Review of

Haruzo Hida's p-adic automorphic forms on Shimura varieties

by

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Three topics figure prominently in the modern higher arithmetic: zeta-functions, Galois representations, and automorphic forms or, equivalently, representations. The zeta-functions are attached to both the Galois representations and the automorphic representations and are the link that joins them. Although by and large abstruse and often highly technical the subject has many claims on the attention of mathematicians as a whole: the spectacular solution of Fermat's Last Theorem; concrete conjectures that are both difficult and not completely inaccessible, above all that of Birch and Swinnerton-Dyer; roots in an ancient tradition of the study of algebraic irrationalities; a majestic conceptual architecture with implications not confined to number theory; and great current vigor. Nevertheless, in spite of major results modern arithmetic remains inchoate, with far more conjectures than theorems. There is no schematic introduction to it that reveals the structure of the conjectures whose proofs are its principal goal and of the methods to be employed, and for good reason. There are still too many uncertainties. I none the less found while preparing this review that without forming some notion of the outlines of the final theory I was quite at sea with the subject and with the book. So ill-equipped as I am in many ways – although not in all – my first, indeed my major task was to take bearings. The second is, bearings taken, doubtful or not, to communicate them at least to an experienced reader and, in so far as this is possible, even to an inexperienced one. For lack of time and competence I accomplished neither task satisfactorily. So, although I have made a real effort, this review is not the brief, limpid yet comprehensive, account of the subject, revealing its manifold possibilities, that I would have liked to write and that it deserves. The review is imbalanced and there is too much that I had to leave obscure, too many possibly premature intimations. A reviewer with greater competence, who saw the domain whole and, in addition, had a command of the detail would have done much better.*

It is perhaps best to speak of L-functions rather than of zeta-functions and to begin not with p-adic functions but with those that are complex-valued and thus – at least in principle, although one problem with which the theory is confronted is to establish this in general – analytic functions in the whole complex plane with only a very few poles. The Weil zeta-function of a smooth algebraic variety over a finite field is a combinatorial object defined

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* These lines will mean more to the reader who consults the supplement to the review that is posted with it on the site

`http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/intro.html`

The supplement can be consulted either as the review is read or later. It contains commentaries by specialists that are especially valuable.

by the number of points on the variety over the field itself and its Galois extensions. The Hasse-Weil zeta-function of a smooth variety over a number field F is the product over all places $\mathfrak{p}$ of the zeta-function of the variety reduced at $\mathfrak{p}$. Of course, the reduced variety may not be smooth for some $\mathfrak{p}$ and for those some additional care has to be taken with the definition. In fact it is not the Hasse-Weil zeta-function itself which is of greatest interest, but rather factors of its numerator and denominator, especially, but not necessarily, the irreducible factors. The zeta-function is, by the theory of Grothendieck, a product, alternately in the numerator and denominator, of an Euler product given by the determinant of the action $\tau_m(\Phi_\mathfrak{p})$ of the Frobenius at $\mathfrak{p}$ on the l-adic cohomology in degree m,

$$\prod_{\mathfrak{p}} \frac{1}{\det\bigl(1-\tau_{\mathrm{m}}(\Phi_\mathfrak{p})/\mathrm{N}\mathfrak{p}^{\mathrm{s}}\bigr)}. \tag{1}$$

Algebraic correspondences of the variety with itself, if they are present, will act on the cohomology and commute with the Frobenius elements, thereby entailing an additional decomposition of τ_m and an additional factorization of the determinants in (1) and thus of (1) itself. These factors are the L-functions that are one of the key concepts of the modern theory. Grothendieck introduced a conjectural notion of motive, as objects supporting these factors. Although there are many major obstacles to creating a notion of motive adequate to the needs of a coherent theory, not least a proof of both the Hodge and the Tate conjectures, it is best when trying to acquire some insight into the theory's aims to think in terms of motives. In practice they are concrete enough.

Since their zeta-functions are major objects to be understood, the first element of a very important nexus, whose four elements will be described one by one, is the collection $\mathfrak{M}$ of motives M over a given finite extension F of $\mathbb{Q}$. With each M is associated an L-function $L(s, M)$ about which, at first, we know little except that it is an Euler product convergent in a right half-plane. The category of motives as envisioned by Grothendieck is Tannakian so that with each M is also associated a reductive algebraic group ${}^{\mu}G_M$ with a projection onto the Galois group of some sufficiently large, but if we prefer finite, extension L of F. The field of coefficients used for the definition of ${}^{\mu}G_M$ lies, according to needs or inclination, somewhere between $\mathbb{Q}$ and $\bar{\mathbb{Q}}$. The μ in the notation is to make it clear that the group ${}^{\mu}G_M$ has a different function than the group G. It is not the carrier of automorphic forms or representations but of motives.

An automorphic representation π is a representation, usually infinite-dimensional, of an adelic group $G(\mathbb{A})$, the group G being defined over F and reductive. With G is associated an L-group ${}^{L}G$, which is a reductive algebraic group over $\mathbb{C}$ that functions in some respects as a dual to G. There is a homomorphism of ${}^{L}G$ onto $\mathrm{Gal(L/F)}$, L being again a sufficiently large finite extension of F. Generalizing ideas of Frobenius and Hecke, not to speak of Dirichlet and Artin, we can associate with π and with almost all primes $\mathfrak{p}$ of F a conjugacy class $\{A(\pi_\mathfrak{p})\}$ in ${}^{L}G$. Then, given any algebraic, and thus finite-dimensional, representation r of ${}^{L}G$, we may introduce the L-function

$$L(s, \pi, r) = \prod_{\mathfrak{p}} 1\det\bigl(1-\mathrm{r}(\mathrm{A}(\pi_\mathfrak{p}))/\mathrm{N}\mathfrak{p}^{\mathrm{s}}\bigr). \tag{2}$$

The usual difficulties at a finite number of places are present.

In principle, and in practice so far, the functions (2) are easier to deal with than (1). Nevertheless, the initial and fundamental question of analytic continuation is still unresolved in any kind of generality. One general principle, referred to as $functoriality$ and inspired by Artin's reciprocity law, would deal with the analytic continuation for (2). Functoriality is the core notion of what is frequently referred to as the Langlands program.

Suppose G and G' are two groups over F and ϕ is a homomorphism from ${}^LG$ to ${}^LG'$. Then if π is an automorphic representation of G there is expected to be an automorphic representation π' of G' such that for each $\mathfrak{p}$ the class $\{A(\pi'_{\mathfrak{p}})\}$ attached to π' is the image under ϕ of $\{A(\pi_{\mathfrak{p}})\}$. To establish this will be hard and certainly not for the immediate future. I have, however, argued in [L] that it is a problem that we can begin to attack.

It is then natural to suppose, once again influenced by Artin's proof of the analytic continuation of abelian L-functions, that each of the Euler products $L(s, M)$ into which (1) factors is equal to one of the Euler products (2). This would of course certainly deal with the problem of its analytic continuation. Better, in [L] it is suggested that we should not only prove functoriality using the trace formula but simultaneously establish that each automorphic representation π on G is attached to a subgroup ${}^\lambda H_\pi$ of ${}^LG$, even to several such subgroups, but the need for this multiplicity is something that can be readily understood. So we are encouraged to believe that the fundamental correspondence is not that between L-functions but that between M and ${}^\mu G_M$ and π and ${}^\lambda H_\pi$. In particular ${}^\mu G_M$ and ${}^\lambda H_\pi$ are to be isomorphic and the Frobenius-Hecke conjugacy classes in ${}^\mu G_M$ attached to M are to be equal to the Frobenius-Hecke conjugacy classes ${}^\lambda H_\pi$ attached to π. Apart from the difficulty that there is little to suggest that ${}^\lambda H_\pi$ is defined over any field but $\mathbb{C}$, it is reasonable to hope that in the long run some correspondence of this nature will be established. The λ in the notation is inherited from [L] and emphasizes that H is a subgroup of ${}^LG$ and not of G.

