Some Holomorphic Semi-Groups*

In this note we state some basic spectral properties of a class of operators associated with a representation of a Lie group. We merely indicate the proofs since we intend to give details in another context.

Let $T(.)$ be a strongly continuous representation of the Lie group $\mathfrak{G}$ in the Banach space $\mathfrak{X}$. For a in $\mathfrak{A}$, the Lie algebra of $\mathfrak{G}$, let $A(a)$ be the infinitesimal generator of the one-parameter group, $T(\exp(ta)), -\infty < t < \infty$, and let $A^*(a)$ be the adjoint of $A(a)$. It is known that

$$\mathfrak{W}_k = \{x \epsilon \mathfrak{X} | x \epsilon \text{Domain}(A(a_1) \dots A(a_k)) \text{ for all } a_1, \dots, a_k \epsilon \mathfrak{A}\}$$

is dense in X and that

$$\mathfrak{W}_k^* = \{x^* \epsilon \mathfrak{X}^* | x^* \epsilon \text{ Domain } (A^*(a_1) \dots A^*(a_k)) \text{ for all } a_1, \dots, a_k \epsilon \mathfrak{A}\}$$

is dense in $\mathfrak{X}^*$ in the weak* topology. Let $\{e_1, \dots, e_n\}$ be a basis for $\mathfrak{A}$ and set $A_i = A(e_i), A_i^* = A^*(e_i)$. If $\alpha = (\alpha_1, \dots, \alpha_k), 1 \le \alpha_i \le n$, is a sequence of integers we denote a product such as $A_{\alpha_1} \dots A_{\alpha_k}$ by A_α; $|\alpha| = k$ is the length of α. The operator $B^\circ = \Sigma_{|\alpha| \le m} a_\alpha A_\alpha$ (the coefficients are complex numbers) is defined on $\mathfrak{W}_m$ and denoting α in reversed order by α^*, the operator $\tilde{B}^\circ = \Sigma_{|\alpha| \le m} a_\alpha A^*_{\alpha^*}$ is defined on $\mathfrak{W}_m^*$. The closure B of B° and the weak* closure $\tilde{B}$ of $\tilde{B}^\circ$ are well-defined operators. If for any real n-vector, $\xi = (\xi_1, \dots, \xi_n), |\Sigma_{|\alpha| \le m} a_\alpha \xi_\alpha| \ge \rho \|\xi\|^m$ ($\rho > 0$ is fixed and $\|\xi\|$ is the ordinary Euclidean norm) then

Theorem 1. *The adjoint B^* of B is $\tilde{B}$.*

To prove this we observe that if $B^* x_1^* = x_2^*$ then[1] for $x = \mathfrak{X}, x_1 * (T(g)x)$ is a weak solution of the equation

$$\Sigma_{|\alpha| \le m} a_\alpha R_{\alpha^*} \{x_1^*(T(g)x)\} = x_2^*(T(g)x);$$

R_i is the infinitesimal transformation

$$R_i f(g) = \lim_{t \to 0} t^{-1} \{f(\exp(te_i)g) - f(g)\}.$$

We use the properties of weak solutions of elliptic equations to show that x_1^* is in $\mathfrak{W}_{m-1}*$ and then apply an adaptation[2] of the mollifier technique.

If the stronger condition

$$(-1)^{j-1} \text{Re}\{\Sigma_{|\alpha| \le m} a_\alpha \xi_\alpha\} \ge \rho \|\xi\|^m,$$

with $2j = m$, is fulfilled then

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Theorem 2. *The operator B is the infinitesimal generator of a semi-group, $S(t)$, of class $H(\phi_1, \phi_2)$, for some ϕ_1 and ϕ_2.*[3]

The theorem is equivalent to an inequality $\|R(\lambda, B)\| \leq M/[\rho(\lambda, S)]$ for the norm of the resolvent of B; M is a constant and $\rho(\lambda, S)$ is the distance of λ from a sector

$$S = \{\zeta | \psi_2 \leq \text{ arg } (\zeta - \zeta_0) \leq \psi_1\}, \frac{\pi}{2} < \psi_2 \leq \pi \leq \psi_1 < 3\pi/2.$$

To obtain the inequality we observe that

$$x^*(T(g)(B - \lambda)x) = \Sigma_{|\alpha| \leq m} a_\alpha L_\alpha x^*(T(g)x) - \lambda x^*(T(g)x),$$

with

$$L_i f(g) = \lim_{t \to 0} t^{-1}\{f(g \exp(te_i)) - f(g)\};$$

partially invert this equation in a neighborhood of the identity using a parametrix; and then establish appropriate estimates[4] for the parametrix. It is of interest to note that if a_α is real when $|\alpha| = m, S(t)$ is holomorphic in a half-plane.

The semi-groups, $S(t)$, have a canonical representation. Let μ be a left-invariant Haar measure on $\mathfrak{G}$, then

Theorem 3. *There is a unique function $h(t, g)$ (defined for t in an open sector containing $(0, \infty)$) in $L_1(\mu)$ for each t and such that for all representations $T(\cdot)$ of $\mathfrak{G}$*

$$S(t)x = \int_{\mathfrak{G}} h(t, g)T(g)x\mu(dg).$$

The integral is a Bochner integral and $h(t, g)$ is analytic in t and g. We remark finally that, when t is in the interior of the domain of $S(.)$ and x is in X, $S(t)x$ is an analytic vector.

The above results generalize theorems of Nelson[5] and Nelson and Stinespring.[1]

[1]Nelson, E., and W.F. Stinespring, *Amer.J.Math.*, **81**, 547-560 (1959).

[2]Langlands, R.P., *On Lie Semi-Groups* (submitted to *Canad.J.Math.*).

[3]For terminology see Hille, E., and R.S. Phillips, *Functional Analysis and Semi-Groups*.

[4]These are essentially estimates for the resolvents of strongly elliptic differential operators with constant coefficients and are obtained from estimates for the fundamental solution of the associated parabolic equation similar to those of Silov G.E., (*Usp.Mat.Nauk.*, **10**, 89-100 (1955)).

[5]*Ann.Math.*, **70**, 572-615 (1959).