$. $W(\mathbf{C}/\mathbf{R})$ is an extension of the Galois group $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ by $\mathbf{C}^\times$. π'_∞ is associated to a two-dimensional complex representation σ_∞ of $W(\mathbf{C}/\mathbf{R})$. Suppose σ_∞ restricted to $\mathbf{C}^\times$ is equivalent to

$$z \to \begin{pmatrix} z^m \bar{z}^n & 0 \\ 0 & z^{m'} \bar{z}^{n'} \end{pmatrix}$$

with m, n, m', and n' in $\mathbf{Z}$. I would then expect that π' is associated to a diophantine object of some sort (a "motive"—a word with as yet no satisfactory definition!) which is (again in a sense not yet made precise) of rank two. In lieu of precise definitions, which would be premature, we can look for the objects themselves.

Since we are working over $\mathbf{Q}$, any such object should yield for each p and ℓ different from p a two-dimensional representation σ'_p of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ over some finite extension of $\mathbf{Q}_\ell$. We can twist both σ'_p and π'_p by a quasi-character of $\mathbf{Q}_p^\times$, at least if it takes values in $\mathbf{Q}^\times$. The twistings define L-functions and ε-factors. If the object is in any sense associated to π' these must be the same for σ'_p as for π'_p. The implications of this demand are discussed at greater length in §4.

In these lectures we take $m' = n$, $n' = m$, the most interesting case, and, an unfortunate but natural restriction, $m \neq n$; and for each ℓ associate to π' a two-dimensional ℓ-adic representation of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$. To obtain a first, and tentative, form of an existence theorem, we must establish the local relations described above.

As was observed this problem cannot yet be completely solved. That it can be treated when there is any ramification whatsoever is only possible because of the recent results of Deligne on the behavior of the modular varieties at primes where they do not have good reduction. Theorems 7.1 and 7.5 are the best we can do at present. To prove them, one has to turn the basic problem, which concerns two very abstract objects, into an elementary assertion. Since the ℓ-adic representations were introduced to study congruences, they contain a great deal of elementary information. To reveal it, all one has to do is unravel the definitions. This is done in §7. However it is a Grothendieck definition we have to unravel; and that is not so easy for ordinary mathematicians.

The most powerful tool available for the study of the representations of $GL\big(2, \mathbf{A}(\mathbf{Q})\big)$ on the space of automorhpic forms is the Selberg trace formula. We exploit it in §6. The elementary manipulations needed to compare the results of §6 and §7 and to prove the theorems are carried out in §5.

As I have just hinted, the analysis of the ℓ-adic cohomology required in §7 is far from trivial. To carry it out properly requires a command of an elaborate theory, which, when I began to prepare this report a year ago, was almost completely foreign to me, and which I have certainly not yet mastered. This has no doubt resulted in many obscurities and lacunae, but, I hope, no mistakes. I observe in particular that Proposition 7.12, which plays an important role in the discussion, is an extension of the available forms of the Lefschetz theorem which should have been, but was not, verified.

If this report is of any value, it will be because of all I have learned from friends in Bonn, Bures, and New Haven. Conversations and correspondence with Casselman, Deligne, Harder, Lang, and Rapoport have been decisive in its genesis and execution.

2. de Rham Cohomology. Let $\mathbf{A}$ be the adèle ring of $\mathbf{Q}$ and let $\mathbf{A}_f$ be the subring of $\mathbf{A}$ formed by the elements with coordinate 0 at ∞. Let $\mathbf{Z}_f$ be the ring of integral elements in $\mathbf{A}_f$. Finally let K be an open compact subgroup of $G(\mathbf{A}_f)$, where $G = GL(2)$, and let

$$K_\infty = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G(\mathbf{R}) \right\}.$$

The map

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{ai+b}{ci+d}$$

identifies $G(\mathbf{R})/K_\infty$ with the complex plane minus the real line.

Since

$$G(\mathbf{A})/K_\infty K = G(\mathbf{R})/K_\infty \times G(\mathbf{A}_f)/K$$

and the second factor is discrete, $G(\mathbf{A})/K_\infty K$ is also a complex manifold. $G(\mathbf{Q})$ acts on it to the left.

Lemma 2.1. *If K is amall enough, $G(\mathbf{Q})$ acts freely.*

If $\gamma \in G(\mathbf{Q})$ has a fixed point it is conjugate to an element of $K_\infty K$. Thus its eigenvalues λ, μ are complex conjugates and, together with their inverses, algebraic integers. Thus they are roots of unity. If, for example,

$$K \subseteq \{k \in G(\mathbf{Z}_f) \mid k \equiv 1 (\bmod 5)\}$$

then $\lambda + \mu = 2$ and $\gamma = 1$.

I always take K so small that $G(\mathbf{Q})$ acts freely. Then $G(\mathbf{Q}) \setminus G(\mathbf{A})/K_\infty K$ is a complex manifold $M_K^o(\mathbf{C})$.

Suppose we are given a rational representation μ of G which is defined over $\mathbf{Q}$ and which acts on the vector space L and thus, in particular, a representation of $G(\mathbf{Q})$ on $L(\mathbf{Q})$ or on $L(\mathbf{C})$.

$$L(\mathbf{C}) \times_{G(\mathbf{Q})} G(\mathbf{A})/K_\infty K$$

is a sheaf of complex vector spaces over $M_K^o(\mathbf{C})$, locally free of rank equal to the dimension of L. This sheaf will be denoted $F_\mu^K(\mathbf{C})$ or, when this is at all compatible with clarity, $F_\mu(\mathbf{C})$.

The de Rham cohomology groups

$$H^i\big(M_K^o, F_\mu(\mathbf{C})\big)$$

as well as the groups with compact support

$$H_c^i\big(M_K^o, F_\mu(\mathbf{C})\big)$$

have been studied by many people. We have to review the results of their efforts. We shall be especially concerned with the image

$$H_p^i\big(M_K, F_\mu(\mathbf{C})\big)$$

of

$$H_c^i\big(M_K^o, F_\mu(\mathbf{C})\big) \to H^i\big(M_K^o, F_\mu(\mathbf{C})\big).$$

By its very definition the sheaf $F_\mu(\mathbf{C})$ lifts to the trivial sheaf

$$L(\mathbf{C}) \times G(\mathbf{A})/K$$

over the manifold $G(\mathbf{A})/K$. Any q-form ω with values in $F_\mu(\mathbf{C})$ lifts to a form ω_0 on $G(\mathbf{A})/K$ with values in $L(\mathbf{C})$. if $g \in G(\mathbf{A})$ let g_∞ be its projection on $G(\mathbf{R})$ and define $\eta = \eta(\omega)$ by

$$\eta(g) = \mu^{-1}(g_\infty)\,\omega_0(g).$$

The forms η obtained in this way are characterized by:

(i) If $\gamma \in G(\mathbf{Q})$ and if L_γ denotes left translation of a form by γ,

$$L_\gamma \eta = \eta.$$

(ii) If $k \in K_\infty$ and if R_k denotes right translation by k,

$$R_k \eta = \mu(k^{-1})\eta.$$

(iii) If $X_1, \cdots, X_q$ belong to $\mathfrak{g}$, the Lie algebra of $G(\mathbf{R})$, and hence define left-invariant vector fields on $G(\mathbf{A})/K$ then

$$\eta(g) : (X_1, \cdots, X_q) \to 0$$

if one of $X_1, \cdots, X_q$ belongs to $\mathfrak{k}$, the Lie algebra of K_∞.

The boundary operator is easily expressed in terms of η. $d\omega$ corresponds to $d\eta$ where

$$d\eta(X_0, \cdots, X_q)$$

is defined to be the sum of

$$\sum_{i=0}^{q} (-1)^i \big(X_i + \mu(X_i)\big)\, \eta(X_0, \cdots, \hat{X}_i, \cdots, X_q)$$

and

$$\sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_q).$$

Suppose now for simplicity that μ is irreducible, so that μ restricted to Z, the centre of G, is of the form $z \to \mu(z)I$, where $\mu(z)$ is a scalar. Let $V(\mu, K)$ be the space of infinitely differentiable functions φ on $G(\mathbf{Q}) \setminus G(\mathbf{A})/K$ satisfying

$$\varphi(zg) = \mu^{-1}(z)\varphi(g)$$

for z in $Z^o(\mathbf{R})$, the connected component of $Z(\mathbf{R})$. The universal enveloping algebra $\mathcal{A}$ of $\mathfrak{g}$ and the group K_∞ both act on $V(\mu, K)$ by right translations. Call this action r. η may be regarded as a function on the q^{th} exterior power $\wedge^q(\mathfrak{g}/\mathfrak{k})$ with values in $L(\mathbf{C}) \otimes V(\mu, K)$. Condition (ii) may be written

$$\eta(kX_1, \cdots, kX_q) = (\mu \otimes r)(k)\big(\eta(X_1, \cdots, X_q)\big). \tag{1}$$

Of course, $X \to kX$ is the adjoint of k. The boundary $d\eta$ becomes the sum of

$$\sum_{i=0}^{q} \big(1 \otimes r(X_i) + \mu(X_i) \otimes 1\big)\, \eta(X_0, \cdots, \hat{X}_i, \cdots, X_q) \tag{2}$$

and

$$\sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_q). \tag{3}$$

These definitions admit of immediate extensions to any space U on which $\mathcal{A}$ and K_∞ act, in a consistent manner, according to a representation s. We introduce namely the space $C^q(U, \mu)$ of functions on $\Lambda^q(\mathfrak{g}/\mathfrak{k})$ with values in $L(\mathbf{C}) \otimes U$ which satisfy (1), with r replaced by s. Since

$$s(kX) = s(k)s(X)s(k^{-1})$$

if $X \in \mathfrak{g}$ and $k \in K_\infty$, the boundary $d\eta$ given by (2) and (3), except that r must again be replaced by s, lies in $C^{q+1}(U, \mu)$. Thus we can introduce the cohomology groups $H^q(U, \mu)$ or, if we want to stress the role of s rather than that of U, $H^q(s,\mu)$. These groups depend covariantly on U. We identify $H^q\big(M_K^o, F_\mu(\mathbf{C})\big)$ with $H^q\big(V(\mu,k),\mu\big)$. If $V_c(\mu,K)$ is the space of functions in $V(\mu,K)$ with compact support modulo $Z^o(\mathbf{R})$, we may also identify $H_c^q\big(M_K^o, F_\mu(\mathbf{C})\big)$ and $H_c^q\big(V_c(\mu,K),\mu\big)$.

Let $V_{sp}(\mu,K)$ be the space of all functions φ in $V(\mu,K)$ for which

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \varphi(ng)\,dn = 0$$

for all g in $G(\mathbf{A})$ and for which

$$|\mu(\det g)|^{1/2} X\varphi(g)$$

is square integrable on $Z^o(\mathbf{R})G(\mathbf{Q}) \setminus G(\mathbf{A})$ for all X in the universal enveloping algebra. N is the group of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Proposition 2.2. *The map*

$$H^1\big(V_{sp}(\mu,K),\mu\big) \to H^1\big(V(\mu,K),\mu\big)$$

is an injection and its image is $H_p^1\big(M_K, F_\mu(\mathbf{C})\big)$.

In order to prove this proposition we have to introduce a large, almost overwhelming, number of auxiliary spaces. $W(\mu,K)$ will be the space of infinitely differentiable functions on $B(\mathbf{Q}) \setminus G(\mathbf{A})/K$ which satisfy

$$\varphi(zg) = \mu^{-1}(z)\,\varphi(g), \qquad z \in Z^o(\mathbf{R}),$$

taken modulo functions which vanish on some set of the form

$$S(M,\Omega) = \Big\{ nak \,\Big|\, n \in N(\mathbf{A}),\ a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A(\mathbf{A}),\ \Big|\frac{\alpha}{\beta}\Big| \geq M,\ k \in \Omega \Big\}.$$

B is the group of super-triangular matrices and Ω is a compact set such that $\Omega K_\infty K = \Omega$. We demand moreover that $G(\mathbf{A}) = B(\mathbf{A})\Omega$. Given M there is an $N > M$ and an infinitely differentiable function ε_M such that

$$\varepsilon_M(hgk) \equiv \varepsilon_M(g)$$

if

$$h \in Z^o(\mathbf{R})N(\mathbf{A})B(\mathbf{Q})$$

and $k \in K$ and such that ε_M is 1 on $S(N,\Omega)$ and 0 off $S(M,\Omega)$. φ and $\varepsilon_M\varphi$ represent the same element of $W(\mu,K)$. In order to avoid clumsy expressions, we speak, taking all necessary care, of the elements of $W(\mu, K)$ as though they were themselves functions. $W_S(\mu, K)$ is the subspace of $W(\mu, K)$ formed by those functions φ for which $|\mu(\det g)|^{1/2} X\varphi(g)$ is square integrable on $Z^o(\mathbf{R})B(\mathbf{Q}) \setminus S(M, \Omega)$ for all X in the unversal enveloping algebra. This condition is satisfied for all pairs M, Ω if it is satisfied for one. $W_p(\mu, K)$ consists of those functions φ in $W(\mu, K)$ for which

$$\int_{N(\mathbf{Q})\setminus N(\mathbf{A})} \varphi(ng)\, dn$$

vanishes, as a function of g, on at least one $S(M, \Omega)$.

$$W_{sp}(\mu, K) = W_s(\mu, K) \cap W_p(\mu, K).$$

Finally we let $V_s(\mu, K)$ be the set of all φ in $V(\mu, K)$ for which $|\mu(\det g)|^{1/2} X\varphi(g)$ is square-integrable on $Z^o(\mathbf{R})G(\mathbf{Q}) \setminus G(\mathbf{A})$ for all X in the unversal enveloping algebra.

Lemma 2.3. *The two maps $V(\mu, K) \to W(\mu, K)$ and $V_s(\mu, K) \to W_s(\mu, K)$ are surjective with kernels $V_c(\mu, K)$.*

The two maps are of course obtained simply be regarding an element of the first space as an element of the second. One knows, from the standard reduction theory, that there exist an M and an Ω such that $\gamma \in G(\mathbf{Q})$, $\gamma S(M, \Omega) \cap S(M, \Omega)$ not empty imply $\gamma \in B(\mathbf{Q})$. Given φ in $W(\mu, K)$, set $\psi = \varepsilon_M \varphi$ and consider

$$\varphi_1(g) = \sum_{B(\mathbf{Q})\setminus G(\mathbf{Q})} \psi(\gamma g).$$

φ_1 lies in $V(\mu, K)$ and determines the same element of $W(\mu, K)$ as φ. Since the complement of the image of $S(M, \Omega)$ in $Z^o(\mathbf{R})G(\mathbf{Q}) \setminus G(\mathbf{A})$ is relatively compact, φ_1 lies in $V_s(\mu, K)$ if it lies in $W_s(\mu, K)$. The last assertion of the lemma follows from reduction theory.

The universal enveloping algebra and the group K_∞ act in a consistent way on the space $W(\mu, K)$ and its variants, so that the cohomology groups such as $H^q\big(W(\mu, K), \mu\big)$ are defined.

Lemma 2.4. *The groups $H^q\big(W_p(\mu, K), \mu\big)$ and $H^q\big(W_{sp}(\mu, K), \mu\big)$ are 0.*

$G(\mathbf{A})$ is a finite disjoint union

$$\cup_i G(\mathbf{Q}) g_i G^o(\mathbf{R}) K.$$

$G^o(\mathbf{R})$ is the connected component of $G(\mathbf{R})$. Let

$$\Gamma_i = G(\mathbf{Q}) \cap g_i G^o(\mathbf{R}) K g_i^{-1}.$$

The group $G(\mathbf{Q})$ is a disjoint union

$$\cup_j \Gamma_i h_{ij} B(\mathbf{Q}).$$

Let $\bar{h}_{ij}$ and $\bar{g}_i$ be the projections of h_{ij} and g_i on $G(\mathbf{R})$ and let

$$S^o(M,\Omega) = S(M,\Omega) \cap G^o(\mathbf{R}).$$

It is a consequence of the reduction theory that, if M is given and N is, for a given Ω, sufficiently large, the map

$$\cup_i \cup_j \Delta_{ij} \setminus S^o(N,\Omega)$$

into $B(\mathbf{Q}) \setminus S(M,\Omega)/K$ given by

$$\Delta_{ij} g \in \Delta_{ij} \setminus S^o(N,\Omega) \to h_{ij}^{-1} g_i \bar{g}_i^{-1} \bar{h}_{ij} g$$

is injective with an image which differs from $B(\mathbf{Q}) \setminus S(M,\Omega)/K$ by a relatively compact set, modulo $Z^o(\mathbf{R})B(\mathbf{Q})$. Here Δ_{ij} is the projection of

$$h_{ij}^{-1} \Gamma_i h_{ij} \cap B(\mathbf{Q}) \cap G^o(\mathbf{R}) G(\mathbf{A}_f)$$

on $G(\mathbf{R})$. Since the operators X in the universal enveloping algebra are left-invariant we can, when proving the lemma, pull back φ in $W(\mu,K)$ or $W_s(\mu,K)$ to a function on $\Delta_{ij} \setminus S^o(N,\Omega)$ and study it there. Thus $W(\mu,K)$ becomes a direct sum

$$\oplus_{i,j} W(i,j).$$

In the same way $W_s(\mu,K)$ is isomorphic to

$$\oplus_{i,j} W_s(i,j).$$

We study these spaces individually. For simplicity we suppress the i and j from the notations. In particular Δ_{ij} is now Δ, a discrete subgroup of $B(\mathbf{R}) \cap G^o(\mathbf{R})$ such that $\Delta \cap N(\mathbf{R}) \setminus N(\mathbf{R})$ is compact and $\Delta \cap N(\mathbf{R}) \setminus \Delta$ is finite. If A is the group of diagonal matrices and $A^o(\mathbf{R})$ is the connected component of $A(\mathbf{R})$,

$$G^o(\mathbf{R}) = N(\mathbf{R}) A^o(\mathbf{R}) K_\infty.$$

W is the space of infinitely differentiable functions φ on $\Delta \setminus G^o(\mathbf{R})$ satisfying

$$\varphi(zg) = \mu^{-1}(z)\varphi(g), \qquad z \in Z^o(\mathbf{R}),$$

taken modulo those which vanish on some set of the form

$$S(M)=\Big\{nak\,\Big|\,n\in N(\mathbf{R}),\ a=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\in A^o(\mathbf{R}),\ \frac{\alpha}{\beta}>M,\ k\in K_\infty\Big\}.$$

W_s is defined in a similar way.

Let $\Delta\cap N(\mathbf{R})$ be generated by

$$\begin{pmatrix}1 & N\\ 0 & 1\end{pmatrix}$$

and if φ belongs to $W(\mu)$ let

$$\varphi(n,a,k)=\varphi(nak)=\sum_{z\in\mathbf{Z}}\exp\Big(\frac{2\pi izx}{N}\Big)\varphi_z(a,k),$$

where

$$n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}.$$

$\varphi_z(a,k)$ is an infinitely differentiable function of a and k. We need to know what conditions to place on the φ_z in order that φ belong to W_s.

The matrices

$$U=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

span $\mathfrak{g}$ modulo $\mathfrak{k}$. Any X in the universal enveloping algebra may be written as a finite sum

$$\sum_{m,\ell}\mathrm{Ad}k^{-1}(U^m)\mathrm{Ad}k^{-1}(H^\ell)Y_{m,\ell}(k)$$

where each $Y_{m,\ell}(k)$ is some finite linear combination of elements of the universal enveloping of $\mathfrak{k}$ with coefficients which are infinitely differentable functions of k with bounded derivatives. Applying

$$\mathrm{Ad}k^{-1}(U^m)\mathrm{Ad}k^{-1}(H^\ell)$$

to φ we obtain

$$\Big(\frac{\alpha}{\beta}\Big)^m\sum_z\Big(\frac{2\pi iz}{N}\Big)^m\exp\Big(\frac{2\pi izx}{N}\Big)H^i\varphi_z(a,k).$$

H belongs to the Lie algebra of $A(\mathbf{R})$ and $H^\ell\varphi_z$ is to be interpreted accordingly. Integrating the square of the absolute value of this expression times $|\mu(\det g)|$ over $S(M)$ modulo $\big(\Delta\cap N(\mathbf{R})\big)Z^o(\mathbf{R})$ we obtain

$$\sum_z\Big(\frac{2\pi z}{N}\Big)^{2m}\int_{\frac{\alpha}{\beta}\ge M}\int_{Z^o(\mathbf{R})\backslash K_\infty}\Big(\frac{\alpha}{\beta}\Big)^{2m-1}|H^\ell\varphi_z(a,k)|^2\,|\mu(\det ak)|\,da\,dk.$$

To be more precise the first integral is taken over the subset of $Z^o(\mathbf{R}) \setminus A^o(\mathbf{R})$ represented by matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

satisfying the indicated condition. If φ lies in W_s this expression must be finite for all ℓ and m. If Δ contains a matrix

$$\begin{pmatrix} -1 & y \\ 0 & -1 \end{pmatrix}$$

then

$$\exp\Big(\frac{-2\pi iyz}{N}\Big)\varphi_z(a,-k) = \varphi_z(a,k) \tag{4}$$

for all z.

If the collection $\{\psi_z(a,k) \,\big|\, z \neq 0\}$ satisfies (4) and if there is a φ in W_s such that for each $\ell \geq 0$ there is a finite set of integers $\ell_j \geq 0$ and constants C_ℓ and r_ℓ such that

$$|H^\ell \psi_z(a,k)| \leq C_\ell \Big|\frac{\alpha}{\beta}\Big|^{r_\ell} \sum |H^{\ell_j}\varphi_z(a,k)|$$

for all $z \neq 0$ and $\big|\frac{\alpha}{\beta}\big| \geq 1$ then

$$\sum_{z \neq o} \exp\Big(\frac{2\pi izx}{N}\Big)\psi_z(a,k)$$

is also in W_s.

We can now begin the proof of Lemma 2.4. It is of course enough to prove that the groups $H^q(W_p,\mu)$ and $H^q(W_{sp},\mu)$ are 0. W_p consists of those φ in W for which $\varphi_0(a,k)$ is identically 0 and W_{sp} is the intersection of W_s and W_p.

We represent $\mathfrak{q}/\mathfrak{k}$ and $\mathfrak{b}/\mathfrak{z}$ where $\mathfrak{b}$ is the Lie algebra of $B(\mathbf{R})$ and $\mathfrak{z}$ that of $Z(\mathbf{R})$. It has a basis formed by U and H. An element of $C^q(W,\mu)$ is represented by a function on $S(M) \times \mathfrak{b}$ with values in $L(\mathbf{C})$, which for convenience we provide with an inner product. Because of (1) it is determined by its values on $N(\mathbf{R})A^o(\mathbf{R}) \cap S(M)$. Expand the function in a Fourier series

$$\eta(na, X_1, \cdots, X_q) = \sum_z \exp\Big(\frac{2\pi izx}{N}\Big)\eta_z(a, X_1, \cdots, X_q).$$

Then

$$\big(U + \mu(U)\big)\eta(na, X_1, \cdots, X_q)$$

is equal to

$$\sum \Big(\frac{\alpha}{\beta}\frac{2\pi iz}{N} + \mu(U)\Big)\exp\Big(\frac{2\pi izx}{N}\Big)\eta_z(a, X_1, \cdots, X_q) \tag{5}$$

and

$$\big(H+\mu(H)\big)\eta(na,X_1,\cdots,X_q)$$

is equal to

$$\sum \exp\Big(\frac{2\pi izx}{N}\Big)\big(H+\mu(H)\big)\eta_z(a,X_1,\cdots,X_q). \tag{6}$$

The operator

$$A_z=\frac{\alpha}{\beta}\frac{2\pi iz}{N}+\mu(U)$$

on $L(\mathbb{C})$ is invertible if $z\neq 0$ because $\mu(U)$ is nilpotent. Moreover for any $\ell\geq 0$ and any $\varepsilon>0$ there is a constant C_ℓ such that

$$\|H^\ell A_z^{-1}\|\leq\frac{C_\ell}{|z|}\Big|\frac{\beta}{\alpha}\Big|$$

for $\big|\frac{\alpha}{\beta}\big|\geq\varepsilon$. If

$$\sum_z \exp\Big(\frac{2\pi izx}{N}\Big)\varphi_z(a,k)$$

lies in W_s the form determined by

$$\nu(na,X_1,\cdots,X_q)=\Big\{\sum_{z\neq 0}\exp\Big(\frac{2\pi izx}{N}\Big)A_z^{-1}\varphi_z(a,1)\Big\}\lambda(X_1,\cdots,X_q),$$

where λ is a linear form on $\mathfrak{g}/\mathfrak{k}$, lies in $C^q(W_{sp},\mu)$.

We show now that $H^q(W_p,\mu)$ and $H^q(W_{sp},\mu)$ are 0. This will prove Lemma 2.4. It is clear from (5) that any cycle in $C^0(W_p,\mu)$, and *a fortiori* in $C^0(W_{sp},\mu)$, is 0. If η is a cycle in $C^1(W_p,\mu)$ set

$$\nu(na)=\sum_{z\neq 0}\exp\Big(\frac{2\pi izx}{N}\Big)A_z^{-1}\eta_z(a,U).$$

If η belongs to $C^1(W_{sp},\mu)$ then the element of $C^0(W_p,\mu)$ corresponding to ν lies in $C^0(W_{sp},\mu)$. Replacing η by $\eta-d\nu$ if necessary, we may suppose $\eta(U)=0$. Since

$$[U,H]=-2U$$

we see, upon computing $d\eta(U,H)$, that $\eta(H)$ is also 0. Finally any form in $C^2(W_p,\mu)$ or $C^2(W_{sp},\mu)$ is clearly the boundary of a form ν which vanishes on U.

Corollary 2.5. *The image of $H^q\big(V_{sp}(\mu,K),\mu\big)$ in $H^q\big(V_s(\mu,K),\mu\big)$ is contained in the image of $H^q\big(V_c(\mu,K),\mu\big)$.*

Given a cycle η in $C^q\big(V_{sp}(\mu,K),\mu\big)$ choose, by the previous two lemmas, a form ν in $C^{q-1}\big(V_s(\mu,K),\mu\big)$ such that $\eta - d\nu$ is 0 in $C^q\big(W_s(\mu,K),\mu\big)$. By Lemma 2.3, $\eta - d\nu$ lies in $C^q\big(V_c(\mu,K),\mu\big)$.

Proposition 2.2 itself will follow from the two lemmas below.

Lemma 2.6. *The map*

$$H^1\big(V_{sp}(\mu,K),\mu\big) \to H^1\big(V_s(\mu,K),\mu\big)$$

is surjective.

Lemma 2.7. *The maps*

$$H^q\big(V_{sp}(\mu,K),\mu\big) \to H^q\big(V(\mu,K),\mu\big)$$

are injective.

Let $W_e(\mu,K)$ be the image in $W(\mu,K)$ of the infinitely differentiable functions on the set $N(\mathbf{A})B(\mathbf{Q}) \setminus G(\mathbf{A})/K$. Set

$$W'(\mu,K) = W_e(\mu,K) + W_{sp}(\mu,K)$$

and let $V'(\mu,K)$ be the inverse image in $V(\mu,K)$ of $W'(\mu,K)$. Since

$$V'(\mu,K) \setminus V(\mu,K) \simeq W_{sp}(\mu,K) \setminus W_p(\mu,K),$$

the maps

$$H^q\big(V'(\mu,K),\mu\big) \to H^q\big(V(\mu,K),\mu\big)$$

are isomorphisms. We shall prove Lemma 2.7 with $V(\mu,K)$ replaced by $V'(\mu,K)$.

There are some standard facts in representation theory that need to be recalled. If s is a representation of $G(\mathbf{R})$ on a Hilbert space and X belongs to the Lie algebra of $G(\mathbf{R})$, a vector u is said to belong to the domain of $s(X)$ if

$$s(X)u = \lim_{t\to 0} \frac{s(\exp tX)u - u}{t}$$

exists in U. In general u is said to be in the domain of $s(X_n \cdots X_1)$ if u is in the domain of $s(X_1)$, $s(X_1)u$ is in the domain of $s(X_2)$, and so on. u is infinitely differentiable if it is in the domain of every

$s(X_1 \cdots X_n)$. The space of such vectors will be denoted U^∞. If s is unitary and u and v belong to the domain of $s(X)$, then

$$\big(s(X)u, v\big) = -\big(u, s(X)v\big).$$

If $U' \subseteq U^\infty$ is a dense $G(\mathbf{R})$-invariant subspace of U and if $u \in U$ is such that

$$v \in U^\infty \to (X_1 \cdots X_n v, u)$$

extends to a continuous function on U for all $X_1, \cdots, X_n$ then $u \in U^\infty$.

Let $L_s(\mu, K)$ be the space of all measurable functions φ on $G(\mathbf{Q}) \setminus G(\mathbf{A})/K$ such that

$$\varphi(zg) = \mu^{-1}(z)\varphi(g), \qquad z \in Z^o(\mathbf{R}),$$

and such that

$$|\mu(\det g)|^{1/2}\varphi(g)$$

is square integrable on $Z^o(\mathbf{R})G(\mathbf{Q}) \setminus G(\mathbf{A})$. The representation r of $G(\mathbf{R})$ on $L_s(\mu, K)$ by right translation is not unitary but r_1 is, where $r_1(h)$ takes φ to φ' with

$$\varphi'(g) = |\mu(\det h)|^{1/2}\varphi(gh).$$

Since r and r_1 have the same infinitely differentiable vectors, it follows from Sobolev's lemma that

$$L_s^\infty(\mu, K) = V_s(\mu, K).$$

If $L_{sp}(\mu, K)$ is the subspace of $L_s(\mu, K)$ consisting of those φ for which

$$\int_{N(\mathbf{Q})\setminus N(\mathbf{A})} \varphi(ng)\, dn = 0$$

for almost all g, then $L_{sp}(\mu, K)$ is closed and invariant and

$$L_{sp}^\infty(\mu, K) = V_{sp}(\mu, K).$$

Let $L_{se}(\mu, K)$ be the orthogonal complement of $L_{sp}(\mu, K)$ in $L_s(\mu, K)$. Then

$$V_s(\mu, K) = V_{se}(\mu, K) \oplus V_{sp}(\mu, K)$$

if

$$V_{se}(\mu, K) = L_{se}^\infty(\mu, K).$$

If φ belongs to $V'(\mu, K)$ write

$$\varphi = \varphi_1 + \varphi_2$$

where φ_1 maps to $W_{sp}(\mu, K)$ and φ_2 has support in some $G(\mathbf{Q})S(M,\Omega)$ and in $S(M,\Omega)$ satisfies

$$\varphi_2(ng) = \varphi_2(g), \qquad n \in N(\mathbf{A}).$$

Of course φ_1 and φ_2 are both to be functions on $G(\mathbf{Q})\setminus G(\mathbf{A})/K$. The possibility of such a construction is assured by Lemma 2.3 If ψ belongs to $L_{sp}(\mu, K)$ then, as long as M is, for a given Ω, sufficiently large

$$(\psi, \varphi_1) = \int_{Z^o(\mathbf{R})G(\mathbf{Q})\backslash G(\mathbf{A})} |\mu(\det g)|\psi(g)\overline{\varphi_1(g)}\, dg$$

is determined by φ alone. There exists a unique φ_p in the space $L_{sp}(\mu, K)$ such that

$$(\psi, \varphi_1) = (\psi, \varphi_p).$$

If $X_1, \cdots, X_n$ belongs to $\mathfrak{q}$ and if

$$\varphi' = r(X_1) \cdots r(X_n)\varphi$$

then φ'_p is equal to $r(X_1)\cdots r(X_n)\varphi_p$ in the sense of distributions; so φ_p lies in $V_{sp}(\mu, K)$. If

$$V_e(\mu, K) = \{\varphi \in V'(\mu, K) \,\big|\, \varphi_p = 0\},$$

then

$$V'(\mu, K) = V_e(\mu, K) \oplus V_{sp}(\mu, K).$$

Lemma 2.7 follows immediately.

To prove Lemma 2.6 we show that

$$H^1\big(V_{se}(\mu, K), \mu\big) = 0.$$

This requires a little more preparation. If s is a representations of G on a Hilbert space U, we set

$$H^q(s, \mu) = H^q(U, \mu) = H^q(U^\infty, \mu).$$

Lemma 2.8. (a) *Suppose the representation*

$$g \to |\mu(\det g)|^{1/2} s(g)$$

is unitary and let ω *be the Casimir operator of* g*. If*

$$s(\omega) - \mu(\omega)$$

has a bounded inverse, then

$$H^q(s,\mu) = 0$$

for all q*.*

(b) *Let* λ^q *be the representation of* K_∞ *on the* q^{th} *exterior power of* $\mathfrak{g}/\mathfrak{k}$ *and let* $\tilde{\mu}$ *be the contragredient of* μ*. If*

$$\mathrm{Hom}_{K_\infty}(\lambda^q \otimes \tilde{\mu}, s) = 0$$

then

$$H^q(s,\mu) = 0.$$

The second part of the lemma is an immediately consequence of the definition, because

$$C^q(U^\infty, \mu) = \mathrm{Hom}_{K_\infty}\big(\Lambda^q \mathfrak{g}/\mathfrak{k}, L(\mathbf{C} \otimes U^\infty)\big)$$

and the right-hand side is isomorphic to

$$\mathrm{Hom}_{K_\infty}(\Lambda^q \mathfrak{g}/\mathfrak{k} \otimes \mathrm{Hom}_{\mathbf{C}}\big(L(\mathbf{C}), \mathbf{C}), U^\infty\big).$$

To prove the first part, we recall a formula of Kuga, or at least the simple case of it we need. As usual,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal with respect to a non-degenerate $G(\mathbf{R})$-invariant form whose restriction to the derived algebra $\mathfrak{g}'$ is the Killing form. Introduce an inner product on $L(\mathbf{C})$ with respect to which $\mu(X)$, $X \in \mathfrak{m}$, is hermitian while $\mu(X)$, $X \in \mathfrak{k} \cap \mathfrak{g}'$ is skew-hermitian. If $X \in \mathfrak{g}'$ the operator $s(X)$ is skew-hermitian. The inner products on $L(\mathbf{C})$ and U yield one on $L(\mathbf{C}) \otimes U^\infty$. Define one on $C^q(U^\infty, \mu)$ by

$$(\eta, \nu) = \sum_i \big(\eta(Y_i), \nu(Y_i)\big)$$

where (Y_i) is an orthonormal basis, with respect to the Killing form, of $\Lambda^q \mathfrak{g}/\mathfrak{k} \simeq \Lambda^q \mathfrak{m}$. There is an operator δ of degree -1 on the complex $C^q(U^\infty, \mu)$ such that

$$(d\eta, \nu) = (\eta, \delta\nu).$$

Set

$$\Delta = d\delta + \delta d.$$

If ω is the Casimir operator then

$$-\Delta = s(\omega) \otimes 1 - 1 \otimes \mu(\omega) = s(\omega) - \mu(\omega).$$

Since μ is assumed irreducible, $\mu(\omega)$ is a scalar.