The Tannakian formalism for motives – when available – suggests that if there is a homomorphism ${}^\mu G_M \subset {}^\mu G'$ then M is also carried by ${}^\mu G'$. If functoriality is available, as is implicit in the constructions, and ${}^\lambda H_\pi \subset {}^\lambda H'$ then, in some sense, π is also carried by ${}^\lambda H'$, but in the form of an automorphic representation π' of a group G' with ${}^\lambda H' \subset {}^LG'$. So if ${}^\mu G'$ and ${}^\lambda H'$ are isomorphic, the couples $\{M, {}^\mu G'\}$ and $\{\pi, {}^\lambda H'\}$ also correspond.

An example, in spite of appearances not trivial, for which the necessary functoriality is available is the unique automorphic representation π of the group $G = \{1\}$ with ${}^LG = {}^\lambda H_\pi = \mathrm{Gal}(L/F)$, where $\mathrm{Gal}(L/F)$ is solvable, together with the motive $M(\sigma)$ of rank 2 and degree 0 attached to a faithful two-dimensional representation σ of $\mathrm{Gal}(L/F)$. They clearly correspond. Moreover ${}^\lambda H_\pi = {}^\mu G_M$ is imbedded diagonally in $GL(2, \mathbb{C}) \times \mathrm{Gal}(L/F)$. The representation π' is given by solvable base-change and the correspondence between $\{\pi', GL(2, \mathbb{C}) \times \mathrm{Gal}(L/F)\}$ and $\{M(\sigma), GL(2, \mathbb{C}) \times \mathrm{Gal}(L/F)\}$ is one of the starting points for the proof of Fermat's Last Theorem.

Although functoriality and its proof are expected to function uniformly for all automorphic representations, when comparisons with motives are undertaken not all automorphic representations are pertinent. The representation π has local factors π_v at each place. At an infinite place v the classification of the irreducible representations π_v of $G(F_v)$ is by homomorphisms of the Weil group at v into ${}^LG$. This Weil group is, I recall, a group that contains $\mathbb{C}^\times$ as a subgroup of index 1 or 2. We say ([Ti]) that the automorphic representation π is $arithmetic$ (or algebraic or motivic) if for each place π_∞ is parametrized by a homomorphism whose restriction to $\mathbb{C}^\times$, considered as an algebraic group over $\mathbb{R}$, is itself algebraic. Thus it is expressible in terms of characters $z \in \mathbb{C}^\times \to z^m \bar{z}^n$, $m, n \in \mathbb{Z}$.

Only arithmetic automorphic representations should correspond to motives. Thus the second element of our nexus is to be the collection $\mathfrak{A}$ of automorphic representations π for F, each attached to a group ${}^\lambda H$. Because of functoriality, in the stronger form described, π is no longer bound to any particular group G.

A central problem is to establish a bijective correspondence between the two elements introduced. Major progress was made by Wiles in his proof of the conjecture of Taniyama and Shimura. Since he had – and still would have – only an extremely limited form of functoriality to work with, his arguments do not appear in exactly the form just suggested. Moreover, there are two further extremely important elements in the nexus in which he works to which we have not yet come.

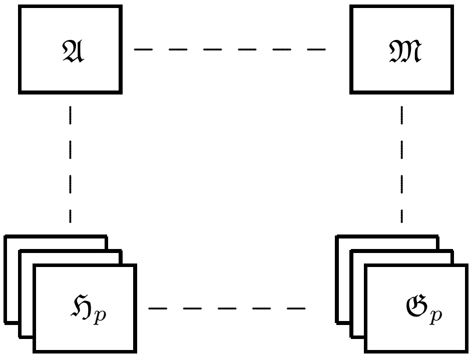
To each motive M and each prime p is attached a p-adic representation of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ of dimension equal to the rank of the motive. The third element of the nexus is not, however, the collection of p-adic Galois representations – subject to whatever constraints are necessary and appropriate. Rather it is a foliated space, in which the leaves are parametrized by p and in which there are passages from one leaf to another, permitted in so far as each p-adic representation is contained in a compatible family of representations, one for each prime. We are allowed to move from one leaf to another provided we move from one element of a compatible family to another element of the same family. The arguments of Wiles and others, those who preceded and those who followed him, rely on an often very deep analysis of the connectivity properties of the third element, either by p-adic deformation within a fixed leaf, in which often little more is demanded than congruence modulo p, or by passage from one leaf to another in the way described (cf. [Kh]) and their comparison with analogous properties of yet a fourth element whose general definition appears to be somewhat elusive.

For some purposes, but not for all, it can be taken to consist of representations of a suitably defined Hecke algebra. For automorphic representations attached to the group G over F, the Hecke algebra is defined in terms of smooth, compactly supported functions f on $G(\mathbb{A}_F^f)$, $\mathbb{A}_F^f$ being the adeles whose components at infinity are 0. They act by integration on the space of any representation π of $G(\mathbb{A}_F)$, in particular on the space of an automorphic representation or on automorphic forms.

Let $\mathbb{A}_F^\infty$ be the product of F_v at the infinite places. When the Lie group $G(\mathbb{A}_F^\infty)$ defines a bounded symmetric domain – or more precisely when a Shimura variety is attached to the group G – then there are quotients of the symmetric domain that are algebraic varieties defined over number fields. There are vector bundles defined over the same field whose de Rham cohomology groups can be interpreted as spaces of automorphic forms for the group G on which the Hecke operators will then act. The images of the Hecke algebra will be finite-dimensional algebras over some number field L and can often even be given an integral structure and then, by tensoring with the ring $\mathcal{O}_{\mathfrak{p}}$ of integral elements at a place $\mathfrak{p}$ of L over p, a p-adic structure, imparted of course to its spectrum. In so far as these rings form the fourth element of the nexus, the leaves are clear, as is the passage from one leaf to another. It seems to correspond pretty much to taking two different places $\mathfrak{p}$ and $\mathfrak{q}$ without changing the homomorphism over L.

The four elements form a square, motives on the upper right-hand side of the diagram, automorphic representations on the upper left, the leaves $\mathfrak{G}_p$ of the p-adic representations on the lower right, and the fourth as yet only partly defined element $\mathfrak{H}_p$ on the lower left-hand side. The heart of the proof of Fermat's theorem is to deduce from the existence of one couple $\{M, {}^{\mu}G_M\} \in \mathfrak{M}$ and $\{\pi, {}^{\lambda}H_\pi\} \in \mathfrak{A}$ of corresponding pairs the existence of other couples. We pass from $\{M, {}^{\mu}G_M\}$ in $\mathfrak{M}$ to some leaf in the element below, thus to the corresponding p-adic Galois representation $\mathfrak{s}_p \in \mathfrak{G}_p$, and from $\{\pi, {}^{\lambda}H_\pi\}$ to an object $\mathfrak{h}_p \in \mathfrak{H}_p$, the fourth element of the nexus. Then the essence of the arguments of Wiles and Taylor-Wiles is to show that movement in $\mathfrak{G}_p$ of the prescribed type is faithfully reflected in permissible movements in $\mathfrak{H}_p$ and that if in $\mathfrak{G}_p$ the movement leads to an image of a pair in $\mathfrak{M}$ then the corresponding movement in $\mathfrak{H}_p$ leads to an element of $\mathfrak{A}$. These two pairs will then necessarily correspond in the sense that the associated Frobenius-Hecke classes will be the same.

