Orbital Integrals on Forms of $\mathrm{SL}(3)$, II

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0. Introduction. In the paper [**6**] we described in a precise fashion the notion of transfer of orbital integrals from a reductive group over a local field to an endoscopic group. We did not, however, prove the existence of the transfer. This remains, indeed, an unsolved problem, although in [**7**] we have reduced it to a local problem at the identity.

In the present paper we solve this local problem for two special cases, the group $\mathrm{SL}(3)$, which is not so interesting, and the group $SU(3)$, and then conclude that transfer exists for any group of type A_2. The methods are those of [**4**], and are based on techniques of Igusa for the study of the asymptotic behavior of integrals on p-adic manifolds. (As observed in [**7**], the existence of the transfer over archimedean fields is a result of earlier work by Shelstad.)

Since the problem is solved in so few cases, some doubts about the value of the method are justified. However, in the hands of Thomas Hales ([**1**], [**2**]) it is revealing itself to be a powerful and suggestive technique, and the simple case treated here may serve as a useful introduction to his work. Moreover, the theory of automorphic forms on $SU(3)$ or $U(3)$ is well worth studying as an example and in its own right ([**9**]); many phenomena that are absent for $\mathrm{GL}(2)$ but present in general appear first with these groups.

1. Local transfer of Shalika germs. Recall from ([**7**], Section 2.1) that the problem of local transfer is to show that for any smooth, compactly supported function f on $G(F)$ there is a function f^H of the same type on the endoscopic group $H(F)$ such that

$$\Phi^{st}(\gamma_H, f^H) = \sum_{\gamma_G} \Delta_{\mathrm{loc}}(\gamma_H, \gamma_G)\Phi(\gamma_G, f) \tag{1.1}$$

for all strongly G-regular γ_H near the identity in $H(F)$. The sum is over a set of representatives for the conjugacy classes in the stable conjugacy class of which γ_H is an image.

Fix one γ_G with γ_H as an image, and let T_G be the centralizer of γ_G, T_H that of γ_H. Choose an admissible embedding $T_H \to T \to T_G$ and let $\gamma \in T \subseteq G^*$ be the image of γ_H. Recall that $\mathcal{E}(T)$ is the image of $H^1(T_{\mathrm{sc}})$ in $H^1(T)$, and may of course be identified with $\mathcal{E}(T_G)$. The element s of the endoscopic data defining H can be transported to $\hat{T}$ or to $\hat{T}_G$ and devines a character, customarily denoted κ, of $X_*(T), X_*(T_G)$, or of $\mathcal{E}(T), \mathcal{E}(T_G)$. Recall that Tate-Nakayama duality implies that $\mathcal{E}(T)$ is a subquotient of $X_*(T)$. If $\delta \in \xi(T_G)$, there is a $g \in G(\bar{F})$ such that the cocycle $\{\sigma(g)g^{-1}\}$ lies in T and belongs to the class of δ. Set

$$\gamma_G^g = g^{-1}\gamma_G g.$$

Since $\Delta(\gamma_H, \gamma_G^g)$ and $\Phi(\gamma_G^g, f)$ depend only on δ and not on g, we write $\Delta(\gamma_H, \gamma_G^\delta)$ and $\Phi(\gamma_G^\delta, f)$. It is a consequence of Section 3.4 of [**6**] that

$$\Delta_{\mathrm{loc}}(\gamma_H, \gamma_G^\delta) = \kappa(\delta)\Delta_{\mathrm{loc}}(\gamma_H, \gamma_G),$$

so that the right side of (1.1) is

$$\Delta_{\mathrm{loc}}(\gamma_H, \gamma_G) \sum_{\delta \in \mathcal{E}(T_G)} \kappa(\delta)\Phi(\gamma_G^\delta, f).$$

For γ_H near the identity we may write $\gamma = \exp X$, and, for any root α, define $\alpha(\gamma)^{1/2}$ to be $\exp \alpha(X)/2$. Then

$$\{a_\alpha = \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}\}$$

is a set of a-data to which we may apply the construction of [**6**, Section 2.3] to obtain a cohomology class

$$\lambda(T) = \mathrm{inv}(T, \gamma)$$

in $\mathcal{E}(T)$. In terms of the classes introduced in [**10**] it is a product

$$\mathrm{inv}(T, \gamma) = \mathrm{inv}(\gamma)\mathrm{inv}(T).$$

Observe that the choice of F-splitting implicit in Section 2.3 is made as in [**10**] and [**6**, Section 5.1].

To calculate $\Delta_{\mathrm{loc}}(\gamma_H, \gamma)$ we may use the a-data just introduced. This yields

$$\Delta_{II}(\gamma_H, \gamma) = 1$$

and

$$\Delta_I(\gamma_H, \gamma) = \kappa(\mathrm{inv}(T, \gamma)).$$

In addition, $\Delta_1(\gamma_H, \gamma)$ is 1 by definition, and $\Delta_2(\gamma_H, \gamma)$ is 1 near the identity. Thus, choosing the overall constant correctly, we have

$$\Delta_{\mathrm{loc}}(\gamma_H, \gamma) = \kappa(\mathrm{inv}(T, \gamma))D_{G^*/H}(\gamma), \tag{1.2}$$

where ([**6**], Section 3.6)

$$D_{G^*/H}(\gamma) = D_{G^*}(\gamma)D_H(\gamma_H)^{-1}.$$

If, with the notation of [**6**, Section 4.2] and [**3**], we set

$$\mathrm{inv}(T_G, \gamma_G) = \mathrm{inv}(T, \gamma)\theta(E, E')$$

then the Local Hypothesis (Corollary 4.2.B of [**6**] yields

$$\Delta_{\mathrm{loc}}(\gamma_H, \gamma_G) = \kappa(\mathrm{inv}(T_G, \gamma_G))D_{G/H}(\gamma_G). \tag{1.3}$$

The usual germ expansion on $G(F)$ is

$$\Phi(\gamma_G, f) \sim \sum_{\mathcal{O}} \Gamma_{\mathcal{O}}(\gamma_G) a_{\mathcal{O}}(f).$$

The germ expansion for

$$\Phi(\gamma_H, f) = \sum_{\gamma_G} \Delta_{\mathrm{loc}}(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

is

$$\Phi(\gamma_H, f) \sim \sum_{\mathcal{O}} \Gamma^{\kappa}_{\mathcal{O}}(\gamma_H) a_{\mathcal{O}}(f)$$

with

$$\Gamma^{\kappa}_{\mathcal{O}}(\gamma_H) = \kappa(\mathrm{inv}(T_G, \gamma_G)) D_{G/H}(\gamma_G) \sum_{\delta} \kappa(\delta) \Gamma_{\mathcal{O}}(\gamma^{\delta}_G)$$

if γ_H is an image of an element of G. Otherwise $\Gamma^{\kappa}_{\mathcal{O}}(\gamma_H) = 0$. It will be referred to as the κ germ-expansion, and the $\Gamma^{\kappa}_{\mathcal{O}}$ as κ-germs, even though the true parameter is the collection of endoscopic data, the character κ changing from Cartan subgroup to Cartan subgroup. Taking $H = G^*$, so that all the κ are trivial, we obtain the stable germs $\Gamma^{\mathrm{st}}_{\mathcal{O}}$.

The existence of the local transfer for a group H and all f is clearly equivalent to the validity of the following assertion for all $\mathcal{O}$.

Assertion A. *The κ-germ $\Gamma^{\kappa}_{\mathcal{O}}$ is a linear combination of stable germs for H.*

2. Some easy general cases. There are a few general cases for which Assertion A is either easy to verify directly or easy to deduce from known results.

The endoscopic group H, or the data defining it, will be said to be *cuspidal* if the maximal split subgroup S_H of the center of H is contained in the center of G (a condition whose precise meaning should be clear enough).

Lemma 2.1. *Suppose H is not cuspidal. If local transfer exists for all endoscopic groups of all Levi factors of proper parabolic subgroups of G then it exists for H.*

Proof. If an admissible embedding $T_H \to T \to T_G$ exists then we say that T_H is an image of T_G. If no Cartan subgroup of H is an image, as can very well happen if, for example, G is anisotropic, then $f^H = 0$ will be the transfer of all f. Thus, suppose that some Cartan subgroup T^0_H is an image of, say, T^0.

We suppose that H is not cuspidal. Then S_H is transported by $T^0_H \to T^0$ to a split group S_G in G that is not central. The centralizer M of S_G is a Levi factor of a proper parabolic subgroup P over F. Any two split tori in G over F that are conjugate in $G(\bar{F})$ are already conjugate in $G(F)$. Thus if T_H is an image of some T_G, it is the image of a T_G contained in M.