There is perhaps no harm in verifying this formula for the group under consideration. Let Y_1 and Y_2 be an orthonormal basis of $\mathfrak{m}$. Then $Y_1 \wedge Y_2$ is an orthonormal basis of $\Lambda^2\mathfrak{m}$ and $[Y_1, Y_2]$ lies in $\mathfrak{k}$. If η is a 0-form

$$\big((d\eta), \nu\big) = \big(\{s(Y_1) + \mu(Y_1)\}\eta, \nu(Y_1)\big) + \big(\{s(Y_2) + \mu(Y_2)\}\eta, \nu(Y_2)\big).$$

So

$$\delta\nu = -\big\{\big(s(Y_1) - \mu(Y_1)\big)\nu(Y_1) + \big(s(Y_2) - \mu(Y_2)\big)\nu(Y_2)\big\}.$$

Here $s(Y_i) - \mu(Y_i)$ is an abbreviation for

$$s(Y_1) \otimes 1 - 1 \otimes \mu(Y_i).$$

If η is a 1-form,

$$(d\eta, \nu) = \big(\{s(Y_1) + \mu(Y_1)\}\eta(Y_2) - \{s(Y_2) + \mu(Y_2)\}\eta(Y_1), \nu(Y_1, Y_2)\big)$$

so that $\delta\nu$ takes

$$Y_1 \to \{s(Y_2) - \mu(Y_2)\}\nu(Y_1, Y_2)$$
$$Y_2 \to -\{s(Y_1) - \mu(Y_1)\}\nu(Y_1, Y_2).$$

Let Z be an element of $\mathfrak{k} \cap \mathfrak{g}'$ on which the Killing form takes the value -1. Then

$$\omega = Y_1^2 + Y_2^2 - Z^2.$$

If η is a 0-form then

$$s(Z)\eta + \mu(Z)\eta = 0$$

and

$$\{s(\omega) - \mu(\omega)\}\eta = \sum_{i=1}^{2}\{s(Y_i^2) - \mu(Y_1^2)\}\eta.$$

The formulas above show that the right-hand side is

$$-\delta d\eta = -(d\delta + \delta d)\eta.$$

Since the value of the Killing form on

$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$$

is $4(x^2 + yz)$ we may take

$$Y_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix},$$

so that

$$[Y_1, Y_2] = Z \quad [Z, Y_1] = -Y_2 \quad [Z, Y_2] = Y_1.$$

If η is a 1-form, condition (1) implies that

$$\{s(Z) + \mu(Z)\}\eta(Y_1) = -\eta(Y_2)$$
$$\{s(Z) + \mu(Z)\}\eta(Y_2) = \eta(Y_1).$$

Now $\delta d\eta$ takes Y_1 to

$$\{s(Y_2) - \mu(Y_2)\}\big(\{s(Y_1) + \mu(Y_1)\}\eta(Y_2) - \{s(Y_2) + \mu(Y_2)\}\eta(Y_1)\big)$$

and $d\delta\eta$ takes Y_1 to

$$-\{s(Y_1) + \mu(Y_1)\}\big(\{s(Y_1) - \mu(Y_1)\}\eta(Y_1) + \{s(Y_2) - \mu(Y_2)\}\eta(Y_2)\big).$$

Adding, we obtain

$$-\sum_{i=1}^{2}\{s(Y_i^2) - \mu(Y_i^2)\}\eta(Y_1)$$

plus

$$\{s([Y_2, Y_1]) - \mu([Y_2, Y_1])\}\eta(Y_2) = \{s(Z) - \mu(Z)\}\{s(Z) + \mu(Z)\}\eta(Y_1)$$

which is simply

$$-s(\omega)\eta(Y_1) + \mu(\omega)\eta(Y_1).$$

The complete verification of the formula proceeds along the same lines.

Returning to the lemma, we observe that $\big(s(\omega)-\mu(\omega)\big)^{-1}$ commutes with $s(g)$, $g\in G(\mathbf{R})$, so that it takes U^∞ to U^∞. If η belongs to $C^q(U^\infty,\mu)$ and $d\eta=0$, set

$$\nu=\big(s(\omega)-\mu(\omega)\big)^{-1}\eta.$$

Since $d\Delta=d\Delta$, we have

$$d\delta d\nu=\Delta d\nu=0.$$

Thus

$$(\delta d\nu,\delta d\nu)=(d\delta\, d\nu,d\nu)=0$$

and both $\delta d\nu$ and $d\nu$ are 0. Consequently $\eta=d\delta\nu$ bounds.

In order to apply Lemma 2.8 we must recall the structure of the space $L_{se}(\mu,K)$. Suppose χ is a quasi-character of $A(\mathbf{Q})\setminus A(\mathbf{A})$ such that

$$\chi(z)=\mu^{-1}(z),\qquad z\in Z^o(\mathbf{R}).$$

Suppose in fact that

$$a=\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\to|\alpha\beta|^{1/2}\chi(a)$$

is a character. Consider the space $B(\chi,K)$ of all measurable functions φ on $G(\mathbf{A})/K$, satisfying

$$\varphi(nag)=\left|\frac{\alpha}{\beta}\right|^{1/2}\chi(a)\varphi(g)$$

for n in $N(\mathbf{A})$ and a in $A(\mathbf{A})$, for which

$$\int_{B(\mathbf{A})\setminus G(\mathbf{A})}|\mu(\det g)|\,|\varphi(g)|^2\rho^{-1}(g)\,d\bar g.$$

Here ρ is a positive function on $G(\mathbf{A})$ such that

$$\rho(nagu)=\left|\frac{\alpha}{\beta}\right|\rho(g)$$

if $n\in N(\mathbf{A}$, $a\in A(\mathbb{A})$, and $u\in K_\infty K$. Moreover $d\bar g$ is the measure associated to ρ as in Theorem 1.1 of [2.2]. The representation $\pi(\chi)$ of $G(\mathbf{R})$ on $B(\chi,K)$ by right translations is unitary. If

$$\chi:\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}\to\alpha^{is}$$

for $\alpha>0$ in $\mathbf{R}$ then

$$\pi(\omega,\chi)=-\frac{s^2+1}{4}.$$

Of course $\mu(\omega)$ is positive or 0; so $\pi(\omega,\chi)-\mu(\omega)$ is a real number less than or equal to $-\frac{1}{4}$.

If φ is an infinitely differentiable function on $N(\mathbf{A})B(\mathbf{A})\backslash G(\mathbf{A})/K$ with compact support modulo $N(\mathbf{A})B(\mathbf{Q})Z^o(\mathbf{R})$ satisfying $\varphi(zg)=\mu^{-1}(z)\varphi(g)$ for $z\in Z(\mathbf{R})$, then

$$\varphi(g,\chi)=\int_{A(\mathbf{Q})Z^o(\mathbf{R})\backslash A(\mathbf{A})}\left|\frac{\alpha}{\beta}\right|^{-\frac{1}{2}}\chi^{-1}(a)\varphi(ag)\,da$$

is a function on $B(\chi,K)$. If $\mathfrak{A}$ is the set of quasi-characters of the form indicated, $\mathfrak{A}$ carries a measure, the translation of the Haar measure, and

$$\int_{\mathfrak{A}}\|\varphi(g,\chi)\|^2\,d\chi<\infty.$$

There is thus a unique direct integral

$$\int_{\mathfrak{A}}B(\chi,K)\,d\chi \tag{7}$$

containing these functions. If

$$s(g)=\int_{\mathfrak{A}}\pi(g,\chi)\,d\chi$$

then $s(\omega)-\mu(\omega)$ is clearly invertible.

From the theory of Eisenstein series one knows that $L_{se}(\mu,K)$ is a direct sum of $G(\mathbf{R})$-invariant subspaces

$$L_{se}(\mu,K)=L^0_{se}(\mu;K)\oplus L^1_{se}(\mu,K)$$

the second of which is $G(\mathbf{R})$-isomorphic to a subspace of (7). The first is a direct sum, over the quasi-characters of the idèle-class group such that $\nu(a)=\mu^{-1}(a)$ if $a>0$ lies in $\mathbf{R}$ and $\nu(\mathrm{det}k)=1$ if $k\in K$, of one-dimensional spaces on which $G(\mathbf{R})$ acts according to $g\to\nu(\mathrm{det}g)$. $H^1\big(L^1_{se}(\mu,K),\mu\big)=0$ because of Lemma 2.8(a) and $H^1\big(L^0_{se}(\mu,K),\mu\big)=0$ because of Lemma 2.8(b). Lemma 2.6 now follows.

Lemma 2.9. *If the representation s of $G(\mathbf{R})$ on U is such that*

$$g\to|\mu(\mathrm{det}g)|^{1/2}s(g)$$

is unitary and if

$$s(\omega)-\mu(\omega)=0$$

then

$$H^q(U,\mu)\simeq\mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{g}/\mathfrak{k}\otimes\mathrm{Hom}_{\mathbf{C}}\big(L(\mathbf{C}),\mathbf{C}\big),U^\infty).$$

The right side is isomorphic to $C^q(U^\infty, \mu)$. If η belongs to $C^q(U^\infty, \mu)$ then

$$0 = (\Delta\eta, \eta) = (d\eta, d\eta) + (\delta\eta, \delta\eta).$$

In particular $d\eta = 0$, so the boundary operator is trivial.

For a given μ there are exactly three irreducible admissible representations of $G(\mathbb{R})$, satisfying $\pi(z) = \mu^{-1}(z)$ for $z \in Z^o(\mathbf{R})$, for which $\pi(\omega) = \mu(\omega)$. There are (i) $\tilde{\mu}$, the contragredient of μ, (ii) $g \to \operatorname{sgn}(\det g)\tilde{\mu}(g)$, (iii) an infinite-dimensional representation $\pi = \pi(\mu)$ such that $g \to |\mu(\deg g)|^{1/2}\pi(g)$ is unitary. If

$$\operatorname{trace} \mu \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \frac{\alpha^n \beta^m - \alpha^m \beta^n}{\alpha - \beta}, \qquad n > m,$$

then $g \to |\det g|^{-1/2}\pi(g)$ is, in the notation of Chapter 12 of Jacquet–Langlands, the representation $\pi(\sigma)$, if σ is the two-dimensional representation of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ induced from the representation $z \to z^{-n}\bar{z}^{-m}$ of $\mathbb{C}^*$. It follows from the previous lemma and the known structure of $\pi(\mu)$ that $H^1\big(\pi(\mu), \mu\big)$ is two dimensional.

We define $L_s(\mu)$ and $L_{sp}(\mu)$ in the same way as we define $L_s(\mu, K)$ and $L_{sp}(\mu, K)$ except that we drop the condition of K-invariance. $G(\mathbb{A})$ acts on $L_{sp}(\mu)$ by right translations. This representation r is a direct sum of irreducible representations

$$\oplus_\pi V^\pi. \tag{8}$$

Each π is a tensor product

$$\pi = \pi_\infty \otimes \pi_f = \pi_\infty \otimes (\otimes_p \pi_p).$$

We write accordingly

$$V^\pi = V^\pi_\infty \otimes V^\pi_f.$$

If $V^\pi_f(K)$ is the space of K-invariant vectors in V^π_f, then $V^\pi_f(K)$ is finite dimensional and

$$L_{sp}(\mu, K) = \oplus_\pi V^\pi_\infty \otimes V^\pi_f(K).$$

Each π_∞ is infinite dimensional and the set $\{\pi_\infty(\omega) \mid \pi \in A,\ V^\pi_f(K) \neq 0\}$ is discrete. A is the set of π occurring in the sum (8). If $A' = \{\pi \in A \mid \pi_\infty \neq \pi(\mu)\}$ and $A'' = \{\pi \in A \mid \pi_\infty = \pi(\mu)\}$, set

$$U^1 = \oplus_{\pi \in A'} V^\pi_\infty \otimes V^\pi_f(K)$$

$$U^2 = \oplus_{\pi \in A''} V^\pi_\infty \otimes V^\pi_f(K).$$

Then

$$H^q\big(V_{sp}(\mu,K)\big) \simeq H^q(U^1,\mu) \oplus H^1(U^2,\mu).$$

By Lemma 2.8 the first of these groups is 0. Because the set of π in A'' for which $V_f^\pi(K) \neq 0$ is finite the second is

$$\oplus_{\pi_\infty} \simeq_{\pi(\mu)} H^q(\pi_\infty,\mu) \otimes V_f^\pi(K).$$

Theorem 2.10. *The group $H_p^1\big(M_K, F_\mu(\mathbf{C})\big)$ is isomorphic to the direct sum over those π occurring in the representation of $G(\mathbf{A})$ on $L_{sp}(\mu)$ for which $\pi_\infty = \pi(\mu)$ of*

$$H^1(\pi_\infty,\mu) \otimes V_f^\pi(K).$$

Moreover for these π, the group $H^1(\pi_\infty,\mu)$ is two dimensional.

Suppose $g \in G(\mathbf{A}_f)$ K' is another open compact subgroup of $G(\mathbf{A}_f)$, and $g^{-1}K'g \subseteq K$. The map $h \to hg$ of $G(\mathbf{A})$ to itself factors to a map

$$R(g) : M^o_{K'}(\mathbf{C}) \to M^o_K(\mathbf{C}).$$

We map the inverse image $R^*(g)F_\mu^K(\mathbf{C})$ isomorphically to $F_\mu^{K'}(\mathbf{C})$ by sending a point $h \times (v \times hg)$ in a fibre of the first to $v \times h$ in a fibre of the second. Thus g defines maps, all denoted by $R(g)$,

$$H^q\big(M^o_K, F_\mu(\mathbf{C})\big) \to H^q\big(M^o_{K'}, F_\mu(\mathbf{C})\big)$$
$$H^q_c\big(M^o_K, F_\mu(\mathbf{C})\big) \to H^q_c\big(M^o_{K'}, F_\mu(\mathbf{C})\big)$$

and, most importantly for us,

$$H^q_p\big(M_K, F_\mu(\mathbf{C})\big) \to H^q_p\big(M_{K'}, F_\mu(\mathbf{C})\big).$$

The corresponding maps

$$H^q\big(V(\mu,K),\mu\big) \to H^q\big(V(\mu,K'),\mu\big)$$
$$H^q\big(V_c(\mu,K),\mu\big) \to H^q\big(V_c(\mu,K'),\mu\big)$$
$$H^1\big(V_{sp}(\mu,K),\mu\big) \to H^1\big(V_{sp}(\mu,K'),\mu\big)$$

are simply those yielded by right translation by g, which takes, for example, $V(\mu,K)$ to $V(\mu,K')$.

The map $R(g) : M_{K'}(\mathbf{C}) \to M_K(\mathbf{C})$ is a local homeomorphism and the inverse image of every point is finite. In fact the inverse image of a point represented by h is represented by $A(h) = \{h' \mid h'g =$

$hk,\ k \in B\}$ if B is a set of coset representatives for $K/g^{-1}K'g$. The fibre of the direct image $R_*(g)F_\mu^{K'}(\mathbf{C})$ at the point corresponding to h is

$$\oplus_{h' \in A(h)} L(\mathbf{C}) \times h'.$$

We map it to the fibre of $F_\mu^K(\mathbf{C})$ at h by sending

$$\oplus v(h') \times h' \to \Big(\sum v(h')\Big) \times h.$$

This yields a map

$$R_*(g)F_\mu^{K'}(\mathbf{C}) \to F_\mu^K(\mathbf{C}) \tag{9}$$

and hence mappings $R(g)$ on the cohomology groups. In particular it yields

$$R(g) : H_p^1\big(M_{K'}, F_\mu(\mathbf{C})\big) \to H_p^1\big(M_K, F_\mu(\mathbf{C})\big).$$

This corresponds to the map

$$H^1\big(V_{sp}(\mu, K'), \mu\big) \to H^1\big(V_{sp}(\mu, K), \mu\big)$$

determined by the linear transformation $V_{sp}(\mu, K') \to V_{sp}(\mu, K)$ which sends φ' to φ with

$$\varphi(h) = \sum_{K/g^{-1}K'g} \varphi(hkg^{-1}).$$

I remark that starting from the isomorphism $F_\mu^{K'}(\mathbf{C}) \to R^*(g)F_\mu^K(\mathbf{C})$ we obtain by functoriality

$$R_*(g)F_\mu^{K'}(\mathbf{C}) \to R_*(g)R^*(g)F_\mu^K(\mathbf{C}). \tag{10}$$

The fibre of $R_*(g)R^*(g)F_\mu^K(\mathbf{C})$ at x in $M_K(\mathbf{C})$ is the direct sum

$$\otimes_{x' \to x} F_\mu^K(\mathbf{C})_x,$$

if $F_\mu^K(\mathbf{C})_x$ is the fibre of $F_\mu^K(\mathbf{C})$ at x. The map $\oplus v(x') \to \sum v(x')$ on fibres yields a map, the trace,

$$R_*(g)R^*(g)F_\mu^K(\mathbb{C}) \to F_\mu^K(\mathbb{C}). \tag{11}$$

The composition of (10) and (11) gives us (9).

Given g we take $K' = K \cap gKg^{-1}$ and let

$$T(g) : H_p^1\big(M_K, F_\mu(\mathbb{C})\big) \to H_p^1\big(M_K, F_\mu(\mathbb{C})\big)$$

be $R(1)R(g)$. In terms of $H^1\big(V_{sp}(\mu, K), K\big)$ it is determined by the linear transformation $T(g) : \varphi \to \varphi'$ with

$$\varphi'(h) = \sum_{K/K'} \varphi(hkg)$$

of $V_{sp}(\mu, K)$.

If f_g is the characteristic function of KgK divided by $\mathrm{meas}(K)$ then $T(g)$ is simply the restriction to $V_{sp}(\mu, K)$ of

$$r(f) = \int_{G(\mathbb{A}_f)} f_g(h) r(h)\, dh$$

which acts on $L_{sp}(\mu)$. Thus the algebra generated over $\mathbf{C}$ by the operators $T(g)$ consists of the restrictions to $V_{sp}(\mu, K)$ of the operators $r(f)$, where $f \in \mathcal{H}_{\mathbf{C}}(K)$, the algebra of compactly supported functions on $G(\mathbf{A}_f)$ biinvariant under K. The corresponding action of $\mathcal{H}_{\mathbf{C}}(K)$ on

$$H^1\big(V_{sp}(\mu, K), \mu\big) \simeq \oplus H^1(\pi_\infty, \mu) \otimes V_f^\pi(K)$$

is given by the actions $\pi_f(f)$ of $f \in \mathcal{H}_{\mathbf{C}}(K)$ on the $V_f^\pi(K)$.

The representation of $\mathcal{H}_{\mathbf{C}}(K)$ on $V_f^\pi(K)$ is irreducible and in fact $\{\pi_f(f) \mid f \in \mathcal{H}_{\mathbb{C}}(K)\}$ is the set of all linear transformations of $V_f^\pi(K)$. By Proposition 11.1.1 of Jacquet–Langlands the representations π of $G(\mathbf{A})$ occurring in the decomposition of r are mutually inequivalent. The set of such π we called A. Therefore the various representations π_f of $\mathcal{H}_{\mathbf{C}}(K)$ are also equivalent. We deduce immediately:

Proposition 2.11. *The algebra of linear transformations of $H_p^1\big(M_K, F_\mu(\mathbf{C})\big)$ commuting with the $T(g)$, $g \in G(\mathbf{A}_f)$ is a direct sum over those π in A for which $\pi_\infty \simeq \pi(\mu)$ and $V_f^\pi(K) \neq 0$ of 2×2 matrix algebras.*

3. ℓ-adic Cohomology. The next step is to define new sheaves, whose cohomology is closely related to that of $F_\mu(\mathbf{C})$ but which will have a meaning in the étale topology. We may of course introduce

$$F_\mu^K(\mathbf{Q}) = F_\mu(\mathbf{Q}) = L(\mathbf{A}) \times_{G(\mathbf{A})} G(\mathbf{A})/K_\infty K.$$

For convenience we introduce

$$L(\mathbf{A}_f) = L(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{A}_f$$

on which $G(\mathbf{A}_f)$ acts via the representation μ. We choose an open compact subgroup of $L(\mathbf{A}_f)$ stable under K and denote it by $L(\mathbf{Z}_f)$. If $L(\mathbf{Z}) = L(\mathbf{Q}) \cap L(\mathbf{Z}_f)$ then $L(\mathbf{Z}_f) = L(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_f$. If $g = g_\infty g_f$ lies in $G(\mathbf{A})$, set $gL(\mathbf{Z}_f) = \mu(g)L(\mathbf{Z}_f) = \mu(g_f)L(\mathbf{Z}_f)$ and set

$$gL(\mathbf{Z}) = L(\mathbf{Q}) \cap gL(\mathbf{Z}_f).$$

$G(\mathbf{Q})$ acts on

$$\cup_{g \in G(\mathbf{A})} gL(\mathbf{Z}) \times g = \cup_{g \in G(\mathbf{A}_f)/K} gL(\mathbf{Z}) \times G(\mathbf{R})gK$$

to the left and $K_\infty K$ acts on it to the right. Dividing out by $K_\infty K$ we obtain

$$\cup_{g \in G(\mathbf{A}_f)/K} gL(\mathbf{Z}) \times G(\mathbf{R})gK/K_\infty K.$$

$G(\mathbf{Q})$ acts freely on this space and if we divide by the action of $G(\mathbf{Q})$ we obtain a sheaf $F_\mu(\mathbf{Z})$ of $\mathbf{Z}$-modules over $M_K^o(\mathbf{C})$. This sheaf is locally free. If we tensor $F_\mu(\mathbf{Z})$ with $\mathbf{Q}$ we obtain $F_\mu(\mathbf{Q})$. If ℓ is a prime and $n \geq 0$ we may tensor with $\mathbf{Z}/\ell^n\mathbf{Z}$ to obtain the sheaf $F(\mathbf{Z}/\ell^n\mathbf{Z})$.

The groups $H^q\big(M_K^o, F_\mu(\mathbf{Z})\big)$ and $H_c^q\big(M_K^o, F_\mu(\mathbf{Z})\big)$ are finitely generated. We have canonical isomorphisms

$$H^q\big(M_K^o, F_\mu(\mathbf{Q})\big) \simeq H^q\big(M_K^o, F_\mu(\mathbf{Z})\big) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and

$$H^q\big(M_K^o, F_\mu(\mathbf{C})\big) \simeq H^q\big(M_K^o, F_\mu(\mathbf{Q})\big) \otimes_{\mathbf{Q}} \mathbf{C},$$

as well as exact sequences

$$0 \to H^q\big(M_K^o, F_\mu(\mathbf{Z})\big) \otimes \mathbf{Z}/\ell^n\mathbf{Z} \to H^q\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^{q+1}\big(M_K^o, F_\mu(\mathbf{Z})\big) * Z/\ell^n\mathbf{Z} \to 0.$$

If A is an abelian group $A * \mathbf{Z}/\ell^n\mathbf{Z}$ is the set of elements of order dividing ℓ^n in $\mathbf{A}$. The map $a \to \ell^{n-m}a$ sends $A * \mathbf{Z}/\ell^n\mathbf{Z}$ to $A * \mathbf{Z}/\ell^m\mathbf{Z}$ if $n \geq m$ and the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^q\big(M_K^o, F_\mu(\mathbf{Z})\big) \otimes \mathbf{Z}/\ell^n\mathbf{Z} & \longrightarrow & H^q\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) & \longrightarrow & H^{q+1}\big(M_K^o, F_\mu(\mathbf{Z})\big) * \mathbf{Z}/\ell^n\mathbf{Z} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^q\big(M_K^o, F_\mu\mathbf{Z})\big) \otimes \mathbf{Z}/\ell^m\mathbf{Z} & \longrightarrow & H^q\big(M_K^o, F_\mu(\mathbf{Z}/\ell^m\mathbf{Z})\big) & \longrightarrow & H^{q+1}(M_K^o, F_\mu(\mathbf{Z})\big) * \mathbf{Z}/\ell^m\mathbf{Z} & \longrightarrow 0
\end{array}$$

in which the first two vertical arrows are yielded by the projection

$$\mathbf{Z}/\ell^n\mathbf{Z} \longrightarrow \mathbf{Z}/\ell^m\mathbf{Z}$$

is commutative. Thus

$$\varprojlim_n H^q\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) \simeq H^q\big(M_K^o, F_\mu(\mathbf{Z})\big) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$$

if $\mathbf{Z}_\ell$ is the ring of ℓ-adic integers. *A fortiori*

$$\varprojlim_n H^q\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \simeq H^q\big(M_K^o, F_\mu(\mathbf{Q})\big) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell.$$

The groups on the right may be identified with $H^q\big(M_K^o, F_\mu(\mathbf{Z}_\ell)\big)$ and with $H^q\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ respectively.

Let A be any of the rings $\mathbf{C}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, or $\mathbf{Z}/\ell^n\mathbf{Z}$. If $g \in G(\mathbf{A}_f)$ and we let $gF_\mu(A)$ be defined in the same way as $F_\mu(A)$ except that $L(\mathbf{Z})$ is replaced by $gL(\mathbf{Z})$, then the analogues of all the above assertions remain true. Of course

$$gF_\mu(A) = F_\mu(A)$$

if A is $\mathbf{C}$, $\mathbf{Q}$, or $\mathbf{Q}_\ell$. If $g^{-1}K'g \subseteq K$, we have, just as in the previous paragraphs, maps

$$R(g) : H^q\big(M_K^o, F_\mu(A)\big) \longrightarrow H^q\big(M_{K'}, gF_\mu(A)\big)$$
$$R(g) : H^q\big(M_{K'}^o, gF_\mu(A)\big) \longrightarrow H^q\big(M_K^o, F_\mu(A)\big).$$

In particular if we take $K' = gKg^{-1} \cap K$ and choose the scalar matrix $a(g)$ so that $a(g)g$ stablizes $L(\mathbf{Z}_f)$, we may define

(1) $$T(g) : H^q\big(M_K^o, F_\mu(A)\big) \longrightarrow H^q\big(M_K^o, a^{-1}(g)F_\mu(A)\big)$$

as the composition

$$H^q\big(M_K^o, F_\mu(A)\big) \xrightarrow{R(g)} H^q\big(M_{K'}^o, gF_\mu(A)\big) \longrightarrow H^q\big(M_{K'}^p, a^{-1}(g)F_\mu(A)\big) \xrightarrow{R(g)} H^q\big(M_K^o, a^{-1}(g)F_\mu(a)\big)$$

in which the middle arrow is obtained from the embedding $gF_\mu(A) \to a^{-1}(g)F_\mu(A)$. These maps are all compatible with the indentifications made above. Exactly analogous assertions are valid for cohomology with compact support. Observe also that in the limit (1) yields

$$H^q\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big) \longrightarrow H^q\big(M_K^o, a^{-1}(g)F_\mu(\mathbf{Q}_\ell)\big) \simeq H^q\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big).$$

In the following we shall be interested in $H^1\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ and $H_c^1\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ which we regard as the tensor product over $\mathbf{Z}_\ell$ of $\mathbf{Q}_\ell$ with

$$\varprojlim H^1\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

and

$$\lim_{\leftarrow} H_c^1\big(M_K^o, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

respectively. These last two groups we shall later identify with groups given by the étale cohomology. We first remark a consequence of Propositions 2.11. Note that the image $H_p^1\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ of $H_c^1\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ in $H^1\big(M_K^o, F_\mu(\mathbf{Q}_\ell)\big)$ may be identified with

$$H_p^1\big(M_K, F_\mu(\mathbf{Q})\big) \otimes \mathbf{Q}_\ell.$$

Let $\mathcal{H}_{\overline{\mathbf{Q}}}(K)$ be the subalgebra of $\mathcal{H}_{\mathbf{C}}(K)$ formed by the linear combinations of the functions f_g with coefficients from $\overline{\mathbf{Q}}$. For this we must have an imbedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}$. We fix once and for all such an imbedding as well as an imbedding of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_\ell$. Set

$$\mathcal{H}_{\overline{\mathbf{Q}}_\ell}(K) = \mathcal{H}_{\overline{\mathbf{Q}}}(K) \otimes_{\overline{\mathbf{Q}}} \overline{\mathbf{Q}}_\ell.$$

The first algebra acts on $H_p^1\big(M_K, F_\mu(\overline{\mathbf{Q}})\big)$; the second acts on $H_p^1\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big)$. Tensoring the first action with $\mathbf{C}$, we obtain the action of $\mathcal{H}_{\mathbf{C}}(K)$ on $H_p^1\big(M_K, F_\mu(\mathbf{C})\big)$. Let $\rho_{\mathbf{C}}^\pi$ be the representation of $\mathcal{H}_{\mathbf{C}}(K)$ on $V_f^\pi(K)$. Proposition 2.11 yields the following assertion immediately.

Proposition 3.1. *For each π in A for which $V_f^\pi(K) \neq 0$ there is a representation $\rho_{\overline{\mathbf{Q}}}^\pi$ of $\mathcal{H}_{\overline{\mathbf{Q}}}(K)$ which extended to $\mathcal{H}_{\mathbf{C}}(K)$ yields $\rho_{\mathbf{C}}^\pi$. The space $H_p^1\big(M_K, F_\mu(\overline{\mathbf{Q}})\big)$ is a direct sum over $\{\pi \in A \,\big|\, \pi_\infty \simeq \pi(\mu)$ and $V_f^\pi(K) \neq 0\}$ of tensor products $U_{\overline{\mathbf{Q}}}^\pi \otimes V_{\overline{\mathbf{Q}}}^\pi(K)$. $U_{\overline{\mathbf{Q}}}^\pi$ is two-dimensional. Moreover each of these subspaces is invariant under the action of $\mathcal{H}_{\overline{\mathbf{Q}}}(K)$ and this algebra acts on $U_{\overline{\mathbf{Q}}}^\pi \otimes V_{\overline{\mathbf{Q}}}^\pi(K)$ according to $1 \otimes \rho_{\overline{\mathbf{Q}}}^\pi$. The same assertions are valid for $\rho_{\overline{\mathbf{Q}}_\ell}^\pi$, the linear extension of $\rho_{\overline{\mathbf{Q}}}^\pi$ to $\mathcal{H}_{\overline{\mathbf{Q}}}(K)$, if $U_{\overline{\mathbf{Q}}_\ell}^\pi = U_{\overline{\mathbf{Q}}}^\pi \otimes \overline{\mathbf{Q}}_\ell$ and $V_{\overline{\mathbf{Q}}_\ell}^\pi(K) = V_{\overline{\mathbf{Q}}}^\pi(K) \otimes \overline{\mathbf{Q}}_\ell$.*

The manifold $M_K^o(\mathbf{C})$ is the set of complex points on a nonsingular algebraic curve defined over $\mathbf{Q}$ which I denote $M_K^o \otimes \mathbf{Q}$. The next step is to interpret the sheaves $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ as sheaves in the étale topology of $M_K^o \otimes \overline{\mathbf{Q}}$.

Choose K_0 normal in K so that K_0 acts trivially on

$$L(\mathbf{Z}_f)/\ell^n L(\mathbf{Z}_f) \simeq L(\mathbf{Z}/\ell^n\mathbf{Z}).$$

I map $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ to

$$L(\mathbf{Z}/\ell^n\mathbf{Z}) \times_{K/K_0} M_{K_0}(\mathbf{C})$$

by means of $hv \times h \to v \times v$ if v lies in $L(\mathbf{Z}/\ell^n\mathbf{Z})$ and h in $G(\mathbf{A})$ projects to x in $M_K(\mathbf{C})$. The action of K/K_0 is, by the way, given by $v \times x \to k^{-1}v \times xk$. The map just introduced is well-defined because

$\gamma hv \times \gamma hk$ with γ in $G(\mathbf{Q})$ and k in $K$is sent to $k^{-1}v \times xk$. It is a local homeomorphism and a bijection. In fact if $hv \times h$ and $h_1v_1 \times h_1$ have the same image $h_1 = \gamma hk$ with γ in $G(\mathbf{Q})$ and k in K, then $v_1 \times xk$ is equivalent to $v \times x$ modulo the action of K/K_0; so $v_1 = k^{-1}v$ and

$$h_1v_1 \times h_1 = \gamma hv \times \gamma hk$$

defines the same element of $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ as $hv \times h$.

The product

$$L(\mathbf{Z}/\ell^n\mathbf{Z}) \times_{K/K_0} (M^o_{K_0} \otimes \mathbf{Q})$$

is defined as a group object in the category of schemes étale over $M^o_K \otimes \mathbf{Q}$. Tensoring with $\overline{\mathbf{Q}}$ we obtain a sheaf for the étale topology of $M^o_K \otimes \overline{\mathbf{Q}}$. Because we have an imbedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}$, the étale cohomology groups of this sheaf, with or without compact support, may be identified with those of $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$. We may, and shall, also regard $H^1_p\big(M_K, F_\mu(\mathbf{Q}_\ell)\big)$ as the tensor product of $\mathbf{Q}_\ell$ with the image of

$$\varprojlim H^1_c\big(M^o_K, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) \longrightarrow \varprojlim H^1\big(M^o_K, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big),$$

where of course the two groups are to be taken in the étale cohomology.