As a summary of the proof of Fermat's Last Theorem, the preceding paragraph is far too brief, but it places two features in relief. There has to be an initial seeding of couples with one term from $\mathfrak{M}$ and one from $\mathfrak{A}$ that are known for some reason or another to correspond and it has to be possible to compare the local structures of the two spaces $\mathfrak{G}$ and $\mathfrak{H}$.



The easiest seeds arise for G an algebraic torus, for then an automorphic representation π is a character of $T(\mathbb{A}_F)$ and if the character is of type A_0, thus if the representation is arithmetic, the process begun in [W] and continued by the construction of the Taniyama group ([LS]), should construct both the p-adic representations and the motive $\{M, {}^{L}T\}$ corresponding to $\{\pi, {}^{L}T\}$. From them others can be constructed by functoriality, a formality for M.

Although they are somewhat technical, it is useful to say a few words about the correspondence for tori, partly because it serves as a touchstone when trying to understand the general lucubrations, partly because the Taniyama group, the vehicle that establishes the correspondence between arithmetic automorphic forms on tori and motives, is not familiar to everyone. Most of what we need about it is formulated either as a theorem or as a conjecture in one of the papers listed in [LS], but that is clear only on close reading. In particular, it is not stressed in these papers that the correspondence yields objects with equal L-functions

The Taniyama group as constructed in the first paper of [LS] is an extension $\mathcal{T} = \mathcal{T}_F = \varprojlim_L \mathcal{T}_F^L$ of the Galois group $\mathrm{Gal}(\bar{F}/F)$, regarded as a pro-algebraic group, by a pro-algebraic torus $\mathcal{S} = \varprojlim_L \mathcal{S}_F^L$ and is defined for all number fields F finite over $\mathbb{Q}$. One of its distinguishing features is that there is a natural homomorphism φ_F of the Weil group W_F of F into $\mathcal{T}_F(\mathbb{C})$. This homomorphism exists because there is a splitting of the image in $\mathcal{S}_f^L$ $(\mathbb{C})$ of the lifting to $W_{L/F}$ of the Galois 2-cocycle in $H^2(\mathrm{Gal}(L/F), \mathcal{S}_f^L(\bar{\mathbb{Q}})$ defining $\mathcal{T}_F^L$. Thus every algebraic homomorphism over $\mathbb{C}$ of $\mathcal{T}_F$ into an L-group ${}^LG$ compatible with the projections on the Galois groups defines a compatible homomorphism of W_F into ${}^LG(\mathbb{C})$. In particular if $G = T$ is a torus, every T-motive over $\mathbb{C}$ (if all conjectures are anticipated, this is just another name for a homomorphism ϕ from $\mathcal{T}_{\mathcal{F}}$ to ${}^LT$) defines a homomorphism $\psi = \varphi \circ \phi$ of the Weil group into ${}^LT$ and thus ([LM]) an automorphic representation π of $T(\mathbb{A}_F)$.

The Weil group can be constructed either at the level of finite Galois extensions L/F as $W_{L/F}$ or as a limit W_F taken over all L. The group $W_{L/F}$ maps onto the Galois group $\mathrm{Gal}(L^{\mathrm{ab}})$. The kernel is the closure of the image of the connected component of the identity in $I_L^\infty = \prod_{v|\infty} L_v^\times$. A key feature of the construction and, especially, of the definition of the group $\mathcal{S}$ that permits the introduction of φ_F is the possibility of constructing certain elements of the group of ideles I_L well-defined modulo the product of I_L^∞ with the kernel of any given continuous character. Moreover in the construction an imbedding of $\bar{\mathbb{Q}}$ in $\mathbb{C}$ is fixed, so that the collection of imbeddings of L in $\mathbb{C}$ may be identified with $\mathrm{Gal}(L/\mathbb{Q})$ or, if the imbedding of F is fixed, with $\mathrm{Gal}(L/F)$. The automorphic representation π associated with ϕ will be arithmetic because of the definition of the group $X^*(\mathcal{S})$ of characters of $\mathcal{S}$ and because of the definition of φ_F.

Conversely every arithmetic automorphic representation π of T arises in this way. Such a representation is attached (cf. [LM]) to a parameter, perhaps to several, $\psi : W_{L/F} \to {}^LT(\mathbb{C})$. The field L is some sufficiently large but finite Galois extension of F. If π is arithmetic this parameter factorizes through φ_F. To verify this, take L so large that all its infinite places are complex and observe first of all that ψ restricted to the idele class group $C_L = I_L/L^\times$ defines a homomorphism of $I_L^\infty \subset I_L$ to $\mathcal{S}^L$ and for any character λ of T, there is a collection of integers $\{\lambda_\tau \mid \tau \in \mathrm{Gal}(L/F)\}$ such that

$$\lambda(\psi(x)) = \prod_{\tau \in \mathrm{Gal}(L/F)} \tau(x)^{\lambda_\tau}.$$

The function $\lambda \to \lambda_\tau$ is a character of $\mathcal{S}^L$ and it defines the homomorphism ϕ from $\mathcal{S}^L$ to the connected component of ${}^LT$, a torus $\hat{T}$. To extend it to $\phi : \mathcal{T}^L \to {}^LT$ all we need do is split the image in $\hat{T}(L^{\mathrm{ab}})$ under ϕ of the cocycle in $H^2(\mathrm{Gal}(L^{\mathrm{ab}}/F), \mathcal{S}^L)$ defining $\mathcal{T}^L$ with the help of ψ. If $w \in W_{L/F}$ maps to τ in $\mathrm{Gal}(L^{\mathrm{ab}}/F)$ and to $\bar{\tau}$ in $\mathrm{Gal}(L/F)$, and $a(\tau)$ the representative of τ in $\mathrm{Gal}(L^{\mathrm{ab}}/F)$ used in the first paper of [LS] to define $\mathcal{T}_F^L$, then $\phi(a(\tau)) = \phi^{-1}(a(\tau)^{-1}\varphi_F(w))\psi(w)$. The right side is well-defined because of the definition of the groups $\mathcal{S}^L$ and $\mathcal{T}^L$.

As emphasized in the first paper of [LS], for each finite place v of F there is a splitting $\mathrm{Gal}(\bar{F}_v/F_v) \to \mathcal{T}^L(F_v)$, thus a v-adic representation of $\mathrm{Gal}(\bar{F}_v/F_v)$ in $\mathcal{T}^L$, in particular a p-adic representation if $F = \mathbb{Q}$ and $v = p$. At the moment, I do not understand how or under what circumstances this representation can be deformed and I certainly do not know which, if any, of the general conjectures about L-values and mixed motives to be described in the following pages are easy for it, which are difficult, or which have been proved (cf. [MW,R]).

The toroidal seeds themselves will be, almost without a doubt, essential factors of any complete theory of the correspondence between motives and arithmetic automorphic forms. There are two conceivable routes: either attempt to establish and use functoriality in general or, as a second possibility, attempt to use only the very little that is known about functoriality at present but to strengthen the other, less analytical and more Galois-theoretic or geometric parts of the argument. Although functoriality in general is not just around the corner, it is a problem for which concerted effort now promises more than in the past. So there is something to be said for reflecting on whether it will permit the correspondence between $\mathfrak{A}$ and $\mathfrak{M}$ to be established in general. I stress, once again, that up until now only simple seeds have been used, perhaps only those for which the group T is the trivial group $\{1\}$.