If $m \in M(F)$ set

$$D_{G/M}(m) = |\det(1 - \mathrm{ad}\ m)\big|_{\mathfrak{g}/\mathfrak{m}}|^{1/2}$$

and

$$\delta_{G/M}(m) = |\det m\big|_{\mathfrak{n}}|^{1/2},$$

where $\mathfrak{n}$ is the Lie algebra of the unipotent radical of P. It follows readily from the definitions that H is also an endoscopic group for M. We denote transfer factors for M by the addition of the superscript M. We can easily arrange [**7**]

$$\Delta(\gamma_H, \gamma) = \Delta^M(\gamma_H, \gamma) D_{G/M}(\gamma).$$

Thus to prove the lemma, we need only verify the existence of a smooth compactly supported function f^M on $M(F)$ such that

$$D_{G/M}(\gamma)\Phi(\gamma, f) = \Phi(\gamma, f^M). \tag{2.1}$$

However, it is well known that for this purpose we can take

$$f^M(m) = c\delta_{G/M}(m) \int_K \int_{N(F)} f(k^{-1}mnk)dndk,$$

with a suitable constant c.

Lemma 2.2. *Assertion A is valid for regular unipotent classes $\mathcal{O}$ and all H.*

This follows readiy from Theorem 5.5.A of [**6**].

Lemma 2.3. *Assertion A is valid for the class $\mathcal{O} = \{1\}$ and all H.*

Proof. Abbreviate $\Gamma_{\{1\}}$ to Γ_1. By results of Howe, Harish-Chandra and Rogawski, Γ_1 vanishes on tori that are not anisotropic, and is a constant c_G on all anisotropic tori. Moreover $c_G = c_{G^*}$ for compatible choices of the measures. Thus it is clear that if $H = G^*$ then

$$\Gamma_1^\kappa = \Gamma_1^{\mathrm{st}} = \Gamma_{1^*}^{\mathrm{st}},$$

where 1^* denotes the identity in G^*.

If $H \neq G^*$ the lemma takes an even stronger form.

Lemma 2.4. *If $H \neq G^*$ then $\Gamma_1^\kappa \equiv 0$.*

Proof. On tori that are not anisotropic this is a consequence of (2.1). On the other hand, for an anistropic torus T the group $\mathcal{E}(T) \simeq \mathcal{E}(T_G)$ is a quotient of $X_*(T_{\mathrm{ad}})$. Since $H \neq G^*$ the character κ is not trivial on $\mathcal{E}(T_G)$ and

$$\sum_\delta \kappa(\delta)\Gamma_1 = 0.$$

Corollary 2.5. *If G is anisotropic modulo its center then local transfer exists for all H.*

With these scraps of general information at our disposal, we now turn to forms of $\mathrm{SL}(3)$, excluding, because of the last corollary, anisotropic groups, and thus confining ourselves to quasi-split groups, so that we may take $G = G^*$. We observe, moreover, that every endoscopic group H of G arises in a natural way from some $H_{(\mathrm{sc})}$ for G_{sc}, and that it follows readily from [**7**] that if local transfer is possible for the pair $G_{\mathrm{sc}}, H_{(\mathrm{sc})}$ then it is possible for G, H. Thus we shall eventually obtain the following statement.

Theorem. *If G is of type A_2 then all pairs (G, H) admit Δ-transfer.*

In treating forms of $\mathrm{SL}(3)$, we need only consider cuspidal endoscopic groups and subregular classes $\mathcal{O}$. We now begin to examine the possible endoscopic groups.

3. Endoscopic groups. Since G is simply connected, we confine ourselves to endoscopic data $(H, \mathcal{H}, s, \xi)$ (see [**6**], (1.2)) for which $\mathcal{H} = {}^L H$, so that ξ is an embedding of ${}^L H$ in ${}^L G$. The most important element among the data is then s. If we realize $\hat{G}$ in the usual way as $PGL(3, \mathbf{C})$, then we may suppose that s is diagonal and that the diagonal matrices form part of a Γ-splitting of ${}^L H$. There are then three possibilities: (i) all eigenvalues of s are equal; (ii) two are equal to each other but not to the third, (iii) no two eigenvalues are equal.

In case (i), the group ${}^L H$ is ${}^L G$ and $H = G^* = G$. In case (ii), we may suppose that it is the first and third eigenvalues that are equal. If $\hat{\alpha}' = x_1 - x_2, \hat{\alpha}'' = x_2 - x_3$, and $\hat{\alpha}''' = x_1 - x_3$ are the usual roots of $\hat{T}$ then $\sigma\hat{\alpha}''' = \hat{\alpha}'''$ for all $\sigma \in \Gamma$. Thus $\sigma\hat{\alpha}' = \hat{\alpha}', \sigma\hat{\alpha}'' = \hat{\alpha}''$ or $\sigma\hat{\alpha}' = \hat{\alpha}'', \sigma\hat{\alpha}'' = \hat{\alpha}'$. The second possibility can occur for some σ if and only if G is a unitary group, and then

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If no two eigenvalues of s are equal then $\hat{T} = \hat{H}$ and H is a torus T.

When discussing a given Cartan subgroup with an image T_H it is convenient to fix a diagram:

$$\begin{array}{ccccc} T_H & \longrightarrow & \mathbf{T} & \longleftarrow & T \\ & & & & \uparrow \\ & & & & T_G \end{array}$$

Then $\hat{T}$ and $\hat{T}_G$ may be identified with $\hat{\mathbf{T}}$, the group of diagonal matrices in $PGL(3, \mathbf{C})$. The group Γ_T acts on $\hat{\mathbf{T}}$.

Lemma 3.1. *If $H \neq G^*$ then the group Γ_T cannot contain all reflections in the Weyl group.*

Proof. Since $H \neq G^*$, we have $\hat{\alpha}(s) \neq 1$ for at least one root α. However, if σ_α is the reflection corresponding to α then there is another root β such that $\hat{\beta} - \sigma_\alpha\hat{\beta} = \hat{\alpha}$. Thus $\hat{\beta}(s) \neq \sigma_\alpha\hat{\beta}(s)$ and σ_α cannot lie in Γ_T.

Corollary 3.2. *If G is an inner form of* $\mathrm{SL}(3), H \neq G^*$*, and T is anisotropic then Γ_T is the cyclic subgroup of the Weyl group of order three and H is a torus.*

Proof. Γ_T must be a proper subgroup of the Weyl group whose only fixed point in $X_*(T)$ is 0. Thus Γ_T is cyclic of order three. Since s is projectively invariant under Γ_T and not a scalar, its three eigenvalues must be of the form $\lambda, \xi\lambda, \xi^2\lambda$ where $\xi^3 = 1, \xi \neq 1$. Thus $\hat{H}$ is a torus, and so is H.

Since $\hat{T}$ has been identified with $\hat{\mathbf{T}}$ by means of the fixed diagram, the group $X_*(T)$ is identified with triples of integers (x, y, z) whose sum is zero.

Corollary 3.3. *If G is* $\mathrm{SU}(3)$*, T is anisotropic, and the element s has exactly two eigenvalues equal then we may suppose that Γ_T is one of the two groups:*

$$(a) \qquad (x,y,z) \to \pm(x,y,z);$$

$$(b) \qquad (x,y,z) \to \pm(x,y,z), \pm(z,y,x).$$

Proof. Choose the diagram so that $(1, 0, -1)$ is a root that is 1 on s.

Lemma 3.4. *If G is* $\mathrm{SU}(3)$ *and $H = T$ is an anisotropic torus, then Γ_T is a group of order six containing the cyclic permutations of order three. It can also be assumed to contain*

$$(x,y,z) \to (-z,-y,-x).$$

Proof. The argument of Lemma 2.1 implies that Γ_T contains no reflections. Thus the intersection of Γ_T with the Weyl group is trivial or cyclic of order three. If it is trivial then Γ_T must be $(x, y, z) \to \pm(x, y, z)$, because T is anisotropic, but then all eigenvalues of s_T have the same square, so that two are equal. Then H is not a torus. Thus Γ_T is of order six and, with a suitable choice of diagram, either contains $(x, y, z) \to (-z, -y, -x)$ or the direct product of the cyclic subgroup of order three with the group of order two generated by $(x, y, z) \to -(x, y, z)$. There is however no s_T invariant under the second group that is not scalar.

There is one more lemma to be noted. Its proof is immediate.