Both maps $R(g)$, and hence $T(g)$, may be defined in the étale cohomology, once we have established that $R^*(g)F^K_\mu(\mathbf{Z}/\ell^m\mathbf{Z})$ and $gF^{K'}_\mu(\mathbf{Z}/\ell^m\mathbf{Z})$ are isomorphic. Recall that there is a map $R(g) : M^o_{K'} \otimes \mathbf{Q} \rightarrow M^o_K \otimes \overline{\mathbf{Q}}$, which when applied to the $\mathbf{C}$-valued points yields the map $R(g)$ already introduced. Tensoring with $\overline{\mathbf{Q}}$, we obtain $R(g) : M^o_{K'} \otimes \overline{\mathbf{Q}} \rightarrow M^o_K \otimes \overline{\mathbf{Q}}$.

$R^*(g)$ applied to $F^K_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$, regarded now as a sheaf in the étale topology and, in fact, at first as a sheaf over $M^o_K \otimes \mathbf{Q}$, gives

$$(M_{K'} \otimes \mathbf{Q}) \underset{M_K \otimes \mathbf{Q}}{\times} \big(L(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K/K_0}{\times} (M^o_{K_0} \otimes \mathbf{Q})\big). \tag{2}$$

The sheaf $gF^{K'}_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ is

$$gL(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K'/K'_0}{\times} (M^o_{K'_0} \otimes \mathbf{Q}). \tag{3}$$

We may suppose $g^{-1}K'_0 g \subseteq K_0$. We may map

$$gL(\mathbf{Z}/\ell^n\mathbf{Z}) \times (M^o_{K'_0} \otimes \mathbf{Q}) \tag{4}$$

to (2) by taking a cartesian product of maps to the two factors. The map to $M^o_{K'} \otimes \mathbf{Q}$ is obtained by composing $R(e) : M^p_{K'_0} \otimes \mathbf{Q} \rightarrow M^o_{K'} \otimes \mathbf{Q}$ with projection on the second factor of (4). The map $g^{-1} \times R(g)$

from (4) to $L(\mathbf{Z}/\ell^n\mathbf{Z}) \times (M^o_{K_0} \otimes \mathbf{Q})$ followed by the projection from this space to its quotient by K/K_0 is the map to the second factor of (2). The map from (4) to (2) is easily seen to factor through (3). The simplest way to see that the resultant map from (3) to (2) is an isomorphism is to look at its effect on the $\mathbf{C}$-valued points and then to invoke the comparison theorem. In terms of our representation of the $\mathbf{C}$-valued points as coset spaces of $G(\mathbf{A})$, the map in question is: $gv \times h \to h \times v \times hg$. It is easily seen to be an isomorphism and, indeed, when the redefinition of the sheaves $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ is taken into account, to be the isomorphism used, explicitly or implicitly, in the original discussion. The isomorphism over $\overline{\mathbf{Q}}$ is obtained by base change.

We have abbreviated an expression like

$$(M^o_K \otimes \mathbf{Q}) \underset{\operatorname{Spec}\mathbf{Q}}{\times} \operatorname{Spec}\overline{\mathbf{Q}}$$

to $M^o_K \otimes \overline{\mathbf{Q}}$. From the proper form one sees that $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on $M_K \otimes \overline{\mathbf{Q}}$ through its action on the second factor. To be specific $s \in \mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on $\operatorname{Spec}\overline{\mathbf{Q}}$ to the left and the corresponding action on $\overline{\mathbf{Q}}$ is $x \to s^{-1}(x)$. We shall denote the action of s on $M^o_K \otimes \overline{\mathbf{Q}}$, for any K, by $L(s)$.

It is clear that the maps $L(s)$ and $R(g)$ commute. Hence we may also let $L(s)$ act on $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$—now pulled back to the scheme $M_K \otimes \overline{\mathbf{Q}}$. Since

$$\begin{array}{ccc} F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) & \xrightarrow{L(s)} & F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) \\ \big\downarrow & & \big\downarrow \\ M_K \otimes \overline{\mathbf{Q}} & \xrightarrow{L(s)} & M_K \otimes \overline{\mathbf{Q}} \end{array}$$

is commutative and $L(s)$ is invertible the upper right corner may be regarded, by means of the diagonal map to the lower left corner, as $L^*(s)F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ and the inverse of the upper horizontal arrow is a map

$$L^*(s)F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) \longrightarrow F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}).$$

These maps yield a representation $s \to \rho(s)$ of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the groups $H^q\big(M^o_K, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)$ and, if one takes into account the completion of $M^o_K \otimes \mathbf{Q}$ introduced in [3.1], on $H^q_c\big(M^o_K, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)$. Hence, because of the obvious consistency, the group $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ also acts on $H^q\big(M^o_K, F_\mu(\mathbf{Q}_\ell)\big)$, $H^q_c\big(M^o_K, F_\mu(\mathbf{Q}_\ell)\big)$, and on $H^q_p\big(M_K, F_\mu(\mathbf{Q}_\ell)\big)$. $\rho(s)$ acts to the right while $T(g)$ acts to the left. The two actions, because of their construction, commute.

We take $H^1_p\big(M_K, F_\mu(\mathbf{Q}_\ell)\big)$, tensor with $\overline{\mathbf{Q}}_\ell$ to obtain $H^1_p\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big)$, and apply Proposition 3.1 to see that ρ, a representation over $\overline{\mathbf{Q}}_\ell$, is a direct sum

$$\otimes_\pi \sigma(\pi) \otimes 1$$

where $\sigma(\pi)$ is a two-dimensional representation on $U^{\pi}_{\overline{\mathbf{Q}}_\ell}$ and 1 acts on $V^{\pi}_{\overline{\mathbf{Q}}_\ell}(K)$.

4. The basic problem. We may, and it is in fact convenient to do so, form the direct limit

$$\varinjlim_K H^1_p\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big).$$

Proposition 3.1 and standard facts about tensor products show that it is a direct sum

$$\oplus_{\{\pi\in A\,|\,\pi_\infty\simeq\pi(\mu)\}} U^{\pi}_{\overline{\mathbf{Q}}_\ell}\otimes V^{\pi}_{\overline{\mathbf{Q}}_\ell} \tag{1}$$

where

$$V^{\pi}_{\overline{\mathbf{Q}}_\ell}=\varinjlim_K V^{\pi}_{\overline{\mathbf{Q}}_\ell}(K).$$

The operators $R(g)$ yield a representation of $G(\mathbf{A})$ on $V^{\pi}_{\overline{\mathbf{Q}}_\ell}$ and $V^{\pi}_{\overline{\mathbf{Q}}_\ell}(K)$ may be regarded as the space of K-invariant vectors in $V^{\pi}_{\overline{\mathbf{Q}}_\ell}$. The space of K-invariant vectors in (1) may be identified with $H^1_p\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big)$. This justifies our failure, in Proposition 3.1, to include the dependence of $U^{\pi}_{\overline{\mathbf{Q}}_\ell}$ on K in the notation. This shows also that the representation $\sigma(\pi)$ depends only on π and not on K and, since K may be chosen arbitrarily small, that it is defined for all π for which $\pi_\infty\simeq\pi(\mu)$ for some μ.

Given such a π let

$$\pi'(g)=|\det g|^{-1/2}\pi(g),\qquad g\in G(\mathbf{Q}).$$

As we observed in the second paragraph, π' is any representation of $G(\mathbf{A})$ occurring in the space of cusp forms such that π'_∞ is a $\pi(\sigma)$ where σ is a two-dimensional representation of $W_{\mathbf{C}/\mathbf{R}}$ whose restriction to $C^\times$ is of the form

$$z\to\begin{pmatrix} z^{-n}\bar z^{-m} & 0\\ 0 & z^{-m}\bar z^{-n}\end{pmatrix}$$

with $m\neq n$.

By restriction $\sigma=\sigma(\pi)$ yields for every prime p and in particular for every $p\neq\ell$, a condition we shall always impose, a representation $\sigma_p=\sigma_p(\pi)$ of the decomposition group $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Of course only the class of σ_p is uniquely determined.

Let $\mathcal{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the subgroup of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ formed by those s which project to an integral power of the Frobenius in $\mathfrak{G}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ if $\mathbf{F}_p=GF(p)$. Recall also that we have fixed an imbedding of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_\ell$. Suppose σ_p is a continuous two-dimensional representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ over some finite extension of $\mathbf{Q}_\ell$ such that

$$\operatorname{trace}\sigma_p(s)\in\overline{\mathbf{Q}}$$

for all s in $\mathcal{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

I want to describe how to associate, in so far as it is possible at present, to such a σ_p an irreducible admissible representation $\pi(\sigma_p)$ of $GL(2,\mathbf{Q}_p)$. There are two cases to consider separately. Either there exists a finite Galois extension F of $\mathbf{Q}_p$ such that σ_p factors through the Galois group, $\mathfrak{G}(F^{un}/\mathbf{Q}_p)$, of the maximal unramified extension of F over $\mathbf{Q}_p$ or it does not. In the second case I say, for lack of a better terminology, that σ_p is special.

Suppose first of all that σ_p is not special. $\mathfrak{G}(F^{un}/F)$ is a normal subgroup of $\mathfrak{G}(F^{un}/\mathbf{Q}_p)$ and is injected into the abelian quotient group $\mathfrak{G}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$. It is therefore central. If $s \in \mathfrak{G}(F^{un}/\mathbf{Q}_p)$ let $\sigma_p(s) = \sigma_1\sigma_2$ where σ_1 is semi-simple and σ_2 is unipotent, the two matrices being supposed to commute. Some power s^n of s is central. Since $\sigma_p(s^n) = \sigma_1^n\sigma_2^n$, σ_2^n commutes with all $\sigma_p(t)$. So does σ_2. Then $s \to \sigma_1$ is a representation σ_p' of $\mathfrak{G}(F^{un}/\mathbf{Q}_p)$ by semi-simple matrices. σ_p' restricted to the finitely generated group $\mathcal{W}(F^{un}/\mathbf{Q}_p)$ is a representation by matrices with entries from a finitely generated subfield k of $\overline{\mathbf{Q}}_p$. Imbedding k in $\mathbf{C}$ we obtain a complex representation of $\mathcal{W}(F^{un}/\mathbf{Q}_p)$ which when combined with the standard homomorphism of the Weil group $W_{F/\mathbf{Q}_p}$ onto $\mathcal{W}(F^{un}/\mathbf{Q}_p)$ yields a representation σ_p'' of $W_{F/\mathbf{Q}_p}$ by complex semi-simple matrices. The irreducible admissible representation $\pi(\sigma_p'')$ of $GL(2,\mathbf{Q}_p)$, in so far as it is known to exist, has been described in Chapter 12 of Jacquet–Langlands. We set $\pi(\sigma_p) = \pi(\sigma_p'')$. It should of course be observed that the class of σ_p'' does not depend on the imbedding of k in $\mathbf{C}$, because it has been assumed that $\operatorname{trace}\sigma(s)$ lies in $\overline{\mathbf{Q}}$ for s in $\mathcal{W}(F^{un}/\mathbf{Q}_p)$.

For convenience I recall how $\pi(\sigma_p)$ is defined for the representations of most interest to us. Suppose σ_p'' is the direct sum of two one-dimensional representations λ and ν, which may be regarded as quasi-characters of $\mathbf{Q}_p^\times$. Let $\rho(\lambda,\nu)$ be the representations of $G(\mathbf{Q}_p)$ by right translations in the space of locally constant functions φ on $G(\mathbf{Q}_p)$ satisfying

$$\varphi\left(\begin{pmatrix}\alpha & x\\ 0 & \beta\end{pmatrix} g\right) = \lambda(\alpha)\nu(\beta)\left|\frac{\alpha}{\beta}\right|^{1/2}\varphi(g).$$

If $\lambda\nu^{-1}$ is not of the form $\alpha \to |\alpha|^{\pm 1}$ then $\rho(\lambda,\nu)$ is irreducible and is in fact $\pi(\sigma_p'')$. Otherwise it has a composition series with two terms, one of which $\pi(\sigma_p'')$ is finite dimensional. The other $\sigma(\lambda,\nu)$ is a so-called special representation.

To see the significance of this terminology, suppose now that σ_p itself is special. Let $\mathbf{Q}_p^t$ be the union of all finite tamely ramified extensions of $\mathbf{Q}_p$. $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p^t)$ is a pro p-group. Thus any ℓ-adic representation is trivial on a subgroup of finite index. There is a finite Galois extension F of $\mathbf{Q}_p$ such that σ_p is trivial on $\mathfrak{G}(\overline{\mathbf{Q}}_p/F^t)$. σ_p is thus a representation of $\mathfrak{G}(F^t/\mathbf{Q}_p)$. Moreover, $\mathfrak{G}(F^t/F^{un})$ is a normal subgroup isomorphic to $\prod_{q\neq p}\mathbf{Z}_q$. Enlarging F is necessary, we suppose that the restriction of σ_p to $\mathfrak{G}(F^t/F^{un})$ factors through the projection on $\mathbf{Z}_\ell$. The action of $\mathfrak{G}(F^{un}/F)$ on $\mathfrak{G}(F^t/F^{un})$ is such that the n^{th} power of the Frobenius, relative to F, sends s to s^{q^n} if q is the number of elements in the finite field corresponding to F. Thus if λ is an eigenvalue of $\sigma_p(s)$ so is λ^{q^n}. Since $\sigma_p(s)$ has only a finite number of eigenvalues they are all roots of unity. Enlarging F is necessary we may suppose that $\sigma_p(s)$ is unipotent if s projects to 1 in $\mathbf{Z}_\ell$ and hence for all s. Because σ_p is special, $\sigma_p(s)$ cannot be 1 for that s projecting to 1. Let it have the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

This may of course be effected by an appropriate choice of basis. Since $\mathfrak{G}(F^t/F^{un})$ is normal in $\mathfrak{G}(F^t/\mathbf{Q}_p)$, σ_p is a representation of the form

$$s \to \begin{pmatrix} \mu(s) & * \\ 0 & \nu(s) \end{pmatrix}$$

where μ and ν are quasi-characters. Restricted to $\mathcal{W}(F^t/\mathbf{Q}_p)$ they yield as before two quasi-characters, which we again call μ and ν, of $\mathbf{Q}_p^\times$. Because of our assumption on the traces these take values in $\overline{\mathbf{Q}}$. Taking the action of $\mathfrak{G}(F^t/\mathbf{Q}_p)$ on $\mathfrak{G}(F^t/F^{un})$ into account we see that $\mu\nu^{-1}$ takes α to $|\alpha|^{-1}$. We take $\pi(\sigma_p)$ to be the special representation $\sigma(\mu,\nu)$.

We consider again a π occurring in the space of cusp forms for which $\pi_\infty \simeq \pi(\mu)$ and let $\pi'(g) = |\det g|^{-1/2}\pi(g)$. π' may be written as a tensor product over the valuations of $\mathbf{Q}$, $\pi' = \otimes_v \pi'_v$.

Conjecture: *If $\sigma_p = \sigma_p(\pi)$ then $\operatorname{trace}\sigma_p(s) \in \overline{\mathbf{Q}}$ for all s in $\mathcal{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and π'_p is $\pi(\sigma_p)$.*

In order to lend some credibility to this conjecture, I shall prove in these lectures that it is valid if $\pi'_p = \pi(\tau_p)$ where τ_p is the direct sum of two one-dimensional representations of the Weil group or if $\pi'_p = \sigma(\lambda,\nu)$ is a special representation.

5. Some auxiliary considerations. p is now fixed. Let $\mathbf{A}_f^p$ be the subring of $\mathbf{A}_f$ formed by those elements whose component at p is 0. $G(\mathbf{A}_f)$ is a direct product $G(\mathbf{A}_f^p)G(\mathbf{Q}_p)$. There is thus a decomposition $V_f^\pi = V_{\hat{p}}^\pi \otimes V_p^\pi$. $G(\mathbf{A}_f^p)$ acts on the first and $G(\mathbf{Q}_p)$ on the second factor. Set $K^p = K \cap G(\mathbf{A}_f^p)$ and $K_p = K \cap G(\mathbf{Q}_p)$. We consider only those K for which $K = K^pK_p$. They form a cofinal family. $V_f^\pi(K)$ is then a tensor product

$$V_{\hat{p}}^\pi(K^p) \otimes V_p^\pi(K_p).$$

The dimension of the second factor is $m(\pi_p, K_p)$, the multiplicity with which the trivial representation of K_p occurs in π_p.

If $g \in G(\mathbf{A}_f^p)$, we may consider $T(g)$ to be acting on $V_{\hat{p}}^{\pi}(K_p)$. The action extends to a representation of $\mathcal{H}_{\mathbf{C}}(K^p)$, the algebra of compactly supported K^p-biinvariant functions on $G(\mathbf{A}_f^p)$, which we denote $\pi_{\hat{p}}$.

Let $A^1(\mu)$ be the set of all π in A such that $\pi_\infty \simeq \pi(\mu)$ and $\pi_p' \simeq \pi(\tau_p)$, where τ_p, a representation of the Weil group of $\mathbf{Q}_p$ by semi-simple matrices, is a direct sum of two one-dimensional representations. I recall that $\pi_p' : g_p \to |\mathrm{det} g_p|^{-\frac{1}{2}}\pi_p(g_p)$. We consider τ^1, a double representation of $\mathcal{H}_{\mathbf{C}}(K^p)$ and the Weil group of $\mathbf{Q}_p$, which is defined as

$$\oplus_{\pi \in A^1(\mu)} m(\pi_p, K_p)\{\pi_{\hat{p}} \otimes \tau_p\}.$$

Observe that the sum is effectively finite and that τ_p is uniquely determined by π_p.

This is not the only double representation we want to consider. Let $A^2(\mu)$ be the set of π in A for which $\pi_\infty \simeq \pi(\mu)$ while π_p' is a special representation $\sigma(\lambda_p, \nu_p)$. The second double representation to be considered is

$$\tau^2 = \oplus_{\pi \in A^2(\mu)} m(\pi_p, K_p)\{\pi_{\hat{p}} \otimes (\lambda_p \oplus \nu_p)\}.$$

In this paragraph we find formulae for $\mathrm{trace}\left(\tau^1(f_g, s)\right)$ and $\mathrm{trace}\left(\tau^2(f_g, s)\right)$ if s is an element of the Weil group mapping to a non-unit in $\mathbf{Q}_p^\times$. Actually the representations of the Weil group considered in this paragraph factor through $\mathbf{Q}_p^\times$. However to treat them as representations of $\mathbf{Q}_p^\times$ would be slightly misleading.

The formulae we obtain are quite pretty. To describe them exactly requires some preparation and, in particular, the introduction of some sets on which $G(\mathbf{A}_f^p)$ acts. They may appear strange at first. I hope their significance becomes clear in the course of this and the following paragraph.

Let F be an imaginary quadratic extension of $\mathbf{Q}$ which splits at p. Choosing a basis of F, a vector space over $\mathbf{Q}$, we may regard G as the group of invertible linear transformations of F. The centralizer of F, which acts on itself by multiplication, is an algebraic subgroup H_f of G. There is a v in $G(\mathbf{Q}_p)$ such that, if A is the group of diagonal matrices,

$$vA(\mathbf{Q}_p)v^{-1} = H(\mathbf{Q}_p).$$

We fix such a $v = v(F)$ and set

$$V(\mathbf{Q}_p) = vN(\mathbf{Q}_p)v^{-1}.$$

We also introduce the set

$$M(p,F) = H_F(\mathbf{Q})V(\mathbf{Q}_p) \setminus G(\mathbf{A}_f)/K_p$$

on which $G(\mathbf{A}_f^p)$ acts to the right. Dividing by K^p we obtain

$$M_K(p,F) = H_F(\mathbf{Q})V(\mathbf{Q}_p) \setminus G(\mathbf{A}_f)/K$$

which we turn into a toplogical space by providing it with the discrete topology. K^p acts on $L(\mathbf{Q}_\ell)$ by means of the projection into $G(\mathbf{Q}_\ell)$ composed with the action of $G(\mathbf{Q}_\ell)$ on $L(\mathbf{Q}_\ell)$ defined by μ. We let it act on

$$L(\mathbf{Q}_\ell) \times M(p,F)$$

by $k:(u,y) \to (k^{-1}u, yk)$. Then

$$L(\mathbf{Q}_\ell) \times_{K^p} M(p,F)$$

is a sheaf over $M_K(p,F)$. I denote it by $F_\mu(\mathbf{Q}_\ell)$, a notation already used in another context. But as it was then used for a sheaf over another space there should be no confusion. The overlapping notation is of course deliberate.

If $g \in G(\mathbf{A}_f^p)$ and $g^{-1}K'g \subseteq K$, with $K'_p = K_p$, we may introduce

$$R(g) : M_{K'}(p,F) \to M_K(p,F)$$

as well as the maps

$$R^*(g)F_\mu(\mathbf{Q}_\ell) \to F_\mu(\mathbf{Q}_\ell)$$

between two sheaves over $M_{K'}(p,F)$ and

$$R_*(g)F_\mu(\mathbf{Q}_\ell) \to F_\mu(\mathbf{Q}_\ell)$$

between two sheaves over $M_K(p,F)$. To be explicit, the first sends the point represented by $(u \times hg) \times h$ to the point represented by $gu \times h$. The second sends

$$\bigoplus_{k \in g^{-1}K'g \backslash K} u \times h'$$

where h' is defined by $h'gk = h$, to

$$\left(\sum k^{-1}g^{-1}u\right) \times h.$$

If s belongs to the Weil group of $\mathbf{Q}_p$ and s maps to b in $\mathbf{Q}_p^\times$, then left translation of $G(\mathbf{A}_f)$ by

$$v\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}v^{-1}$$

factors to yield maps $L(s)$ from $M_K(p,F)$ to itself or from $M(p,F)$ to itself. There is also a map

$$L^*(s)F_\mu(\mathbf{Q}_\ell) \to F_\mu(\mathbf{Q}_\ell).$$

Both ends of the arrow are sheaves on $M_K(p,F)$. The map is obtained by factoring

$$\Big(u\times v\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}v^{-1}h\Big)\times h \to u\times h.$$

Now suppose $K' = K\cap gKg^{-1}$. We form the map

$$\varphi(g,s) = R(e)\times R(g)L(s) = R(e)\times L(s)R(g) : M_{K'}(p,F)\to M_K(p,F)\times M_K(p,F).$$

A point x' which is the inverse image of (x,x) on the diagonal will be called a fixed point of $\varphi(g,s)$. The maps of sheaves introduced earlier yield in the present circumstances a linear transformation first from the fibre $F_\mu(\mathbf{Q}_\ell)_x$ of $F_\mu(\mathbf{Q}_\ell)$ at x, which is also the fibre of $R^*(g)L^*(s)F_\mu(\mathbf{Q}_\ell)$ at x', to the fibre $F_\mu(\mathbf{Q}_\ell)_{x'}$ and then from this space, a subspace of the fibre of $R_*(e)F_\mu(\mathbf{Q}_\ell)$ at x, to $F_\mu(\mathbf{Q}_\ell)_x$. This composition we denote

$$\varphi_{x'}(g,s) : F_\mu(\mathbf{Q}_\ell)_x \to F_\mu(\mathbf{Q}_\ell)_x.$$

Lemma 5.1. *If s maps to a non-unit b in $\mathbf{Q}_p^\times$ then the set of fixed points of $\varphi(g,s)$ is finite. It is in fact empty for all but a finite number of isomorphism classes of imaginary quadratic extensions which split at p.*

We have chosen a basis of F. We may therefore identify $\mathbf{A}_f(F) = F\otimes\mathbf{A}_f$, the ring of finite adèles of F, with $\mathbf{A}_f\oplus\mathbf{A}_f$. For our purposes a module is just an open compact subgroup $M(\mathbf{Z}_f)$ of $\mathbf{A}_f(F)$. It must be of the form

$$\prod_p M(\mathbf{Z}_p)$$

with $M(\mathbf{Z}_p)\subseteq F\otimes\mathbf{Q}_p$. A module is an order if it is a subring of $\mathbf{A}_f(F)$ containing 1. $G(\mathbf{A}_F)$ acts transitively on the modules. Two modules will be said to lie in the same genus if one can be transformed to the other by an element of $H(\mathbf{A}_f)$ and to the same class if one can be transformed to the other by an element of $H(\mathbf{Q})$. As on p. 251 of Jacquet–Langlands every genus contains a unique order. Since the stablizer of a module is open and $H(\mathbf{Q}) \setminus H(\mathbf{A}_f)$ is compact every genus contains only a fininte number of classes.

Let n be a positive integer. It follows readily from Lemma 7.3.1 of Jacquet–Langlands that if $\gamma \in H(\mathbf{Q}) = F^\times$ and γ is not a scalar matrix, that is not in $\mathbf{Q}^\times$, then the set of orders containing $n\gamma$ is finite. Thus the number of genera Γ such that $n\gamma : M(\mathbf{Z}_f) \to M(\mathbf{Z}_f)$ for $M(\mathbf{Z}_f)$ in Γ is finite.

Returning to the lemma we represent a fixed point x' by h in $G(\mathbf{A}_f)$. Set

$$g(s) = v \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} v^{-1} \in G(\mathbf{Q}_p).$$

There is a k in K, a u in $V(\mathbf{Q}_p)$, and a γ in $H(\mathbf{Q})$ such that

(1) $$g(s)hgk = u\gamma h.$$

Let

$$h = h_{\hat{p}} h_p$$

with $h_{\hat{p}}$ in $G(\mathbf{A}_f^p)$ and h_p in $G(\mathbf{Q}_p)$. Decompose k in the same manner. The equation (1) yields

$$g(s)h_p k_p = u\gamma h_p$$

or

(2) $$k_p = h_p^{-1} g^{-1}(s) u\gamma h_p$$

and

$$h_{\hat{p}} g k_{\hat{p}} = \gamma h_{\hat{p}}$$

or

(3) $$g k_{\hat{p}} = h_{\hat{p}}^{-1} \gamma h_{\hat{p}}.$$

Considering eigenvalues, we infer from (2) and (3) that γ must lie in a compact subset of $H(\mathbf{A}_f)$ and hence in a finite subset of $H(\mathbf{Q})$.

Choose a module $M(\mathbf{Z}_f)$ fixed by K. The map $h \to hM(\mathbf{Z}_f)$ maps $H(\mathbf{Q}) \setminus G(\mathbf{A}_f)/K$ to the set of classes of modules. The inverse image of each class is finite. We choose a set of representtives for the fixed points. We have to show that this set is finite. Since $H(\mathbf{Q}_p)V(\mathbf{Q}_p) \setminus G(\mathbf{Q}_p)/K_p$ is finite we may assume h_p lies in a fixed finite set. Choose n in $\mathbf{Z}$ such that ng takes $M(\mathbf{Z}_f)$ into itself. Suppose also that for all γ which can enter into the equation (1), and for the fixed finite set of h_p, $n\gamma$ takes $h_pM(\mathbf{Z}_p)$ into itself. From (3) we deduce that $n\gamma$ takes $hM(\mathbf{Z}_f)$ into itself. Since there are only a finite number of γ under consideration and, because of (2) and our assumption on s, none of them lie in $\mathbf{Q}^\times$, the set of classes in which $hM(\mathbf{Z}_f)$ can lie is finite. It follows that the set of representatives is itself finite.

The second assertion of the lemma follows from the observation that (2) and (3) force the trace and determinant of γ to lie in a compact subset of $\mathbf{A}_f$ and thus in a finite subset of $\mathbf{Q}$.

Set

$$A(F) = \sum \operatorname{trace} \varphi_{x'}(g, s).$$

The sum is taken over the fixed points of $\varphi(g, s)$. It will be important to us but we should write it in a more useful form.

It was agreed at the beginning to take K so small that no element of $G(\mathbf{Q})$ is conjugate to an element of $K_\infty K$. If K_p is given we can choose K^p so small that for no element γ of an imaginary quadratic field can γ be integral at p and such that the equations

$$\operatorname{trace} k_q = \operatorname{trace} \gamma$$

$$\det k_q = \operatorname{Nm} \gamma$$

are solvable for all $q \neq p$ with $k = (k_q)$ in K^p. This we assume henceforth. Since K plays only an auxiliary role, we can afford to make such an assumption.

Given h in $G(\mathbf{A}_f)$ which represents a fixed point, suppose that in addition to equation (1) an equation

$$g(s)hg\bar{k} = \bar{u}\bar{\gamma}h$$

of the same type also holds. Then

$$\bar{k}_{\hat{p}}^{-1} k_{\hat{p}} = h_{\hat{p}}^{-1} \bar{\gamma}^{-1} \gamma h^{\hat{p}}$$

and

$$\bar{k}_p^{-1} k_p = h_p^{-1} \bar{\gamma}^{-1} \bar{u}^{-1} u \gamma h_p.$$

The eigenvalues of $\bar{\gamma}^{-1}\bar{u}^{-1}uV$ are those of $\bar{\gamma}^{-1}\gamma$. Our assumption implies therefore that $\gamma = \bar{\gamma}$. Thus γ is uniquely determined by h, and $k_{\hat{p}}$ is in its turn uniquely determined by γ and h.

B is the group of supertriangular matrices. Let $G(\mathbf{Q}_p)$ be the disjoint union

$$\cup_i B(\mathbf{Q}_p) g_i K_p.$$

If $h_i = vg_i$, with $v = v(F)$, then $G(\mathbf{Q}_p)$ is also the disjoint union

$$\cup_i V(\mathbf{Q}_p)H(\mathbf{Q}_p)h_iK_p.$$

Let U_i be the projection of

$$g_iK_pg_i^{-1} \cap B(\mathbf{Q}_p)$$

on $A(\mathbf{Q}_p)$. If $h_p \in V(\mathbf{Q}_p)H(\mathbf{Q}_p)h_iK_p$ the equation (2) can be satisfied with some u if and only if $g^{-1}(s)\gamma$ lies in vU_iv^{-1}.

We observe next that at a fixed point representated by h the trace of $\varphi_{x'}(g,s)$ is

$$\operatorname{trace}\mu(\gamma)$$

if γ satisfies (1). To see this observe that the corresponding map of $F_\mu(\mathbf{Q}_\ell)_x$ to $F_\mu(\mathbf{Q}_\ell)_{x'}$ is the composition

$$\Big((u \times g(s)hg) \times hg\Big) \times h \to (u \times hg) \times h \to gu \times h.$$

Now passing from $F_\mu(\mathbf{Q}_\ell)_{x'}$ tp $F_\mu(\mathbf{Q}_\ell)_x$ we send

$$(gu \times h) \to kgu \times hk^{-1}$$

because of equation (1). Thus

$$\operatorname{trace}\varphi_{x'}(g,s) = \operatorname{trace}\mu(k_\ell g_\ell) = \operatorname{trace}\mu(g_\ell k_\ell) = \operatorname{trace}\mu(\gamma)$$

because of equation (3). k_ℓ and g_ℓ are the components of k and g at ℓ.

Lemma 5.2. *Let f_g be the characteristic function K^pgK^p, a subset of $G(\mathbf{A}_f^p)$, divided by the measure of K^p and let χ_i be the characteristic function of U_i. Then $A(F)$ is the measure of $H(\mathbf{Q}) \setminus H(\mathbf{A}_f)$ times the sum over γ in $H(\mathbf{Q})$ of*

$$\operatorname{trace}\mu(\gamma)\left\{\sum_i \frac{\chi_i\big(v^{-1}g^{-1}(s)\gamma v\big)}{\operatorname{meas}(vU_iv^{-1})}\right\}\int_{H(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f_g(h^{-1}\gamma h)\,dh.$$

Our discussion to this point shows that $A(F)$ is equal to

$$\sum_\gamma\sum_i\sum_h \chi_i\big(v^{-1}g^{-1}(s)\gamma v\big)\operatorname{trace}\mu(\gamma). \tag{4}$$

The outer sum is over γ in $H(\mathbf{Q})$. The inner sum is over a set of coset representatives for $H(\mathbf{Q})V(\mathbf{Q}_p)\setminus G(\mathbf{A}_f)/K'$ for which

$$h_p \in vB(\mathbf{Q}_p)g_iK_p$$

and

$$h_{\hat{p}}^{-1}\gamma h_{\hat{p}} \in gK^p.$$

Let $\{k_\alpha\}$ be a set of coset representatives for K/K', each of which is taken to lie in K^p. I claim that it is possible to replace the sum over coset representatives of $H(\mathbf{Q})V(\mathbf{Q}_p)\setminus G(\mathbf{A}_f)/K'$ by a sum over hk_α, α varying and h varying over a collection of coset representatives of $H(\mathbf{Q})V(\mathbf{Q}_p)\setminus G(\mathbf{A}_f)/K$. Suppose in fact that

$$hk_\alpha = \gamma u h k_\beta k' \tag{5}$$

with γ in $H(\mathbf{Q})$, u in $V(\mathbf{Q}_p)$, and k' in K'. Then γu lies in hKh^{-1}. Since γ and γu have the same eigenvalues at each place, our basic assumption implies that $\gamma = 1$. Examining equation (5) away from p and recalling that k_α and k_β lie in K^p, we conclude that $\alpha = \beta$, as claimed.