The principal merit of the second route is perhaps that it quickly confronts us with a difficulty carefully skirted in the above presentation, an adequate definition of the fourth element $\mathfrak{H}$. In addition, starting with known couples, the method can also arrive at other couples, of which the first element, thus the element in $\mathfrak{A}$ can, because of the element in $\mathfrak{M}$ with which it is paired, be identified with the functorial image of a representation of a second group. Such examples are a feature of the work of Richard Taylor and his collaborators ([Ta], but see also [Ki]) on odd icosahedral representations or on the Sato-Tate conjecture. Although their present forms were suggested by functoriality, these problems are of great independent interest and can be presented with no reference to it – and sometimes are. Nevertheless, functoriality is expected to be valid for all automorphic representations, not just for arithmetic automorphic representations, and is indispensable for analytic purposes such as the Selberg conjecture. So proofs of it that function only in the context of arithmetic automorphic representations are not enough.

I have so far stressed the correspondence between the four elements $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{G}$ and $\mathfrak{H}$ partly because the research of the most popular appeal as well as much of the wave that arose in the wake of the proof of Fermat's Last Theorem involves them all. There is nevertheless a good deal to be said about the relation between the deformations in $\mathfrak{G}_p$ and those in $\mathfrak{H}_p$ that bear more on the structure of the elements of $\mathfrak{M}$ and on the problematic definition of $\mathfrak{H}$ than on the relation between $\mathfrak{M}$ and $\mathfrak{G}$. The notion of a deformation in $\mathfrak{H}$ or $\mathfrak{H}_p$ remains imprecise and it is not at first clear when two elements of $\mathfrak{H}$ or $\mathfrak{G}$ are potentially in the same connected component. By definition there is attached to each arithmetic automorphic representation a family $\{\varphi_v\}$, v running over the infinite places, of homomorphisms of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ into an L-group ${}^LG$. The restriction of φ_v to $\mathbb{C}^\times$ can be assumed to have an image in any preassigned Cartan subgroup $\hat{T}$ of the connected component $\hat{G}$ of ${}^LG$ and will be of the form $z \to z^\lambda \bar{z}^\mu$, where $\lambda, \mu \in X_*(\hat{T})$ are cocharacters of $\hat{T}$. The homomorphism φ_v is then determined by a choice of w in the normalizer of $\hat{T}$ in ${}^LG$ of order two modulo T itself whose image in the Galois group is complex conjugation at v and which satisfies $w^2 = e^{\pi i(\lambda-\mu)}$, $w\lambda = \mu$. Since φ_v is only determined up to conjugation, there are equivalence relations on the triples $\{w, \lambda, \mu\}$, but the essential thing is that for each v the homomorphisms fall into families, defined by w and the linear space in which λ lies. These spaces may intersect, and in the intersection there is ambiguity. For example, if the ground field is $\mathbb{Q}$, $G = GL(2)$, ${}^LG = GL(2,\mathbb{C})$ and

$$T = \{t(a,b)\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}$$

then the action of w on T is either trivial or it takes $t(a,b)$ to $t(b,a)$. Moreover

$$z^\lambda \bar{z}^\mu = t(z^k \bar{z}^l, z^m \bar{z}^n)$$

with $k = l$, $m = n$ if the action is trivial and $k = n$, $l = m$ if it is not. In the first case, $w^2 = 1$, so that w can be taken as $t(\pm 1, \pm 1)$. In the second, at least if $k \neq l$, w can be taken in the form

$$w = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad \alpha = e^{\pi i(k-l)}. \tag{3}$$

This is a possibility even if $k = l$. It is equivalent to the particular choice $w = \pm t(1,-1)$. Thus, even though the parameters λ and μ are discrete and not continuous it is natural to distinguish two components in the space of parameters. In each λ is arbitrary and $\mu = w\lambda$, but in the first $w = \pm t(1,1)$ and in the second w is given by (3). These two families reappear in $\mathfrak{G}$ as even and odd Galois representations, the odd being apparently readily deformable, while the even seem to admit at best trivial deformations, as happens for reducible representations. There are similar families for other groups. Formally the number of parameters will be the dimension of the space of λ, thus the rank of the group G.

Although deformation in these parameters is not possible in $\mathfrak{A}$ or $\mathfrak{M}$, the deformations in $\mathfrak{G}_p$ or $\mathfrak{H}_p$ appear in some sense as deformations within families like those just described. Nevertheless the most important step in the proof of Wiles is a comparison of the local structure of $\mathfrak{H}_p$ and $\mathfrak{G}_p$ that does not involve a variation of the parameter, which we should think of as a Hodge type, or rather as the source of the Hodge type, motives being objects that are realized by a linear representation of the associated group, and the Hodge type being affected by the realization. The parameter maps to the L-group ${}^LG$, so that representations r of ${}^LG$ are an important source of realizations.

It is the deformations within $\mathfrak{H}_p$ and their structure that are central to much of Hida's efforts over the past two decades. His early work on the infinitesimal structure of $\mathfrak{H}_p$ for modular forms or for Hilbert modular forms appears to my untutored eye to have been a serious influence, but of course by no means the only one, on developments that ultimately led to a proof of Fermat's theorem.

As the notation indicates the spaces $\mathfrak{H}_p$ are related to Hecke algebras, but these algebras cannot be exactly those that are defined by the algebra of compactly supported functions on $G(\mathbb{A}_f)$ acting on automorphic forms, thus on complex-valued functions on $G(F)\backslash G(\mathbb{A}_F)$, because the algebras defining $\mathcal{H}_p$ must be algebras over a number field or, at least, over an extension of $\mathbb{Q}_p$.

For classical automorphic forms or representations the difficulty is not so egregious, since from the subject's very beginning the modular curves were present. The forms appeared as sections of line bundles on them, so that a structure of vector space over $\mathbb{Q}$ or some other number field or of module over $\mathbb{Z}$ was implicit in their very definition. In general, however, an adequate definition of $\mathfrak{H}_p$ remains problematic.

Suggestions can be made. It is natural to look for actions of the Hecke operators on cohomology groups, for these can be taken either in the Betti form so that they have a $\mathbb{Q}$-structure or, if the group G defines a Shimura variety, in the de Rham form so that they are defined over a number field. Not only can both, or at least their tensor products by $\mathbb{R}$ or $\mathbb{C}$, be calculated in representation-theoretic terms ([BW,GS]) but there are general theorems to compare the two, of which I suppose the Eichler-Shimura map favored by Hida is a particular manifestation. Although he intimates both possibilities before finally favoring a presentation in the de Rham form, Hida does not undertake a description of the general background. A brief but thorough account of it would have been of great benefit to the reader, the reviewer, and perhaps to the author as well.

To continue, I take, by restriction of scalars if necessary, the group G to be defined for simplicity over $\mathbb{Q}$. Suppose, as in [GS], that B is a Borel subgroup of G over $\mathbb{C}$ and that $B(\mathbb{C}) \cap G(\mathbb{R})$ is a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ whose projection on the derived group is compact. Then $B(\mathbb{C})\backslash G(\mathbb{C})$ is a projective variety and the complex manifold $F = T(\mathbb{R})\backslash G(\mathbb{R})$ is imbedded in it as an open subset. If K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$ then $T(\mathbb{R})\backslash G(\mathbb{R})$ is a fiber space over $D = K_\infty\backslash G(\mathbb{R})$. Sometimes it can be realized as a bounded symmetric domain and then for any open compact subgroup K_f of $G(\mathbb{A}^f)$ the complex manifold $D \times G(\mathbb{A}^f)/K_f$ can carry the structure of a Shimura variety, whose exact definition demands a little additional data that it is not useful to describe here. Of paramount importance, however, is that the variety is defined over a specific number field, the reflex field, and that if K_f is sufficiently small it is smooth, although not necessarily complete.