Lemma 3.5. *If H is cuspidal but is neither G^* nor a torus, then G is* $\mathrm{SU}(3)$ *and s is of type* (ii).

4. Recapitulation. Up to conjugacy there are only finitely many Cartan subgroups of H over F. Suppose T_H is one of them, and $\mathfrak{t}_H$ its Lie algebra. As a consequence of a theorem of Harish-Chandra, there is a neighborhood V_H of 0 in $\mathfrak{t}_H$ such that for regular $X \in V_H$ and $t \in F^*$, $|t| < 1$,

$$\Gamma^{\mathrm{st}}_{\mathcal{O}}(\exp t^2 X) = |t|^{d-l} \Gamma^{\mathrm{st}}_{\mathcal{O}}(\exp X),$$

if $\mathcal{O}$ is a unipotent conjugacy class and $d = d_{\mathcal{O}}$ is defined by the condition that $2d + l$ is the dimension of the centralizer of any element in it. The integer l is the rank of H, and $2d_I + l$ is the dimension of H.

A similar assertion is valid for the κ-germ. Thus if for a given unipotent class $\mathcal{O}$ in $G(F)$ there is a collection of unipotent classes $\mathcal{O}_1, \dots, \mathcal{O}_n$ in $H(F)$ and complex numbers $a_1, \dots, a_n$ such that for every Cartan subgroup T_H and every regular ray $\{tX \mid t \in F^*\}$ in $\mathfrak{t}_H$ we have

$$\Gamma^{\kappa}_{\mathcal{O}}(\exp tX) = \sum_{i=1}^{n} a_i \Gamma^{\mathrm{st}}_{\mathcal{O}_i}(\exp\ tX)$$

for t sufficiently small, then there is an equality of germs

$$\Gamma^{\kappa}_{\mathcal{O}} = \sum_{i=1}^{n} a_i \Gamma^{\mathrm{st}}_{\mathcal{O}_i}$$

with

$$d_{\mathcal{O}_i} = d_{\mathcal{O}}$$

for all i. Hence the theorem of Harish-Chandra allows us in principle to apply the methods of Igusa.

For forms of $\mathrm{SL}(3)$ only the subregular classes $\mathcal{O}$ are not dealt with by the remarks of Section 2; so we suppose henceforth that $\mathcal{O}$ is subregular. There are two lemmas to be proven. The first is simple to state.

Lemma 4.1. *If H is a torus and $\mathcal{O}$ is subregular then*

$$\Gamma^{\kappa}_{\mathcal{O}} \equiv 0.$$

Otherwise we may suppose that G is $\mathrm{SU}(3)$ and

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the derived group of H is $\mathrm{SL}(2)$. Moreover $d_{\mathcal{O}} = 1$ if $\mathcal{O}$ is subregular and $d_{\mathcal{O}_i} = 1$ if and only if $\mathcal{O}_i = \{1\}$.

Lemma 4.2. *If G is $\mathrm{SU}(3)$, H is cuspidal but neither G^* nor a torus, and if $\mathcal{O}$ is subregular then for some $a \in \mathbf{C}$*

$$\Gamma^{\kappa}_{\mathcal{O}} = a\Gamma^{\mathrm{st}}_{1}.$$

Recall that Γ^{st}_1, a stable germ for H, has been calculated in **[5]**. It is 0 on Cartan subgroups whose intersection with H_{der} is not anisotropic. If, however, $T_0 = T_H \cap H_{\mathrm{der}}$ is anisotropic then Γ_{T_0} is a group of order two:

$$\Gamma_{T_0} = \{1, \sigma\}.$$

Take $Q = Q_{T_H}$ to be the form of $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the cocycle:

$$a_\sigma : (x, y) \to (y, x).$$

Here x, y are inhomogeneous coordinates on $\mathbf{P}^1 \times \mathbf{P}^1$. Then, apart from a constant that is the same for all Cartan subgroups, Γ_1^{st} is the principal-value integral ([**5**])

$$|\alpha(X)| \oint_{Q(F)} \frac{dxdy}{|x-y|^2} \tag{4.1}$$

times

$$D_H^{-1}(\gamma_H).$$

Here $\gamma_G = e^X$ and α is one of the two roots of H_{der}. Thus, we prefer to study

$$D_H(\gamma_H)\Gamma_{\mathcal{O}}^{\kappa}(\gamma_H) = \kappa(\mathrm{inv}(T_G, \gamma_G))D_G(\gamma_G)\Sigma\kappa(\delta)\Gamma_{\mathcal{O}}(\gamma_G^{\delta}).$$

5. The method of Igusa. For a given T_G the method of Igusa [**4**] yields (for forms of $\mathrm{SL}(3)$) an asymptotic expression for

$$D_G(\gamma_G)\Sigma\kappa(\delta)\Gamma_{\mathcal{O}}(\gamma_G^{\delta}) = \Sigma\kappa(\delta)D_G(\gamma_G^{\delta})\Gamma_{\mathcal{O}}(\gamma_G^{\delta})$$

along rays. One begins with

$$\Sigma\kappa(\delta)D_G(\gamma_G^{\delta})\Phi(\gamma_G^{\delta}, f),$$

which in the notation of [**4**] is $F^{\kappa}(\gamma, f)$, and applies Proposition 1.1 of [**4**].

The theorem of Harish-Chandra implies that the terms appearing in that proposition are 0 for $r > 1$ or $\beta < 1$. For $r = 1, \beta \geqq 1$ the factor $F_r(\theta, \beta, f)$ is a principal-value integral

$$\oint_{E(\theta,\beta)} h_E|\nu_E|$$

over the union $E(\theta, \beta)$ of the varieties E in $E(\theta, \beta)$. Observe that the factor $A_r(M)$ appearing in Lemma 1.3 of [**4**] has been incorporated into the function h_E.

The morphism π of [**4**] maps E onto the closure $\bar{\mathcal{O}}_E$ of a unipotent orbit $\mathcal{O}_E$ in G. Departing from the notation of [**4**], we let $\hat{E}$ be the inverse image of $\mathcal{O}_E$ in E. The set $\mathcal{O}_E(F)$ is a stable orbit $\mathcal{O}^{\mathrm{st}}$ and thus a finite union of conjugacy classes.

For a given β, the sum over θ

$$\kappa(\mathrm{inv}(T, \gamma_G)) \sum_{\theta} \theta(\lambda)|\lambda|^{\beta-1} \oint_{E(\theta,\beta)} h_E|\nu_E| \tag{5.1}$$

is equal to

$$\sum_{d_{\mathcal{O}}=\beta-1} \left\{ \oint f(\eta)|\eta| \right\} D_H\Gamma_{\mathcal{O}}^{\kappa}. \tag{5.2}$$

It was observed to us by Thomas Hales that, when we are concerned with a particular $\mathcal{O}$, we may as well suppose that f vanishes on all unipotent orbits $\mathcal{O}'$ with $d_{\mathcal{O}'} \geqq d_{\mathcal{O}}$ other than $\mathcal{O}$ itself. This has two advantages at present, when we are dealing with forms of $\mathrm{SL}(3)$ and a subregular $\mathcal{O}$. First of all, the blow-up otherwise entailed by Lemma 4.5 of [**4**] is unnecessary. Secondly, if $\beta = d_{\mathcal{O}} + 1$ and $E \in \mathfrak{C}(\theta, \beta)$ then, by Lemma 4.7 and 4.8 of [**4**], necessarily $\mathcal{O} \subseteq \mathcal{O}_E$. Hence in (5.1) the integrals may be taken over

$$\hat{E}(\theta, \beta) = \cup_{E \in \mathfrak{C}(\theta,\beta)} \hat{E}.$$

Moreover $\mathcal{O}_E$ is smooth and, as we shall see, $\hat{E}(\theta, \beta) \to \mathcal{O}_E$ is flat, so that (5.1) is equal to

$$\sum_{\theta} \theta(\lambda)|\lambda|\kappa(\mathrm{inv}(T, \gamma_G)) \int_{\mathcal{O}_E(F)} f(n) \left\{ \int_{E_n(\theta,\beta)} h_n |\nu_n| \right\}, \tag{5.3}$$

where $E_n(\theta, \beta)$ is the inverse image of n in $\hat{E}(\theta, \beta)$.