The set K^pgK^p is the disjoint union

$$\cup k_\alpha g K^p.$$

Thus

$$h_{\hat{p}}^{-1}\gamma h_{\hat{p}} \in K^pgK^p$$

if and only if

$$k_\alpha^{-1}h_{\hat{p}}^{-1}\gamma h_{\hat{p}}k_\alpha \in gK^p$$

for some α, and this α is then unique. $A(F)$ is thus equal to

$$\sum_\gamma\sum_i\sum_h \operatorname{trace}\mu(\gamma)\chi_i\big(v^{-1}g^{-1}(s)\gamma v\big)\,f_g(h_{\hat{p}}^{-1}\gamma h_{\hat{p}})\operatorname{meas}K^p.$$

The inner sum is taken over a set of representatives for the double cosets

$$H(\mathbf{Q})V(\mathbf{Q}_p)vU_iv^{-1}\setminus G(\mathbf{A}_f^p)V(\mathbf{Q}_p)H(\mathbf{Q}_p)/K^p = H(\mathbf{Q})vU_iv^{-1}\setminus G(\mathbf{A}_f^p)H(\mathbf{Q}_p)/K^p.$$

$A(F)$ may be written then as

$$\sum_\gamma\sum_i \operatorname{trace}\mu(\gamma)\frac{\chi_i\big(v^{-1}g^{-1}(s)\gamma v\big)}{\operatorname{meas}(vU_iv^{-1})}\int_{H(\mathbf{Q})\setminus G(\mathbf{A}_f^p)H(\mathbf{Q}_p)} f_g(h_{\hat{p}}^{-1}\gamma h_{\hat{p}})\,dh.$$

The step from here to the assertion of the lemma is short.

There are still more spaces like $M_K(p,F)$ to be constructed. Let D be the quaternion algebra over $\mathbf{Q}$ split everywhere but at p and ∞ and let G' be the multiplicative group of D. G' and G are isomorphic as algebraic groups over $\mathbf{Q}_q$ if $q \neq p$. This observation yields isomorphisms $G'(\mathbf{Q}_q) \to G(\mathbf{Q}_q)$ as well as $G'(\mathbf{A}_f^p) \to G(\mathbf{A}_f^p)$. We choose one such isomorphism. Any two would differ by an inner automorphism, so it does not really matter which one we take. Let

$$V_i = \{\det k \,\big|\, k \in U_i\}$$

and

$$W_i = \{g \in G'(\mathbf{Q}_p) \,\big|\, \mathrm{Nm} g \in V_i\}.$$

Then $M_K^i(p,D)$ is the space

$$G'(\mathbf{Q}) \setminus G'(\mathbf{A}_f)/K^p W_i.$$

I have taken the liberty of using the isomorphism introduced, to imbed K^p in $G(\mathbf{A}_f^p)$. We may also introduce

$$M^i(p,D) = G'(\mathbf{Q}) \setminus G'(\mathbf{A}_f)/W_i$$

as well as the sheaf $F_\mu(\mathbf{Q}_\ell)$ on $M_K^i(p,D)$ defined by

$$L(\mathbf{Q}_\ell) \times_{K^p} M^i(p,D).$$

$G(\mathbf{A}_f^p)$ acts on $M^i(p,D)$ to the right by means of the isomorphism between $G(\mathbf{A}_f^p)$ and $G'(\mathbf{A}_f^p)$, which we might as well use to identify the two groups. We may also define the maps $R(g)$. If s belongs to the Weil group and maps to b in $\mathbf{Q}_p^\times$ let $g(s)$ now be any element of $G'(\mathbf{Q}_p)$ such that

$$\mathrm{Nm} g(s) = b.$$

Let $L(s)$ be the map of $M_K^i(p,D)$ to itself obtained by factoring the map $h \to g(s)h$ on $G'(\mathbf{A}_f)/W_i$.

We can again introduce the necessary maps from one sheaf to another, as well as the correspondence $\varphi(g,s)$ and, if x' is a fixed point of the correspondence, the linear transformation $\varphi_{x'}(g,s)$. The introduction of

$$A(D) = \sum_i \sum \mathrm{trace}\, \varphi_{x'}(g,s)$$

where the inner sum is over the fixed points of $\varphi(g,s)$ on $M_{K'}^i(p,D)$ is more than justified by the next lemma.

Lemma 5.3. *The set $M^i_K(p,D)$ is finite.*

We may regard Z, the centre of G, also as the centre of G'. $G'(\mathbf{Q})$ is discrete in $G(\mathbf{A}_f)$ and

$$G'(\mathbf{Q})Z^o(\mathbf{R}) \setminus G'(\mathbf{A}) \simeq Z^o(\mathbf{R}) \setminus G'(\mathbf{R}) \times G'(\mathbf{Q}) \setminus G'(\mathbf{A}_f).$$

The left side is known to be compact. Since K^pW_i is open in $G(\mathbf{A}_f)$ the lemma follows.

Lemma 5.4. *Let δ_i be the characteristic function of W_i. If $\gamma \in G'(\mathbf{Q})$ let $G'(\gamma)$ be its centralizer in G' and let $\{\gamma\}$ be the conjugacy class of γ in $G'(\mathbf{Q})$. $A(D)$ is equal to the sum over all $\{\gamma\}$ of*

$$\operatorname{trace}\mu(\gamma)\,\operatorname{meas}\big(G'(\gamma,\mathbf{Q}) \setminus G'(\gamma,\mathbf{A}_f)\big)\,\operatorname{meas}\big(G'(\gamma,\mathbf{Q}_p) \setminus G'(\mathbf{Q}_p)\big)$$

times

$$\left\{\sum_i \frac{\delta_i\big(g^{-1}(s)\gamma\big)}{\operatorname{meas} W_i}\right\}\int_{G(\gamma,\mathbf{A}^p_f)\setminus G(\mathbf{A}^p_f)} f_g(h^{-1}\gamma h)\,dh.$$

Suppose h in $G'(\mathbf{A}_f)$ represents a fixed point on $M'_K(p,D)$. There is then an equation

$$g(s)hgkk_p = \gamma h$$

with γ in $G'(\mathbf{Q})$, k in K^p, and k_p in W_i. This equation, for a given h and γ, can be solved for k and k_p if and only if $g^{-1}(s)\gamma \in W_i$ and

$$h_{\hat p}^{-1}\gamma h_{\hat p} \in gK^p.$$

If h represents the fixed point x' and

$$x = L(s)R(g)x' = R(e)x'$$

the map from the fibre of $L^*(s)R^*(g)F_\mu(\mathbf{Q}_\ell)$ at x', which is also $F_\mu(\mathbf{Q}_\ell)_{x'}$ to $F_\mu(\mathbf{Q}_\ell)_{x'}$ is given by composition

$$\Big(\big(u \times g(s)hg\big) \times hg\Big) \times h \to (u \times hg) \times h \to gu \times h.$$

Passing from $F_\mu(\mathbf{Q}_\ell)_{x'}$, which is contained in the fibre of $R_*(e)F_\mu(\mathbf{Q}_\ell)$ at x, to $F_\mu(\mathbf{Q}_\ell)_x$ we send

$$gu \times h \to kk_pgu \times hk_p^{-1}k^{-1} = kgu \times hk_p^{-1}k^{-1}.$$

Thus

$$\operatorname{trace}\varphi_{x'}(g,s) = \operatorname{trace}\mu(k_\ell g_\ell) = \operatorname{trace}\mu(\gamma).$$

Let me observe also that an equation

$$g(s)hg\bar{k}\bar{k}_p = \bar{\gamma}h$$

implies that

$$h^{-1}\bar{\gamma}^{-1}\gamma h = \bar{k}_p^{-1}\bar{k}^{-1}kk_p$$

and hence, by our assumption, that $\gamma = \bar{\gamma}$. $A(D)$ is thus equal to

$$\sum_\gamma \sum_i \sum_h \operatorname{trace} \mu(\gamma).$$

γ runs over $G(\mathbf{Q})$. h runs over those elements of a set of representatives for the double coset space $M^i_{K'}(p, D)$ which satisfy $g^{-1}(s)\gamma \in W_i$ and $h_{\hat{p}}^{-1}\gamma h_{\hat{p}} \in gK^p$. As before we replace the set of representatives for $M^i_{K'}(p, D)$ by hk_α, α varying as before and h varying over a set of representatives for $G'(\mathbf{Q}) \setminus G'(\mathbf{A}_f)/K^pW_i$. The sum is then

$$\sum_\gamma \sum_i \sum_h \operatorname{trace} \mu(\gamma)\delta_i\big(g^{-1}(s)\gamma\big) f_g(h_{\hat{p}}^{-1}\gamma h_{\hat{p}}) \operatorname{meas} K^p$$

which is equal to

$$\sum_\gamma \sum_i \operatorname{trace} \mu(\gamma) \frac{\delta_i\big(g^{-1}(s)\gamma\big)}{\operatorname{meas} W_i} \int_{G'(\mathbf{Q})\setminus G'(\mathbf{A}_f)} f_g(h_{\hat{p}}^{-1}\gamma h_{\hat{p}})\, dh.$$

This expression is readily transformed into that of the lemma.

$A(F)$ and $A(D)$ will occur in our formula for $\operatorname{trace} \tau^1(f_g, s)$. However to express this trace completely, still more supplementary terms must be introduced. N is the group of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and $A^+(\mathbf{Q})$ is the group of diagonal matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

with α and β in $\mathbf{Q}$ and $\alpha\beta > 0$. We form the space

$$C(p) = N(\mathbf{A}_f)A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/K_p$$

as well as

$$C_K = C_K(p) = N(\mathbf{A}_f)A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/K.$$

If $g \in G(\mathbf{A}_f^p)$, $R(g)$ is as usual the transformation of C yielded by right translation of $G(\mathbf{A}_f)$ by g. If $g^{-1}K'g \subseteq K$ and $K'_p = K_p$, in particular if $K' = K \cap gKg^{-1}$, then $R(g) : C_{K'} \to C_K$. If s is in the Weil group and maps to b in $\mathbf{Q}_p^\times$, we again take

$$g(s) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

and let $L(s) : C_K \to C_K$ be left translation by $g(s)$.

We introduce once again

$$\varphi(g, s) = R(e) \times L(s)R(g) : C_{K'} \to C_K \times C_K.$$

x' is a fixed point of $\varphi(g, s)$ if it lies in the inverse image of some point (x, x) on the diagonal. h in $G(\mathbf{A}_f)$ is the representative of a fixed point if there is a γ in $A^+(\mathbf{Q})$, a k in K, and an n in $N(\mathbf{A}_f)$ such that

$$g(s)hgk = n\gamma h. \tag{6}$$

If a similar equation

$$g(s)hg\bar{k} = \bar{n}\bar{\gamma}h$$

also obtains, then

$$k^{-1}\bar{k} = h^{-1}(\gamma^{-1}n^{-1}\bar{n}\bar{\gamma})h.$$

Since $\gamma^{-1}n^{-1}\bar{n}\bar{\gamma}$ and $\gamma^{-1}\bar{\gamma}$ have the same eigenvalues, our basic assumption implies that $\gamma = \bar{\gamma}$. Let

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

It is determined for a given g and s solely by x'. Since

$$hkh^{-1} = g^{-1}(s)n\gamma$$

we have $|a|_p = 1$ and $|d|_p = |b|_p$. In particular if, as we assume, b is a non-unit, then $|a| \neq |d|$. Let a' and d' be a and d cleared of any common divisor or sign and let

$$\operatorname{trace} \mu \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \sum_{\substack{1+j=m \\ i \geq \ell \\ j \geq \ell}} a^i d^j.$$

If $|a| > |d|$, set

$$\psi(\gamma) = (d' - a') \operatorname{trace} \mu(\gamma) + a' a^{m-\ell} d^\ell$$

and if $|a| < |d|$ set

$$\psi(\gamma) = a' a^{m-\ell} d^\ell.$$

Take B to be the sum over all fixed points in $C_{K'}$ of $\psi(\gamma)$.

Lemma 5.5. *Assume that the image b of s in $\mathbf{Q}_p^\times$ is not a unit. Then B is equal to the measure of $A^+(\mathbf{Q}) \setminus A(\mathbf{A}_f)$ times the sum over γ in $A^+(\mathbf{Q})$ of the eigenvalue $\mu(\gamma)$ smallest in absolute value multiplied by*

$$|b|_p^{-\frac{1}{2}} \left\{ \sum_i \frac{\chi_i\big(g^{-1}(s)\gamma\big)}{\operatorname{meas} U_i} \right\} \min\left\{ \left|\frac{a}{d}\right|^{\frac{1}{2}}, \left|\frac{d}{a}\right|^{\frac{1}{2}} \right\}$$

times

$$\left\{ \prod_{q \neq p} \left| \frac{(a-d)^2}{ad} \right|_q^{\frac{1}{2}} \int_{A(\mathbf{A}_f^p)\setminus G(\mathbf{A}_f^p)} f_g(h^{-1}\gamma h)\, dh \right\}.$$

If h_p lies in $B(\mathbf{Q}_p)g_iK_p$ then (6) has a solution for a given h and γ if and only if $g^{-1}(s)\gamma$ lies in U_i and $n_{\hat{p}}$ in $N(\mathbf{A}_f^p)$ exists such that

$$h_{\hat{p}}^{-1} n_{\hat{p}} \gamma h_{\hat{p}} \in gK^p. \tag{7}$$

If

$$V = V^pV_p = \gamma h\big(K \cap h^{-1}\gamma^{-1}N(\mathbf{A}_f)\gamma h\big)h^{-1}\gamma^{-1}$$

then $n_{\hat{p}}$ is uniquely determined modulo V^p and $k_{\hat{p}}$ is determined by $n_{\hat{p}}$.

The sum B is thus equal to

$$\sum_\gamma \sum_i \sum_{h, n_{\hat{p}}} \chi_i\big(g^{-1}(s)\gamma\big)\psi(\gamma).$$

The inner sum is taken over the coset representatives of $N(\mathbf{A}_f)A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/K'$ and $N(\mathbf{A}_f^p)/V^p$ which satisfy $h_p \in B(\mathbf{Q}_p)g_iK_p$ and relation (7). The eigenvalue $\lambda(\gamma)$ of $\mu\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ of smallest absolute value is $a^{m-\ell}d^\ell$ if $|a| < |d|$ and $d^{m-\ell}a^\ell$ if $|d| < |a|$. Thus

$$\psi(\gamma) = \lambda(\gamma)\min\{a', d'\}.$$

We cannot immediately indulge in the usual device of replacing the coset representatives of $C_{K'}$ by hk_α, where h runs over a set of coset representatives of C_K and k_α over a set of representatives of K/K'. To see what modification is necessary, we suppose hk_α and hk_β determine the same coset in $C_{K'}$, so that

$$hk_\alpha = n\gamma hk_\beta\bar{k}$$

with n in $N(\mathbf{A}_f)$, γ in $G(\mathbf{Q})$, and $\bar{k}$ in K'. The equation implies that $n\gamma$ is conjugate to an element of K and it follows then, from our assumption, that $\gamma = 1$. Hence

$$k_\beta^{-1}k_\alpha\bar{k} = k_\beta^{-1}h^{-1}nhk_\beta \in k_\beta^{-1}h^{-1}N(\mathbf{A}_f)hk_\beta \cap K.$$

For a given β the number of α satisfying this condition is the index

$$[k_\beta^{-1}h^{-1}N(\mathbf{A}_f)hk_\beta \cap K : k_\beta^{-1}h^{-1}N(\mathbf{A}_f)hk_\beta \cap K'].$$

Since $K_p = K'_p$, $K = K^p K_p$, and $K' = \big(K' \cap G(\mathbf{A}_f^p)\big)K_p$, we may replace h by $h_{\hat{p}}$, $\mathbf{A}_f$ by $\mathbf{A}_f^p$, K by K^p, and K' by $K' \cap G(\mathbf{A}_f^p)$. There is also no harm in supposing that $k_\beta = 1$. Set

$$W = N(\mathbf{A}_f^p) \cap h_{\hat{p}} K^p h_{\hat{p}}^{-1}.$$

We want to compute

$$[W : W \cap h_{\hat{p}}\big(K' \cap G(\mathbf{A}_f^p)\big)h_{\hat{p}}^{-1}].$$

The smaller group is

$$W \cap h_{\hat{p}} g K^p g^{-1} h_{\hat{p}}^{-1} \tag{8}$$

because K' is $gKg^{-1} \cap K$. We are also supposing the relation (7) is satisfied. When that is so

$$h_{\hat{p}} g K^p g^{-1} h_{\hat{p}}^{-1} = n_{\hat{p}} \gamma h_{\hat{p}} K^p h_{\hat{p}}^{-1} \gamma^{-1} n_{\hat{p}}^{-1}$$

and the intersection (8) is

$$W \cap n_{\hat{p}} \gamma W \gamma^{-1} n_{\hat{p}}^{-1} = W \cap \gamma W \gamma^{-1}.$$

The index $[W : W \cap \gamma W \gamma^{-1}]$ is clearly $\prod_{q \neq p} |a'|_q^{-1}$.

We may therefore replace the sum over coset representatives of $C_{K'}$ by a sum over hk_α provided we replace $\lambda(\gamma)\min\{a', d'\}$ by $\lambda(\gamma)\min\{1, \frac{d}{a}\}$ that is, provided we divide by $\prod_{q \neq p} |a'|_q^{-1}$. As usual we drop the sum over α and replace condition (7) by

$$h_{\hat{p}}^{-1} n_{\hat{p}} \gamma h_{\hat{p}} \in K^p g K^p.$$

It must of course be observed that the group V or V^p, which at first sight depends on h and k_α, depends in fact only on h.

Taking the above discussion into account, I rewrite the sum B as

$$\sum_\gamma \sum_i \sum_h \sum_{n_{\hat{p}}} \chi_i\big(g^{-1}(s)\gamma\big) \min\{1, \frac{d}{a}\}\lambda(\gamma).$$

Here $h = h_{\hat{p}} h_p$ runs over those elements in a set of coset representatives for C_K for which h_p lies in $B(\mathbf{Q}_p) g_i K_p$ and $n_{\hat{p}}$ runs over a set of coset representatives of $N(\mathbf{A}_f^p)/V^p$ for which

$$h_{\hat{p}}^{-1} n_{\hat{p}} \gamma h_{\hat{p}} \in K^p g K^p.$$

The sum over $n_{\hat{p}}$ may be written as

$$\frac{\operatorname{meas} K^p}{\operatorname{meas} V^p} \int_{N(\mathbf{A}_f^p)} f_g(h_{\hat{p}}^{-1} n_{\hat{p}} \gamma h_{\hat{p}})\, dn_{\hat{p}} \tag{9}$$

multiplied by

$$\chi_i\big(g^{-1}(s)\gamma\big) \min\Big\{1, \frac{d}{a}\Big\} \lambda(\gamma) |b|_p^{-1}. \tag{10}$$

Observe that

$$\gamma^{-1} V^p \gamma = h K^p h^{-1} \cap N(\mathbf{A}_f^p) = U^p$$

depends only on h and not on γ. Changing variables one sees that (9) is equal to

$$\frac{\operatorname{meas} K^p}{\operatorname{meas} U^p} \int_{N(\mathbf{A}_f^p)} f_g(h_{\hat{p}}^{-1} \gamma n_{\hat{p}} h_{\hat{p}})\, dn_{\hat{p}}. \tag{11}$$

We may take our coset representatives for C_K to be of the form hg_i, where $h = h_{\hat{p}} h_p$ and h_p lies in $A(\mathbf{Q}_p)$. For each i, h then varies over a set of coset representatives for

$$N(\mathbf{A}_f^p) A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f^p) A(\mathbf{Q}_p) / K^p U_i. \tag{12}$$

The expression (10) depends on hg_i only through g_i. Hence to compute B we sum (9), or rather (11), over coset representatives for (12). Choose a product measure on $G(\mathbf{A}_f^p) A(\mathbf{Q}_p)$. The given measure on $N(\mathbf{A}_f^p)$, combined with the measure on $A^+(\mathbf{Q})$ which assigns the measure 1 to each point, yields a measure on $N(\mathbf{A}_f^p) A^+(\mathbf{Q})$. Taking quotients we obtain a measure on $N(\mathbf{A}_f^p) A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f^p) A(\mathbf{Q}_p)$. The measure of the double coset in (12) represented by h is

$$\frac{\operatorname{meas} K^p \cdot \operatorname{meas} U_i}{\operatorname{meas} U^p}.$$

The sum of (11) over coset representatives for (12) is therefore

$$\frac{1}{\operatorname{meas} U_i} \int_{N(\mathbf{A}_f^p) A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f^p) A(\mathbf{Q}_p)} \Big\{ \int_{N(\mathbf{A}_f^p)} f_g(h_{\hat{p}}^{-1} \gamma n h_{\hat{p}})\, dn \Big\}\, dh. \tag{13}$$

The map

$$n \to n' = \gamma^{-1}n^{-1}\gamma n$$

is a homeomorphism of $N(\mathbf{A}_f^p)$ with itself and

$$dn' = \Big\{\prod_{q\neq p}\Big|1-\frac{d}{a}\Big|_1\Big\}\,dn.$$

Since

$$\Big|1-\frac{d}{a}\Big|_q = \Big|\frac{d}{a}\Big|_q^{1/2}\Big|\frac{(a-d)^2}{ad}\Big|_q^{1/2}$$

and

$$\Big|\frac{d}{a}\Big|_p\Big\{\prod_{q\neq p}\Big|\frac{d}{a}\Big|_q\Big\} = \Big|\frac{a}{d}\Big|$$

while

$$\Big|\frac{a}{d}\Big|_p = |b|_p^{-1}$$

a change of variables in (13) yields the product of

$$|b|_p^{-1/2}\Big|\frac{a}{d}\Big|^{1/2}\prod_{q\neq p}\Big|\frac{(a-d)^2}{ad}\Big|_q^{1/2}$$

and

$$\frac{1}{\operatorname{meas} U_i}\int_{N(\mathbf{A}_f^p)A^+(\mathbf{Q})\backslash G(\mathbf{A}_f^p)A(\mathbf{Q}_p)}\Big\{\int_{N(\mathbf{A}_f^p)} f_g(h_{\hat{p}}^{-1}n^{-1}\gamma n h_{\hat{p}})\,dn\Big\}\,dh.$$

The double integral is

$$\int_{A^+(\mathbf{Q})\backslash G(\mathbf{A}_f^p)A(\mathbf{Q}_p)} f_g(h_p^{-1}\gamma h_{\hat{p}})\,dh$$

or

$$\operatorname{meas}\big(A^+(\mathbf{Q})\setminus A(\mathbf{A}_f)\big)\int_{A(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f_g(h^{-1}\gamma h)\,dh.$$

We obtain the lemma simply by reassembling the above data.

There are further spaces, simpler than the previous ones, to be introduced. Let $\bar{G} = GL(1)$ and let

$$\bar{K} = \bar{K}^p\bar{K}_p = \{\det k \,\big|\, k \in K\}.$$

We introduce the space

$$\bar{M}_K = \bar{G}^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/\bar{K}$$

where $\bar{G}^+(\mathbf{Q})$ is the set of positive elements in $\bar{G}(\mathbf{Q})$. If as before V_i is the set of $\det k$, $k \in U_i$, we set

$$\bar{M}^i_K = \bar{G}^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/\bar{K}^p V_i.$$

Each of these spaces is finite.

$\bar{g}$ is $\det g$ and $\bar{g}(s)$ is just b, the image of s in $\mathbf{Q}_p^\times = \bar{g}(\mathbf{Q}_p)$. The maps $R(g) : \bar{M}^i_K \to \bar{M}^i_K$ are defined by right multiplication by $\bar{g}$ and $L(s) : \bar{M}^i_K \to \bar{M}^i_K$ by left multiplication by $\bar{g}(s)$. We do not introduce any sheaves unless μ is one-dimensional. Assume for now that this is so. μ is then of the form $g \to \nu(\det g)$. If $\bar{M}^i = \bar{G}^+(\mathbf{Q}) \setminus \bar{G}(\mathbf{A}_f)/V_i$ we set

$$F_\nu(\mathbf{Q}_\ell) = L(\mathbf{Q}_\ell) \times_{\bar{K}^p} \bar{M}^i$$

where $k : v \times y \to \nu(k_\ell^{-1}) v \times yk$. Once this is done we can introduce

$$\varphi(g,s) : \cup \bar{M}^i_K \to \cup \bar{M}^i_K \times \bar{M}^i_K$$

as well as the various maps between sheaves. We let $\bar{A}$ be the product of

$$[K : K'] \,\big|\, (|b|_p^{-1} + 1)$$

with the sum over the fixed points x' of $\varphi(g,s)$ on $\cup \bar{M}^i_K$ of $\operatorname{trace} \varphi_{x'}(g,s)$.

Lemma 5.6. *If μ is one-dimensional, $\bar{A}$ is equal to*

$$(|b|_p^{-1} + 1) \sum_i \Big\{ \sum \chi(b) \int_{G(\mathbf{A}_f^p)} f_g(h) \chi(\det h)\, dh \Big\}.$$

The inner sum is over all quasi-characters χ of $\bar{G}(\mathbf{Q}) \setminus \bar{G}(\mathbf{A})$ which are trivial on $\bar{K}^p V_i$ and such that $\chi(z)\nu(z) = 1$ of $z \in \bar{G}(\mathbf{R})$ is positive.

If h in $\bar{G}(\mathbf{A}_f)$ represents a fixed point x'

$$bh\bar{g}\bar{k} = \delta h$$

with $\bar{k}$ in $\bar{K}^p V_i$ and δ in $\bar{G}^+(\mathbf{Q})$. In general the map from the fibre at $bh\bar{g}$ to that at h may be represented by

$$v \times bh\bar{g} \to \bar{g} v \times h = \nu(\bar{g}_\ell) v \times h.$$

When $bh\bar{g}\bar{k} = \delta h$ the point represented by the expression on the right is also represented by

$$\nu(\bar{k}_\ell \bar{g}_\ell) v \times bh\bar{g}.$$

Thus

$$\mathrm{trace}\, y_{x'}(g,s) = \mathrm{trace}\, \bar{\nu}(\bar{k}_\ell \bar{g}_\ell) = \mathrm{trace}\, \nu(\delta).$$

δ is uniquely determined by b and $\bar{g}$. Thus

$$\bar{A} = \sum_i \bar{A}_i$$

where $\bar{A}_i$ is 0 if there are no fixed points on $\bar{M}^i_{K'}$, that is, if $b\bar{g}$ does not lie in $\bar{G}^+(\mathbf{Q})\bar{K}^p V_i$, but where $\bar{A}_i$ is otherwise $[K : K'](|b|_p^{-1} + 1)$ times

$$[\bar{G}(\mathbf{A}_f) : \bar{G}^+(\mathbf{Q})\bar{K}^p V_i]\nu(\delta).$$

Let ε_i be the characteristic function of $\bar{G}^+(\mathbf{Q})\bar{K}^p V_i$. Then

$$\sum \chi(b) \int_{G(\mathbf{A}_f^p)} f_g(h)\chi(\det h)\, dh$$

is equal to

$$[\bar{G}(\mathbf{A}_f) : \bar{G}^+(\mathbf{Q})\bar{K}^p V_i] \int_{G(\mathbf{A}_f^p)} |\nu(b \det h)|^{-1} f_g(h)\varepsilon_i(b \det h)\, dh.$$

Recall that f_g is the characteristic function of $K^p g K^p$ divided by the measure of K^p. If $h = k_1 g k_2$ then

$$\varepsilon_i(b \det h) = \varepsilon_i(b\bar{g}).$$

Thus the integrand vanishes identically if $\bar{A}_i = 0$. Otherwise the integral is equal to

$$\frac{\mathrm{meas}(K^p g K^p)}{\mathrm{meas}\, K^p} = [K : K']$$

multiplied by

$$|\nu(b \det h)|^{-1} = |\nu(\delta)|.$$

This equality is a consequence of the product formula applied to $\nu(\delta)$. The lemma is an immediate consequence of these calculations.

The following proposition will be proved in the next paragraph.

Proposition 5.7. *If K_p is given and K^p is then chosen sufficiently small and if s in the Weil group maps to an element of absolute value less that 1 in $\mathbf{Q}_p^\times$, then the trace of $\tau^1(f_g, s)$ is equal to*

$$-\Big\{\sum_F A(F)\Big\} - A(D) - B + \bar{A}$$

where $\bar{A}$ is taken to be 0 if μ is not one-dimensional. The sum is over a set of representatives for the isomorphism classes of imaginary quadratic fields.

We shall also find a formula for $\operatorname{trace}\tau^2(f_g, s)$. Define $\bar{A}_0$ just as $\bar{A}$ was defined except that the union of $\bar{M}_K^i$ is replaced by $\bar{M}_K$. Let

$$W = \{g' \in G'(\mathbf{Q}_p) \,\big|\, Nmg' \in \bar{K}_p\}$$

and set

$$M_K(p, D) = G'(\mathbf{Q}) \setminus G'(\mathbf{A}_f)/K^pW.$$

Define $A_o(D)$ in the same way as $A(D)$ except that $M_K(p, D)$ replaces the union of $M_K^i(p, D)$.

Proposition 5.8 *With the assumptions of the previous proposition the trace of $\tau^2(f_g, s)$ is equal to*

$$(|b|_p^{-1} + 1)\big(A(D) - A_o(D)\big) - (\bar{A} - \bar{A}_o).$$

6. The trace formula. This formula, which will be used to prove Propositions 5.7 and 5.8 has been described, in a form pretty close to that needed, in [4.1] and [6.3]. The preprint [6.1] is also a very good reference. I write it out again here, or at least the part of it we require, with the appropriate changes and comments.

It will save a lot of trouble if we fix some conventions for measures. Let F be a global field, which to avoid irrelevant explanations I take of characteristic zero, and let ψ be a non-trivial character of $\mathbf{A}(F)$, trivial on F. For every place v, ψ_v defines a non-trivial character ψ_v for F_v. Let dx_v be the measure on F_v self-dual with respect to ψ_v. If Ω is an n-dimensional analytic manifold over F_v and $\omega = \varphi(x^1, \cdots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$ is an n-form we set $|\omega|_v = |\varphi(x^1, \cdots, x^n)|\, dx_v^1 \cdots dx_v^n$. In particular, if G is a connected algebraic group over F and ω is a right-invariant form of highest degree also defined over F, then $|\omega|_v$ is a Haar measure on $G(F_v)$. If λ is the representation of $\mathfrak{G}(\bar{F}/F)$ on the lattice of rational characters of G and $L(s, \lambda_v)$ is defined as, for example, in [6.4], then

$$dg_v = L(1, \lambda_v)|\omega|_v$$

is called the Tamagawa measure on $G(F_v)$ determined by ω. It will be convenient to write $L(s, G/F_v)$ for $L(s,\lambda_v)$. The product measure $\prod_v dg_v$ will be called the unnormalized Tamagawa measure on $G(\mathbf{A})$. If r is the multiplicity of the trivial representation in λ, the quotient of the unnormalized Tamagawa measure by $\lim_{s\to 1}(s-1)^r L(s,\lambda)$ is called the Tamagawa measure. $L(s,\lambda)$ is defined by a product over all places. We shall use Tamagawa measures locally and unnormalized Tamagawa measures globally. On discrete groups we take the measure which assigns the value 1 to each point. On quotient spaces we take, unless otherwise stated, quotient measures, at least when this is possible, and on the Pontrjagin dual $D(M)$ of a locally compact group M we take the measure dual to that in M.

If Φ is a function on $G(\mathbf{A})$ which satisfies

$$\Phi(zg) = \mu(z)\Phi(g)$$

for z in $Z^o(\mathbf{R})$, has compact support modulo $Z^o(\mathbf{R})$, is bi-invariant with respect to some open subgroup of $G(\mathbf{A}_f)$, and infinitely differentiable as a function of its coordinate at infinity, then

$$r(\Phi) = \int_{Z^o(\mathbf{R})\backslash G(\mathbf{A})} \Phi(h) r(h)\, dh$$

operates on $L_{sp}(\mu)$ and is of trace class. The trace formula gives its trace.

The reader will notice that in [4.1] and [6.3] $Z(\mathbf{A})$ played the role here assumed by $Z^o(\mathbf{R})$. This has an effect on the formula. Moreover the trace formula as described in [4.1] gives the trace not of the representation r on $L_p(\mu)$ but of the representation on the sum of $L_p(\mu)$ and $L^o_{se}(\mu)$. Thus we have to add to it the negative of the trace on $L^o_{se}(\mu)$. This is

$$-\sum_{\chi} \int_{Z^o(\mathbf{R})\backslash G(\mathbf{A})} \chi(\det h)\Phi(h)\, dh. \tag{1}$$

The sum is over all quasi-charaacters of $\bar{G}(Q)\setminus \bar{G}(\mathbf{A})$ such that $\chi(\det z)\mu(z)$ is 1 for z in $Z^o(\mathbf{R})$.

We now rewrite, in so far as we need them, the terms (i) to (viii) of the trace formula given on p. 516–517 of Jacquet–Langlands. The term (i) is

$$\operatorname{meas}\big(Z^o(\mathbf{R})G(\mathbf{Q})\setminus G(\mathbf{A})\big) \sum_{\gamma\in Z(\mathbf{Q})} \Phi(\gamma). \tag{2}$$

The term (ii) becomes

$$\sum_{\{\gamma\}} \operatorname{meas}\big(Z^o(\mathbf{R})G(\gamma,\mathbf{Q})\setminus G(\gamma,\mathbf{A})\big) \int_{G(\gamma,\mathbf{A})\backslash G(\mathbf{A})} \Phi(h^{-1}\gamma h)\, dh. \tag{2$'$}$$

The sum is over conjugacy classes in $G(\mathbf{Q})$ whose eigenvalues do not lie in $\mathbf{Q}$. The term (iii) does not occur for a field of characteristic 0. The terms (vi) and (vii), as well as the first part of (v), will vanish for the Φ to which we shall apply the trace formula, so there is no need to write them out explicitly.