In particular, if the adjoint group of $G(\mathbb{R})$ is compact then D is a point and trivially a bounded symmetric domain. This is perhaps significant because the arguments of, for example, [T,Ki,Kh], not to speak of those in §4.3 of the book under review, often appeal to the JL-correspondence, at least in the special case of $GL(2)$, but because of the recent work by Laumon and Ngo on the fundamental lemma a proof of the correspondence, a special, comparatively easy case of functoriality, is – with time and effort because difficulties will arise ([B])– within reach for many groups. The correctly formulated correspondence relates automorphic representations on a group and an inner twisting of it and any group over $\mathbb{R}$ with a compact Cartan subgroup has an inner twisting that is compact.

Any character of T defines a line bundle on F, but also a cocharacter of type A_0 of a Cartan subgroup of the connected component $\hat{G}$ of ${}^LG$. The cocharacter can be extended to a homomorphism of the Weil group $W(\mathbb{C}/\mathbb{R})$ and this homomorphism defines a parameter ϕ_∞ and an L-packet of representations in the discrete series of $G(\mathbb{R})$; moreover, according to [GS] a substantial part of the cohomology of the line bundle is yielded by the automorphic forms associated with these L-packets. For a given group, just as for the example of $GL(2)$, it appears that all these parameters are expected to define the same connected component of $\mathfrak{H}_p$ or $\mathfrak{G}_p$.

Although Hida recognizes clearly the need for general definitions of $\mathfrak{H}_p$, he concentrates on groups G that define a Shimura variety. As their designation suggests these varieties were introduced and studied by Shimura in a long series of papers. Although their importance was quickly recognized, these papers were formulated in the algebro-geometric language created by Weil, not in the more supple and incisive language of Grothendieck that is especially suited to moduli problems, and were not easily read. An influential Bourbaki report by Deligne in 1971 clarified both the basic definitions and the proofs, although he like Shimura only treated those varieties, a very large class, which are essentially solutions of moduli problems. The remaining varieties were eventually treated in papers of Borovoi and of Milne ([BM]) by different methods. Neither their papers nor the investigations that preceded them are mentioned by Hida and the reader of his book is strongly advised to turn elsewhere for an introduction to the modern theory of Shimura varieties, for example to the lectures of Milne ([BM]). For the purposes of the book, only special Shimura varieties are invoked, but that is presumably a reflection of the limitations of current methods.

At present, to give any definition whatsoever of $\mathfrak{H}_p$, one has either to work with groups with $G(\mathbb{R})$ compact modulo its center or with groups for which the associated Shimura varieties can be defined over the ring of integers in some finite extension of $\mathbb{Q}$. The first possibility has not, so far as I know, been examined, except in some low-dimensional cases, and the second requires, for the moment, that the variety be the solution of a moduli problem. Then H_p is the algebra of Hecke operators acting on p-adic automorphic forms. Classically these automorphic forms had been investigated by others earlier (cf. [KS]), but Hida discovered even for classical forms some remarkable features that seem to appear for general groups as well.

The present book is an account of that part of the theory developed by him for several important types of Shimura varieties: modular curves, Hilbert modular varieties, and Siegel modular varieties. The publishers recommend it as a text for graduate students, but that is irresponsible. Although many of Hida's early papers and a number of his books are very well written, neither expository flair nor a pedagogical conscience are evident in the present text. The style is that of rough lecture notes, cramped pages replete with formulas and assertions that run one into the other, largely obscuring the threads of the argument, and with an unchecked flood of notation. The meaning of essential symbols is variable and not always transparent so that the reader is occasionally overcome by a disconcerting uncertainty.

On the other hand, Hida's goals, both those realized in the book and those still unrealized, are cogently formulated in his introduction and, so far as I can appreciate, of considerable interest. Experts or even experienced mathematicians in neighboring domains, for example the reviewer, will, I believe, be eager to understand his conclusions, but they, and the author as well, might have been better served either by a series of normal research papers or by a frankly pedagogical monograph that assumed much less facility with the technical apparatus of classical and contemporary algebraic geometry. The material is difficult and in the book the definitions and arguments come at the reader thick and fast, in an unmitigated torrent in which I, at least, finally lost my footing.

Although as I have emphasized, the parameters ϕ_∞ of the discrete series or of the Hodge structure seem to lie in the same connected component, we cannot expect to pass continuously from one to another. Indeed, for Shimura varieties, fixing the parameter corresponds approximately to fixing the weight of the form. The dimension of the space of automorphic forms being, according to either the trace formula or the Riemann-Roch formula, pretty much a polynomial in λ, we cannot expect it to be constant and independent of λ. For p-adic forms, however, there are large families of constant dimension that interpolate, in a space with p-adic parameters, a certain class of arithmetic automorphic forms.

The Hecke algebra and its actions are just another expression of the automorphic representations or of the automorphic forms. Fixing imbeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and into $\bar{\mathbb{Q}}_p$ and taking all fields F to be subfields of $\bar{\mathbb{Q}}$, at a finite place p we replace the collection of local parameters $\phi_\infty = \{\phi_v\}$, $v|\infty$, by a collection of homomorphisms of the local Weil groups W_{F_v}, into the L-group over $\mathbb{C}$. For those representations $\pi = \otimes\pi_v$ that are associated with motives, these parameters will presumably be given by homomorphisms σ_v, $v|p$, of the Galois groups $\mathrm{Gal}(\bar{F}_v/F_v)$, $v|p$, into the L-group over $\bar{Q}_l$, where l may or may not be equal to p.

If $l \neq p$, such a homomorphism will be tamely ramified and the restriction to the decomposition group is strongly limited and does not offer much room for deformation. It may as well be fixed, so that the deformations will take place over the image of the Frobenius which there seems to be no attempt to constrain. If, however, $p = l$, the possibilities for the σ_v are at first manifold but when the representations σ_v, $v|p$, arise from a motive they are constrained in an important way first discovered by Tate. They can be assigned a Hodge-Tate type whose basic description in terms of parameters λ subject to an integrality condition is much like that attached to the Hodge structures at infinity. Since Tate's paper [T] a very great deal has been learned about the restrictions of the p-adic representations associated with motives to the decomposition groups at places v dividing p ([FI]) that appears to be indispensable for the study the spaces $\mathfrak{H}_p$ or $\mathfrak{G}_p$, but what the reader of the present book will discover is that at p the Hodge type seems to control the possible deformations just as it did at infinity in combination with the elements w of order two. In the much studied case of the group $GL(2)$, a w with the two eigenvalues $+1$ and -1 can allow many deformations but a w with equal eigenvalues does not appear to do so. At p the analogous dichotomy seems to be between ordinary and extraordinary or – more colloquially expressed – nonordinary, although I suppose that there will ultimately be a whole spectrum of possibilities each permitting some kinds of deformation and forbidding others. The ordinary case is presumably the optimal case and is the one on which Hida concentrates.