To prove Lemma 4.1 it is enough to show that

$$\oint_{E_n(\theta,2)} h_n |\nu_n| = 0 \tag{5.4}$$

for the pertinent T and κ; to prove Lemma 4.2 it is enough to show that

$$\sum_{\theta} \theta(\lambda)|\lambda|\kappa(\mathrm{inv}(T, \gamma_G)) \oint_{E_n(\theta,2)} h_n |\nu_n| = a|\alpha(X)| \oint_{Q(F)} \frac{dxdy}{|x-y|^2}, \tag{5.5}$$

a being a constant that depends on the endoscopic data, and perhaps on n and θ, but not on the particular T under consideration. Moreover, $\gamma_G = \exp \lambda X$ and α is a root of H_{der}.

The remainder of the note is devoted to the explicit description of h_n, ν_n, and the varieties $E_n(\theta, 2)$, based, of course, on the construction of [**4**], and to the calculation of the principal-value integrals. Observe that if $\beta(E) = 2$ then, in the notation of [**4**], E is one of E_1', E_1'', or E_6.

6. The fibres. If G is $\mathrm{SL}(3)$ then there is no supplementary blow-up, and the divisor E_6 does not appear; if G is $\mathrm{SU}(3)$ the divisors E_1', E_1'' have no rational points and thus may be discarded. Therefore it is convenient to treat the two cases separately, beginning with $\mathrm{SL}(3)$.

Associated to a subregular n over F in $\mathrm{SL}(3)$ are a distinguished point and a distinguished line in $\mathbf{P}^2$. The point P is the range of $n-1$ and the line l its null space. On the other hand, a Borel subgroup B of $\mathrm{SL}(3)$ is determined by a point q on a line m in $\mathbf{P}^2$, and $n \in B$ if and only if $p \in m$ or $q \in l$. Let M_n be the join at $p = l$ of the variety $\mathbf{P}_l$ of points on l with the variety $\mathbf{P}_p$ of lines through p, so that M_n is the union of two projective lines crossing normally at $p = l$, and parameterizes the Borel subgroups containing n.

There is a set-theoretical mapping of $E_n = E_n(2)$, the inverse image of n in $\hat{E}_1' \cup \hat{E}_2'$, to M_n. To a point in E_n there is associated a star and the Borel subgroup $B = B(W_+)$ in this star lies in M_n. To investigate $E_n(2)$

over the inverse image of a neighborhood of B, we choose B_0 to be defined by $p \in l$ and B_∞ to be opposite to B. Then the inverse image is contained in

$$G \times S_1(B_\infty, B_0) \times N_\infty.$$

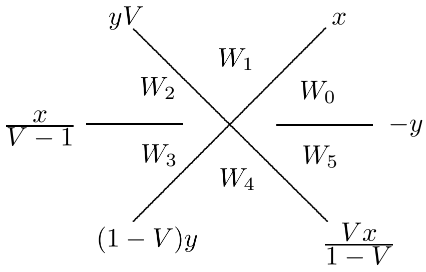
We may choose $X_{\alpha'}, X_{\alpha''}$ so that

$$n = \exp w_0[X_{\alpha'}, X_{\alpha''}].$$

Observe that $w_0 \neq 0$.

If, as in Section 4 of [**4**], we take x, y, V as coordinates on $S_1(B_\infty, B_0)$ then the coordinates z_i', z_i'' figuring in the diagram on p. 489 of [**4**] can be calculated in terms of x, y, V. The results can be summarized in a similar diagram

(6.1)



Thus there is a coordinate x, attached to each of six rays r. The extended Weyl group, viz, the Weyl group together with outer automorphisms, acts by replacing x_r by

$$\epsilon(\omega) x \omega_r^{-1}.$$

An element σ of the Galois group acts by replacing x_r by

$$\epsilon(\sigma_T)\sigma(x\sigma_T^{-1}r).$$

Here $\epsilon(\omega)$ is the determinant of ω, and $\epsilon(\sigma_T)$ the determinant of σ_T. Thus the varieties $S'(B_\infty, B_0)$, $S_1(B_\infty, B_0), R(B_\infty, B_0)$ are all defined over F.

On the open set $Y_1^{11}(B_\infty, B_0)$ of Section 4 of [**4**] we have

$$uU = b(\lambda) yVw, \quad \nu U = c(\lambda) xVw. \tag{6.2}$$

If

$$n' = \exp uX_{\alpha'} \exp vX_{\alpha''} \exp w[X_{\alpha'}, X_{\alpha''}]$$

and if

$$n = (n')^{n_\infty}, n_\infty \in N_\infty$$

then $m \sim B' = B_0^{n_\infty}$ is the image in M_n and $w = w_0$. The coordinates u, v serve as coordinates of n_∞ or as coordinates on M_n. We conclude from Equation 6.2 and Equation 4.6 of [**4**] that $E_n \to M_n$ is a morphism, and that it is smooth off $V = 0$.

The condition $V = 0$ is more invariantly defined as the condition that the point in S_1 is defined by a point of type B_3 (see Lemmas 3.13, 3.14 of [**4**]). Since the B_j are cyclically permuted by the Galois group, no such point has coordinates in F and $E_n \to M_n$ is smooth at every point of $E_n(F)$. Moreover, as we shall see in the next section, $\kappa_1' \neq \kappa_2'$. Consequently

$$\oint_{E_n(\theta,2)} h_n|\nu_n| = \oint_{M_n} \left\{ \oint_{F_m} k_m|\nu_m'| \right\} |\nu''|. \tag{6.3}$$

Here ν'' is any non-vanishing form on the base and k_m, ν_m' are determined by restriction and division.

The conditions for rationality are deduced from the diagram and the equation following (3.4) of [**4**]. For the F-valued points in F_m, the coordinate V is neither $0, 1$, nor ∞ and the map that sends a point to $(-1, V, 1-V)$ or simply to V identifies $F_m(F)$ with the $\mathbf{P}^1$ associated to elements of trace zero in the cyclic cubic extension K associated to the torus T.

Passing to SU(3) we observe first that if $n \in G(F)$ is subregular then there is a unique Borel subgroup B_0 over F containing it. If B_∞ is defined over F and opposed to B_0 then $E_n = E_n(2)$ is contained in

$$G \times S_1(B_\infty, B_0) \times N_\infty.$$

The only pertinent divisor E is now E_6; it is obtained by a blow-up along the intersection of E_1' and E_1''. Hence it fibres over this intersection with fibre $\mathbf{P}^1$. The fibre E_n is contained in $n \times S_1(B_\infty, B_0) \times 1$ and the projection on $S_1(B_\infty, B_0)$ of its intersection I_n with $E_1' \cap E_1''$ is (see Section 3 of [**4**])

$$S'(B_\infty, B_0 \times_{T_0} U) - 0 \simeq R(B_\infty, B_0).$$

Let ρ be the element of the Galois group such that

$$\rho_T : (x, y, z) \to (-z, -y, -x)$$

as given by Lemma 3.4. Thus a condition for rationality is that

$$\rho(V-1) \cdot (V-1) = 1, \tag{6.4}$$

so that $V - 1 = \rho(R)R^{-1}$. The map that sends a point with coordinate V to

$$R(-1, V, 1-V) \tag{6.5}$$

identifies $I_n(F)$ with the $\mathbf{P}^1$ defined by elements of trace zero.

In the case of Corollary 3.3 (a), the element ρ does not exist, and the condition for rationality to be imposed on V is that $V \in F$. In the case of Corollary 3.3 (b), the number R can be taken to be fixed by the element σ with

$$\sigma_T : (x, y, z) \to -(x, y, z),$$

and the coordinate V lies in K_0, the fixed field of σ.

It is easy to see that

$$\oint_{E_n(\theta,2)} h_n|\nu_n| = \oint_{l_n} \left\{ \oint_{F_x} k_x|\nu'_x| \right\} |\nu''|, \tag{6.6}$$

x denoting a point of I_n, and F_x the fibre over x. Observe that at a point of I_n where V is not $0, 1$, nor ∞ we may use as coordinate $z = x/y$. The condition for rationality is somewhat elaborate to describe.

Think of the chambers, $W_i, 0 \leqq i \leqq 5$, as oriented in the counterclockwise direction for i even and in the clockwise direction for i odd. Every element of the extended Weyl group takes W_0 either to a W_i or to a $-W_i$, the chamber W_i with the opposite orientation. We assign to a chamber a function $\varphi(W_i)$ of V as follows:

$$\begin{aligned} &\varphi(W_0) = 1; \varphi(W_1) = \frac{-1}{V}; \varphi(W_2) = V^{-1}(1-V)^{-1};\\ &\varphi(W_3) = (V-1)^2;\\ &\varphi(W_4) = -V(1-V)^{-2}; \varphi(W_5) = V(1-V)^{-1};\\ \varphi(-W_i) &= \varphi(W_i)^{-1}. \end{aligned}$$

The positive chamber is $W_0 = W_+$. The condition for rationality of a point in I_n with coordinate V is that

$$\rho(\varphi(\sigma_T W))\varphi(\rho_T W) = \varphi(\rho_T \sigma_T W) \quad \forall \rho, \sigma, W. \tag{6.7}$$

The condition for rationality of a point on the fibre over a rational base point is that

$$\sigma(z) = \varphi(\sigma_T W_+) z^{\epsilon_\sigma} \quad \forall \sigma. \tag{6.8}$$

Here ϵ_σ is 1 if σ fixes E and -1 if it does not. The field E is the quadratic field over which the hermitian form giving the group is defined.