Each term mentioned so far, including the ones not written out explicitly, has been invariant, in the sense that it does not change if Φ is replaced by Φ' with $\Phi'(h) = \Phi(x^{-1}hx)$. This is not true of (iv), the last part of (v), and (viii). Their sum can however be written as a sum of invariant terms. Let me describe the form taken by these terms when Φ is of the form

$$\Phi(h) = \prod_v \Phi_v(h_v).$$

If $a \in A(\mathbf{Q}_v)$ is a diagonal matrix set

$$F^A(a, \Phi_v) = \Delta(a) \int_{A(\mathbf{Q}_v)\backslash G(\mathbf{Q}_v)} \Phi_v(h^{-1}ah)\, dh$$

with

$$\Delta(a) = \left| \frac{(\alpha-\beta)^2}{\alpha\beta} \right|_v^{\frac{1}{2}}$$

if α and β are the eigenvalues of a. I observe that if the representation $\rho(\lambda_v, \eta_v)$, where λ_v and η_v are two quasi-characters of $\mathbf{Q}_v^\times$, is defined as the previous paragraph, then

$$\operatorname{trace} \rho(\Phi_v; \lambda_v, \eta_v) = \int_{A(\mathbf{Q}_v)} F^A(a, \Phi_v) \lambda_v(\alpha)\eta_v(\beta)\, da$$

if v is finite. If $\mathbf{Q}_v$ is $\mathbf{R}$, the right side must be replaced by an integral over $Z^o(\mathbf{R}) \setminus A(\mathbf{R})$ and the formula is only valid when $\lambda_v\eta_v$ equals μ^{-1} on $Z^o(\mathbf{R})$.

The distribution $\Phi_v \to F^A(a, \Phi_v)$ is invariant. There is for each v and each γ in $A(\mathbf{Q})$ another invariant distribution $\Phi_v \to D(\gamma, \Phi_v)$, rather complicated to write out explicitly, such that the remaining parts of the trace formula may be combined into

$$\text{(3)} \qquad \operatorname{meas}\big(Z^o(\mathbf{R})Z(\mathbf{Q}) \setminus Z(\mathbf{A})\big) \sum_{\gamma \in A(\mathbf{Q})} \sum_v \Big\{ D(\gamma, \Phi_v) \prod_{w \neq v} F^A(\gamma, \Phi_w) \Big\}.$$

All but finitely many of the terms in this double sum are 0.

Given g in $G(\mathbf{A}_f^p)$ and an s in the Weil group which maps to an element of absolute value less than 1 in $Q_p^\times$, we shall choose Φ^1 and Φ^2 so that

$$\text{(4)} \qquad \operatorname{trace} r(\Phi^1) = -\operatorname{trace} \tau^1(f_g, s)$$

and

$$\text{(5)} \qquad \operatorname{trace} r(\Phi^2) = -\operatorname{trace} \tau^2(f_g, s).$$

Write h in $G(\mathbf{A})$ as $h_\infty h_p h_f^p$ with h_∞ in $G(\mathbf{R})$, h_p in $G(\mathbf{Q}_p)$, and h_f^p in $G(\mathbf{A}_f^p)$. The functions Φ^1 and Φ^2 will have the form

$$\text{(6)} \qquad \Phi^1(h) = \Phi_\infty(h_\infty)\Phi_p^1(h_p, s) f_g(h_f^p)$$

and

$$\text{(7)} \qquad \Phi^2(h) = \Phi_\infty(h_\infty)\Phi_p^2(h_p, s) f_g(h_f^p).$$

The function Φ_∞ is of course to be infinitely differentiable, with compact support modulo $Z^o(\mathbf{R})$, and to satisfy

$$\Phi_\infty(zg) = \mu(z)\Phi_\infty(g)$$

for z in $Z^o(\mathbf{R})$. If π_∞ is an irreducible admissible representation of $G(\mathbf{R})$ which agrees with μ^{-1} in $Z^o(\mathbf{R})$ then $\pi_\infty(\Phi_\infty)$ is defined. We demand that the trace of $\pi_\infty(\Phi_\infty)$ be zero unless $\pi_\infty \simeq \pi(\mu)$, when it is to be -1, unless $\pi_\infty \simeq \tilde{\mu}$ or $\operatorname{sgn}(\det g)\tilde{\mu}$, when it is to be 1. $\tilde{\mu}$ is as before the representation contragredient to μ. The existence of Φ_∞, at least for μ trivial on Z, is proved by Duflo–Labesse as an application of a Paley–Wiener theorem for the projective group. The existence in general is proved in a similar manner.

The conditions imposed on Φ_∞ entail several properties that will be important to us. First of all

$$F^A(\gamma, \Phi_\infty) = 0$$

if $\gamma \in A(\mathbf{R})$. This equation is what forces the terms of the trace formula that we have not written out explicitly to vanish. It means moreover that (3) is equal to

$$\text{(8)} \qquad \operatorname{meas}\big(Z^o(\mathbf{R})Z(\mathbf{Q}) \backslash Z(\mathbf{A})\big) \sum_{\gamma \in A(\mathbf{Q})} D(\gamma, \Phi_\infty) \prod_{w \neq \infty} F^A(\gamma, \Phi_w).$$

Let $G'(\mathbf{R})$ be the multiplicative group of a quaternion algebra over $\mathbf{R}$. As in §15 of Jacquet–Langlands, the measure on $G(\mathbf{R})$ determines one on $G'(\mathbf{R})$. The centre of $G'(\mathbf{R})$, which to simplify the notation we identify with $Z(\mathbf{R})$, is also provided with a measure. If $z \in Z(\mathbf{R})$

$$\Phi_\infty(z) = \frac{-1}{\operatorname{meas}\big(Z^o(\mathbf{R}) \backslash G'(\mathbf{R})\big)} \operatorname{trace} \mu(z).$$

If γ in $G(\mathbf{R})$ has two distinct real eigenvalues

$$\int_{G(\gamma,\mathbf{R})\backslash G(\mathbf{R})} \Phi_\infty(h^{-1}\gamma h)\, dh = 0$$

but if γ has complex eigenvalues

$$\int_{G(\gamma,\mathbf{R})\backslash G(\mathbf{R})} \Phi_\infty(g^{-1}\gamma g)\, dg = \frac{1}{\operatorname{meas}\big(Z^o(\mathbf{R}) \setminus G(\gamma,\mathbf{R})\big)} \operatorname{trace}\mu(\gamma).$$

All these facts are consequences of the general theory of harmonic analysis on reductive Lie groups. We also need to evaluate $D(\gamma, \Phi_\infty)$. For μ trivial on Z this has been done in Duflo–Labesse [6.3]; otherwise one has either to carry out their proof for general μ or to appeal to a forthcoming paper (tentatively entitled "The Fourier transform of some tempered distributions") by J. Arthur, who discusses the problem for all groups of real rank 1. In any case, $D(\gamma, \Phi_\infty)$ is zero if γ has eigenvalues α and β of opposite sign. Otherwise let

$$\varepsilon\Big|\frac{\beta}{\alpha}\Big| d\Big(\Big|\frac{\alpha}{\beta}\Big|\Big)$$

where ε is some constant, be the measure on $Z(\mathbf{R}) \setminus A(\mathbf{R})$. Then $D(\gamma, \Phi_\infty)$ is

$$\frac{1}{4\varepsilon} \min\Big\{\Big|\frac{\alpha}{\beta}\Big|^{\frac{1}{2}}, \Big|\frac{\beta}{\alpha}\Big|\Big\}$$

times the eigenvalue of $\mu(\gamma)$ of the smallest absolute value.

$\Phi_p^1(h,s)$ and $\Phi_p^2(h,s)$ will be locally constant functions on $G(\mathbf{Q}_p)$ with compact support. They will be chosen to satisfy the following conditions, in which π_p denotes an irreducible admissible representation of $G(\mathbf{Q}_p)$ and b is the image of s in $\mathbf{Q}_p^\times$.

(i) If π_p is infinite dimensional and $\pi = \pi(\tau_p)$, where τ_p, a complex representation of the Weil group, is the direct sum of two one-dimensional representations, then

$$\operatorname{trace}\pi_p\big(\Phi_p^1(s)\big) = |b|_p^{-\frac{1}{2}} m(\pi_p, K_p) \operatorname{trace}\tau_p(s)$$
$$\operatorname{trace}\pi_p\big(\Phi_p^2(s)\big) = 0.$$

(ii) If $\pi_p = \sigma(\lambda_p, \eta_p)$ is a special representation then

$$\operatorname{trace}\pi_p\big(\Phi_p^1(s)\big) = 0$$
$$\operatorname{trace}\pi_p\big(\Phi_p^2(s)\big) = |b|_p^{-\frac{1}{2}} m(\pi_p, K_p)\big(\lambda_p(b) + \eta_p(b)\big).$$

(iii) If $\pi_p = \pi_p(\tau_p)$ where τ_p is the direct sum of λ_p and η_p, two quasi-characters of $\mathbf{Q}_p^\times$ with $\lambda_p^{-1}\eta_p(x) = |x|_p$, so that π_p is one-dimensional and if $\pi'_p = \sigma(\lambda_p, \eta_p)$, then

$$\operatorname{trace}\pi_p\big(\Phi_p^1(s)\big) = |b|_p^{-\frac{1}{2}}\{m(\pi_p, K_p) + m(\pi'_p, K_p)\}\operatorname{trace}\tau_p(s)$$
$$\operatorname{trace}\pi_p\big(\Phi_p^2(s)\big) = -|b|_p^{-\frac{1}{2}} m(\pi'_p, K_p)\big(\lambda_p(b) + \eta_p(b)\big).$$

(iv) If π_p is absolutely cuspidal

$$\operatorname{trace}\pi_p\big(\Phi_p^1(s)\big) = \operatorname{trace}\pi_p\big(\Phi_p^2(s)\big) = 0.$$

$\Phi_p^1(s)$ and $\Phi_p^2(s)$ are by no means unique. To establish their existence, as well as some of their additional properties, requires some preparation. I recall that if γ in $G(\mathbf{Q}_p)$ has eigenvalues γ_1, γ_2, which need not lie in $\mathbf{Q}_p$, then

$$\Delta(\gamma) = \left|\frac{(\gamma_1 - \gamma_2)^2}{\gamma_1\gamma_2}\right|_p^{\frac{1}{2}}.$$

If T is a Cartan subgroup of G, over $\mathbf{Q}_p$, and f a locally constant function on $G(\mathbf{Q}_p)$ with compact support, we set as usual

$$F^T(\gamma, f) = \Delta(\gamma)\int_{T(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} f(g^{-1}\gamma g)\,dg$$

for γ regular in $T(\mathbf{Q}_p)$.

The collection of functions $\gamma \to F^T(\gamma, f)$ on the regular elements in the Cartan subgroups is not arbitrary. To characterize them we apply a result of Shalika; but first an observation. A regular element in G is an element whose centralizer is of dimension 2. Thus h is regular if and only if it is not a scalar matrix. We introduce the map φ from the variety of $\hat{G}$ of regular elements in G to the affine plane X given by

$$\varphi : h \to (\operatorname{trace} h, \det h).$$

It is possible to introduce the notion of a regular two-form on $\hat{G}$ relative to X. First of all a two-form on a Zaariski-open subset Y of $\hat{G}$ will be called relatively trivial if Y can be covered by Zariski-open sets Y_α, mapping X_α in X, such that on Y_α the form is a sum of $\beta\, d\lambda\, d\eta$ where β and λ are regular on Y_α and η is the pullback to Y_α of a regular function on X_α. To give a two-form on $\hat{G}$ regular relative to X, one gives a Zariski-open covering $\{Y_\alpha\}$ of $\hat{G}$, and on each Y_α a regular two-form ω_α, defined only modulo relatively trivial two-forms, such that $\omega_\alpha - \omega_\beta$ is trivial on $Y_\alpha \cap Y_\beta$.

The fibres of φ are smooth and two-dimensional. The restriction of a regular relative two-form ω to a fibre yields a well-defined two-form on the fibre. If γ is regular, the map

$$h \to h^{-1}\gamma h$$

defines an isomorphism of $G(\gamma)\setminus G$ with the fibre over $(\operatorname{trace}\gamma, \det\gamma)$. $G(\gamma)$ is the centralizer of γ. Let ω_γ be the pullback to $G(\gamma)\setminus G$ of the restriction of ω to this fibre.

Suppose on the other hand that λ is a non-zero invariant form on degree 4 on G and η a non-zero invariant form of degree 2 on $G(\gamma)$. It is possible to find a regular two-form ν on $G(\gamma)\setminus G$, with pullback ν' to G, and a two-form η' on G regular relative to $G(\gamma)\setminus G$ and right-invariant under G such that the restriction to η' to $G(\gamma)$ is η and such that $\lambda = \eta'\nu'$. ν is uniquely determined by λ and η and we denote it $\frac{\lambda}{\eta}$.

Lemma 6.1. *Suppose non-zero invariant forms λ and η, of degrees 4 and 2 respectively, are given on G and A. There is then a unique two-form ω on $\hat{G}$ regular relative to X which is invariant under the adjoint action of G and satisfies*

$$(x_o - y_o)\frac{\lambda}{\eta} = \omega_\gamma$$

if

$$\gamma = \begin{pmatrix} x_o & 0 \\ 0 & y_o \end{pmatrix}$$

is a regular element of A. The restriction of ω to any fibre is non-zero.

It is enough to verify this for

$$\eta = \frac{dX}{X}\frac{dY}{Y}$$

and

$$\lambda = \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(\alpha\delta - \beta\gamma)^2}$$

when

$$h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

I define ω as follows. If $\beta \neq 0$

$$\omega = -\frac{d\alpha\, d\beta}{\beta}.$$

If $\gamma \neq 0$

$$\omega = \frac{d\alpha\, d\gamma}{\gamma}$$

and if $\alpha - \delta \neq 0$

$$\omega = \frac{1}{\alpha - \delta} d\gamma\, d\beta.$$

Since one of β, γ and $\alpha - \delta$ is non-zero at any point of G, ω is defined. The compatability is a consequence of the relative triviality of

$$d\alpha + d\delta$$

and

$$\alpha\, d\delta + \delta\, d\alpha - \beta\, d\gamma - \gamma\, d\beta.$$

The entire lemma can now be verified by direct, although somewhat lengthy, calculations. I omit them.

Suppose λ and η, and hence ω, have been chosen. If γ is the regular let η_γ be the two-form on $G(\gamma)$ defined by

$$\frac{\lambda}{\eta_\gamma} \omega_\gamma.$$

If the centralizer of γ is a Cartan subgroup and x and y are the rational characters given by some diagonalization, and if η and λ are chosen as in the proof of the lemma, then

$$\eta_\gamma = \frac{1}{x(\gamma) - y(\gamma)} \frac{dx}{x} \frac{dy}{y}.$$

If

$$\gamma = \begin{pmatrix} z_o & x_o \\ 0 & z_o \end{pmatrix}$$

then

$$\eta_\gamma = \frac{1}{x_o} \frac{dz\, dx}{z^2}.$$

Suppose then λ and η have been fixed, as above or in some other way. If for each Cartan subgroup T defined over $\mathbf{Q}_p$ we have introduced on T a non-zero invariant two-form η_T, also defined over $\mathbf{Q}_p$, then for any locally constant function f with compact support we may consider the functions $F^T(\gamma, f)$. Let $c(\gamma)$ in $\mathbf{Q}_p$ be the constant defined by

$$c(\gamma)\eta_\gamma = \eta_T$$

and let

$$E^T(\gamma, f) = \frac{|c(\gamma)|_p}{\Delta(\gamma)|\mathrm{det}\gamma|_p^{\frac{1}{2}}} \frac{L(1, T/\mathbf{Q}_p)}{L(1, G/\mathbf{Q}_p)} F^T(\gamma, f).$$

As before λ determines a measure not only on $G(\mathbf{Q}_p)$ but also on $G'(\mathbf{Q}_p)$, if G' is the multiplicative group of a quaternion algebra which is not split at p.

We shall call a family $\{a^T\}$, where a^T is a locally constant function on $\hat{T}(\mathbf{Q}_p) = \hat{G}(\mathbf{Q}_p) \cap T(\mathbf{Q}_p)$ a Shalika family if the following conditions are satisfied.

(i) If $T' = h^{-1}Th$ with h in $G(\mathbf{Q}_p)$ and $\gamma \in T(\mathbf{Q}_p)$, then

$$a^{T'}(h^{-1}\gamma h) = A^T(\gamma).$$

(ii) The support of a^T is relatively compact on $T(\mathbf{Q}_p)$.

(iii) There is a locally constant function ξ on $Z(\mathbf{Q}_p)$ with compact support such that if $z \in Z(\mathbf{Q}_p)$

$$a^A(\gamma) = \xi(z)$$

for γ in the intersection of some neighborhood of z in $A(\mathbf{Q}_p)$ with $\hat{A}(\mathbf{Q}_p)$.

(iv) There is another locally constant function ζ on $Z(\mathbf{Q}_p)$ with compact support such that if T is a Cartan subgroup associated to a quadratic extension of $\mathbf{Q}_p$ then for each z in $Z(\mathbf{Q}_p)$

$$a^T(\gamma) = \xi(z) = \frac{|c(\gamma)|_p}{|\det\gamma|_p^{\frac{1}{2}}} \frac{L(1, T/\mathbf{Q}_p)}{L(1, G/\mathbf{Q}_p)} \operatorname{meas}\big(T(\mathbf{Q}_p) \setminus G'(\mathbf{Q}_p)\big)\zeta(z)$$

on the intersection of some neighborhood of z in $T(\mathbf{Q}_p)$ with $\hat{T}(\mathbf{Q}_p)$. As in §15 of Jacquet–Langlands, we have felt free to regard T also as a Cartan subgroup of G'.

Lemma 6.2. *There exists a locally constant function f on $G(\mathbf{Q}_p)$ with compact support such that*

$$a^T \equiv E^T(f)$$

for all T if and only if $\{a^T\}$ is a Shalika family. $\xi(z)$ is then the integral of f over the orbits of $\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with respect to the measure defined by ω and

$$\zeta(z) = f(z).$$

I observe that

$$\frac{|c(\gamma)|_p}{\Delta(\gamma)|\det\gamma|_p^{\frac{1}{2}}}$$

is a constant which depends only on T and not on γ. Moreover $E^T(\gamma, f)$ depends on ω but not on η_T.

Consider a family $\{E^T(f)\}$. The necessity of (i) is clear. Condition (iii), as well as condition (ii) together with the local constancy for $T = A$, is a consequence of the familiar relation

$$\Delta(\gamma) \int_{N(\mathbf{Q}_p)} \int_{G(\mathbf{Z}_p)} f(k^{-1}n^{-1}\gamma nk)\, dk\, dn = \left|\frac{\alpha}{\beta}\right|^{\frac{1}{2}} \int_{N(\mathbf{Q}_p)} \int_{G(\mathbf{Z}_p)} f(k^{-1}\gamma nk)\, dn\, dk \tag{9}$$

for

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and for $dn = dx$ when

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

because the right-hand side is equal to

$$\iint f\left(k^{-1}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} k\right)\frac{d\alpha}{|\alpha|^2}\,dk \tag{10}$$

when γ is close to z. We should of course recall that

$$dh = \left|\frac{\beta}{\alpha}\right| dx\,\frac{d\alpha}{|\alpha|}\,\frac{d\beta}{|\beta|}\,dk \tag{11}$$

is, for

$$h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k$$

a Haar measure on $G(\mathbf{Q}_p)$. To be more precise, the double integral on the left side of (9) is really an integral over $A(\mathbf{Q}_p)\setminus G(\mathbf{Q}_p)$ with respect to the quotient of the measure of (11) by $\frac{d\alpha}{|\alpha|}\,\frac{d\beta}{|\beta|}$. The integral in (10) is however an integral over $G(\gamma_o,\mathbf{Q}_p)\setminus G(\mathbf{Q}_p)$, where

$$\gamma_o = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix},$$

with respect to the quotient of the measure in (11) by the measure

$$\frac{du\,dv}{|u|^2}$$

on

$$G(\gamma_o) = \left\{\begin{pmatrix} u & v \\ 0 & u \end{pmatrix}\right\}.$$

Recalling the definition of the local Tamagawa measures as well as the explicit form for η_{γ_o}, we obtain (ii).

The condition (ii) and the local constancy of $E^T(f)$ for a Cartan subgroup corresponding to a quadratic extension of $\mathbf{Q}_p$ follows readily from, for example, the proof of Lemma 7.3.2 of Jacquet–Langlands. Their Lemma 7.3.1, together with a little calculation, implies (iv) when f is the characteristic function of $G(\mathbf{Z}_p)$. Its validity in general is then a consequence of a theorem of Shalika [6.6].

When $\{a^T\} = \{F^T(f)\}$ we write $\xi(z,f)$ and $\zeta(z,f)$ to stress the dependence of ξ and ζ on f. It is clear that $\zeta(z,f)$ can be specified arbitrarily. Consider the family of functions on $Z(\mathbf{Q}_p)$ formed by the $\xi(z,f)$ corresponding to those f for which $\zeta(z,f)$ vanishes identically. This family is linear and translation invariant. It contains moreover a positive function with support in an arbitrarily small neighborhood of 1. To see this, take f to be a function which is positive at $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and vanishes at

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if $|a+d-2| \geq \varepsilon$, or $|ad-bc-1| \geq \varepsilon$ or $|b|+|c| \leq \varepsilon$ where ε is a small positive number. It follows easily that the family contains every locally constant function with compact support.

To complete the proof of the lemma, it has only to be verified that f exists when $\{a^T\}$ is a Shalika family for which both ξ and ζ vanish. This is easy because there are only a finite number of conjugacy classes of Cartan subgroups and the map

$$\hat{T}(\mathbf{Q}_p) \times T(\mathbf{Q}_p) \setminus G(\mathbf{Q}_p) \to G(\mathbf{Q}_p)$$

which sends $t \times h$ to $h^{-1}th$ is a local homeomorphism and in fact either a double or simple covering of an open subset of $G(\mathbf{Q}_p)$.

We now take $\Phi_p^1(s)$, when s maps to a non-unit b, to be any locally constant function with compact support for which

$$F^A\big(\gamma, \Phi_p^1(s)\big) = |b|_p^{-\frac{1}{2}} \sum_i \frac{\chi_i\big(g^{-1}(s)\gamma\big) + \chi_i\big(g^{-1}(s)\tilde{\gamma}\big)}{\operatorname{meas} U_i}$$

if

$$g(s) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

and

$$\tilde{\gamma} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

for

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and for which

$$F^T\big(\gamma, \Phi_p^1(s)\big) = \Delta(\gamma)\Big\{\sum_i \frac{\chi_{V_i}(b^{-1}\mathrm{det}\gamma)}{\operatorname{meas} W_i}\Big\} \operatorname{meas}\big(T(\mathbf{Q}_p) \setminus G'(\mathbf{Q}_p)\big)$$

if T is a Cartan subgroup associated to a quadratic extension of $\mathbf{Q}_p$. That $\Phi_p^1(s)$ exists is assured by the previous lemma. $\xi\big(z, \Phi_p^1(s)\big)$ is zero and

$$\Phi_p^1(z,s) = -\sum_i \frac{\chi_{V_i}(b^{-1}\mathrm{det} z)}{\operatorname{meas} W_i}.$$

This relation will be needed for the trace formula.

$\Phi_p^2(s)$ is specified by demanding that

$$F^A\big(\gamma, \Phi_p^2(s)\big) = 0$$

while $F^T\big(\gamma, \Phi_p^2(s)\big)$ equals

$$\Delta(\gamma)(|b|_p^{-1}+1)\Big\{-\sum_i \frac{\chi_{V_i}(b^{-1}\mathrm{det}\gamma)}{\mathrm{meas}\, W_i} + \frac{\chi_V(b^{-1}\mathrm{det}\gamma)}{\mathrm{meas}\, W}\Big\}\, \mathrm{meas}\,\big(T(\mathbf{Q}_p)\setminus G'(\mathbf{Q}_p)\big)$$

if T is not split. For symmetry we denote the group introduced as $\bar{K}_p$ by V. We remark explicitly that

$$\Phi_p^2(z,s) = -(|b|^{-1}+1)\Big\{-\sum_i \frac{\chi_{V_i}(b^{-1}\mathrm{det}z)}{\mathrm{meas}\, W_i} + \frac{\chi_V(b^{-1}\mathrm{det}z)}{\mathrm{meas}\, W}\Big\}.$$

In order to verify that these functions satisfy the required conditions we recall that if π is an irreducible admissible representation of $G(\mathbf{Q}_p)$ then its character is a function χ_π and for every locally constant function f on $G(\mathbf{Q}_p)$ with compact support the trace of $\pi(f)$ is given by

$$\frac{1}{2}\sum \int_{T(\mathbf{Q}_p)} F^T(\gamma, f)\chi_\pi(\gamma)\Delta(\gamma)\, d\gamma.$$

The sum is taken over a set of representatives for the conjugacy classes of Cartan subgroups of $G(\mathbf{Q}_p)$. This fact is also valid if π admits a finite composition series whose terms are irreducible and admissible.

In particular for one of the representations $\rho = \rho(\lambda,\nu)$ introduced in §4 we have as a consequence of Proposition 7.6 of Jacquet–Langlands

$$\mathrm{trace}\,\rho(f) = \frac{1}{2}\int_{A(\mathbf{Q}_p)} F^A(\gamma, f)\{\lambda(\alpha)\nu(\beta)+\nu(\alpha)\lambda(\beta)\}\, d\gamma$$

if α and β are the eigenvalues of γ. It follows immediately that $\mathrm{trace}\,\rho\big(\Phi_p^2(s)\big) = 0$ and that $\mathrm{trace}\,\rho\big(\Phi_p^1(s)\big)$ is $|b|_p^{-\frac{1}{2}}\big(\lambda(b)+\nu(b)\big)$ times the number of i for which $\gamma \to \lambda(\alpha)\nu(\beta)$ is trivial on U_i. This latter number is also equal to the number of i for which $\nu(\alpha)\lambda(\beta)$ is trivial on U_i. The conditions on $\mathrm{trace}\,\pi\big(\Phi_p^1(s)\big)$ and $\mathrm{trace}\,\pi\big(\Phi_p^2(s)\big)$ for π infinite-dimensional and of the form $\pi(\tau_p)$, $\tau_p = \lambda\oplus\nu$, follow from:

Lemma 6.3. *The multiplicity with which the trivial representation of K_p occurs in $\rho(\lambda,\nu)$ is equal to the number of i for which the quasi-character $\eta:\gamma\to\lambda(\alpha)\nu(\beta)$ of $A(\mathbf{Q}_p)$ is trivial on U_i.*

Extend η to $B(\mathbf{Q}_p)$ by making it trivial on $N(\mathbf{Q}_p)$. Recall that

$$G(\mathbf{Q}_p)=\cup B(\mathbf{Q}_p)g_iK_p.$$

Thus $\rho(\lambda,\nu)$ restricted to K_p is the direct sum of the representations of K_p induced from the representations $\eta_i:k\to\eta(g_ikg_i^{-1})$ of $K_p\cap g_i^{-1}B(\mathbf{Q}_p)g_i$. The relevant multiplicity is just the number of i for which η_i is trivial and η_i is trivial if and only if η is trivial on U_i.

If α is the quasi-character $x\to|x|_p$ then, for any quasi-character χ of $\mathbf{Q}_p^\times$, $\rho(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$ has a composition series of length 2 in which the special representation $\sigma(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$ and the one-dimensional representation $\pi(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi):h\to\chi(\det h)$ appear.

If $\pi=\pi(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$ then

$$\frac12\int_{A(\mathbf{Q}_p)}F^A\big(\gamma,\Phi_p^1(s)\big)\chi_\pi(s)\Delta(\gamma)\,d\gamma$$

is, since $\Delta(\gamma)$ is equal to $|b|_p^{-\frac12}$ on the support of $F^A\big(\gamma,\Phi_p^1(s)\big)$, equal to

$$|b|_p^{-1}\Big\{\sum_i\delta_i(\chi)\Big\}\chi(b).$$

Here $\delta_i(\chi)$ is 1 or 0 according as χ is or is not trivial on V_i. The previous lemma, applied to the pair $\lambda=\alpha^{\frac12}\chi$, $\nu=\alpha^{-\frac12}\chi$, shows that $\sum_i\delta_i(\chi)$ is the multiplicity with which the trivial representation of K_p occurs in $\rho(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$.

The character of $\sigma(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$ is the difference of the characters of $\rho(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$ and $\pi(\alpha^{\frac12}\chi,\alpha^{-\frac12}\chi)$. Thus

$$\frac12\int_{A(\mathbf{Q}_p)}F^A\big(\gamma,\Phi_p^1(s)\big)\chi_\pi(s)\Delta(\gamma)\,d\gamma \tag{12}$$

is equal to $\sum_i\delta_i(\chi)\chi(b)$ if π is this special representation. The following lemma implies that (12) vanishes when π is absolutely cuspidal.

Lemma 6.4. *If π is an absolutely cuspidal representation the support of its character does not contain any matrix whose eigenvalues lie in $\mathbf{Q}_p$ and have different absolute values.*

For $p \neq 2$, this has been known for some time (cf. [6.5] and [6.7]). In general it is a simple consequence of a recent theorem of Casselman [6.2]. I omit the proof.

If $\Phi_p^1(s)$ is replaced by $\Phi_p^2(s)$ then (12) vanishes for every π. The condition (iii) is a consequence of condition (ii), which we shall now verify together with condition (iv). We have to show that

$$\frac{1}{2}\sum_T{}' \int_{T(\mathbf{Q}_p)} F^T(\gamma, f)\chi_\pi(\gamma)\Delta(\gamma)\, d\gamma \tag{13}$$

is equal to $-\sum_i \delta_i(\chi)\chi(b)$ if $\pi = \sigma(\alpha^{\frac{1}{2}}\chi, \alpha^{-\frac{1}{2}}\chi)$ and $f = \Phi_p^1(s)$ and that it is equal to 0 if f is $\Phi_p^1(s)$ or $\Phi_p^2(s)$ and π is absolutely cuspidal. If $\delta(\chi)$ is 1 or 0 acording as χ is or is not trivial on V, then the multiplicity of the trivial representation of K_p in $\sigma(\alpha^{\frac{1}{2}}\chi, \alpha^{-\frac{1}{2}}\chi)$ is clearly

$$\sum_i \delta_i(\chi) - \delta(\chi). \tag{14}$$

We have to show that when π is this special representation and f is $\Phi_p^2(s)$ the expression (13) is equal to $(|b|_p^{-1} + 1)\chi(b)$ times (14). I observe that the prime indicates that we omit the split Cartan subgroup from the summation.

If π is special or absolutely cuspidal and if π' is, as in Theorem 15.1 of Jacquet–Langlands, the corresponding representation of $G'(\mathbf{Q}_p)$, then the expression (13) is equal to

$$-\frac{1}{2}\sum_T{}' \int_{T(\mathbf{Q}_p)} F^T(\gamma, f)\chi_{\pi'}(\gamma)\Delta(\gamma)\, d\gamma. \tag{15}$$

We may regard this as a sum over conjugacy classes of Cartan subgroups of $G'(\mathbf{Q}_p)$. If f' is a function on $G'(\mathbf{Q}_p)$ it is possible to define $F^T(\gamma, f')$ in the same way as we defined $F^T(\gamma, f)$. If $\Psi_p^1(s)$ is the sum over i of the characteristic functions of the sets $\{h \in G'(\mathbf{Q}_p) \bigm| \mathrm{Nm}h \in bV_i\}$ divided by their measures, then

$$F^T\big(\gamma, \Phi_p^1(s)\big) = F^T\big(\gamma, \Psi_p^1(s)\big).$$

Take $\Psi_p^2(s)$ to be $(|b|_p^{-1} + 1)$ times the difference of the characteristic function of the set $\{h \in G'(\mathbf{Q}_p) \bigm| \mathrm{Nm}h \in bV\}$ divided by its measure and the function $\Psi_p^1(s)$. Then (15), for $f = \Phi_p^k(s)$, is equal to

$$-\int_{G'(\mathbf{Q}_p)} \Psi_p^k(h, s)\chi_{\pi'}(h)\, dh.$$

The required relations follow immediately.

It is clear that the functions Φ^1 and Φ^2 defined by (6) and (7) satisfy (4) and (5). To prove Propositions 5.7 and 5.8 we apply the trace formula to calculate the left sides of (4) and (5).

Let χ be a quasi-character appearing in the sum (1).

$$\int_{Z^o(\mathbf{R})\backslash G(\mathbf{R})} \chi_\infty(\det h)\Phi_\infty(h)\,dh = \operatorname{trace}\pi(\Phi_\infty)$$

if $\pi : h \to \chi_\infty(\det h)$. Our conditions on Φ_∞ are such that this is 0 if μ is not one-dimensional and 1 if it is. For similar reasons

$$\int_{G(\mathbf{Q}_p)} \chi_p(\det h)\Phi^1_p(h,s)\,dh = (|b|_p^{-1}+1)\Big\{\sum_i \delta_i(\chi_p)\Big\}\chi_p(b)$$

and

$$\int_{G(\mathbf{Q}_p)} \chi_p(\det h)\Phi^2_p(h,s)\,dh$$

is equal to

$$(|b|_p^{-1}+1)\Big\{\delta(\chi_p)-\sum_i \delta_i(\chi_p)\Big\}\chi_p(b).$$

Since

$$\int_{Z^o(\mathbf{R})\backslash G(\mathbf{A})} \chi(\det h)\Phi^k(h)\,dh$$

is equal to

$$\int_{Z^o(\mathbf{R})\backslash G(\mathbf{R})} \chi_\infty(\det h)\Phi_\infty(h)\,dh \int_{G(\mathbf{Q}_p)} \chi_p(\det h)\Phi^k_p(s,h)\,dh \int_{G(\mathbf{A}^p_f)} \chi(\det h) f_g(h)\,dh,$$

it follows easily from Lemma 5.6 that the contribution of (1) to the trace formula for Φ^1 is $-\bar{A}$. Its contribution to the trace formula for $\Phi^2$is $\bar{A}-\bar{A}_o$.