For the types at ∞ there was no possibility of real deformation because λ was constrained by an integrality condition. At p it is possible to abandon the integrality condition because the Galois group of the infinite cyclotomic extension $\mathbb{Q}_{\mu_{p^\infty}}$ is $\mathbb{Z}_p^\times$ which is isomorphic to the product of the group $\mathbb{F}_p^\times$ with $1 + p\mathbb{Z}_p$ and the second factor admits a continuous family of characters $x \to x^a$, $a \in \mathbb{Z}_p$, interpolating the characters given by integral a. This allows for deformation or interpolation in the space $\mathfrak{G}_p$ which is, it turns out, accompanied by possible deformations in the space $\mathfrak{H}_p$. The new parameter is usually not just an open subset of $\mathbb{Z}_p^\times$ but, as for abelian G, of some subspace of $X^*(T) \otimes \mathbb{Z}_p$, $X^*(T)$ being the character module of a Cartan subgroup of G.

This discovery by Serre (cf. [KM]), whose work was followed by that of Katz and preceded by that of Swinnerton-Dyer, can perhaps be regarded as a second point where Ramanujan influenced the course of the general theory of automorphic forms in a major way, for Swinnerton-Dyer was dealing with congruences conjectured by him. The first point was of course the Ramanujan conjecture itself, which led, through Mordell and Hecke, to the general theory of automorphic L-functions. Hida appreciated that in the p-adic theory, where the weight was no longer integral, there was a possibility of the uniform deformation of whole families of modular forms, the ordinary forms, to a rigid-analytic parameter space, thus to an open subset of $\mathbb{Z}_p^n$ for some integer n. It would be surprising if this possibility were limited to $GL(2)$ and Hida has devoted a great deal of time, energy and space to the admirable design of creating a general theory. To read his books and papers grows increasingly difficult; to read them alone without consulting those of other authors, Katz or Fontaine for example, or, in a different optic, Taylor or Khare, is ill-advised, even impossible for some of us. Nevertheless, although no-one, neither Hida nor anyone else, appears to have broken through to a clear and comprehensive conception of the ultimate theory, there is a great deal to be learnt from his writings, both about goals and about techniques. In spite of Hida's often trying idiosyncrasies, to follow his struggles for a deep and personal understanding of the resistant material is, as Tilouine observed in a briefer review, not only edifying but also challenging, although it appears to be easier to begin with the earlier papers, for they are often more concrete and in them some key ideas are less obscured by technical difficulties and general definitions.

Hida has also been preoccupied with two problems parallel to that of constructing deformations of p-adic forms: parametrized families of p-adic Galois representations; p-adic L-functions. Although the theory of parametrized families of Galois representations is not developed in the book under review and, indeed, so far as I know, unless very recently, has hardly been developed beyond $GL(2)$, it is adumbrated in the introduction as one of the ultimate goals of the author. In earlier papers of Hida ([Hi]), the elaborate "infinitesimal" structure, whose appearance in $\mathfrak{H}_p$ is for $GL(2)$ a manifestation of congruences between the Fourier expansions of automorphic forms and whose coupled appearance in $\mathfrak{H}_p$ and $\mathfrak{G}_p$ is a key feature of the proof of Fermat's theorem, appears and is investigated not only for fixed weight and central character but also for entire parametrized families. There is much more number-theoretical information in these investigations than I have been able to digest.

The elements of $\mathfrak{A}$ or of $\mathfrak{H}_p$ are attached to automorphic representations or forms, thus to a particular group G and to a particular L-group ${}^LG$, but to the extent that functoriality is available the group G can be replaced by others G' and the representation π of $G(\mathbb{A}_F)$ by another π' of $G'(\mathbb{A}_F)$. The p-adic Galois representations can be modified in the same way, and without any ado. It might be worth reflecting on how the passage to the primed objects should be interpreted in $\mathfrak{H}_p$.

A final, major goal described briefly in the introduction to the book and of concern to many people (*cf.* [Gr]) is the construction of p-adic L-functions. They seem to me of such importance both to Hida's project and to all mathematicians with an interest in number theory that I cannot end this review without a very brief and even more superficial description of the attendant questions. I have no clear idea of their current state. I believe that we can safely assume that they are largely unanswered.

The complex L-functions attached to $\pi \in \mathfrak{A}$ or to a motive $M \in \mathfrak{M}$ are specified only when in addition a finite-dimensional complex representation r of ${}^LG$ is given, π being an automorphic representation of G and M a motive of type ${}^LG$. It is of the form $L(s,\pi,r)$ or $L(s,M,r)$ although both can be – in principle! – written as $L(s,\pi')$ or $L(s,M')$, $\pi' = \pi_r$ an automorphic representation of $GL(n)$, $M' = M_r$ a motive of rank d, $d = dim\, r$. Of course, if M is attached to π then $L(s,\pi,r) = L(s,M,r)$. These somewhat speculative remarks are meant only to emphasize that all problems related to the p-adic L-functions will have to incorporate r. They will also have to incorporate the parameter space of the deformations, which appears to be, the elaborate local structure aside, at its largest, an open subset A of $X^*(T) \otimes \mathbb{Z}_p$, T being a Cartan subgroup of G over the chosen ground field F, but it is of this size only in unusual situations. As we noticed for tori, there are important constraints on the subspace in which A is to be open. Of course $X^*(T) \otimes \mathbb{Z}_p$ has a Galois action.

The functions are to be p-adic analytic functions $L_p(s,r)$ on the parameter space, thus on a subset of $\mathfrak{H}_p$ or $\mathfrak{G}_p$ identified with the set A in $X^*(T) \otimes \mathbb{Z}_p$. Elements $s = \mu \times z$ of A define equivariant homomorphisms of open subgroups of $K \otimes \mathbb{Z}_p$ into $\hat{T}(\mathbb{Q}_p)$ in the form $a \to \prod_{\varphi|p} \varphi(a)^{z\varphi(\mu)}$. Moreover at points in $\lambda \in X^*(T) \cap A$ (or at least at a large subset of this space, perhaps defined by a congruence condition) the element of $\mathfrak{G}_p$ is to be the image of a motive $M(\lambda)$. So $M'(\lambda) = M_r(\lambda)$ is defined. The p-adic function $L_p(s,r)$ is to interpolate in an appropriate form values $R(M'(\lambda))$ of the complex L-functions $L(z, M'(\lambda))$ at $z = 0$.

There are many important conjectures pertinent to the definition of $R(M'(\lambda))$. Unfortunately we do not have the space to describe them fully ([Mo,Ha]), but something must be said. For this it is best to simplify the notation and to suppose $M = M'(\lambda)$. When discussing the L-function $L(z,M)$ it is also best to suppose that M is pure, thus that all its weights are equal, for otherwise there is no well-defined critical strip, and no well-defined center. Since every motive will have to be a sum of pure motives, this in principle presents no difficulty.

Motives are defined (in so far as they are well-defined) by projections constructed from linear combinations of algebraic correspondences with coefficients from a field K of characteristic zero. It is customary to take K to be a finite extension of $\mathbb{Q}$. The field K is not the field over which the correspondences are defined. That field is F, the base field, or, more generally, a finite-dimensional extension L of it. It is probably best, for the sake of simplicity, to take at this point K and F both to be $\mathbb{Q}$. Once the ideas are clear, it is easy enough to transfer them to general F and K, but not necessary to do so in a review.