7. The forms. It is sufficient to calculate them where V is finite and not 0 nor 1. Beginning with the form ν_n, we make use of Lemmas 2.8 and 2.12 of [**4**]. We may suppose that the form $\wedge\varphi_i$ appearing in Lemma 2.8 is $\omega_T \wedge du \wedge dv \wedge dw$. Then the form ω^0 on Y^0 used to define $F^\kappa(\gamma, f)$ is the product of $d\lambda \wedge du \wedge dv \wedge dw$ with the restriction of $\wedge\omega_j$ to N_∞.

The form on N_∞ need not be investigated further. To describe the first factor in other coordinates we need to calculate the jacobians:

$$\frac{\partial(\lambda, u, v, w)}{\partial(u, v, V, w)}; \quad \frac{\partial(\lambda, u, v, w)}{\partial(x, y, V, w)}; \quad \frac{\partial(\lambda, u, v, w)}{\partial(z, y, V, w)}.$$

This is a straightforward matter. If $G = G^*$ is so defined that we may take B^* to be the group of upper-triangular matrices and T^* to be the group of triangular matrices, then the diagram (2.0) of [**4**] identifies the tangent space to the curve C with a line

$$F \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}.$$

We choose α, β, γ so that λ is also a natural coordinate on the line; then $b = b(0) = \alpha - \beta, c = c(0) = \beta - \gamma, d = d(0) = \alpha - \gamma$. By assumption, none of these numbers is zero. The three determinants are the product of a factor of the form $1 + E(\lambda), E(0) = 0$, with

$$-\frac{uv(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)V^2w}; \quad AxyV^2w^3; \;\; Azy^3V^2w^3.$$

Here

$$A = -\frac{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}{((\alpha-\gamma)-V(\beta-\gamma))^4} = -\frac{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}{U^4}.$$

Thus the form γ'_m that appears in (6.3) may be taken to be (see Section 1 of [**4**])

$$-(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)\frac{dV}{U^2} = (\alpha-\beta)(\alpha-\gamma)\frac{dU}{U^2}, \tag{7.1}$$

for the form ν appearing on p. 469 of [**4**] is the product of this with wv^{-1} (at the divisor $u = 0$) or wu^{-1} (at the divisor $v = 0$) and the invariant form on N_∞.

It is convenient to choose $(\alpha_0, \beta_0, \gamma_0) = (\alpha_0, \sigma(\alpha_0), \sigma^2(\alpha_0))$ of trace zero and linearly independent of $(c, -d, b)$, and to set

$$U_1 = \frac{d - cV}{\gamma_0 + \beta_0 V}.$$

Then U_1 takes rational values at rational points, and (7.1) becomes

$$-(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(c\gamma_0 + d\beta_0)\frac{dU_1}{U_1^2}. \tag{7.2}$$

On the divisor E_6 the prescription of Section 1 of [**4**] leads to the form

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\frac{dV}{U^2}\cdot\frac{dz}{z} \tag{7.3}$$

on the fibre, the form on the base being the product of wdw with the invariant form on N_∞. Thus we have the form (7.2) on I_n and the form dz/z on the fibre F_x. If the endoscopic group is a torus, we may once again use the coordinate U_1. At a rational point,

$$U_1 \in E \text{ and } U_1 = U_1(U_1 - 1)^{-1},$$

the bar denoting conjugation over F.

It is also convenient to replace z by

$$z_1 = \frac{z}{V-1},$$

for then the condition for rationality becomes $z_1 \in E, \bar{z}_1 = z_1^{-1}$.

8. The functions. The functions h_n are essentially the functions on m_κ of Section 2 of **[4]** and are to be calculated as in Section 5 of that note; or rather as in Section 5.4 and Section 5.5 of **[6]**. Since we have multiplied by $D_H(\gamma_H)$, the function $m_k(e(\cdot))$, of **[4]** becomes the function $\Delta(\cdot)$ of **[6]**. We calculate with $\Delta(\cdot)$, obtaining thereby not the functions h_n of (5.4) and (5.5) but the functions

$$\kappa(\mathrm{inv}(T,\gamma)h_n,$$

without for the moment the factor $A_r(M)$. Since we are close to the identity, $\Delta(\cdot)$ may be taken to be

$$\Delta_I(\gamma_H, \pi(\cdot))\Delta_{II}(\gamma_H, \pi(\cdot))\Delta_1(\gamma_H, \pi(\cdot)). \tag{8.1}$$

On putting the first and third factors together as in Section 5.5 of **[6]** we obtain the value of s_T on the cocycle

$$\sigma \to \zeta_\sigma = \prod_{1,\sigma}^{\rho} \left[\frac{-a_\alpha}{z(\sigma,\alpha)}\right]^{\alpha^\nu}. \tag{8.2}$$

We choose, as in Section 1,

$$a_\alpha = \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2},$$

so that the second factor of (8.1) becomes 1. (Notice that the symbols α, γ have two conflicting meanings.)

We begin with SL(3), and the endoscopic groups given by anisotropic tori. Then Γ_T is cyclic of order three, and we take the generator given by

$$\sigma : (\alpha, \beta, \gamma) \to (\beta, \gamma, \alpha).$$

It is enough to calculate the cocycle (8.2) on this generator. We write $\sigma = \omega''\omega'$, and find that the roots appearing in (8.2) are α'' and $\alpha''' = \alpha' + \alpha''$. The factors $z(\sigma, \alpha)$ are calculated in terms of the coordinates in (6.1). Thus

$$\xi_\sigma = \left[\frac{-a_{\alpha''}(V-1)}{x}\right]^{\hat{\alpha}''} \left[\frac{-a_{\alpha'''}}{yV}\right]^{\hat{\alpha}'''}.$$

In terms of the coordinates u, v this is

$$\xi_\sigma = \left[\frac{-a_{\alpha''}(V-1)Vwc(\lambda)}{vU}\right]^{\hat{\alpha}''} \left[\frac{-a_{\alpha'''}wb(\lambda)}{uU}\right]^{\hat{\alpha}'''}.$$

Since λ is close to 0, we may replace $c(\lambda)$ by $c(0) = c$ and $b(\lambda)$ by $b(0) = b$. Moreover, on a fibre over a point of C close to the identity, we may replace $a_{\alpha'}, a_{\alpha''}, a_{\alpha'''}$ by λb, λc, and λd respectively, so that ζ_σ becomes

$$\left[\frac{-c}{b}(V-1)u\right]^{\hat{\alpha}''} \left]\frac{-d}{c}\frac{v}{V}\right]^{\hat{\alpha}'''}.$$

Let K be the splitting field of T. The standard identification

$$F^\times/Nm_{K/F}K^\times \simeq H^1(T) \tag{8.3}$$

is obtained by choosing for a given $k \in F^*$ a triple e, f, g in K^* with $efg = k$ and then taking the cocycle whose value at σ is

$$\begin{bmatrix} \frac{e}{\sigma(g)} & & \\ & \frac{f}{\sigma(e)} & \\ & & \frac{g}{\sigma(f)} \end{bmatrix}. \tag{8.4}$$

Of course, T is identified with the group of diagonal matrices of determinant 1. The contributions of u, v, w to ξ_σ may be factored out as a cocycle of this type with $e = v$, $f = uv$, $g = 1$, for u, v, w lie in $F^\times$.

The identification (8.3) allows us to identify κ with a character of $F^\times$, necessarily non-trivial. The contribution of u, ν, w to $\Delta(\cdot)$ is then

$$\kappa(u\nu^2) = \kappa(u)\kappa(\nu)^{-1}.$$

Thus $\kappa_1' = \kappa^{-1}, \kappa_1'' = \kappa$. In particular, $\kappa_1' \neq \kappa_1''$.