If the eigenvalues of γ do not lie in $\mathbf{Q}$

$$\int_{G(\gamma,\mathbf{A})\backslash G(\mathbf{A})} \Phi^k(h^{-1}\gamma h)\,dh$$

is

$$\int_{G(\gamma,\mathbf{R})\backslash G(\mathbf{R})} \Phi_\infty(h^{-1}\gamma h)\,dh \int_{G(\gamma,\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} \Phi^k_p(s,h^{-1}\gamma h)\,dh \int_{G(\gamma,\mathbf{A}^p_f)\backslash G(\mathbf{A}^p_f)} f_g(h^{-1}\gamma h)\,dh.$$

The first integral is 0 if γ has real eigenvalues, otherwise it is

$$\frac{\operatorname{trace}\mu(\gamma)}{\operatorname{meas}\big(Z^o(\mathbf{R})\setminus G(\gamma,\mathbf{R})\big)}.$$

Consider the second for those matrices γ with eigenvalues on $\mathbf{Q}_p$. It is 0 for $k=2$. If $k=1$ we choose a set of representations for the conjugacy classes of γ which split in $\mathbf{Q}_p$ but not in $\mathbf{R}$ as follows. For each of the imaginary quadratic fields occurring in Proposition 5.7, we consider the corresponding group H. For each F we choose from each pair $\{\gamma,\tilde\gamma\}$ in $H(\mathbf{Q})$ corresponding to an element of F not in $\mathbf{Q}$ and its conjugate we choose one element γ. This yields the required set of representatives. For such a γ the centralizer $G(\gamma)$ is H and the second integral equals

$$\frac{\sum_i \chi_i\big(v^{-1}g^{-1}(s)\gamma_v\big)+\chi_i\big(v^{-1}g^{-1}(s)\tilde\gamma_v\big)}{\operatorname{meas} vU_iv^{-1}}$$

if $v=v(F)$ is defined as before. Observe that

$$\chi_i\big(v^{-1}g^{-1}(s)\gamma_v\big)=0$$

if γ is a scalar matrix—we are assuming b is not a unit.

Since

$$\operatorname{meas}\big(Z^o(\mathbf{R})H(\mathbf{Q})\setminus H(\mathbf{A})\big)=\operatorname{meas}\big(H(\mathbf{Q})\setminus H(\mathbf{A}_f)\big)\operatorname{meas}\big(Z^o(\mathbf{R})\setminus H(\mathbf{R})\big)$$

the contribution of the terms in (2′) corresponding to a γ split in $\mathbf{Q}_p$ but not in $\mathbf{R}$ to the trace formula for Φ^1 is $\sum_F A(F)$. They contribute nothing to the trace formula for Φ^2.

If γ is not split in $\mathbf{Q}_p$ the second integral equals

$$\left\{\frac{\sum_i \chi_{V_i}(b^{-1}\det\gamma)}{\operatorname{meas} W_i}\right\}\operatorname{meas}\big(G'(\gamma,\mathbf{Q}_p)\setminus G'(\mathbf{Q}_p)\big)$$

if $k=1$ and

$$(|b|_p^{-1}+1)\left\{\frac{\chi_v(b^{-1}\det\gamma)}{\operatorname{meas} W}-\sum_i\frac{\chi_{V_i}(b^{-1}\det\gamma)}{\operatorname{meas} W_i}\right\}\operatorname{meas}\big(G'(\gamma,\mathbf{Q}_p)\setminus G'(\mathbf{Q}_p)\big)$$

if $k=2$. The contributions of the terms in (2′) corresponding to a γ split neither in $\mathbf{Q}_p$ nor in $\mathbf{R}$ to the trace formula for Φ^1 is, since these conjugacy classes may be identified with the conjugacy classes of non-scalars in $G'(\mathbf{Q})$, just that part of the sum in Lemma 5.4 corresponding to such classes.

Consider the contribution of (2) to the trace formula for Φ^1. If γ belongs to $Z(\mathbf{Q})$

$$\Phi^1(\gamma)\left\{\frac{-\operatorname{trace}\mu(\gamma)}{\operatorname{meas}\big(Z^o(\mathbf{R})\setminus G'(\mathbf{R})\big)}\right\}\left\{\frac{-\sum_i \chi_{V_i}(b^{-1}\det\gamma)}{\operatorname{meas} W_i}\right\}f_g(\gamma).$$

If we also regard γ as an element of $G'(\mathbf{Q})$ then $G'(\gamma) = G'$. Since

$$\operatorname{meas}\big(Z^o(\mathbf{R})G(\mathbf{Q})\setminus G(\mathbf{A})\big) = \operatorname{meas}\big(Z^o(\mathbf{R})G'(\mathbf{Q})\setminus G'(\mathbf{A})\big)$$

and

$$\operatorname{meas}\big(Z^o(\mathbf{R})G'(\mathbf{Q})\setminus G'(\mathbf{A})\big) = \operatorname{meas}\big(G'(\mathbf{Q})\setminus G'(\mathbf{A}_f)\big)\operatorname{meas}\big(Z^o(\mathbf{R})\setminus G'(\mathbf{R})\big),$$

the contribution of (2) to the trace formula is just the remaining part of the sum in Lemma 5.4.

Using similar formulae and an obvious variant of Lemma 5.4 we show that the contribution of (2) and that part of (2′) corresponding to γ which split neither in $\mathbf{R}$ nor in $\mathbf{Q}_p$ to the trace of Φ^2 is

$$(|b|_p^{-1}+1)\big(A(D)-A_o(D)\big).$$

To complete the proofs of Propositions 5.7 and 5.8 all we have to do is show that (8) contributes nothing to the trace formula for Φ^2 and that its contribution for Φ^1 is B. Since we are no longer dealing with a function given by a product we must replace

$$\prod_{w\neq\infty} F^A(\gamma,\Phi_w)$$

by

$$F^A\big(\gamma,\Phi_p^k(s)\big)\Big\{\prod_{q\neq p}\Big|\frac{(a-d)^2}{ad}\Big|_q^{\frac12}\Big\}\int_{A(\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f_g(h^{-1}\gamma h)\,dh.$$

The eigenvalues of γ are a and d. The first term of this product vanishes if $k=2$. If $k=1$ it is equal to

$$|b|_p^{-\frac12}\sum_i \frac{\chi_i\big(g^{-1}(s)\gamma\big)+\chi_i\big(g^{-1}(s)\tilde\gamma\big)}{\operatorname{meas} U_i}.$$

After we sum over γ we may replace

$$\chi_i\big(g^{-1}(s)\gamma\big)+\chi_i\big(g^{-1}(s)\tilde\gamma\big)$$

by

$$\chi_i\big(g^{-1}(s)\gamma\big)$$

and multiply by 2. Since $D(\gamma,\phi_\infty)$ is

$$\frac{1}{4\varepsilon}\min\Big\{\Big|\frac{\alpha}{\beta}\Big|^{\frac12},\Big|\frac{\beta}{\alpha}\Big|^{\frac12}\Big\}$$

times the eigenvalue of $\mu(\gamma)$ of smallest absolute value, we may apply Lemma 5.5 to conclude that the required contribution is B, provided we show that

$$\frac{2\,\mathrm{meas}\,\big(Z^o(\mathbf{R})Z(\mathbf{Q})\setminus Z(\mathbf{A})\big)}{4\varepsilon} = \mathrm{meas}\,\big(A^+(\mathbf{Q})\setminus A(\mathbf{A}_f)\big). \tag{16}$$

This assertion is independent of the choice of forms on Z and A giving the Tamagawa measures. To avoid fuss, I cheat a little and take them to be $\frac{dz}{z}$ and $\frac{da}{a}\frac{db}{b}$. Then $\varepsilon = 1$,

$$\mathrm{meas}\,\big(Z^o(\mathbf{R})Z(\mathbf{Q})\setminus Z(\mathbf{A})\big) = 1$$

and

$$\mathrm{meas}\,\big(A^+(\mathbf{Q})\setminus A(\mathbf{A}_f)\big) = \frac{1}{2}.$$

We had as a matter of fact already cheated, because the unnormalized Tamagawa measure of the set of $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $(Z\setminus A)(\mathbf{A})$ for which $|\frac{a}{b}| = 1$ taken modulo $(Z\setminus A)(\mathbf{Q})$ should have stood in front of (3) and (8). We have in effect made a tautologous use of the fact that this measure is 1. It would have been better not to do so.

7. The Lefschetz formula. We begin by recalling those cases of the conjecture that we are trying to prove. Suppose π is an irreducible representation of $G(\mathbf{A})$ occurring in $L_{sp}(\mu)$ for which $\pi_\infty \simeq \pi(\mu)$. We define π' as in §4.

Theorem 7.1. *If $\pi'_p \simeq \pi(\tau_p)$, where τ_p is the direct sum of two one-dimensional complex representations of the Weil group, then trace $\sigma_p(s) \in \overline{\mathbf{Q}}$ for all s in $\mathfrak{W}(\mathbf{Q}_p/\mathbf{Q}_p)$ and $\pi'_p \simeq \pi(\sigma_p)$.*

In order to reduce the theorem to an assertion we are in a position to prove we digress.

Lemma 7.2. *Suppose λ is a continuous finite-dimensional representation of the Galois group $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ over a finite extension of $\mathbf{Q}_\ell$. If* trace $\lambda(s) \in \overline{\mathbf{Q}}$ *whenever $s \in \mathfrak{W}(\mathbf{Q}_p/\mathbf{Q}_p)$ and the image of s in $\mathbf{Q}_p^\times$ has an absolute value which is sufficiently small then it lies in $\overline{\mathbf{Q}}$ for all s in $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.*

If the absolute value of the image of s in $\mathbf{Q}_p^\times$ is small, so are the absolute values of the images of s^n, $n \geq 1$. Thus trace $\lambda(s^n) \in \overline{\mathbf{Q}}$ for $n \geq 1$. It follows that all eigenvalues of $\lambda(s)$ lie in $\overline{\mathbf{Q}}$. In proving the lemma we may therefore suppose that λ is absolutely irreducible. If the absolute value of the image of s is different from 1, then the absolute value of the image of some power s^n of s, with n positive or negative, is very small so that the eigenvalues of $\lambda(s^n)$ and hence those of $\lambda(s)$ lie in $\overline{\mathbf{Q}}$.

Supposing λ absolutely irreducible, we may apply the argument of §4 to show that λ factors, through $\mathfrak{G}(F^{un}/\mathbf{Q}_p)$ where F is a finite Galois extension of $\mathbf{Q}_p$. If $s \in \mathfrak{W}(F^{un}/F)$ then $\lambda(s)$ is a scalar. Moreover if $s \neq 1$ the image of s has absolute value different from 1. In any case the eigenvalues of $\lambda(s)$ lie in $\overline{\mathbf{Q}}$. Since some power of any element of $\mathfrak{W}(F^{un}/\mathbf{Q}_p)$ lies in $\mathfrak{W}(F^{un}/F)$ this is true in general.

Lemma 7.3. *Suppose λ is a continuous finite-dimensional representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ over a finite extension of $\mathbf{Q}_\ell$ such that $\operatorname{trace}\lambda(s) \in \overline{\mathbf{Q}}$ for all $s \in \mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Suppose ν is a continuous finite-dimensional complex representation of $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and suppose that*

$$\operatorname{trace}\lambda(s) = \operatorname{trace}\nu(s)$$

for those s in $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ whose images in $\mathbf{Q}_p^\times$ have an absolute value which is sufficiently small. Then the equality is valid for all s in $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Since we are only dealing with traces we may suppose that both λ and ν are direct sums of absolutely irreducible representations. Choosing the finite Galois extension F of $\mathbf{Q}_p$ sufficiently large, we may suppose that both the restriction of λ to $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and ν factor through $\mathfrak{W}(F^{un}/\mathbf{Q}_p)$ and that

$$\operatorname{trace}\lambda(s) = \operatorname{trace}\nu(s)$$

if $s \in \mathfrak{W}(F^{un}/F)$ maps to an element of $\mathbf{Q}_p^\times$ of absolute value less than 1. Choose an s_o of this type which generates $\mathfrak{W}(F^{un}/F)$.

If p is a polynomial with coefficients in $\overline{\mathbf{Q}}$ and s maps to an element of sufficiently small absolute value, then

$$\operatorname{trace} p\big(\lambda(s_o)\big)\lambda(s) = \operatorname{trace} p\big(\nu(s_o)\big)\nu(s).$$

Let λ_j, $1 \leq j \leq r$, be the restrictions of λ to the various eigenspaces of $\lambda(s_o)$ and let the polynomials p_j be such that $p_j\big(\lambda(s_o)\big)$ are the corresponding projections. We suppose, as we may, that they have constant term 0. Then

$$\operatorname{trace}\lambda_j(s) = \operatorname{trace} p_j\big(\lambda(s_o)\big)\lambda(s)$$

for those s which map to an element in $\mathbf{Q}_p^\times$ of small absolute value. Moreover

$$\operatorname{trace} p_j^m\big(\lambda(s_o)\big)\lambda^n(s_o) = \operatorname{trace} p_j^m\big(\nu(s_o)\big)\nu^n(s_o), \qquad m,\ n \geq 0.$$

Thus $p_j\big(\nu(s_o)\big)$ is also a projection and corresponds to the same eigenvalue as $p_j\big(\lambda(s_o)\big)$. Since

$$\operatorname{trace}\nu_j(s) = \operatorname{trace} p_j\big(\nu(s_o)\big)\nu(s)$$

if s maps to an element of small absolute value we may consider the λ_j separately. In other words we may suppose that $\lambda(s_o)$ and $\nu(s_o)$ are the same scalar ε.

Given s in $\mathfrak{W}(F^{un}/\mathbf{Q}_p)$ choose n sufficiently large that

$$\operatorname{trace}\lambda(ss_o^n) = \operatorname{trace}\nu(ss_o^n).$$

Then

$$\varepsilon^n \operatorname{trace}\lambda(s) = \varepsilon^n \operatorname{trace}\nu(s).$$

The lemma follows.

Suppose $K = K^p K_p$ is given. Both $\mathcal{H}_{\overline{\mathbf{Q}}}(K^p)$, the algebra of $\overline{\mathbf{Q}}$-linear combinations of the functions f_g, $g \in G(\mathbf{A}_f^p)$, and $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$

$$H_p^1\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big) = H_p^1\big(M_K, F_\mu(\mathbf{Q}_\ell)\big) \underset{\mathbf{Q}_\ell}{\otimes} \overline{\mathbf{Q}}_\ell.$$

This double representation we call ρ.

Lemma 7.4. *Suppose there are two subspaces $X \subseteq Y$ of $H_p^1\big(M_K, F_\mu(\overline{\mathbf{Q}}_\ell)\big)$ invariant under ρ such that if ρ' is the double representation on the quotient*

$$\operatorname{trace}\rho'(f_g, s) = \operatorname{trace}\tau^1(f_g, s) \tag{1}$$

whenever s maps to an element in $\mathbf{Q}_p^\times$ of absolute value less than 1. Suppose moreover that ρ' factors through some $\mathfrak{G}(F^{un}/\mathbf{Q}_p)$, where F is a finite Galois extension of $\mathbf{Q}_p$. Then Theorem 6.1 is valid if π_f contains the trivial representation of K.

Proposition 5.7 implies that the right side of (1) is rational. Thus the left side is also. Choose a π satisfying the conditions of Theorem 7.1. By a theorem of Casselman [7.1] and Miyake [7.3], π is uniquely determined by $\{\pi_q \mid q \neq p\}$. Thus we can find $\alpha_1, \cdots, \alpha_r$ in $\overline{\mathbf{Q}}$ and $g_1, \cdots, g_r$ in $G(\mathbf{A}_f^p)$ such that

$$\sum_i \alpha_i \rho(f_{g_i})$$

is, in the notation of Proposition 3.1, the projection on $U_{\overline{\mathbf{Q}}}^\pi \otimes V_{\overline{\mathbf{Q}}}^\pi(K)$. Then

$$\sum_i \alpha_i \operatorname{trace}\rho(f_{g_i}, s) = \operatorname{trace}\Big(\sum_i \alpha_i \rho(f_{g_i}, s)\Big) = \dim V_{\overline{\mathbf{Q}}}^\pi(K) \operatorname{trace}\sigma_p(s)$$

if $\sigma = \sigma(\pi)$. If σ'_p is the representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ or of $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on

$$U^{\pi}_{\overline{\mathbf{Q}}} \otimes V^{\pi}_{\overline{\mathbf{Q}}}(K) \cap X \setminus U^{\pi}_{\overline{\mathbf{Q}}} \otimes V^{\pi}_{\overline{\mathbf{Q}}}(K) \cap Y$$

then

$$\sum_i \alpha_i \operatorname{trace} \rho'(f_{g_i}, s) = \operatorname{trace} \sigma'_p(s).$$

Moreover if

$$\deg \sigma'_p = \dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \cdot \deg \sigma_p = 2 \dim V^{\pi}_{\overline{\mathbf{Q}}}(K)$$

then σ'_p is equivalent to $\dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \cdot \sigma_p$.

Suppose π^1 lies in $A(\mu)$ and $\pi^1_{\hat{p}}$ is defined as at the beginning of §5. If $V^{\pi_1}_f(K) \neq 0$ then

$$\sum \alpha_i \pi^1_{\hat{p}}(f_{g_i})$$

is 0 if $\pi^1 \neq \pi$ and is the identity if $\pi^1 = \pi$. By the very definition of the double representation τ^1

$$\sum_i \alpha_i \operatorname{trace} \tau^1(f_{g_i}, s) = \dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \operatorname{trace} \tau_p(s).$$

Lemmas 7.2 and 7.3 imply that $\operatorname{trace} \sigma'_p(s)$ lies in $\overline{\mathbf{Q}}$ for all s in $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and is equal to $\dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \operatorname{trace} \tau_p(s)$. In particular the degrees of σ'_p and $\dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \cdot \tau_p$ are the same. Since τ_p is of degree 2 we conclude that $\sigma'_p = \dim V^{\pi}_{\overline{\mathbf{Q}}}(K) \sigma_p$ and that $\operatorname{trace} \sigma_p(s) = \operatorname{trace} \tau_p(s)$ for $s \in \mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q})$. However we have assumed that no subrepresentation of σ'_p is special. In particular σ_p is not special. Referring to the definitions, we conclude that $\pi' = \pi(\sigma_p)$.

There is only one other case in which we can do anything about the conjecture at present.

Theorem 7.5. *If π'_p is a special representation $\sigma(\lambda_p, \nu_p)$ then $\operatorname{trace} \sigma_p(s) \in \overline{\mathbf{Q}}$ for $s \in \mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and $\pi'_p = \pi(\sigma_p)$.*

To verify this theorem we shall use the following lemma.

Lemma 7.6. *Suppose $X \subseteq Y$ are two subspaces of $H^1_p\big(M_k, F_\mu(\overline{\mathbf{Q}}_\ell)\big)$ invariant under the double representation ρ and let ρ' be the double representation on the quotient. Suppose*

$$\operatorname{trace} \rho'(f_g, s) = \operatorname{trace} \tau^2(f_g, s)$$

whenever s in $\mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ maps to b in $\mathbf{Q}_p^\times$ with $|b|_p < 1$. Suppose moreover that ρ' as a representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is a direct sum of two-dimensional special representations. Then $\pi'_p = \pi(\sigma_p)$ for every π in $A^2(\mu)$ for which $V_f^\pi(K) \neq 0$.

If $\pi'_p = \sigma(\lambda_p, \nu_p)$ the conclusion of the lemma amounts to the assertion that σ_p is special, and that $\operatorname{trace} \sigma_p(s) \in \overline{\mathbf{Q}}$ and is equal to $\lambda_p(b) + \nu_p(b)$ if $s \in \mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

The structure of two-dimensional special representations is such that for some sufficiently large finite Galois extension F of $\mathbf{Q}_p$ the space of vectors invariant under the restriction of ρ' to $\mathfrak{G}(\overline{\mathbf{Q}}_p/F^{un})$ has dimension $\frac{1}{2}\deg \rho'$. For no F is the dimension of this space larger. It follows that if ρ' is in any way the direct sum of two-dimensional representations then each of them is special.

Choose $\alpha_1, \cdots, \alpha_r$ in $\overline{\mathbf{Q}}$ and $g_1, \cdots, g_r$ in $G(\mathbf{A}_f^p)$ so that if π^1 lies in $A(\mu)$ and $V_f^{\pi^1}(K) \neq 0$ then

$$\sum \alpha_i \pi^1_{\hat{p}}(f_{g_i})$$

is 0 if $\pi^1 \neq \pi$ and is the identity if $\pi^1 = \pi$. The proof of Lemma 6.4 may be imitated to show that

$$\operatorname{trace} \sigma_p(s) = \lambda_p(b) + \nu_p(b)$$

for $s \in \mathfrak{W}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. It is of course implicit in this equality that both sides lie in $\overline{\mathbf{Q}}$. The proof also shows that the direct sum over those π in $A^2(\mu)$ for which $V_f^\pi(K) \neq 0$ of $\dim V_{\overline{\mathbf{Q}}}^\pi(K)\sigma_p$ is a subrepresentation of ρ' regarded as a representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ alone. However it follows from Lemma 7.3 that $\deg \rho' = \deg \tau^2$. Consequently this direct sum is not merely a subrepresentation but in fact all of ρ'. In particular each σ_p is special.

Since we are only interested in the action of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ we may replace $M_K^o \otimes \mathbf{Q}$ by $M_K^o \otimes \mathbf{Q}_p = (M_K^o \otimes \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $M_K^o \otimes \overline{\mathbf{Q}}$ by $M_K^o \otimes \overline{\mathbf{Q}}_p = (M_K^o \otimes \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$. Moreover for our purposes it is sufficient to take K of the form $K^p K_p$ where

$$K^p \subseteq \{h \in G(\mathbf{Z}_p^f) \mid h \equiv 1 \,(\operatorname{mod} M)\}$$
$$K_p = \{h \in G(\mathbf{Z}_p) \mid h \equiv 1 \,(\operatorname{mod} p^m)\}.$$

However we must be able to take M, which is prime to p, and m arbitrarily large. It will be useful to assume that K^p is so small that if $K_1 = K^p G(\mathbf{Z}_p)$ then each component of $M_{K_1} \otimes \overline{\mathbf{Q}}$ has genus greater than zero.

We choose then m arbitrarily and fix it in the discussion to follow. Let R be the ring of integers in $\mathbf{Q}_p(\zeta)$ where ζ is a p^mth root of unity. $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on R and hence on $\operatorname{Spec} R$. I recall that the action of s on $\operatorname{Spec} R$ is dual to that of s^{-1} on R. Thus the group acts on the left. In [3.1] it was seen that there is a scheme $M_K^o\overline{\otimes}R$ over $\operatorname{Spec} R$ and an action of $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on this scheme as well as an isomorphism

$$(M_K^o\overline{\otimes}R)\otimes_R\overline{\mathbf{Q}}_p\simeq M_K^o\otimes\overline{\mathbf{Q}}_p$$

compatible with the actions of the Galois group on both sides. The action of the Galois group on the left is given by its action on the two factors. Another scheme $M_K\overline{\otimes}R$ proper and flat over $\operatorname{Spec} R$ was also introduced. The Galois group acted on $M_K\otimes R$ and there was an open immersion $M_K^o\overline{\otimes}R\to M_K\overline{\otimes}R$ compatible with the actions. Moreover the complement of $M_K^o\overline{\otimes}R$ in $M_K\overline{\otimes}R$ with its induced subscheme structure is also flat over $\operatorname{Spec} R$. Finally the map $M_K\overline{\otimes}R\to\operatorname{Spec} R$ is smooth except at a finite number of points in the special fibre, all of which lie in $M_K^o\overline{\otimes}R$.

There is a family of sheaves $F_\mu(\mathbf{Z}/\ell^m\mathbf{Z})$ on $M_K^o\overline{\otimes}R$ whose pullbacks to $M_K^o\otimes\overline{\mathbf{Q}}_p$ yield the family $\{F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\}$, defined by pulling back the corresponding family on $M_K^o\otimes\mathbf{Q}$. Let $F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$ be the extension of $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ by zero to $M_K\overline{\otimes}R$ or $M_K\otimes\overline{\mathbf{Q}}_p$. We were interested in the ℓ-adic cohomology of the family $\{F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\}$ on $M_K^o\otimes\overline{\mathbf{Q}}$ and we denoted it by $H^i\big(M_K^o,F_\mu(\mathbf{Q}_\ell)\big)$. For reasons which will soon become apparent, we denote it now by $H^i\big(M_K^o\otimes\overline{\mathbf{Q}},F_\mu(\mathbf{Q}_\ell)\big)$; but replace it immediately by the isomorphic group $H^i\big(M_K^o\otimes\overline{\mathbf{Q}}_p,F_\mu(\mathbf{Q}_\ell)\big)$. The group formerly denoted $H^i_c\big(M_K^o,F_\mu(\mathbf{Q}_\ell)\big)$ is $H^i_c\big(M_K^o\otimes\overline{\mathbf{Q}},F_\mu(\mathbf{Q})\big)$, which is isomorphic to $H^i_c\big(M_K^o\otimes\overline{\mathbf{Q}}_p,F_\mu(\mathbf{Q}_\ell)\big)$. I recall that this is by definition $H^i\big(M_K\otimes\overline{\mathbf{Q}}_p,F_{\mu!}(\mathbf{Q}_\ell)\big)$.

Let φ be the immersion of $M_K^o\overline{\otimes}R$ in $M_K\overline{\otimes}R$ or of $M_K^o\otimes\overline{\mathbf{Q}}_p$ in $M_K\otimes\overline{\mathbf{Q}}_p$. The isomorphism

$$\varphi^*F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\simeq F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$$

yields by adjunction

$$F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\to\varphi_*F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}).$$

More generally, we have in the derived category

$$F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\to R_{\varphi_*}F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}). \tag{4}$$

This yields a map from

$$H^i_c\big(M_K^o\otimes\overline{\mathbf{Q}}_p,F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)\simeq H^i\big(M_K\otimes\overline{\mathbf{Q}}_p,F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

to

$$H^i\big(M_K\otimes\overline{\mathbf{Q}}_p,R_{\varphi_*}F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)\simeq H^i\big(M_K^o\otimes\overline{\mathbf{Q}}_p,F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

These are of course the maps used implicitly throughout the report; so in the limit the image of the left side in the right is the group formerly denoted $H_p^i\big(M_K, F_\mu(\mathbf{Q}_\ell)\big)$, but which will now be denoted $H_p^i\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big)$.

Let $G(\mathbf{Z}/\ell^m\mathbf{Z})$ be the mapping cone of (4). This is a complex

$$\cdots 0 \to F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \to G^0(\mathbf{Z}/\ell^n\mathbf{Z}) \to G^1(\mathbf{Z}/\ell^n\mathbf{Z}) \to \cdots$$

in which $F_{\mu!}(\mathbf{Z}/\ell^m\mathbf{Z})$ is the term of degree -1. We have a long exact sequence

$$\cdots \to H^i\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^i\big(M_K^o \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^i\big(M_K \otimes \overline{\mathbf{Q}}_p, G(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to \cdots .$$

The complex $G(\mathbf{Z}/\ell^n\mathbf{Z})$ is exact on $M_K^o \otimes \overline{\mathbf{Q}}_p$. If $M_K^\infty \otimes \overline{\mathbf{Q}}_p$ is the complement of $M_K^o \otimes \overline{\mathbf{Q}}_p$ in $M_K \otimes \overline{\mathbf{Q}}_p$ with its reduced subscheme structure, let i be the imbedding $M_K^\infty \otimes \overline{\mathbf{Q}}_p \to M_K \otimes \overline{\mathbf{Q}}_p$. In the derived category the complex $G(\mathbf{Z}/\ell^n\mathbf{Z})$ is isomorphic to

$$\cdots \to 0 \to i_*i^*G^o(\mathbf{Z}/\ell^n\mathbf{Z}) \to \cdots .$$

We need to know the cohomology of this complex. This is the same as that of the complex

$$0 \to i^*G^o(\mathbf{Z}/\ell^n\mathbf{Z}) \to \cdots . \tag{5}$$

Suppose $\widetilde{M}_K \otimes \overline{\mathbf{Q}}_p$ is the sum of the strict henselisations of $M_K \otimes \overline{\mathbf{Q}}_p$ at the points of $M_K^\infty \otimes \overline{\mathbf{Q}}_p$. We know [3.1] that $\widetilde{M}_K \otimes \overline{\mathbf{Q}}_p$ is a direct sum over

$$C_K = N(\mathbf{A}_f)A^+(\mathbf{Q}) \setminus G(\mathbf{A}_f)/K$$

of schemes $M_K \otimes \overline{\mathbf{Q}}_p|h$ each of which is the spectrum of a strict henselian ring which is also a discrete valuation ring, with residue field $\overline{\mathbf{Q}}_p$. h is to be thought of as a representative of the given double coset in $G(\mathbf{A}_f)$ or as the double coset itself, according to the context. $x(K,h)$ will denote a uniformizing parameter for this ring. $\widetilde{M}_K^o \otimes \overline{\mathbf{Q}}_p$, with summands $M_K^o \otimes \overline{\mathbf{Q}}_p|h$, will be the fibre product of $\widetilde{M}_K \otimes \overline{\mathbf{Q}}_p$ with $M_K^o \otimes \overline{\mathbf{Q}}_p$. It is the spectrum of a field $A(K,h,\overline{\mathbf{Q}}_p)$.

Suppose $K_o = K_o^pK_p$ with $K_o^p \subseteq K^p$. The fibre over $M_K \otimes \overline{\mathbf{Q}}_p|h$ in $\widetilde{M}_{K_o}^o \otimes \overline{\mathbf{Q}}_p$ is

$$\coprod_{\substack{h_o \in C_{K_o} \\ h_o \to h}} M_K^o \otimes \overline{\mathbf{Q}}_p|h_o.$$

Let $r = r(h)$ be the order of the cyclic group

$$h_o^{-1}N(\mathbf{A}_f)h_o \cap K/h_o^{-1}N(\mathbf{A}_f)h_o \cap K_o. \tag{6}$$

The map $M_{K_o} \otimes \overline{\mathbf{Q}}_p|h_o \to M_K^o \otimes \overline{\mathbf{Q}}_p|h$ takes $x(K,h)$ to a unit times $x(K_o,h_o)^r$. If K_o is normal in K then K/K_o acts in a simple way. An element k of K takes $M_{K_o} \otimes \overline{\mathbf{Q}}_p|h_o$ to $M_{K_o}^o \otimes \overline{\mathbf{Q}}_p|h_o'$ if $h_o k$ and h_o' represent the same element of C_{K_o}. With this action the group (6) becomes the Galois group of $A(K_o,h_o,\overline{\mathbf{Q}}_p)$ over $A(K,h,\overline{\mathbf{Q}}_p)$.

To compute the cohomology groups of (5) we make use of the fact (SGA 4. VIII.5.2) that they are the cohomology groups of the pullback of $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ to $\widetilde{M_K^o} \otimes \overline{\mathbf{Q}}_p$ viz., the direct sum over C_K of the cohomology groups of the pullbacks to $M_K^o \otimes \overline{\mathbf{Q}}_p|h$. These may be computed as Galois cohomology and, indeed, since the order of the Galois module defined by $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ is prime to p as cohomology groups for

$$\mathfrak{G}^p(K,h,\overline{\mathbf{Q}}_p) = \mathfrak{G}\big(A^p(K,h,\overline{\mathbf{Q}}_p)/A(K,h,\overline{\mathbf{Q}}_p)\big)$$

if $A^p(K,h,\overline{\mathbf{Q}}_p)$ is the direct limit of all finite Galois extensions of $A(K,h,\overline{\mathbf{Q}}_p)$ of degree prime to p. We may identify $\mathrm{Spec}\,\big(A^p(K,h,\overline{\mathbf{Q}}_p)\big)$ with

$$\varprojlim_{K_o} M_{K_o}^p \otimes \overline{\mathbf{Q}}_p|h$$

and hence $\mathfrak{G}^p(K,h,\overline{\mathbf{Q}}_p)$ with $h^{-1}N(\mathbf{A}_f^p)h \cap K$. There is of course a K_o such that the pullback of $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ is

$$L(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K/K_o}{\times} M_{K_o}^o \otimes \overline{\mathbf{Q}}_p|h.$$

Every map of the inverse limit to this scheme factors through a map of $M_{K_o}^o \otimes \overline{\mathbf{Q}}_p|h$ of the form $v \times \textit{identity}$ with $v \in L(\mathbf{Z}/\ell^n\mathbf{Z})$. The Galois module associated to $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ is thus $L(\mathbf{Z}/\ell^n\mathbf{Z})$ with $h^{-1}N(\mathbf{A}_f^p)h \cap K$ acting in the usual way.

Since $\mathfrak{G}^p(K,h,\overline{\mathbf{Q}}_p)$ is isomorphic to $\mathbf{Z}_f^p$, the cohomology groups are easily calculated. In degree 0 we obtain the invariants $L_h^o(\mathbf{Z}/\ell^n\mathbf{Z})$ of $L(\mathbf{Z}/\ell^n\mathbf{Z})$ with respect to the actions of $h^{-1}N(\mathbf{A}_f^p)h \cap K$. In degree 1 we obtain $L_o^h(\mathbf{Z}/\ell^n\mathbf{Z})$, the quotient of $L(\mathbf{Z}/\ell^n\mathbf{Z})$ by the subgroup generated by elements of the form $(k-1)v$, $k \in h^{-1}N(\mathbf{A}_f^p)h \cap K$. In higher degrees the groups vanish.

If we apply the properties of the mapping cone, then take a limit and tensor with $\mathbf{Q}_\ell$, we obtain the following lemma.

Lemma 7.7. *There is an exact sequence*

$$
\begin{aligned}
&0 \to H^o\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Q}_\ell)\big) \to H^o\big(M_K^o \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to \\
&\to \oplus_{C_K} L_h^o(\mathbf{Q}_\ell) \to H^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Q}_\ell)\big) \to H^1\big(M_K^o \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to \\
&\to \oplus_{C_K} L_o^h(\mathbf{Q}_\ell) \to H^2\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Q}_\ell)\big) \to H^2\big(M_K^o \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to 0
\end{aligned}
$$

in which $L_h^o(\mathbf{Q}_\ell)$ *is the set of invariants of* $h^{-1}\big(N(\mathbf{Q}_\ell)\big)h$ *in* $L(\mathbf{Q}_\ell)$ *and* $L_o^h(\mathbf{Q}_\ell)$ *is the quotient of* $L(\mathbf{Q}_\ell)$ *by the sum of the ranges of* $k-1$, $k \in h^{-1}N(\mathbf{Q}_\ell)h$.