The expectation is that the order $n = n(k, M)$ of the zero of $L(z, M)$ at $z = k$, k an integer, will be expressible directly in terms of geometric and arithmetic properties of M, and so will

$$R(M) = \lim_{z \to k} \frac{L(z, M)}{(z-k)^n}. \tag{4}$$

These geometric and arithmetic properties are defined by the mixed motives attached to the pure motive M. Mixed motives appear in the theory of L-functions as extensions of powers $\mathbb{T}(m)$ of the Tate motive by M and in the simplest cases are determined by, say, divisors over the ground field F (for example, $\mathbb{Q}$) on a curve or, indeed, on any smooth projective variety over F. The most familiar examples are rational points on elliptic curves. Of importance are the extensions N of the form

$$0 \to M \to N \to \mathbb{T}(-k) \to 0, \tag{5}$$

as well as similar extensions in related categories defined by various cohomology theories for varieties, motives and mixed motives, by de Rham theories, by the Betti theory for varieties over the real and complex fields, and by p-adic theories that attach to the motive M a p-adic Galois representation of dimension equal to the rank of M.

The motive M has a weight $w(M)$ that is the degree in which it appears if M is a piece of the cohomology of a smooth projective variety. So, by the last of the Weil conjectures, for almost all finite places $\mathfrak{p}$ of F there are attached to M algebraic numbers $\alpha_1(\mathfrak{p}), \ldots, \alpha_d(\mathfrak{p})$ of absolute value $\mathrm{N}\mathfrak{p}^{\mathrm{w(M)}/2}$. The integer d is the rank of M. Thus $L(z, M)$ which is essentially

$$\prod_{\mathfrak{p}} \frac{1}{\prod_1^d (1 - \alpha_i(\mathfrak{p})/\mathrm{N}\mathfrak{p}^{\mathrm{z}})}$$

does not vanish for $\Re z > w(M) + 1$.

Suppose that it can be analytically continued with a functional equation of the expected type, thus

$$\Gamma(z, M)L(z, M) = \epsilon(z, M)\Gamma(1 - z, \hat{M})L(1 - z, \hat{M}),$$

where $\epsilon(z, M)$ is a constant times an exponential in z and thus nowhere vanishing, $\hat{M}$ a dual motive, which will be of weight $-w(M)$ and of the same rank as M, and $\Gamma(z, M)$ a product of Γ-factors. The product is taken over the infinite places v of the basic field F. If the weights in the Hodge structure of the Betti cohomology associated with M at v are $\{(p_1, q_1), \ldots, (p_d, q_d)\}$ and v is complex, the Γ-factor is $\prod_i \Gamma(s - \min(p_i, q_i))$, if v is real it is $\prod_i \Gamma(s/2 + \epsilon_i/2 - \min(p_i, q_i)/2)$, where ϵ_i is either 0 or 1.

The center of the critical strip is $w(M)/2 + 1/2$. Suppose $w(M)$ is even, then for integral $k > w(M)/2 + 1$, $L(k, M) \neq 0$ and for integral $k > -w(M)/2 + 1$, $L(k, \hat{M}) \neq 0$. The functional equation allows us to deduce from this the order of the zero of $L(z,M)$ at all integral $k < w(M)/2$. Moreover the order of the pole of $L(k,M)$ at $w(M)/2+1$ is presumably equal to the multiplicity with which M contains the Tate motive $\mathbb{T}(-w(M)/2)$. Applying this to $\hat{M}$ we deduce the order of the zero of $L(z,M)$ at $w(M)/2$. So there is, in principle, no mystery about the order of the zero of $L(z,M)$ at any integer when $w(M)$ is even. When $w(M)$ is odd, the same arguments deal with all integral points except $w(M)/2+1/2$, but this point is very important, being for example the one appearing in the conjecture of Birch and Swinnerton-Dyer. So the order of vanishing of $L(z,M)$ at the BSD-point $z = w(M)/2+1/2$ is related to much more recondite geometric information. According to the conjectures of Beilinson and Deligne, the irrational factor of (4) is determined topologically by the motive over the infinite places of the field F (cf. [Ha, Fo]). We first consider $k \geq w(M)/2+1$, supposing that M does not contain the Tate motive $\mathbb{T}(-w(M)/2)$ as a factor.

The motive has, on the one hand, a Betti cohomology $H_{\mathrm{B}}(M)$ over $\mathbb{Q}$ that when tensored with $\mathbb{C}$ has a Hodge structure

$$H_{\mathrm{B}}(M) \otimes \mathbb{C} = \oplus_{p+q=w(M)} H^{p,q}(M)$$

and, on the other, a de Rham cohomology $H_{\mathrm{dR}}(M)$ over $\mathbb{Q}$ with a filtration

$$\dots F^{p-1}(M) \supset F^{p}(M) \supset F^{p+1}(M) \dots$$

that terminates above at $H_{\mathrm{dR}}(M)$ and below at 0. Moreover the two spaces $H_{\mathrm{B}}(M) \otimes \mathbb{C}$ and $H_{\mathrm{dR}}(M) \otimes \mathbb{C}$, identified with de Rham cohomology over $\mathbb{C}$, are canonically isomorphic. Under the canonical isomorphism

$$F^{p}(M) \otimes \mathbb{C} \simeq \oplus_{p' \geq p} H^{p',q}(M).$$

There is an involution ι_1 on $H_{\mathrm{B}}(M)$ that arises from the complex conjugation of varieties over $\mathbb{Q}$. It can be extended to $H_{\mathrm{B}}(M) \otimes \mathbb{C}$ linearly. There is a second involution $\iota_2 : x \otimes z \to x \otimes \bar{z}$ on this tensor product. On the other hand, complex conjugation defines an involution ι of the de Rham cohomology over $\mathbb{C}$. Under the canonical isomorphism $\iota_1 \circ \iota_2$ becomes ι. The particular pair $H_{\mathrm{B}}(M)$ and $H_{\mathrm{dR}}(M)$ with the auxiliary data described define a structure that we denote M_{HdR}, but we can also consider the category of all such structures, referred to in [Ha] as the category of Hodge-de Rham structures and here as HdR-structures. This category also contains extensions

$$0 \to M_{\mathrm{HdR}} \to N_{\mathrm{HdR}} \to \mathbb{T}_{\mathrm{HdR}}(-k), \tag{6}$$

in which N_{HdR} may not be associated with a motive. Nevertheless extensions (5) in the category of mixed motives presumably give rise to extensions (6) in the category of HdR-structures.

It follows readily from the definition of HdR-structures, mixed or not, that the sequence (6) splits if $k \leq w(M)/2$. Otherwise the group $\mathrm{Ext}^1_{\mathrm{HdR}}(\mathbb{T}_{\mathrm{HdR}}(-k), M_{\mathrm{HdR}})$ formed by classes of the extensions (6) can be calculated readily. If, as can be achieved by a simple twisting, we suppose $k=0$, then it is

$$H_{\mathrm{dR}}(M) \otimes \mathbb{R} / \{H_{\mathrm{B}}(M)^{+} + F^{0}(M)\}.$$

The vector space $H_{\rm B}(M)^+$ is the plus eigenspace of ι_1 in $H_{\rm B}(M)$.