The remaining contribution to ζ_σ is more disagreeable. Fortunately, we need not calculate it, for it is independent of u, ν, w, as is the form (7.2). Thus the inner integral appearing in (6.3) becomes

$$\oint_{F_m} k_m(\nu_m') = c\kappa^2(\nu)$$

for $\theta = \kappa^{-1}$,

$$\oint_{F_m} k_m(\nu_m') = c\kappa^{-2}(u)$$

for $\theta = \kappa$. The constant c does not depend on n. Then the integral over the base becomes, apart from this constant,

$$|w| \oint_{\mathbf{P}^1} \kappa^2(\nu)\frac{|d\nu|}{|\nu|} = |w| \oint_{\mathbf{P}^1} \kappa^{-2}(u)\frac{|du|}{|u|} = 0.$$

(These simple principal values are calculated in Lemma 1.C of [**5**].)

We observe in passing that when we are dealing with table orbital integrals, the cocycle is no longer pertinent because $\kappa = 1$. Then the fibre integrals become essentially

$$\oint_{\mathbf{P}^1} \frac{|dU_1|}{|U_1|^2} = 0.$$

However, the formula (6.3) is no longer valid, for $\kappa_1' = \kappa_2' = 1$. Now the calculation must be made as in the appendix. The total contribution of the term (A.3) is 0, but the total contribution of the terms (A.2) is given by an integral of the form (A.4).

Lemma 4.1 is now proven for groups of type $\mathrm{SL}(3)$. We turn to $\mathrm{SU}(3)$ beginning with endoscopic groups that are tori. The following lemma is easy and simplifies the calculations.

Lemma 8.1. *If G is* $\mathrm{SU}(3)$ *and H is an anisotropic torus, then the restriction map*

$$H^1(\mathrm{Gal}(K/F), T(K)) \to H^1(\mathrm{Gal}(K/E), T(K))$$

is injective.

The field E appearing here is the quadratic extension of F associated to G. To prove the lemma, one observes that the order of $H^{-1}(\mathrm{Gal}(K/F), X(T))$ clearly divides three, and then appeals to Tate-Nakayama theory.

This lemma allows us to calculate the value of κ on the cocycle (8.2) as before, except that u and v are no longer the appropriate coordinates. They are to be replaced by z_1, y. Thus if σ is chosen in the same way, then

$$\zeta_\sigma = \left[\frac{-a_{\alpha''}}{z_1 y}\right]^{\hat\alpha''} \left[\frac{-a_{\alpha'''}}{yV}\right]^{\hat\alpha'''}.$$

When λ is close to 0, we may replace $a_{\alpha''}$ by λc, $a_{\alpha'''}$ by λd, and express λ with the help of (4.3) of **[4]** to obtain

$$\zeta_\sigma = \left[\frac{-cV(V-1)yw}{U}\right]^{\hat\alpha''} \left[\frac{-dz_1(V-1)yw}{U}\right]^{\hat\alpha'''}. \tag{8.5}$$

If ρ is such that $\rho_T : \alpha', \alpha'', \alpha''' \to \alpha'', \alpha', \alpha'''$ then the rationality conditions for y are:

$$\sigma(y) = yV;\ \rho(y) = zy \text{ or } \rho(yR^{-1}) = z_1 yR^{-1}.$$

Since $\bar z_1 = \rho(z_1) = z_1^{-1}$, we can find $r \in E^\times$ such that $\rho(r) = z_1 r$. Let $y = y_0 Rr$. If K_0 is the fixed field of ρ then $y_0 \in K_0$ and

$$\sigma(y_0) = y_0 rR\sigma(rR)^{-1}V = y_0 R\sigma(R)^{-1}V.$$

Choose one such y_0. Then all others are of the form $ty_0, t \in F$. It is t rather than y that is the local coordinate defining E_6.

The contribution of t to (8.5) is of the form (8.3) with $e = t^{-1}, f = t^{-2}, g = 1$. Thus $\kappa_6 = 1$ and the θ appearing in (6.6) are of order two.

The inner integral of (6.6) is taken over the projective line with coordinate z_1 and rationality condition $\bar z_1 = z_1^{-1}$. The dependence of (8.5) on z_1 is through z_1 and r, and

$$\sigma \to \begin{bmatrix} z_1 r & & \\ & r & \\ & & z_1^{-1} r^{-2} \end{bmatrix}$$

is of the form (8.4) with $e = z_1 r, f = z_1 r^2, g = 1, efg = z_1^2 r^3$.

K is the extension $K = K_0 E$ of F, and there is a diagram

$$\begin{tikzcd}[column sep=small] & K \arrow[dl, dash] \arrow[dr, dash] & \\ K_0 \arrow[dr, dash] & & E \arrow[dl, dash] \\ & F & \end{tikzcd}$$

The character κ may be regarded as a non-trivial character on the group of order three,

$$E^*/Nm_{K/E}K^*,$$

and

$$z_1 \to \kappa(z_1^2 r^3) = \kappa(z_1)^{-1}$$

as a character on

$$\{z_1 \in E^* \mid \bar{z}_1 = z_1^{-1}\} = E^1.$$

Since

$$\lambda = z_1 y^2 V(V-1)U^{-1}w$$

the procedure of Section 1 of **[4]** leads to an inner integral in (6.6) that is the product of a function on the base and

$$\oint_{\mathbf{P}^1} \theta(r^2 z_1)^{-1}\kappa(z_1)^{-1}\frac{|dz_1|}{|z_1|}. \tag{8.6}$$

This is the integral of $\theta(r^2 z_1)\kappa(z_1)^{-1}$ over the group E^1 with respect to the Haar measure. To formulate the result of the procedure this way it has been necessary to extend θ from $F^\times$ to $E^\times$.

Observe that $\theta(r^2 z_1)$ is independent of the choice of r, and that it is a character on E^1 of order two. Thus to show that (8.6) vanishes we need only verify that κ is not trivial on E^1. By Lemma 3.4 the field E is the maximal abelian subfield of the Galois extension K. Thus, by local classfield theory

$$Nm_{K/F}K^\times = Nm_{E/F}E^\times.$$

Moreover,

$$[E^\times; Nm_{K/E}K^\times] = 3.$$

Consequently

$$[E^1 : E^1 \cap Nm_{K/E}K^\times] = 3,$$

and κ is not trivial on E_1.

It remains to treat the case that H is cuspidal but not a torus. We may suppose that $\pm\alpha'''$ are the roots that take the value 1 on s. Only anisotropic tori are to be considered. According to Corollary 3.3 there are two possibilities. Let σ now be the element such that

$$\sigma_T : (x, y, z) \to -(x, y, z).$$

In case (b) let K_0 be the fixed field of σ.

Lemma 8.2. *The restriction map*

$$H^1(\mathrm{Gal}(K/F), T(K)) \to H^1(\mathrm{Gal}(K/K_0), T(K))$$

is injective.

Let $\rho_T : (x, y, z) \to -(z, y, x)$. Then under the restriction map

$$H^{-1}(\mathrm{Gal}(K/F), X_*(T)) \to H^{-1}(\mathrm{Gal}(K/K_0), X_*(T)) \tag{8.7}$$

the element $\mu = (1, -1, 0)$ in $X_*(T)$ is sent to

$$\mu + \rho\mu = (1, 0, -1).$$

The group on the left of (8.7) is of order two and is generated by the class of μ^*, the group on the right is of order four and the class of $(1, 0, -1)$ is not zero.

In case (a) the field K_0 is F and $H^1(\mathrm{Gal}(K/K_0), T(K))$ consists of all

$$\begin{bmatrix} e & & \\ & f & \\ & & g \end{bmatrix} \tag{8.8}$$

with $e, f, g \in K_0^\times, efg = 1, e, f, g$ modulo $Nm_{K/K_0} K^\times$.

Consider the number

$$\frac{-bw}{Vd/c}.$$

Since $\sigma(b) = -b, \sigma(c) = -c, \sigma(d) = -d$ and $\sigma(w) = -w$, it belongs to K_0 in case (b) and to F in case (c). Moreover in case (b), $\rho(w) = -w$ and

$$\rho\left(\frac{-bw}{Vd/c}\right) = cw\left(\frac{V}{V-1}\right)^{-1}\left(-\frac{d}{b}\right) = (V-1)\left(\frac{-bw}{Vd/c}\right).$$

Thus in both cases we may take

$$R = \frac{-bw}{Vd/c}.$$

The two rationality conditions are:

$$V_1 = RV \in F; \bar{z}_1 = z_1^{-1}.$$

The third is

$$\sigma(y) = \frac{x}{1-V} = -z_1 y,$$

and in case (b)

$$\rho(y) = x = zy = z_1(V-1)y.$$

Thus

$$-\sigma(yR^{-1}) = \rho(yR^{-1}) = z_1 yR^{-1}.$$

Choose $r \neq 0$ such that

$$\sigma(r) = \rho(r) = z_1 r$$

and $a \neq 0$ such that

$$\rho(a) = a = -\sigma(a).$$

Then

$$y = atrR,$$

and $t \in F$ is the correct local coordinate at $y = 0$. Notice that $a/d \in F$, indeed, that a can be taken equal to d.