The terms of higher degree have been left out of this exact sequence because they vanish anyway. The complex $G^\bullet(\mathbf{Z}/\ell^n\mathbf{Z})$ may also be taken as a complex of sheaves over $M_K\overline{\otimes}R$ as may, if i now denotes the imbedding $M_K^\infty\overline{\otimes}R \to M_K\overline{\otimes}R$, $i_*i^*G^\bullet(\mathbf{Z}/\ell^n\mathbf{Z})$. We can also regard it, by taking an inverse image, as a sheaf on the special fibre $(M_K\overline{\otimes}R) \otimes \overline{\mathbf{F}}_p$, which we denote $M_K\overline{\otimes}\,\overline{\mathbf{F}}_p$ or on $(M_K\overline{\otimes}R) \otimes R^{un} = M_K\overline{\otimes}R^{un}$. R^{un} is the maximal unramified extension of R.

The diagram

$$M_K\overline{\otimes}\,\overline{\mathbf{F}}_p \to M_K\overline{\otimes}R^{un} \leftarrow M_K \otimes \overline{\mathbf{Q}}_p \tag{7}$$

yields

$$
\begin{gathered}
H^i\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, i_*i^*G^\bullet(\mathbf{Z}/\ell^n\mathbf{Z})\big) \\
\uparrow \\
H^i\big(M_K\overline{\otimes}R^{un}, i_*i^*G^\bullet(\mathbf{Z}/\ell^n\mathbf{Z})\big) \\
\downarrow \\
H^i\big(M_K \otimes \overline{\mathbf{Q}}_p, i_*i^*G^\bullet\,(\mathbf{Z}/\ell^n\mathbf{Z})\big).
\end{gathered}
$$

One establishes, using standard techniques, that both arrows are isomorphisms. The groups on the left may be calculated in the same way as those on the right. We employ, *mutatis mutandis*, the same notations for the objects required in the course of calculations.

We can now seriously begin to set the stage for the application of Lemma 7.4. The arrow on the left of (7) determines, for all sheaves we are considering, isomorphisms of cohomology groups—before or after limits are taken. We have therefore a commutative diagram

$$
\begin{array}{ccccccc}
\oplus_{C_K} L_h^o(\mathbf{Q}) & \longrightarrow & H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, F_{\mu!}(\mathbf{Q}_\ell)\big) & \longrightarrow & H_p^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\oplus_{C_K} L_h^o(\mathbf{Q}_\ell) & \longrightarrow & H^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Q}_\ell)\big) & \longrightarrow & H_p^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big) & \longrightarrow & 0
\end{array} \tag{8}
$$

in which the rows are exact and the left hand arrow is the identity. To show that the arrow on the right is injective, we have just to establish this for the middle arrow. We employ the results [7.2] and the works therein cited.

There is a finite extension of the quotient field of R^{un} such that $M_K \otimes \overline{\mathbf{Q}}_p$ has a stable model over its ring of integers S. Set $(M_K \overline{\otimes} R^{un}) \underset{R^{un}}{\otimes} S = M_K \overline{\otimes} S$. $M_K \overline{\otimes} S \to \operatorname{Spec} S$ is smooth except perhaps at those points in the special fibre lying over a point of $\bar{M}_K \overline{\otimes} R$ at which $\bar{M}_K \overline{\otimes} R \to \operatorname{Spec} R$ is not smooth. In particular the arithmetical surface $M_K \overline{\otimes} S$ is regular except at a finite number of points in the special fibre. Let $M'_K \overline{\otimes} S$ be a regular scheme obtained by a desingularization of $M_K \overline{\otimes} S$ over this finite set. Because of our assumption on K^p, no irreducible component of $M_K \overline{\otimes} \mathbf{F}_p$ is rational. We may therefore suppose that $M'_K \overline{\otimes} S$ is a minimal model. Fixing a commutative diagram

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \overline{\mathbf{F}}_p & & \overline{\mathbf{Q}}_p \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

we obtain another

$$\begin{tikzcd} M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p \arrow[r, "x"] \arrow[d, "f"] & M'_K \overline{\otimes} S \arrow[d, "g"] & M'_K \overline{\otimes} \overline{\mathbf{Q}}_p \arrow[l, "y"'] \arrow[d, "h"] \\ M_K \overline{\otimes}\, \overline{\mathbf{F}}_p \arrow[r, "u"] & M_K \overline{\otimes} R^{un} & M_K \overline{\otimes} \overline{\mathbf{Q}}_p \arrow[l, "v"'] \end{tikzcd} \tag{9}$$

in which the right hand arrow is an isomorphism.

There is a spectral sequence

$$H^i\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, u^* R^j v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \Longrightarrow H^{i+j}\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

As a consequence there is an exact sequence

$$\begin{aligned} 0 \to H^1\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, u^* R^o v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to \\ \to H^o\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, u^* R^1 v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^2\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, u^* R^o v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big). \end{aligned} \tag{10}$$

To establish the asserted injectivity in the diagram (8) we need only show that

$$u^* v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

Let $F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$ be the pullback of $F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$ by means of any of the vertical arrows in (9). Since h is an isomorphism

$$u^* v_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq u^* v_* h_* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq u^* g_* y_* F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

Since g is proper, the base change theorem for proper morphisms shows that the group on the right is isomorphic to

$$f_* x^* y_* F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

The proof of Lemma 1.12 of [7.2] shows that on $M'_K\overline{\otimes}S$ the special fibre $M'_K\overline{\otimes}\,\overline{\mathbf{F}}_p$ is a divisor with normal crossings. Just as in SGA 7.1.3 one sees that

$$x^* y_* F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

It remains to be shown that

$$f_* F'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

The left side is $f_* f^* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$. Since $M_K\overline{\otimes}R^{un}$ is normal and g is birational, this final isomorphism is a consequence of Zariski's main theorem.

Let $\hat{M}_K$ be the desingularization $M_K\overline{\otimes}\overline{\mathbf{F}}_p$. It can be obtained by tensoring $\overline{\mathbf{F}}_p$ with a desingularization of $\hat{M}_K\overline{\otimes}\mathbf{F}_p$ of $M_K\overline{\otimes}\mathbf{F}_p$. Let

$$q : \hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p \to M_K\overline{\otimes}\,\overline{\mathbf{F}}_p$$

be the map giving the desingularization and let $\hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$ be $q^* F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$. From the Leray spectral sequence we obtain

$$0 \to H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, R^o q_* \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^1\big(\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^o\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, R^1 q_* \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

Since $R^1 q_* = 0$ we have

$$H^1\big(\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \simeq H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, q_* \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

But we have an exact sequence

$$0 \to F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \to q_* \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \to E(\mathbf{Z}/\ell^n\mathbf{Z}) \to 0$$

in which $E(\mathbf{Z}/\ell^n\mathbf{Z})$ has support at the set of singular points. Thus we have an exact sequence

$$H^0\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, E(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \to H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, q_* \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to 0$$

and hence a surjection

$$H^1\big(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^1\big(\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big). \tag{11}$$

This yields a surjection

$$H_p^1\bigl(M_K\overline{\otimes}\,\overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\bigr) \to H_p^1\bigl(\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\bigr).$$

The notation on the right is to be interpreted in the obvious way.

$L(s)$ and $R(g)$ continue to act on $\hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p$, the inverse image of $M_K^o\overline{\otimes}\,\overline{\mathbf{F}}_p$ in $\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p$. They also act on $M_K\overline{\otimes}\mathbf{F}_p$ and $\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p$. Moreover one defines a map

$$g\hat{F}_\mu^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \to R^*(g)\hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z})$$

as in the third paragraph and verifies, by examining its effect on fibres, that it is an isomorphism. Thus $T(g)$ acts on $H_p^1(\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\bigr)$. So does $L(s)$. Since all the necessary compatabilities are satisfied, Theorem 7.1 will be a consequence of the next lemma.

Lemma 7.8. *Let ρ' be the double representation of $\mathcal{H}_{\mathbf{Q}}(K^p)$ and $\mathfrak{G}(\overline{\mathbf{Q}}_p/\overline{\mathbf{Q}}_p)$ on $H_p^1\bigl(\hat{M}_K\overline{\otimes}\overline{\mathbf{F}}_p,$ $\hat{F}_\mu(\mathbf{Q}_\ell)\bigr)$. If $g \in G(\mathbf{A}_f^p)$ and s in $\mathfrak{W}(\overline{\mathbf{Q}}/\mathbf{Q}_p)$ maps to b in $\mathbf{Q}_p^\times$ with $|b|_p < 1$ then the trace of $\rho'(f_g, s)$ is*

$$-\Bigl\{\sum_F A(F)\Bigr\} - A(D) - B + \bar{A}.$$

We begin with an observation about the operator $L(s)$ on cohomology. We have set

$$(M_K\overline{\otimes}R)\otimes_R \mathbf{F}_p = M_K\overline{\otimes}\mathbf{F}_p$$

and we may write

$$M_K\overline{\otimes}\,\overline{\mathbf{F}}_p = (M_K\overline{\otimes}R)\otimes_R \overline{\mathbf{F}}_p = (M_K\overline{\otimes}\mathbf{F}_p)\otimes_{\mathbf{F}_p}\overline{\mathbf{F}}_p.$$

If we use the representation in the middle the action $L(s)$ of s in $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on $M_K\overline{\otimes}\,\overline{\mathbf{F}}_p$ is defined by its action on the factors $M_K\overline{\otimes}R$ and $\overline{\mathbf{F}}_p$. Howover the action on $M_K\overline{\otimes}R$, together with the trivial action on $\operatorname{Spec}\mathbf{F}_p$, yields an action on $M_K\overline{\otimes}\mathbf{F}_p$. This can be combined with the trivial action on $\mathbf{F}_p$ to yield, in terms of the representation on the right an action $L_1(s)$ on $M_K\overline{\otimes}\,\overline{\mathbf{F}}_p$. The trivial action on $M_K\overline{\otimes}\mathbf{F}_p$ combined with the action on $\operatorname{Spec}\overline{\mathbf{F}}_p$ yields an action $L_2'(s)$ on $M_K\overline{\otimes}\overline{\mathbf{F}}_p$. $L(s)$ is the product of $L_1(s)$ and $L_s'(s)$. Similar remarks apply to the actionon $\hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p$ or on $\hat{M}_K\overline{\otimes}\,\overline{\mathbf{F}}_p$ on which we now concentrate our attention.

To make $L_1(s)$ act on the cohomology we observe that the broken arrow in the diagram below yields a sheaf over $\hat{M}_K^o \overline{\otimes} \overline{\mathbf{F}}_p$ which may be taken as $L_1^*(s)F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$. The inverse of the upper horizontal arrow yields

$$\begin{array}{ccc} F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) & \xrightarrow{L_1(s)} & F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) \\ \Big\downarrow & & \Big\downarrow \\ \hat{M}_K^o \otimes \overline{\mathbf{F}}_p & \longrightarrow & \hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p \end{array}$$

an isomorphism between $L_1^*(s)F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$ and $F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})$.

Suppose $s \to b \in \mathbf{Q}_p^\times$ and $|b|_p < 1$ so that s maps to a positive power, the mth, of the Frobenius. Because of our definitions $L_2'(s^{-1})$, the inverse of $L_2'(s)$, acts as as the mth power of the Frobenius endomorphism of the second factor $\operatorname{Spec} \overline{\mathbf{F}}_p$. Therefore, in so far as its effect on the cohomology is concerned, $L_2'(s)$ may be replaced by $L_2(s)$, the mth power of the Frobenius endomorphism of the first factor $\hat{M}_K^o \overline{\otimes} \mathbf{F}_p$, the geometric Frobenius (SGA 5 XV). The same remarks apply to $\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ and the sheaves $\hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$.

The actions on cohomology groups are defined by the correspondence: $\varphi_1 \times \varphi_2 = R(e) \times R(g)L_1(s)L_2(s)$ which may be regarded as mapping $\hat{M}_{K'}^o \overline{\otimes}\, \overline{\mathbf{F}}_p$ to $\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p \times \hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p$ or as a mapping $\hat{M}_{K'} \overline{\otimes}\, \overline{\mathbf{F}}_p$ to $\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p \times \hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p$. In addition one needs the map

$$\varphi_2^* \hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to \varphi_1^* a^{-1}(g) \hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z})$$

which is obtained as a composition

$$\begin{aligned} &R^*(g)L_1^*(s)L_2^*(s)\hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to R^*(g)\hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to g\hat{F}_\mu^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \to \\ &\to a^{-1}(g)\hat{F}_\mu^K(\mathbf{Z}/\ell^n/\mathbf{Z}) \to R^*(e)a^{-1}(g)\hat{F}_\mu^L(\mathbf{Z}/\ell^n\mathbf{Z}) \end{aligned}$$

as well as the trace map (SGA 4 XVII)

$$R_*(e)R^*(e)\big(a^{-1}(g)\hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to a^{-1}(g)\hat{F}_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}).$$

If we apply the analogue of Lemma 7.7 for the space $\hat{M}_K \otimes \overline{\mathbf{F}}_p$ we see that the trace of $\rho'(f_g, s)$ is

$$\operatorname{trace} \rho_1(f_g, s) - \operatorname{trace} \rho_2(f_g, s) + \operatorname{trace} \rho_3(f_g, s) - \operatorname{trace} \rho_4(f_g, s).$$

Here ρ_1 and ρ_4 are the representations on $H^i\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Q}_\ell)\big)$ with $i = 1$ and 0 respectively. The representation on $H^o\big(\hat{M}_K^o \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\big)$ is ρ_3 and ρ_2 is the representation on the tensor product of $\mathbf{Q}_\ell$ with

$$\varprojlim H^o\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, i_* i^* G^\bullet(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

which is a group we have explicitly calculated. Notice also that we are exploiting the circumstance that $\hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p$ is isomorphic to $M_K \overline{\otimes} \overline{\mathbf{F}}_p$ in a neighborhood of $M_K^{\infty} \overline{\otimes} \overline{\mathbf{F}}_p$. Let ρ_5 be the representation on $H^2\big(\hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Q}_\ell)\big)$. The trace of $\rho'(f_g, s)$ is the sum of

$$\operatorname{trace} \rho_1(f_g, s) - \operatorname{trace} \rho_4(f_g, s) - \operatorname{trace} \rho_5(f_g, s)$$

and

$$\operatorname{trace} \rho_3(f_g, s) + \operatorname{trace} \rho_5(f_g, s)$$

and

$$-\operatorname{trace} \rho_2(f_g, s).$$

Lemma 7.9. *Under the assumptions of Lemma 7.8, the sum of* $\operatorname{trace} \rho_3(f_g, s)$ *and* $\operatorname{trace} \rho_5(f_g, s)$ *is equal to* $\bar{A}$.

Let $T(\mathbf{Z}/\ell^n\mathbf{Z})$ be the sheaf of ℓ^nth roots of unity. As usual we introduce the dual complexes

$$D\hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}) = \underline{\underline{R}}\operatorname{Hom}\big(\hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}), T(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

If φ denotes, for any K, the imbedding $\hat{M}_K^o \overline{\otimes} \overline{\mathbf{F}}_p \to \hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p$ and $\tilde{\mu}$ the representation contragredient to μ, then

$$D\hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \simeq \underline{\underline{R}}\,\varphi_*\big(\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

The map

$$\varphi_2^* \hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to \varphi_1^* a^{-1}(g) \hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \tag{12}$$

yields

$$D\big(\varphi_1^* g \hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to D\big(\varphi_2^* \hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

or

$$\underline{\underline{R}}\,\varphi_* \varphi_1^* g \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to \underline{\underline{R}}\,\varphi_* \varphi_2^* \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \tag{13}$$

or (SGA 5.I.1.12)

$$\varphi_1^* D a^{-1}(g) \hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to \underline{\underline{R}}^! \varphi_2 D\hat{F}_{\mu!}^K(\mathbf{Z}/\ell^n\mathbf{Z}). \tag{14}$$

There is a map of sheaves over $\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p$

$$\varphi_1^*\big(a^{-1}(g)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to \varphi_2^*\big(\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes (\mathbf{Z}/\ell^n\mathbf{Z})\big). \tag{15}$$

which can be constructed in the following manner. We start with the isomorphism (SGA 5.XV)

$$\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \to L_2^*(s)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

Then we lift by means of $L_1^*(s)$ and compose with the map $\tilde{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \to L_1^*(s)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})$ defined by the upper horizontal arrow in the following diagram

$$\begin{tikzcd}
\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[r, "L_1(s)"] \arrow[d] & \hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[d] \arrow[dl] \\
\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p \arrow[r, "L_1(s)"] & \hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p\, .
\end{tikzcd}$$

Finally we lift by $R^*(g)$ and compose with the resultant of

$$R^*(e)a^{-1}(g)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} a^{-1}(g)\hat{F}^{K'}_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \to g\hat{F}^{K'}_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} R^*(g)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})$$

to obtain

$$\varphi_1^*\big(g\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to \varphi_2^*\big(\bar{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

To complete the construction of (15) we observe that the processes of taking an inverse image and tensoring with $T(\mathbf{Z}/\ell^n\mathbf{Z})$ commute. The map (13) is obtained from (15) by applying $\underline{\underline{R}}\,\varphi_*$.

There is a lack of symmetry between the formulae (12) and (14) that should be commented upon and corrected. The map $R(g) : \hat{M}_{K'} \overline{\otimes}\, \overline{\mathbf{F}}_p \to \hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ is unramified over $\hat{M}^o_K \overline{\otimes}\, \mathbf{F}_p$ so that as long as we were working with sheaves over the latter space or their extensions by zero, we did not need to distinguish beteen $\underline{\underline{R}}^! R(g)$ and $R^*(g)$. In (14) we do. To restore the symmetry we write (12) as

$$\varphi_2^*\hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}) \to \underline{\underline{R}}^!\varphi_1 a^{-1}(g)\hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z}). \tag{16}$$

The preceding discussion will be of use again later, but its immediate purpose is the proof of Lemma 7.9. The map (16) defines (SGA 5.III)

$$\Phi_i : H^i\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^i\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

and (14) defines

$$\Psi_i : H^i\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, Da^{-1}(g)\hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^i\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, D\hat{F}^K_{\mu}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

which may also be written as

$$\Psi_i : H^i\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^i\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

When we take the limit and tensor with $\mathbf{Q}_\ell$, Φ_2 becomes $\rho_5(f_g, s)$. By the duality theorem

$$H^o\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, Da^{-1}(g)\hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big) \simeq \mathrm{Hom}\Big(H^2\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}^K_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})\big), \mathbf{Z}/\ell^n\mathbf{Z}\Big)$$

and Φ_2 and Ψ_0 are adjoint. It will be enough to see what happens to Ψ_0 upon passage to the limit.

The group

$$H^o\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}^K_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

is the group of sections

$$\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p \to a^{-1}L(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K/K_o}{\times} (\hat{M}^o_{K_o} \overline{\otimes}\, \overline{\mathbf{F}}_p).$$

When restricted to a connected component Y of $\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p$, such a section must be of the form

$$Y \xrightarrow{\sim} a^{-1}(g)v \underset{K_1/K_0}{\times} X$$

where X is a connected component of $\hat{M}^o_{K_o} \overline{\otimes}\, \overline{\mathbf{F}}_p$ whose stabilizer K_1/K_o and where $a^{-1}(g)v$ is fixed by every element of K_1/K_o.

It is shown in [3.1] that the set of connected components of $M^o_{K_o} \overline{\otimes}\, \overline{\mathbf{F}}_p$ is the union over the double cosets of $B(\mathbf{Q}_p) \setminus G(\mathbf{Q}_p)/K_p$, which as usual we index by i, of

$$\bar{M}^i_{K_o} = \bar{G}^+(\mathbf{Q}) \setminus \bar{G}(\mathbf{A}_f)/\bar{K}^p_o V_i$$

where each V_i is $\{\alpha \in \mathbf{Q}^\times_p \mid \alpha \equiv 1 \,(\mathrm{mod}\, p^m)\}$. The action of $k \in K/K_o \simeq K^p/K^p_o$ is left multiplication by $\det k$. In particular the stablizer of any connected component contains all cosets represented by k with $\det k = 1$. If $\deg \mu > 1$ there can thus be cohomology in degree 0 at a finite stage but none in the limit so the trace of $\rho_5(f_g, s)$ is zero. If $k \in K$ stabilizes any component then $\det k \in \bar{G}^+(\mathbf{Q})\det K_o$ and hence lies in $\det K_o$. Consequently if $\deg \mu = 1$ the action of k on $a^{-1}(g)L(\mathbf{Z}/\ell^n\mathbf{Z})$ is equal to that of some k_o and is therefore trivial. Thus if $\deg \mu = 1$

$$H^o\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

is isomorphic to the direct sum over $\cup_i \bar{M}_K^i = \bar{M}_K$ of the fibres of the sheaf

$$a^{-1}(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) = \cup_i a^{-1}(g)\tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{\bar{K}^p}{\times} \bar{M}^i \tag{17}$$

defined in the same was as $F_\nu(\mathbf{Q}_\ell)$, in the prelude to Lemma 5.6, except that $\mathbf{Z}/\ell^n\mathbf{Z}$ replaces $\mathbf{Q}_\ell$ and that $\tilde{\nu}$, which is defined by $\tilde{\nu}(\det g) = \tilde{\mu}(g)$, replaces ν. In other words it is the group of sections of this sheaf over $\bar{M}_K^\wedge$.

The operator Ψ_o is determined by (15). Its effect on a section of $a^{-1}(g)\hat{F}_{\tilde{\mu}}^L(\mathbf{Z}/\ell^n\mathbf{Z})$ has to be determined in steps. The first step is to pull back a section to $R^*(e)a^{-1}(g)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ and then to use the isomorphism of this sheaf with $a^{-1}(g)\hat{F}_{\tilde{\mu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z})$. This corresponds to pulling back a section of $a^{-1}(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ to $R^*(e)a^{-1}(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ and then using the following Cartesian diagram, where $R(h) : \bar{M}_{K'}^\wedge \to \bar{M}_K^\wedge$ is multiplication by $\det h$, with $h = e$,

$$\begin{array}{ccc} ha^{-1}(g)F_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) & \longrightarrow & a^{-1}(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \\ \downarrow & & \downarrow \\ \bar{M}_{K'}^\wedge & \xrightarrow{R(h)} & \bar{M}_K^\wedge \end{array} \tag{18}$$

where the upper arrow is defined by the action of $R(h)$ on the second factors and h^{-1} on the first in

$$ha^{-1}(g)F_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) = \cup_i ha^{-1}(g)\tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K'}{\times} \bar{M}^i$$

to obtain a section of $a^{-1}(g)F_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z})$ over $\bar{M}_{K'}^1$.

The next step is an application of the embedding $a^{-1}(g)\hat{F}_{\tilde{\mu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \to g\hat{F}_{\tilde{\mu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z})$ which in terms of sections over $\bar{M}_{K'}^\wedge$ is an application of the imbedding $a^{-1}(g)F_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \to gF_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z})$. In case it has been puzzling the reader, I observe that $F_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})$ is defined by means of the lattice dual to $L(\mathbf{Z}_f)$. Since $a(g)g$ fixes $L(\mathbf{Z}_f)$, $\big(a(g)g\big)^{-1}$ fixes the dual lattice and the imbedding is defined. The next step is to use the isomorphism

$$g\hat{F}_{\tilde{\mu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} R^*(g)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$$

to obtain a section of the sheaf on the right. This corresponds in terms of sections over $\bar{M}_{K'}^\wedge$ to using the isomorphism

$$gF_{\tilde{\nu}}^{K'}(\mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} R^*(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$$

given by (18).

The map defined by $L_1(s)$ is no problem. If $b = p^m x$ with x a unit, let $L_1(s)$ be the map on $\bar{M}_K^{\wedge}$ and on the sheaves $F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ defined by left multiplication by x. The broken arrow in the diagram

(19)
$$\begin{tikzcd} F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[r, "L_1(s)"] \arrow[d] & F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[d] \arrow[dl, dashed] \\ \bar{M}_K^{\wedge} \arrow[r, "L_1(s)"] & \bar{M}_K^{\wedge} \end{tikzcd}$$

turns the upper right corner into $L_1^*(s)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$. The map defined by $L_1(s)$ is the upper horizontal arrow.

The action of $L_2(s)$ on $\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ is given (SGA 4.XV.1) by the broken arrow in the commutative diagram.

(20)
$$\begin{tikzcd} \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[d] & L_2^*(s)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[l] \arrow[d] & \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[l, dashed] \arrow[dl] \arrow[ll, bend right, "L_2(s)"'] \\ \hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p & \hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p \arrow[l, "L_2(s)"'] & \end{tikzcd}$$

We must pull back the action of $L_2(s)$ by $L_1^*(s)$, compose with the action of $L_1(s)$, pull back the result by $R^*(g)$, and then take the trace with respect to $R(g)L_1(s)L_2(s)$. Taking the trace with respect to $R(g)$ erases the pullback by $R^*(g)$ and replaces it by multiplication by

$$\frac{[K : K']}{[\bar{K} : \bar{K}']}$$

where, for example, $\bar{K} = \{\det k \mid k \in K\}$. This number is the degree of $R(g)$ on a connected component. Of course we must then apply the trace at the level of sheaves over $\bar{M}_{K'}^{\wedge}$, and $\bar{M}_K^{\wedge}$. The trace with respect to $L_1(s)$ of the pullback of the action of $L_2(s)$ by $L_1^*(s)$ gives this action back again. The trace with respect to $L_1(s)$ applied to the action of $L_1(s)$ replaces a section of the sheaf defined by the left vertical arrow in (19) by the section of the sheaf defined by the right vertical arrow obtained by applying $L_1(s)$.

Taking the trace with respect to $L_2(s)$ of the actions of $L_2(s)$ has the effect of replacing a section of the sheaf defined by the skew arrow on the right in (20) by the unique section of the sheaf defined by the arrow on the left which makes

$$\begin{tikzcd} \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) & \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \arrow[l, "L_2(s)"'] \\ \hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p \arrow[u] \arrow[ur] & \hat{M}_K^o\overline{\otimes}\overline{\mathbf{F}}_p \arrow[l, "L_2(s)"'] \end{tikzcd}$$

commutative.

If we let $L_2(s)$ acting on $\bar{M}_K^\wedge$ be multiplication by $p^m \in \bar{G}(\mathbf{Q}_p)$ then, when we represent a section of $\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$ by a section of $F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})$, the effect of $L_2(s)$ is to take the section defined by the arrow on the right in the diagram below and replace it by the one defined by the arrow on the left which makes the diagram commutative

$$\begin{array}{ccc} F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) & \xleftarrow{L_2(s)} & F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \\ \downarrow & & \downarrow \\ \bar{M}_K^\wedge & \xleftarrow{L_2(s)} & \bar{M}_K^\wedge \end{array}$$

If we observe that under the map $\bar{M}_{K'}^\wedge \to \bar{M}_K^\wedge$ the inverse image of every point contains $[\bar{K} : \bar{K}']$ points, we may summarize the preceding discussion as follows. We have shown that

$$H^o\big(\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \simeq H^o\big(\hat{M}_K^\wedge, F_{\tilde{\nu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

and that the map

$$H^o\big(\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p, a^{-1}(g)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^o\big(\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

defined by the map

$$\varphi_1^*\big(a^{-1}(g)\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to \varphi_2^*\big(\hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big), \tag{21}$$

itself defined as (15) was, except that $T(\mathbf{Z}/\ell^n\mathbf{Z})$ need not be taken into account, corresponds to $[K : K']$ times the map

$$H^o\big(\bar{M}_K^\wedge, a^{-1}(g)F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big) \to H^o\big(\bar{M}_K^\wedge, F_{\tilde{\nu}}^K(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

which assigns to the section defined by the arrow in the right of the diagram below that defined by the arrow on the left

$$\begin{array}{ccc} F_{\tilde{\nu}}(\mathbf{Z}/\ell^n\mathbf{Z}) & \longleftarrow & a^{-1}(g)F_{\tilde{\nu}}(\mathbf{Z}/\ell^n\mathbf{Z}) \\ \uparrow & & \uparrow \\ \bar{M}_K^\wedge & \longleftarrow & \bar{M}_K^\wedge \end{array}$$

The horizontal arrows are defined by the maps on $\bar{M}_K^\wedge$ and $\bar{M}^\wedge$ given by multiplication by $b\,\det g$.

The elements of

$$H^o\big(\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}_{\tilde{\mu}}^K(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

are obtained from those of

$$H^o\big(\hat{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}^K_{\hat{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

by tensoring with a fixed non-zero section of $T(\mathbf{Z}/\ell^n\mathbf{Z})$. The two groups are in particular isomorphic. The action defined by (15) differs from that defined by (21) only in its effect on this section: There is none until the last stage, when we apply $L_2(s)$. This introduces an additional factor $p^m = |b|_p^{-1}$. Taking the limit and tensoring with $\mathbf{Q}_\ell$, we see readily that

$$\operatorname{trace} \rho_f(f_g, s) = \frac{|b|_p^{-1}}{1 + |b|_p^{-1}} \bar{A}.$$

A similar but easier discussion, since there is no need to introduce duality, shows that

$$\operatorname{trace} \rho_3(f_g, s) = \frac{1}{1 + |b|_p^{-1}} \bar{A}$$

and proves Lemma 7.9.

The number B discussed in Lemma 5.5 was defined as a sum of certain expressions $\psi(\gamma)$. With the same notations as there, set

$$\psi_1(\gamma) = (d' - a') \operatorname{trace} \mu(\gamma)$$

if $|a| > |d|$ and let it be 0 otherwise. Set

$$\psi_2(\gamma) = a' a^{m-\ell} d^\ell.$$

Then $\psi(\gamma) = \psi_1(\gamma) + \psi_2(\gamma)$ and B is $B_1 + B_2$.

Lemma 7.10. *Under the assumptions of Lemma 7.8, the trace of $\rho_2(f_g, s)$ is B_2.*

$\rho_2(f_g, s)$ may be taken to be the double representation on the tensor product with $\mathbf{Q}_\ell$ of the projective limit of the groups

$$H^o\big(\tilde{M}^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F_\mu(\mathbf{Z}/\ell^n\mathbf{Z})\big)$$

which are equal to

$$\oplus_{C_K} H^o\big(\mathfrak{G}^p(K, h, \overline{\mathbf{F}}_p), L(\mathbf{Z}/\ell^n\mathbf{Z})\big).$$

To be more precise, the sum is over a set of representatives in $G(\mathbf{A}_f)$ of the double cosets in C_K.

If $K_1 = K_1^p K_p$ and $g^{-1}K_1 g \subseteq K$ then $R(g) : \tilde{M}^o_{K_1} \overline{\otimes}\, \overline{\mathbf{F}}_p \to \tilde{M}^o_K \overline{\otimes} \overline{\mathbf{F}}_p$. It takes $M^o_{K_1} \overline{\otimes}\, \overline{\mathbf{F}}_p | h$ to $M^o_K \overline{\otimes}\, \overline{\mathbf{F}}_p | hg$. Our conventions are such that it yields a well-defined map from

$$\mathfrak{G}^p(K_1, h, \overline{\mathbf{F}}_p) \simeq h^{-1} N(\mathbf{A}_f^p) h \cap K_1$$

to

$$\mathfrak{G}^p(K, hg, \overline{\mathbf{F}}_p) \simeq g^{-1}h^{-1}N(\mathbf{A}_f^p)hg \cap K.$$

Namely, it sends the element of the Galois group represented by the matrix n to that represented by $g^{-1}ng$.

The action of $\rho_2(f_g, s)$ is defined by

$$\varphi_2^* F_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}) \to \varphi_1^* a^{-1}(g) F_\mu^K(\mathbf{Z}/\ell^n\mathbf{Z}) \tag{22}$$

which has been defined as the composite of a sequence of operations. Without repeating their definition, we shall describe their effect on the Galois cohomology in degree 0. Of course the operations over $\tilde{M}_{K'}^o \overline{\otimes} \overline{\mathbf{F}}_p$ have to be preceded by a lifting by the inverse image functor and followed by the trace map.

If $K_o = K_o^p K_p$ is normal in K and $K_o' = K_o \cap gK_o g^{-1}$ and $h_1 = g(s)hg$ we have

$$M_{K_o'}^o \overline{\otimes} \overline{\mathbf{F}}_p \big| h \xrightarrow{R(g)L_1(s)L_2(s)} M_{K_o}^o \overline{\otimes} \overline{\mathbf{F}}_p \big| h_1.$$

These maps define an imbedding

$$\mathfrak{G}^p(K', h, \overline{\mathbf{F}}_p) \to \mathfrak{G}(K, h_1, \overline{\mathbf{F}}_p)$$

which in terms of matrices takes n in $h^{-1}N(\mathbf{A}_f^p)h \cap K'$ to $g^{-1}ng$ in $h_1^{-1}N(\mathbf{A}_f^p)h_1 \cap K$. The lifting by $R^*(g)L_1^*(s)L_2^*(s)$ is, in terms of Galois cohomology, the restriction map defined by this imbedding. The effect of (22) is to map $L(\mathbf{Z}/\ell^n\mathbf{Z})$ to $a^{-1}(g)L(\mathbf{Z}/\ell^n\mathbf{Z})$, on which $h^{-1}N(\mathbf{A}_f^p)h \cap K'$ acts in the usual way, by $v \to gv$. The last step is the trace with respect to $R(e)$ and it is just corestriction with respect to the imbedding $\mathfrak{G}^p(K', h, \overline{\mathbf{F}}_p) \to \mathfrak{G}^p(K, h, \overline{\mathbf{F}}_p)$.