The group in the (hypothetical) category of mixed motives formed by classes of the extensions in (5) is denoted $\rm Ext^1(\mathbb{T}(-k), M)$. The functor $M \to M_{\rm HdR}$ leads to

$$\rm Ext^1(\mathbb{T}(-0), M) \to Ext^1_{HdR}(\mathbb{T}_{HdR}(-0), M_{HdR}) \equiv H_{dR}(M) \otimes \mathbb{R}/\{H_B(M)^+ + F^0(M)\}.$$

in which $H_{\rm B}(M)^+$ is the fixed point set of ι_1. The combined conjectures of Beilinson and Deligne affirm not only that the resulting map

$$\rm Ext^1(\mathbb{T}(-0), M) \to H_{dR}(M) \otimes \mathbb{R}/\{H_B(M)^+ + F^0(M)\}$$

is injective but also that it yields an isomorphism of $\rm Ext^1(\mathbb{T}(-0), M) \otimes \mathbb{R}$ with the quotient of $H_{\rm dR}(M) \otimes \mathbb{R}$ by $(H_{\rm B}(M)^+ + F^0(M)) \otimes \mathbb{R}$. Thus the product of the determinant of a basis of $\rm Ext^1(\mathbb{T}(-0), M)$ with the determinant of a basis of $H_{\rm B}(M)^+$ can be compared with the determinant of a basis of the rational vector space $H_{\rm dR}(M)/F^0(M)$, the quotient being an element of $\mathbb{R}^\times/\mathbb{Q}^\times \subset \mathbb{C}^\times/\mathbb{Q}^\times$ that is supposed, as part of the Beilinson-Deligne complex of conjectures, to be the image of (4).

Although the notion of a mixed motive is somewhat uncertain and little has been proved, the theory is in fact strongly geometric with, I find, considerable intuitive appeal. Moreover when developed systematically, it permits a clean description of the integers n appearing in (4), even when k is the BSD-point, and of the limits $R(M)$, not simply up to a rational number as in the conjectures of Beilinson-Deligne, but precisely as in the conjectures of Bloch-Kato. Although clean, the description is neither brief nor elementary. It is expounded systematically in [FP].

The general form of the Main Conjecture of Iwasawa can also be profitably formulated in the context of mixed objects. Recall that part of Hida's program is to attach to π a p-adic representation in ${}^LG$ and thus to each representation r a p-adic family of representations σ_r. The principal objective of the book is the algebro-geometrical constructions that enable him to transfer to Siegel varieties, thus to the Shimura varieties associated with symplectic groups in higher dimensions, the techniques developed by him earlier for $GL(2)$ over $\mathbb{Q}$ and over totally real fields and to construct for them a theory of p-adic automorphic forms, from which a construction of p-adic L-functions might be deduced. This is a well-established tradition. The p-adic L-functions are constructed either directly as interpolating functions or indirectly from the Fourier expansions of p-adic automorphic forms. Then the main conjecture affirms that they are equal to the characteristic function of a Selmer group defined by a parametrized family of p-adic Galois representations, essentially, if I am not mistaken, by showing that this characteristic function does interpolate the modified values of the complex automorphic L-function. The first, easiest, yet extremely difficult cases of the Riemann zeta-function and Dirichlet L-functions are in [MW].

The main conjecture could therefore be formulated directly in terms of the complex L-function and the p-adic representation were it not that, at present, the only way to construct the parametrized Galois representations is often, as in Hida's books and papers, through the mediating family of p-adic automorphic forms.

The p-adic space A on which the p-adic L-function was to be defined could – since we agreed to take both fields K and F to be $\mathbb{Q}$ – be the continuous $\mathbb{Q}_p$-valued spectrum of a commutative ring R over $\mathbb{Z}_p$, thus the continuous homomorphisms of R into $\mathbb{Q}_p$. The ring R will be chosen such that these homomorphisms all have image in $\mathbb{Z}_p$. The ring R could be a power series ring in finitely many variables. For example, the extension of $\mathbb{Q}$ generated by all p^nth roots of unity contains a subfield $\mathbb{Q}_\infty$ over which it is of finite index and for which $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_p \equiv 1 + p\mathbb{Z}_p$. Let $\Lambda = \lim \mathbb{Z}_p[\Gamma']$ be the limit over finite quotients of Γ. Of course $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. The ring Λ is a common choice of R and is isomorphic to a power series ring $\mathbb{Z}_p[[T]]$. Its continuous $\mathbb{Q}_p$-valued spectrum may be identified with the continuous homomorphisms of Γ into $\mathbb{Q}_p^\times$.

Certain isolated points λ in the spectrum of R are to correspond to motives $M(\lambda)$. If the primary object is not the p-adic L-function but a family $\{\sigma_s\}$ of p-adic representations parametrized by $A = \mathrm{SpecR}$ and given by representations into $GL(d, R)$ then at $s = \lambda$ the representation $\sigma_{M(\lambda)} = \lambda \circ \sigma$ is to be that attached to $M(\lambda)$, thus that on the p-adic étale cohomology $H_p(M(\lambda))$.

The p-adic representation σ_M of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the étale cohomology $H_p(M)$ of a motive M over $\mathbb{Q}$ or perhaps better the restriction of σ_M to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is an object whose theory ([FI]) I do not yet understand and do not try to describe. Perhaps the most important thing to recall is that its Hodge type, which describes the action on the tensor product of $H_p(M)$ with the completion $\mathbb{C}_p$ of $\bar{\mathbb{Q}}_p$, is a sequence of integers $h_1, \ldots, h_d$, with d equal to the dimension of M, supposed pure.

In [Gr] very tentative, yet very appealing conjectures are formulated. They are difficult to understand, but are a benchmark with which to compare the aims and results of Hida. First of all, the representation σ is supposed to take values in $GL(d, R)$. Then the parametrized representations arise on taking a continuous homomorphism $\phi = \phi_s : R \to \mathbb{Z}_p$, $s \in A$ and composing it with σ.

Denote the space of the representation σ by $V = R^d$. The appropriate analogue for p-adic representations of the mixed objects (6) would appear at first to be extensions

$$0 \to V \to W \to T \to 0, \tag{7}$$

in which $T = T(0)$ is the one-dimensional trivial representation, so that W stands for a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of degree $d + 1$. Thus k in (7) has been taken to be 0, a formal matter because the sequence can be twisted. If we write the representation on W in block form, the first diagonal block $d \times d$ and the second 1×1, only the upper-diagonal $d \times 1$ block is not determined and it defines an element of $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), V)$.

Not this group appears in [Gr] but the group

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \tilde{V}), \qquad \tilde{V} = V \otimes \mathrm{Hom_R(R, \mathbb{Q}_p/\mathbb{Z}_p)}. \tag{8}$$

More precisely, it is a subgroup of this group, the Selmer group S, that is pertinent. It is defined as an intersection over primes q of subgroups defined by local conditions. If $q \neq p$ the subgroup is the kernel of the restriction to the decomposition group. To define the subgroup at $q = p$, Greenberg imposes a condition that he calls the Panchishkin condition, a condition that I do not understand, although the notion of an ordinary form or Galois representation seems to be an expression of it.

Thus the group S is defined by extensions that are a reflection at the p-adic level of extensions of motives. The ring R acts on it and on its dual $\hat{S} = \mathrm{Hom}(\mathrm{S}, \mathbb{Q}_\mathrm{p}/\mathbb{Z}_\mathrm{p})$. The general form of the main conjecture would be that the characteristic ideal of $\hat{S}$, an element in the free abelian group on the prime ideals of R of height one, is – apart from some complications related to those that arose at $k = w(M)/2 + \epsilon$, $\epsilon = 0, 1, 2$ – essentially the interpolating p-adic L-function. This is vaguely expressed both by Hida and Greenberg and even more vaguely by me, because I understand so little, but, as a general form of the Main Conjecture of Iwasawa, it is, in concert with the Fontaine/Perrin-Riou form of the Beilinson-Deligne-Bloch-Kato conjectures, of tremendous appeal.

As a valediction I confess that I have learned a great deal about automorphic forms while preparing this review, but not enough. It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task. Obtaining proofs of serious results is another, even more difficult matter and each success demands an enormous concentration of forces.

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