Since σ_T has changed, the formula (8.5) for the value of the cocycle (8.2) at σ has to be modified. It becomes

$$\zeta_\sigma = \left[\frac{-a_{\alpha'}}{(1-V)y}\right]^{\hat{\alpha}'} \left[\frac{-a_{\alpha'''}(1-V)}{Vx}\right]^{\hat{\alpha}'''} \left[\frac{a_{\alpha''}}{y}\right]^{\hat{\alpha}''}.$$

For λ close to 0, this may be replaced by

$$\left[\frac{bz_1 Vyw}{U}\right]^{\hat{\alpha}'} \left[\frac{-d(1-V)yw}{U}\right]^{\hat{\alpha}''} \left[\frac{cz_1 ywV(V-1)}{U}\right]^{\hat{\alpha}''}. \tag{8.9}$$

The characters on $H^1(\mathrm{Gal}(K/K_0), T(K))$ are given by choosing two characters η_1, η_2 of

$$K_0^\times / Nm_{K/K_0} K^\times$$

and then sending the class given by e, f, g to $\eta_1(e)\eta_2(g)$. It is the character κ defined by s (or its extension, in case (b)) that is of concern. Since

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Tate-Nakayama isomorphism sends (l, m, n) to the class of (8.8) with

$$e = h^l, f = h^m, g = h^n, h \in K_0^\times, h \notin Nm_{K/K_0} K^\times,$$

the character κ corresponds to $\eta_1 = \eta_2 \neq 1$ in case (a). In case (b) the injection (8.7) is such that we must have $\eta_1 \neq \eta_2$ in order that κ not be trivial, for the class of $(l, m, n) = (1, 0, -1)$ is in the image of (8.7).

Since the contribution of t to (8.9) is of the form (8.8) with $e = t^2, f = 1, g = t^{-2}$, the character $\kappa_6 = 1$, and the θ that appear in (6.6) are quadratic. The contribution of z_1 is through z_1 and r and is given by

$$e = z_1 r^2 = r\sigma(r), f = 1, g = e^{-1}.$$

Thus the inner integral of (6.6) is, as before, essentially the integral over the group E^1 of the character

$$\theta(r^2 z_1) = \theta(r\sigma(r)).$$

If this is not to be 0, then θ must be 1 or the character η associated to the quadratic extension E/F. Suppose henceforth that this is so. Then we may replace the inner integral in (6.6) by 1.

For the integral over the base it is convenient to introduce the variable

$$V_1 = -bw \cdot \frac{V}{V - d/c} = bcw\frac{V}{U} = RV.$$

According to (7.3) the pertinent form to take on the base is still (7.1). In terms of V_1 it becomes $w^{-1}dV_1$. The factor w^{-1} may be incorporated into the integral over the unipotent orbit, and therefore ignored, although it will in fact be cancelled in the next section. Moreover,

$$\lambda = xy\frac{V}{U}w = z_1(atrR)^2 b^{-1}c^{-1}V_1(V-1)$$

so that the function to be integrated is the product of

$$\theta^{-1}(a^2R^2(V-1)V_1b^{-1}c^{-1}) \tag{8.10}$$

with the value of κ on the cocycle given by

$$\sigma \to \left[\frac{-aRV_1}{c(1-V)}\right]^{\hat{\alpha}'} \left[\frac{-adR(1-V)V_1}{bcV}\right]^{\hat{\alpha}'''} \left[\frac{aRV_1}{b}\right]^{\hat{\alpha}''}. \tag{8.11}$$

When expressing (8.11) in the form (8.8) the factors R^2 and V_1^2 may be dropped, as may $-a^2 = a\sigma(a)$ or $-c^2 = c\sigma(c)$ and $-b^2$. This yields

$$e = \frac{d}{b}V^{-1}, f = \frac{c}{b}(V-1), g = -\frac{c}{d}\cdot\frac{V}{1-V}.$$

Consider first the case that θ is η. Then we obtain in case (a)

$$\oint \eta(V_1)|dV_1|. \tag{8.12}$$

To treat the case (b), we observe that

$$\eta_1\eta_2 = \eta \circ Nm_{K_0/F},$$

and that

$$\eta_1(e)\eta_2(g) = \eta_1(e)\eta_2(e) = \eta\left(\frac{d^2}{bc}\right)\eta(Nm_{K_0/F}V),$$

because

$$b\rho(b) = bc, d\rho(d) = d^2.$$

Moreover

$$Nm_{K_0/F}(V-1) = (V-1)\rho(V-1) = 1$$

and

$$Nm_{K_0/F}V = (V/(V-1)R)^2R^2(V-1) \in (F^\times)^2R^2(V-1).$$

Thus we again obtain (8.12). It is one of the simple principal-value integrals calculated in Lemma 1.C of **[5]** and equals 0. Hence we may take $\theta = 1$.

Until now, we have been able to ignore the factor $A_r(M)$ of Lemma 1.3 of **[4]** that is to be incorporated into h_E. However, to obtain the correct expression for (5.5), the integral (8.12) has to be multiplied by $A_r(M)$. Since $a_6 = 2$ and the principal part of $(1-t^2)^{-1}$ at $t=1$ is $(1-t)^{-1}/2$, the procedure of **[4]** yields

$$A_r(M) = \frac{1}{2}.$$

9. Final calculations. | To put the left side of (5.5) in a form that can be compared directly with the right, we first observe that (8.12) is also 0 when η is replaced by 1. Thus we may add a constant times

$$\frac{1}{2}\oint\oint \frac{|dz_1|}{|z_1|}|dV_1|$$

to our integral representation of (5.5). This yields $|\lambda|$ times

$$|bcd|\oint\oint \frac{|dz_1|}{|z_1|}\frac{dV}{|U|^2}. \tag{9.1}$$

The $|w|$ that appears in the change of variables has cancelled the factor $|w|^{-1}$ from the previous section. The integral is, however, to be taken over

$$\left\{(z_1,V) : z_1 \in E^1, \frac{b}{c}(V-1) \in Nm_{E/F}E^\times\right\} \tag{9.2}$$

in case (a), and over

$$\{(z_1,V) : z_1 \in E^1, \sigma(V) = V, \rho(V-1) = (V-1)^{-1}, bcd^{-2}V\rho(V) \in Nm_{E/F}E^\times\} \tag{9.3}$$

in case (b). Thus we have implicitly extended, but in a rather simple way, the notion of principal value. Since $|\lambda d|$ is the factor $|\alpha(X)|$ occurring on the right side of (5.5) we suppress the $|d|$ and show that what remains is a constant times

$$\oint_{Q(F)} \frac{dxdy}{|x-y|^2}. \tag{9.4}$$

The symbols x, y are now free of any earlier meaning and refer solely to coordinates on Q. Choose $\delta \in E^\times, \delta \notin F^\times$ and set

$$x' = \frac{x-\delta}{x-\bar{\delta}},\ \ y' = \frac{y-\delta}{y-\bar{\delta}}.$$

Then the rationality conditions on s', y' are

$$\sigma(x') = y'^{-1},\ \sigma(y') = x'^{-1}$$
(9.5)
$$\rho(x') = x'^{-1},\ \rho(y') = y'^{-1}.$$

In case (a), the second condition is replaced by $x', y' \in E$.

Choose $\mu \in E^1$ and consider the morphism

(9.6) $$z_1 = \mu x'y',\ \ W_1 = \frac{c}{b}(V-1) = \frac{x'}{y'}.$$

It is not defined at $x' = y' = 0$ or $x' = y' = \infty$. So we blow up these two points. Since they are not rational this has no effect on (9.4). The morphism is a double covering ramified along the two curves introduced by the blow-up. A simple calculation shows that the form

$$\frac{dx'dy'}{(x'-y')^2}$$

is the pull-back of

$$-\frac{1}{2}bc\frac{dz_1}{z_1}\ \frac{dV}{U^2}.$$

It is evident that the rationality conditions on x', y' imply that

$$z_1 \in E,\ \ \bar{z}_1 = z_1^{-1},\ \ \sigma(V-1) = V-1,$$

and that $\rho(V-1) = (V-1)^{-1}$ in case (b). In case (a), $V \in E$.