Suppose h_1 and h represent the same element of C_K, that is, h represents a fixed point of $\varphi(g, s)$ in $C_{K'}$, so that $g(s)hgk = n\gamma h$ with $k \in K$, $n \in N(\mathbf{A}_f)$, $\gamma \in \mathbf{A}^+(\mathbf{Q})$. To pass from cohomology with respect to $\mathfrak{G}^p(K, h, \overline{\mathbf{F}}_p)$ to that with respect to $\mathfrak{G}^p(K, h_1, \overline{\mathbf{F}}_p)$, we have to use the map from $L(\mathbf{Z}/\ell^n\mathbf{Z}) \to L(\mathbf{Z}/\ell^n\mathbf{Z})$ given by k. Recall that it was shown during the proof of Lemma 5.5 that when h represents a fixed point the index

$$[h^{-1}N(\mathbf{A}_f^p)h \cap K : h^{-1}N(\mathbf{A}_f^p)h \cap K'] \tag{23}$$

is equal to a'.

To compute the trace of $\rho_2(f_g, s)$ we have to take the sum over a set of representatives of the fixed points of $\varphi(g, s)$ on $C_{K'}$ of the trace on

(24)
$$\lim_{\leftarrow} H^o\big(\mathfrak{G}^p(K, h, \overline{\mathbf{F}}_p), L(\mathbf{Z}/\ell^n\mathbf{Z})\big) \otimes \mathbf{Q}_\ell = L_h^p(\mathbf{Q}_\ell).$$

Under the present circumstances, corestriction in degree zero becomes in the limit simply multiplication by the index (23). Thus the trace of $\rho_2(g, s)$ on (24) is a' times the trace gk on $L_h^o(\mathbf{Q}_\ell)$, the invariants in $L(\mathbf{Q}_\ell)$ of $h^{-1}N(\mathbf{Q}_\ell)h$. This is a' times the trace of $n\gamma$ on the invariants of $N(\mathbf{Q}_\ell)$, which is in turn $a'a^{m-\ell}d^\ell = \psi_2(\gamma)$.

To complete the proof of Lemma 6.8 we have only to establish one more fact.

Lemma 7.11. *Under the conditions of Lemma 7.8, the alternating sum of the traces of the operators on $H^i\big(\hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_{\mu!}(\mathbf{Q}_\ell)\big)$ defined by g and s is*

$$\sum_F A(F) + A(D) + B_1.$$

As explained in the introduction we use a variant of the Lefschetz fixed point formula [7.4] to prove this. Suppose X and Y are complete, non-singular, but not necessarily connected, curves over an algebraically closed field and $\varphi = \varphi_1 \times \varphi_2$ is a morphism $Y \to X \times X$. Let $F(\mathbf{Q}_\ell) = \{F(\mathbf{Z}/\ell^n\mathbf{Z})\}$ be a constructible étale sheaf of $\mathbf{Q}_\ell$ vector spaces over X. If a is a non-zero ℓ-adic integer $aF(\mathbf{Q}_\ell)$ is by definition the sheaf defined by $\{aF(\mathbf{Z}/\ell^n\mathbf{Z})\}$. The map $aF(\mathbf{Q}_\ell) \to F(\mathbf{Q}_\ell)$ given by $aF(\mathbf{Z}/\ell^n\mathbf{Z}) \hookrightarrow F(\mathbf{Z}/\ell^n\mathbf{Z})$ yields an isomorphism of cohomology groups. If b is another non-zero ℓ-adic integer define $\frac{1}{b}F(\mathbf{Q}_\ell)$ by $\{\frac{1}{b}F(\mathbf{Z}/\ell^n\mathbf{Z})\}$ where by definition

$$\frac{1}{b}F(\mathbf{Z}/\ell^n\mathbf{Z}) = F(\mathbf{Z}/\ell^n\mathbf{Z}).$$

Map $F(\mathbf{Q}_\ell)$ to $\frac{1}{b}F(\mathbf{Q}_\ell)$ by $b : F(\mathbf{Z}/\ell^n\mathbf{Z}) \to F(\mathbf{Z}/\ell^n\mathbf{Z})$. This map also yields an isomorphism of cohomology groups. In general if $\alpha = \frac{a}{b}$ is a non-zero ℓ-adic number we may combine the two operations to obtain $\alpha F(\mathbf{Q}_\ell)$ and well-defined isomorphisms between the cohomology groups of this sheaf and those of $F(\mathbf{Q}_\ell)$. We also have isomorphisms between the fibres $F(\mathbf{Q}_\ell)_x \simeq \alpha F(\mathbf{Q}_\ell)_x$ where, for example,

$$F(\mathbf{Q}_\ell)_x = \lim_{\leftarrow} F(\mathbf{Z}/\ell^n\mathbf{Z})_x \otimes \mathbf{Q}_\ell.$$

Fix $\alpha \in \mathbf{Q}_\ell^*$. Suppose we have a consistent collection of maps

$$\Phi : \varphi_2^* F(\mathbf{Z}/\ell^n\mathbf{Z}) \to R^!\varphi_1 \alpha F(\mathbf{Z}/\ell^n\mathbf{Z}).$$

As was observed earlier this yields maps

$$\Phi^i : H^i\big(X, F(\mathbf{Q}_\ell)\big) \to H^i\big(X, \alpha F(\mathbf{Q}_\ell)\big) \simeq H^i\big(X, F(\mathbf{Q}_\ell)\big).$$

A fixed point of φ is a closed point y which maps to a point (x, x) in the diagonal. Let X_x and Y_y be the strict localizations of X and Y at x and y respectively. We have a commutative diagram

$$\begin{array}{ccc} Y_y & \longrightarrow & X_x \times X_x \\ \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \psi} \\ Y & \longrightarrow & X \times X \end{array}$$

$\psi^*(\Phi)$ defines a map Φ_y from $H^o\big(X_x, \psi^* F(\mathbf{Q}_\ell)\big)$ to itself and, by definition, $H^o\big(X_x, \psi^* F(\mathbf{Q}_\ell)\big)$ is $F(\mathbf{Q}_\ell)_x$.

Φ also defines

$$D\Phi : \varphi_1^* \frac{1}{\alpha} DF(\mathbf{Z}/\ell^n\mathbf{Z}) \to R^!_{\varphi_2} DF(\mathbf{Z}/\ell^n\mathbf{Z})$$

and homomorphisms

$$D\Phi_y : \frac{1}{\alpha} DF(\mathbf{Z}/\ell^n\mathbf{Z})_x \to DF(\mathbf{Z}/\ell^n\mathbf{Z})_x.$$

This is a homomorphism not from one $\mathbf{Z}/\ell^n\mathbf{Z}$ module to another but from one complex of modules to another. We may still take a direct limit to obtain $DF(\mathbf{Q}_\ell)_x \simeq \frac{1}{\alpha} DF(\mathbf{Q}_\ell)_x$ as well as

$$D\Phi_y : DF(\mathbf{Q}_\ell)_x \longrightarrow DF(\mathbf{Q}_\ell)_x.$$

The trace of $D\Phi_y$ is the alternating sum of the traces on the cohomology groups.

We do not prove the following proposition.

Proposition 7.12. *Suppose that at every fixed point, $y \to (x, x)$, $\varphi = \varphi_1 \times \varphi_2$ has the form $\varphi_1^*(t_x) = u t_y^a$, $\varphi_2^*(t_x) = v t_y^d$, with $a \neq d$, where t_x and t_y are uniformizing parameters at x and y and u and v are units in the local ring of Y at y. Then the alternating sum of the traces of the Φ^i is equal to the sum over those fixed points y with $d > a$ of* $\operatorname{trace} \Phi_y$ *plus the sum over the fixed points with $a > d$ of* $\operatorname{trace} D\Phi_y$.

The correspondence which figures in Lemma 7.11 certainly satisifes the conditions of this proposition. At a fixed point in $\hat{M}_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p$, a is 1 and d is p^m with, by assumption, $m \geq 1$. Since $\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ and $M_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ are isomorphic away from the singular points of $M_K \overline{\otimes}\, \overline{\mathbf{F}}_p$, all of which lie in $M_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p$, it follows from [3.1] that at a fixed point in $\hat{M}_K^\infty \overline{\otimes}\, \overline{\mathbf{F}}_p$ the integer a is prime to p and d is divisible by p^m.

To prove Lemma 7.11, we examine the contributions of the fixed points in $M_{K'}(p,F)$, $\cup_i M^i_{K'}(p,D)$, and $C_{K'}$ separately.

We start with a point x in $M_K(p,F)$ represented by h in $G(\mathbf{A}_f)$. Let $\hat{M}_K\overline{\otimes}\overline{\mathbf{F}}_p|h$ be the strict localization of $\hat{M}_K\overline{\otimes}\overline{\mathbf{F}}_p$, or, what is the same, of $M_K\overline{\otimes}\overline{\mathbf{F}}_p$ at this point. If $K_o = K^p_o K_p$ with $K^p_o \subseteq K^p$ the map $\hat{M}^o_{K_o}\overline{\otimes}\overline{\mathbf{F}}_p \to \hat{M}^o_K\overline{\otimes}\overline{\mathbf{F}}$ is étale. Consequently if h_o and h represent the same point in $M_K(p,F)$ the map

$$\hat{M}_{K_o}\overline{\otimes}\overline{\mathbf{F}}_p|h_o \to M_K\overline{\otimes}\overline{\mathbf{F}}_p|h$$

is an isomorphism. If K_o is normal in K and sufficiently small

$$F(\mathbf{Z}/\ell^n\mathbf{Z})_x = \big(L(\mathbf{Z}/\ell^n\mathbf{Z}) \underset{K/K_o}{\times} \hat{M}^o_{K_o}\overline{\otimes}\overline{\mathbf{F}}_p\big) \underset{M^o_K\overline{\otimes}\overline{\mathbf{F}}_p}{\times} \hat{M}^o_K\overline{\otimes}\overline{\mathbf{F}}_p|h$$

is canonically isomorphic to

$$L(\mathbf{Z}/\ell^n\mathbf{Z}) \times \hat{M}^o_{K_o}\overline{\otimes}\overline{\mathbf{F}}_p|h \simeq L(\mathbf{Z}/\ell^n\mathbf{Z}) \times \hat{M}^o_K\overline{\otimes}\overline{\mathbf{F}}_p|h.$$

The isomorphism here is also canonical and we may identify $F(\mathbf{Z}/\ell^n\mathbf{Z})_x$ with $L(\mathbf{Z}/\ell^n\mathbf{Z})$ and $F(\mathbf{Q}_\ell)_x$ with $L(\mathbf{Q}_\ell)$. Since this identification depends upon h we write $F(\mathbf{Z}_\ell)_x$ as $L(\mathbf{Q}_\ell) \times h$.

It is shown in [3.1] that the correspondence $R(e) \times R(g)L_1(s)L_2(s)$ acts on $M_K(p,F)$ as the correspondence $\varphi(g,s)$ used in §5 to define $A(F)$. Moreover if (h_1,h_2) with $n\gamma h_2 = g(s)h_1gk$, represents a point in the image, the map $L(\mathbf{Q}_\ell) \times h_2 \to L(\mathbf{Q}_\ell) \times h_1$ on fibres is $v \times h_2 \to gkv \times h_1$ and this is the map used there. It follows that the contribution of the fixed points in $M_K(p,F)$ to the alternating sum of the traces of the Φ^i is $A(F)$.

One sees in the same way that the contribution of the fixed points in $\cup_i M^i_K(p,D) = \hat{M}_K(p,D)$ to the alternating sum of the traces is $A(D)$.

It is also shown in [3.1] that $R(e)\times R(g)L_1(s)L_2(s)$ acts on the points of C_K as the correspondence $\varphi(g,s)$ used to define the number B. If h represents a fixed point, there is an equation $g(s)hgk = n\gamma h$ with

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

a and d have the same sign. Let a' and d' be relatively prime positive integers with $a' : d' = a : d$. If h represents x' and $x' \to (x,x)$ let t_x and $t_{x'}$ be uniformizing parameters at x and x'. Then $\varphi_1^*(t_{x'}) = ut_x^{a'}$, $\varphi_2^*(t_{x'}) = vt_x^{d'}$ where u and v are units in the local ring at x'. If $a' < d'$ there is no contribution to the alternating sum of the traces because the fibre of $F(\mathbf{Q}_\ell)$ is 0 at a point of C_K.

To compute the trace of the map

$$D\hat{F}^K_{\mu!}(\mathbf{Q}_\ell)_x \to D\hat{F}^K_{\mu!}(\mathbf{Q}_\ell)_x$$

defined by (14), we use the description of (14) in terms of (15). If $M_K\overline{\otimes}\,\overline{\mathbf{F}}_p|h$ is the strict localization of $M_K\overline{\otimes}\,\overline{\mathbf{F}}_p$ at the point represented by h and

$$M_K^o\overline{\otimes}\,\overline{\mathbf{F}}_p|h = M_K^o\overline{\otimes}\,\overline{\mathbf{F}}_p \underset{M_K\overline{\otimes}\,\overline{\mathbf{F}}_p}{\times} M_K\overline{\otimes}\,\overline{\mathbf{F}}_p|h,$$

it is a question of determining the effect of (15) on the cohomology groups

$$H^i\bigl(M_K^o\overline{\otimes}\,\overline{\mathbf{F}}_p|h, a^{-1}(g)F_{\tilde{\mu}}(\mathbf{Z}/\ell^n\mathbf{Z})\bigr)$$

at a fixed point. This involves considerations in the Galois cohomology discussed already in the proof of Lemma 7.10; so we can be brief. Only degrees 0 and 1 need be considered.

In degree 0 the difference between the present situation and that of Lemma 7.10 is that μ is replaced by $\tilde{\mu}$ and (22) by (15). The steps involved are restriction by $\mathfrak{G}^p(K',h,\overline{\mathbf{F}}_p) \to \mathfrak{G}^p(K,h,\overline{\mathbf{F}}_p)$, which is defined in terms of matrices by the imbedding

$$h^{-1}N(\mathbf{A}_f^p)h \cap K' \hookrightarrow h^{-1}N(\mathbf{A}_f^p)h \cap K.$$

This is followed by

$$a^{-1}(g)\tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z}) \hookrightarrow g\tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z})$$

and then by

$$k^{-1}g^{-1} : g\tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \otimes T(\mathbf{Z}/\ell^n\mathbf{Z}) \to \tilde{L}(\mathbf{Z}/\ell^n\mathbf{Z}) \times T(\mathbf{Z}/\ell^n\mathbf{Z}).$$

At this stage an element of $\mathfrak{G}^p(K',h,\overline{\mathbf{F}}_p)$ represented by n acts on the right as $k^{-1}g^{-1}ngk$. The last step is corestriction with respect to the imbedding $\mathfrak{G}^p(K',h,\overline{\mathbf{F}}_p) \to \mathfrak{G}^p(K'h,\overline{\mathbf{F}}_p)$ defined by $n \to k^{-1}g^{-1}ngk$ tensored with multiplication by p^m on $T(\mathbf{Z}/\ell^n\mathbf{Z})$.

An easy calculation shows that the index

$$\bigl[h^{-1}N(\mathbf{A}_f)h \cap K : k^{-1}g^{-1}\bigl(h^{-1}N(\mathbf{A}_f)h \cap K'\bigr)gk\bigr]$$

is equal to $\prod_{q\neq p}|d'|_q^{-1}$. The trace in degree 0 is thus d' times the trace of $\tilde{\mu}(\gamma^{-1})$ on the invariants of $N(\mathbf{Q}_\ell)$ and this product is, in the notation of Lemma 5.5, $d'a^\ell d^{m-\ell}$. The restriction in degree 1 amounts to multiplication by

$$[h^{-1}N(\mathbf{A}_f^p)h \cap K : h^{-1}N(\mathbf{A}_f^p)h \cap K'] = a'.$$

Corestriction, when we interpret the limit of the first cohomology groups as $L_o^h(\mathbf{Q}_\ell)$, has no effect. Thus the trace in degree 1 is $a'a^{m-\ell}d^\ell$. Since

$$\psi_1(\gamma) = d'd^{m-\ell}a^\ell a - a'a^{m-\ell}d^\ell$$

we are done with Lemma 7.11.

Theorem 7.5 remains to be proven. We have an injection

$$0 \to H_p^1\big(M_K \overline{\otimes} \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to H_p^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big)$$

and a surjection

$$H_p^1\big(M_K \overline{\otimes} \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to H_p^1\big(\hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\big) \to 0.$$

On all spaces involved we have a double representation of $\mathcal{H}_{\mathbf{Q}}(K^p)$ and $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let ρ^o be the double representation on the kernel of the arrow in the second diagram.

Lemma 7.13. *If $s \to b$ in $\mathbf{Q}_p^\times$ and $|b|_p < 1$ the trace of $\rho^o(f_g, s)$ is equal to*

$$\big(A(D) - A_o(D)\big) - (|b|_p^{-1} + 1)^{-1}(\bar{A} - \bar{A}_o).$$

There is an exact sequence

$$\begin{aligned} &0 \to H^o\big(M_K^o \overline{\otimes} \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to H^o\big(\hat{M}_K^o \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\big) \to \\ &\to H^o\big(M_K^o \overline{\otimes} \overline{\mathbf{F}}_p, E(\mathbf{Q}_\ell)\big) \to H^1\big(M_K^o \overline{\otimes} \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) \to H^1\big(\hat{M}_K^o \overline{\otimes} \overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\big) \end{aligned}$$

ρ^o is also the representation on the kernel of the arrow in the second line. The trace in which we are interested is therefore the alternating sum of the traces on the groups in the first row. The representation on the group in the middle is ρ_3; and we have already seen, during the proof of Lemma 7.9, that its trace is $(|b|_p^{-1} + 1)^{-1}\bar{A}$. One proves in the same way that the trace on the first group is $(|b|_p^{-1} + 1)^{-1}\bar{A}_o$. It is only necessary to take into consideration the different structure of the set of connected components of $M_{K_o} \overline{\otimes} \overline{\mathbf{F}}_p$, for $K_o = K_o^p K_p \subseteq K$.

The sheaves $E(\mathbf{Z}/\ell^n\mathbf{Z})$ are supported on the finite set $M_K(p, D)$ of closed points in $M_K \overline{\otimes} \overline{\mathbf{F}}_p$ [3.1]. We also use $M_K(p, D)$ to denote this set with its reduced subscheme structure and let i denote the corresponding immersion. We do the same for $\hat{M}_K(p, D)$ in $\hat{M}_K \overline{\otimes} \overline{\mathbf{F}}_p$. $\hat{M}_K(p, D)$ is the fibre of $M_K(p, D)$ in $\hat{M}_K \overline{\otimes} \mathbf{F}_p$. Let q be the map: $\hat{M}_K(p, D) \to M_K(p, D)$. There is an exact sequence

$$0 \to i_* i^* F_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) \to i_* q_* i^* \hat{F}_\mu(\mathbf{Z}/\ell^n\mathbf{Z}) \to E(\mathbf{Z}/\ell^n\mathbf{Z}) \to 0$$

from which it is easily seen that the trace on $H^o\big(M_K^o \overline{\otimes}\, \overline{\mathbf{F}}_p, E(\mathbf{Q})\big)$ is $A(D) - A_o(D)$.

The representation ρ^o acts in effect on an invariant subspace of $H_p^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big)$. Call this subspace X^o. Suppose that, over a finite extension E_ℓ of $\mathbf{Q}_\ell$, we can find another subspace Y invariant at least under the restriction of ρ to $\mathfrak{G}(\mathbf{Q}_p/F^{un})$, where F is a finite extension of $\mathbf{Q}_p$, so that on Y this restriction is a direct sum of two-dimensional indecomposable representations. Let X be the sum of the one-dimensional subspaces of Y invariant under $\mathfrak{G}(\mathbf{Q}_p/F)$. Then $2\dim X = \dim Y$. Suppose also that $X^o \subseteq X$.

By assumption we can find a decomposition $Y = \oplus Y_j$ of Y as a direct sum of two-dimensional subspaces, on each of which the representation of $\mathfrak{G}(\mathbf{Q}_p/F^{un})$ is indecomposable. If $X_j = X \cap Y_j$ then $X = \oplus_j X_j$. We may also suppose that

$$X^o \otimes F = \oplus_{X_j \subseteq X^o \otimes F} X_j.$$

On the other hand, by an obvious refinement of Proposition 3.1, if E_ℓ is suffficiently large

$$H_p^1\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(E_\ell)\big) = \oplus_\pi \oplus_k U_{E_\ell}^\pi \otimes v_k^\pi$$

where $\{v_k^\pi\}$ is a basis of $V_{E_\ell}^\pi(K)$. Since $X^o \otimes F$ is invariant under the double representation, it is the direct sum of its intersections with the spaces $U_{E_\ell}^\pi \otimes V_{E_\ell}^\pi(K)$. Moreover if, for a given π, it intersects $U_{E_\ell}^\pi \otimes V_{E_\ell}^\pi(K)$ in a non-zero subspace, then its intersection with each $U_{E_\ell}^\pi \otimes v_k^\pi$ is non-zero.

If the projection of Y^j to $U_{E_\ell}^\pi \otimes v_k^\pi$ is non-zero on X^j it must be surjective. It follows that whenever

$$(X^o \otimes F) \cap \big(U_{E_\ell}^\pi \otimes V_{E_\ell}^\pi(K)\big) \neq 0 \tag{25}$$

the representation $\sigma_p(\pi)$ is special. Moreover the dimension of this intersection is one-half the dimension of $U_{E_\ell}^\pi \otimes V_{E_\ell}^\pi(K)$. Let Y^o be the sum over those π satisfying (25) of $U_{E_\ell}^\pi \otimes V_{E_\ell}^\pi(K)$. Since ρ acts to the right, it follows easily from Lemma 7.13 and the structure of special representations that the trace of the restriction of $\rho(f_g, s)$ to Y^o is

$$(|b|_p^{-1} + 1)\big(A(D) - A_o(D) - (\bar{A} - \bar{A}_o)\big).$$

Theorem 7.5 therefore follows from Proposition 5.8 and Lemma 7.9 once we have proved the existence of such a subspace Y.

Let S be the finite extension of R^{un} figuring in the diagram (9). By [7.2], we may suppose that

$$M'_K \overline{\otimes} S = (M'_K \overline{\otimes} S_o) \otimes_{S_o} S$$

where $S_o \subseteq S$ is a finite extension of R and $M'_K \overline{\otimes} S_o$ is a desingularization of $M_K \overline{\otimes} R$. The field F is to contain S_o. Let $\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ be the desingularization of $M'_K \overline{\otimes}\, \mathbf{F}_p$. We have a commutative diagram

$$\begin{array}{ccc} \hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p & \longrightarrow & M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p \\ \downarrow & & \downarrow \\ \hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p & \longrightarrow & M_K \overline{\otimes}\, \overline{\mathbf{F}}_p \end{array}$$

Let, for example, $\hat{F}'_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$ be the inverse image on $\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ of the sheaf $F_{\mu!}(\mathbf{Z}/\ell^n\mathbf{Z})$. There is a commutative diagram

$$\begin{array}{ccc} H^1_p\big(\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}'_\mu(\mathbf{Q}_\ell)\big) & \longleftarrow & H^1_p\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_\mu(\mathbf{Q}_\ell)\big) \\ \uparrow & & \uparrow \\ H^1_p\big(\hat{M}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}_\mu(\mathbf{Q}_\ell)\big) & \longleftarrow & H^1_p\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) \end{array} \tag{26}$$

in which the horizontal arrows are surjections and the right vertical arrow is an injection. Let X_o be the kernel of the lower horizontal arrow and X that of the upper.

There is another commutative diagram which may be attached to the diagram (26)

$$\begin{array}{ccc} H^1_p\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_\mu(\mathbf{Q}_\ell)\big) & \longrightarrow & H^1_p\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_\mu(\mathbf{Q}_\ell)\big) \\ \uparrow & & \uparrow \\ H^1_p\big(M_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F_\mu(\mathbf{Q}_\ell)\big) & \longrightarrow & H^1_p\big(M_K \otimes \overline{\mathbf{Q}}_p, F_\mu(\mathbf{Q}_\ell)\big) \end{array} \tag{27}$$

The horizontal arrows are injective and the vertical arrow on the right is an isomorphism. We may therefore regard X_o and X, with $X_o \subseteq X$, as subspaces of $H^1_p\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_\mu(\mathbf{Q}_\ell)\big)$.

The group $\mathfrak{G}(\overline{\mathbf{Q}}_p/F)$ acts, in a compatible way, on all the spaces appearing in (26) and (27). With the following lemma the proof of Theorem 7.5 is complete.

Lemma 7.14. *There is a subspace Y of $H^1_p\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_\mu(\mathbf{Q}_\ell)\big)$ with $X \subseteq Y$ and with $2\dim X = \dim Y$ such that over some finite extension of $\mathbf{Q}_\ell$ the representation of $\mathfrak{G}(\overline{\mathbf{Q}}_p/F^{un})$ on Y is the direct sum of two-dimensional indecomposable representations. Moreover X is the sum of all one-dimensional subspaces of Y invariant under $\mathfrak{G}(\overline{\mathbf{Q}}_p/F)$.*

The proof that follows is simply a bowdlerization of a conversation with Deligne. It relies of course on the Picard–Lefschetz theory of SGA 7. If

$$q' : \hat{M}^{o\prime}_K \overline{\otimes}\, \overline{\mathbf{F}}_p \to M^{o\prime}_K \overline{\otimes}\, \overline{\mathbf{F}}_p$$

let $E'(\mathbf{Q}_\ell)$ be defined by

$$0 \to F'_\mu(\mathbf{Q}_\ell) \to q'_* \hat{F}'_\mu(\mathbf{Q}_\ell) \to E'(\mathbf{Q}_\ell) \to 0.$$

There is an exact sequence

$$\begin{aligned} &0 \to H^o\big(M^{o'}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_\mu(\mathbf{Q}_\ell)\big) \to H^o\big(\hat{M}^{o'}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}'_\mu(\mathbf{Q}_\ell)\big) \to \\ &\to H^o\big(M^{o'}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, E'(\mathbf{Q}_\ell)\big) \to X \to 0. \end{aligned} \tag{28}$$

Moreover

$$H^o\big(M^{o'}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, E'(\mathbf{Q}_\ell)\big) = \oplus_x E'(\mathbf{Q}_\ell)_x$$

and

$$E'(\mathbf{Q}_\ell)_x \simeq F'(\mathbf{Q}_\ell)_x = F'(\mathbf{Q}_\ell)_x \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell. \tag{29}$$

The sum is over all double points of $M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ and the isomorphism (29) is determined up to a sign which depends upon the choice of an order for the pair of branches passing through x.

Let q' also be the map $\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p \to M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ and let $X_!$ be the kernel of

$$H^1\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big) \to H^1\big(\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, \hat{F}'_{\mu!}(\mathbf{Q}_\ell)\big).$$

After extending by zero, we may regard $E'(\mathbf{Q}_\ell)$ as a sheaf on $M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$. There is an analogue of (28). Moreover

$$H^o\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, E'(\mathbf{Q}_\ell)\big) = H^o\big(M^{o'}_K \overline{\otimes}\, \overline{\mathbf{F}}_p, E'(\mathbf{Q}_\ell)\big),$$

so that we have a commutative diagram

$$\begin{array}{ccc} \oplus F'(\mathbf{Q}_\ell)_x \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell & \longrightarrow & X_! \\ & \searrow & \downarrow \\ & & X \end{array} \tag{30}$$

in which all arrows are surjective.

A spectral sequence similar to the one which led to (10) yields

$$\begin{aligned} &H^1\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big) \to H^0\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, u^* R^1 v_* F'_{\mu!}(\mathbf{Q}_\ell)\big) \to \\ &\to H^2\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big) \to H^2\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_{\mu!}(\mathbf{Q})\big) \to 0. \end{aligned}$$

The meaning of the maps u and v is more or less the same as in (10). The sheaf $u^* R^1 v_* F'_{\mu!}(\mathbf{Q}_\ell)$ has support in the set of double points and, since $F'_\mu(\mathbf{Q}_\ell)$ is locally constant in $M_K^{o'} \overline{\otimes} S$, its fibre at x is, once an order has been chosen for the branches at this point, isomorphic to

$$F'_\mu(\mathbf{Q}_\ell)_x \otimes_{Z_\ell} T_\ell^{-1}$$

T_ℓ is $\varprojlim T(\mathbf{Z}/\ell^n\mathbf{Z})$ and $T(\mathbf{Z}/\ell^n\mathbf{Z})$ is the group of the ℓ^nth roots of unity. $\mathfrak{G}(\overline{\mathbf{Q}}_p/F)$, and in fact $\mathfrak{G}(F^{un}/F)$, act on T_ℓ to the right. The action is the direct limit of $\zeta \to \tau^{-1}(\zeta)$, $\zeta \in T(\mathbf{Z}/\ell^n\mathbf{Z})$. T_ℓ^{-1} is the contragredient module.

If F' is the limit of the finite Galois extensions of F^{un} whose order is a power of ℓ the action of $\mathfrak{G}(\mathbf{Q}_p/F^{un})$ on $H^1\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big)$ can be factored through $\mathfrak{G}(F'/F^{un})$. With respect to the action of $\mathfrak{G}(F^{un}/F)$ to the right given by $\sigma \to \tau^{-1}\sigma\tau$, $\mathfrak{G}(F'/F^{un})$ is canonically isomorphic to T_ℓ. There is therefore a pairing

$$\mathfrak{G}(F'/F^{un}) \times T_\ell^{-1} \to \mathbf{Z}_\ell,$$

and hence for each $\sigma \in \mathfrak{G}(F'/F^{un})$ a linear map

$$\varepsilon(\sigma) : T_\ell^{-1} \to \mathbf{Z}_\ell.$$

$\sigma - 1$ annihilates $H^1\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big)$ and the diagram

$$\begin{array}{ccc}
\oplus F'_\mu(\mathbf{Q}_\ell)_x \otimes T_\ell^{-1} & \xrightarrow{\oplus 1\otimes\varepsilon(\sigma)} & F'_\mu(\mathbf{Q}_\ell)_x \otimes Z_\ell \to X \\
\uparrow & & \downarrow \\
H^1\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big) & \xrightarrow{\sigma-1} & H^1\big(M'_K \otimes \overline{\mathbf{Q}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big)
\end{array}$$

is commutative. All we need do to verify the lemma is to show that if $\sigma \neq 1$ so that $\varepsilon(\sigma) \neq 0$ then the composite of the top arrows is surjective on the kernel of

$$\oplus_x F_\mu(\mathbf{Q}_\ell)_x \otimes T_\ell^{-1} \to H^2\big(M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p, F'_{\mu!}(\mathbf{Q})\big). \tag{31}$$

For this purpose the action of $\mathfrak{G}(F^{un}/F)$ is irrelevant so we may as well identify T_ℓ^{-1} with $\mathbf{Z}_\ell$ and $\varepsilon(\sigma)$ with a constant, which by an appropriate choice of σ we may take to be 1.

The remainder of the discussion will be easier to follow if we make use of a graph of Z whose edges are formed by the set U of double points of $M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ and whose vertices are formed by the set V of irreducible components of $M'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$ or, what is the same, of $\hat{M}'_K \overline{\otimes}\, \overline{\mathbf{F}}_p$. A vertex lies on an edge if the corresponding double point lies in the component represented by the vertex.

If μ is one-dimensional let W be empty, otherwise let it be the set of vertices, or irreducible components, which do not contract to a point in $M_K\overline{\otimes}\overline{\mathbf{F}}_p$. All the vertices and edges lying in the same connected component of $Z-W$ must map to the same irreducible component of $M_K\overline{\otimes}\overline{\mathbf{F}}_p$ and if μ is not one-dimensional to a single point. We saw during the proof of Lemma 7.9 that if μ is one-dimensional then $F_\mu(\mathbf{Q}_\ell)$ is constant on each connected component of $M_K\overline{\otimes}\overline{\mathbf{F}}_p$.

These observations allow us to identify

$$\oplus_x F(\mathbf{Q}_\ell)_x \otimes \mathbf{Z}_\ell$$

with

$$L(\mathbf{Q}_\ell)\otimes(\oplus_x \mathbf{Q}) = \oplus_x L(\mathbf{Q}_\ell). \tag{32}$$

The identification is unique up to a composition with $\oplus_x A_x$ where A_x is an isomorphism of $L(\mathbf{Q}_\ell)$ and $x\to A_x$ is constant on the edges lying in the same connected component of $Z-W$.

It follows from the proof of Lemma 7.9 that if $\deg\mu>1$ then

$$H^o\big(N, F'_\mu(\mathbf{Q}_\ell)\big)=0$$

for each component N of $\hat{M}^{o'}\overline{\otimes}\overline{\mathbf{F}}_p$ lying in W. Thus

$$H^o\big(\hat{M}^{o'}\overline{\otimes}\overline{\mathbf{F}}_p, F'_\mu(\mathbf{Q}_\ell)\big)\simeq \oplus_{N\in V-W} L(\mathbf{Q}_\ell).$$

The map of this space into (32) is given by

$$\oplus v_N \to \oplus_x\big(v_{N_1(x)}-v_{N_2(x)}\big)$$

where $N_1(x)$ and $N_2(x)$ are the components containing the two branches passing through x. Recall that these branches were ordered. It can happen that $N_1(x)$ and $N_2(x)$ are the same. In any case the image is of the form $L(\mathbf{Q}_\ell)\otimes P$ with $P\subseteq\oplus_x\mathbf{Q}$.

The group

$$H^2\big(M'_K\overline{\otimes}\overline{\mathbf{F}}_p, F'_{\mu!}(\mathbf{Q}_\ell)\big)$$

is isomorphic to

$$\oplus_{N\in V-W} L(\mathbf{Q}_\ell)\otimes T_\ell^{-1}.$$

According to our argument, we may ignore the T_ℓ^{-1}. The map (31), after identification of its source with (32), is

$$\oplus_x v_x \to \oplus_N \Big(\sum_{N=N_i(x)} (-1)^i v_x \Big).$$

Thus the kernel is of the form $L(\mathbf{Q}_\ell) \otimes Q$, where $Q \subseteq \oplus_x \mathbf{Q}$ is the orthogonal complement of P with respect to the standard inner product on $\oplus_x \mathbf{Q}$.

With this the lemma is proved and the report is concluded.

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