In case (a)

$$\frac{x'}{y'} = x'\bar{x}'.$$

Thus we can solve (9.6) for x', y' if and only if the condition (9.2) is satisfied and

$$z_1 \in \{\mu x'/\bar{x}' | x'\bar{x}' = W_1\}.$$

Hence if we let μ run over a set of representatives for the cosets of $(E^1)^2$ in E^1, the equations (9.5) have exactly two solutions. (Points with $S_1 = 0$ or ∞ are exceptional.) We conclude that (9.1) is equal to $|2|N$ times (9.4) if $2N = [E^1 : (E^1)^2]$.

The conditions on V in (9.3) are:

$$\sigma(V) = F;\ W_1 = cV\rho(cV)^{-1};\ cd^{-1}V \in Nm_{K/K_0}K^\times.$$

If

$$cd^{-1}V = t\sigma(t),\ \ t \in K^\times$$

and if we set

$$x' = ut\rho(t)^{-1}, \;\; y' = \sigma(x')^{-1},$$

then the second set of conditions in (9.5) amounts to

$$\rho(u) = u^{-1},$$

and the equations (9.6) become

$$z_1 = \mu u\sigma(u)^{-1}t\sigma\rho(t)\rho(t)^{-1}\sigma(t)^{-1}; \;\; \sigma(u) = u^{-1}.$$

If we again let μ run over a set of representatives for the cosets of $(E^1)^2$ in E^1 this equation has a solution unique in μ and in u up to a sign for a given $z_1 \in E^1$.

Conversely, if we can solve the equations (9.6) subject to (9.5) then μ is uniquely determined and

$$W_1 = e\rho(e)^{-1}\sigma(e)\sigma\rho(e)^{-1}.$$

It follows that $V \in K_0$ and that $(V-1)\rho(V-1) = 1$, so that

$$W_1 = cV\rho(cV)^{-1}$$

and $cVe^{-1}\sigma(e)^{-1}$ is fixed by ρ. Since it is also fixed by σ, it lies in $Nm_{K/K_0}K_0^\times$ and so does cV. We conclude once again that (9.1) is equal to $|2|N$ times (9.4).

Appendix I. Although it would be out of place to elaborate on the formal properties of the principal values of [**4**], there is one calculation that will be useful. Before describing it, we observe that in the definition of $A(x)$ in Section 1 of [**4**], the coefficients c_j should be divided by $(j-1)!$

Suppose that in a coordinate patch U defined by

$$|\mu_i| \leqq q^{-m}, 0 \leqq i \leqq r,$$

we have:

$$\lambda = \alpha\mu_0^{a_0}\mu_1\mu_2; \;\; \omega = W\mu_0^{b_0-1}\mu_1\mu_2 \wedge d\mu_i; \;\; f = \lambda\kappa_0(\mu_0)\kappa(\mu_1\mu_2).$$

Suppose also that $b_0 \neq 2a_0$ or that $\kappa^{a_0} \neq \kappa_0$. We take $D_r = D_r(\kappa, 2)$ and calculate

$$\oint_{D_1\cap U} h_1|\nu_1|, \tag{A.1}$$

in the sense of Section 1 of [**4**], using the notation of [**5**] for a principal value.

It is the sum of two terms. The first is obtained by setting

$$\omega' = W(\mu_0, 0, 0, \mu_3, \cdots)\alpha^{-\beta}\mu_0^{b_0-2a_0-1}d\mu_0 \wedge d\mu_3 \wedge \cdots$$
$$f' = (\ln_q|\alpha\mu_0^{a_0}|)(\lambda\kappa_0(\mu_0)\kappa^{-a_0}(\mu_0)\kappa^{-1}(\alpha))$$

and taking

$$\text{(A.2)} \qquad \left(1-\frac{1}{q}\right)\int_{D_2\cap U} f'|\omega'|.$$

The second is obtained by setting:

$$\begin{aligned}
\lambda'' &= \mu_1\mu_2;\\
\omega'' &= \mu_1\mu_2 d\mu_1\wedge d\mu_2;\\
f'' &= \int \lambda|W||\alpha|^{-2}|\mu_0|^{b_0-2a_0-1}\kappa_0(\mu_0)\kappa^{-a_0}(\mu_0)\kappa^{-1}(\alpha)\times|d\mu_0\wedge d\mu_3\cdots|;
\end{aligned}$$

and then taking

$$\text{(A.3)} \qquad \oint_{D_1''\cap U''} h_1''|\nu_1''|,$$

the data for this integral being defined by λ'', ω'', f''.

This is of course an easy calculation, for in the present circumstances the only c_j that is not 0 is c_2 and $c_2 = 1$. Then

$$A_1(y) = 1-y$$

and the decomposition of (A.1) into the sum of (A.2) and (A.3) is given by

$$\begin{aligned}
&1-m-M_1-M_2-a_0m(\mu_0)\\
&\quad=(-m-a_0m(\mu_0))+(1-M_1-M_2).
\end{aligned}$$

In the text, the integral to which the term (A.2) leads is

$$\text{(A.4)} \qquad \oint_{P^1} \ln|x-\xi|\frac{dx}{|x|^2},$$

in which ξ lies in a cubic extension of F. If, for example, the extension if ramified then, with no loss of generality, we may suppose that the order of ξ is $r+n$ with r equal to $1/3$ or $2/3$. Then the integral is calculated to be

$$-rq^{-n-1}+\frac{q^{-n}}{q-1}=\frac{q^{-n}}{q-1}\left((1-r)+\frac{r}{q}\right).$$

In the context of Section 8, this is to be compared with Section 10 of [**8**].

Finally we note two more corrections to [**4**]. The exponent n' should be removed from the middle term of (3.4) and "non-zero" in Lemma 3.13 (ii) should be "zero".

Appendix II. Although not necessary for proving the existence of the transfer it is sometimes useful [**9**] to know the value of the constant appearing in Lemma 4.2. Of course, the constant only has a meaning after the measure defining the integral over the class $\mathcal{O}$ has been fixed.

If, to be explicit, we take G to be the group attached to the hermitian matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the quadratic extension E, then every subregular conjugacy class over F has a representative of the form

$$n(w) = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad w \in E^\times$$

with $w + \bar{w} = 0$, the bar denoting conjugation of E over F. It is easy to see that $n(w)$ and $n(aw), a \in F^*$, are conjugate over F if and only if $a \in Nm_{E/F}E^\times$.

We take N_∞ to be the group of lower-triangular unipotent matrices in G. Then

$$(n_\infty, n(w)) \to n_\infty^{-1} n(w) n_\infty, n_\infty \in N_\infty(F), w \in E^\times, w + \bar{w} = 0,$$

yields a parameterization of a dense open subset of the manifold of F-valued points on the variety of subregular unipotent elements. If, as in Section 7, we take the measure dn_∞ on $N_\infty(F)$ to be that associated to $\wedge\omega_j$, then

$$\int f(n_\infty^{-1} n(w) n_\infty)|w| dw dn_\infty$$

defines a G-invariant integration on the manifold of subregular unipotent elements over F and thus on each G-orbit in it. We take

$$a_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(n_\infty^{-1} n(w) n_\infty)|w||dw||dn_\infty|.$$

It is best to observe that the absolute value on E is to be taken to be an extension of the normalized absolute value on F.

The form $\nu_n, n = n(w)$, appearing in (5.5) is then that given by (7.3). Thus, according to the discussion at the end of Section 7, the end of Section 8, and Section 9, the constant a appearing in (5.5) and therefore in Lemma 4.2 is $|2|N$.

Lemma. *For any local field of characteristic zero, $|2|N = 1$.*

Since

$$N = \frac{1}{2} \cdot [E^1 : (E^1)^2]$$

the lemma is clear for a field with odd residual characteristic, for then $N = 1$. If the residual characteristic is even, let 2^r be the number of elements in the residue field, and let $2 = \tilde{\omega}^k$ where $\tilde{\omega}$ is a uniformizing parameter.

Let $E^j, j \geqq 2$, be

$$\{1 + a\tilde{\omega}^{j-1} | a \text{ integral}\}.$$

Since $(1+a\tilde{\omega}^j)^2 = 1 + 2a\tilde{\omega}^j + a^2\tilde{\omega}^{2j}$, the index

$$[E^j : E^{j+1}(E^j \cap (E^1)^2)]$$

is equal to 1 for j odd and different from $2k+1$ and for all j greater than $2k+1$. If j is even and $j \leqq 2k$ it is 2^r, and it is 2 for $j = 2k+1$. The lemma follows.

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