Descent for Transfer Factors

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Dedicated to A. Grothendieck on his 60th Birthday

Introduction

In [**I**] we introduced the notion of transfer from a group over a local field to an associated endoscopic group, but did not prove its existence, nor do we do so in the present paper. Nonetheless we carry out what is probably an unavoidable step in any proof of existence: reduction to a local statement at the identity in the centralizer of a semisimple element, a favorite procedure of Harish Chandra that he referred to as descent.

The principal difficulty is to show that the transfer factors of [I] for the original group G are compatible with those on the connected centralizer G_ϵ of the semisimple element ϵ. After some preliminary explanations in Section 1, the compatibility is stated as Theorem 1.6.A. In Section 2 we show that this compatibility indeed reduces the problem of existence to a local problem at the identity on the groups G_ϵ, and in passing we note some other applications.

The remaining four sections are devoted to the proof of Theorem 1.6.A. The transfer factors are defined in a rather elaborate manner as the product of five factors that mix group-theoretical data with Galois cohomology. The first steps are to reduce to quasisplit groups and then to discard two of the five factors, leaving only three, one of which is defined in a simple fashion, and two of which involve group-theoretic and cohomological data. It is these two that are difficult to compare for G and G_ϵ. The principal tools are the two comparison lemmas of Section 3.

The contributions to the factors are labelled by orbits of the Galois group in sets of roots, and the first use made of the comparison lemmas is to deal in Section 4, and rather quickly, with all orbits except those lying outside both G_ϵ and the endoscopic group.

This leaves a rather concise but still far from trivial statement that is proved partly by an analysis of the structure of semisimple groups and partly by explicit cohomological calculations. The structural analysis is possible only after the critical lemma of Section 5.1 has been established. This lemma allows us to introduce an inductive component into the argument, and then to assume that both the element ϵ and the datum s defining the endoscopic group are essentially of order two, and moreover, that all roots are of the same length. This done, the burden of the rest of the proof is carried by explicit arguments with the constructs of local class-field theory. They all appear in Section 6.

We cannot hope that the groping, pedestrian style of the paper will appeal to Grothendieck, for it lacks the force and penetration that he achieved so readily, like Nietzche's *Philosoph der Zukunft, erfinderisch in Schematen, mitunter stolz auf Kategorien-Tafeln*. Nonetheless, it is a great pleasure for us to express our admiration of his magnificient contributions to the mathematics of our time.

§1. Descent Principles

1.1. Notation

We follow closely the notation of [I]. In particiular, G is a connected reductive group over a field F of characteristic zero, now assumed local.

As in [I, Sect. 1.2.] G^*, ψ are quasisplit data and ${}^LG = \widehat{G} \rtimes W_F$ is the L-group. To conserve notation we fix an F-splitting $(\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ of $\widehat{G}$ and given a class of endoscopic data choose a representative $(H, \mathcal{H}, s, \xi)$ with $\xi : \mathcal{H} \hookrightarrow {}^LG$ as inclusion and s an element of $\mathcal{T}$. It is also convenient to fix an F-splitting $(\mathcal{B}_H, \mathcal{T}_h, \{Y_{\beta^\vee}\})$ of $\widehat{H} = \ \mathrm{Cent}(s, \widehat{G})^0$ and assume that $\mathcal{B}_H = \mathcal{B} \cap \widehat{H}, \mathcal{T}_n = \mathcal{T}$.

For the moment we refer directly to [I] for the definition of the factor Δ. Measures also remain as there. If $\epsilon \in G(F)$ is semisimple we choose an invariant differential form of highest degree on

Cent$(\epsilon, G)^0$ in order to fix a Haar measure on the F-rational points of this group. We require that differential forms on inner forms be obtained by transport.

1.2. Images of semisimple elements

For ϵ in G the *identity component of* Cent(ϵ, G) will be denoted G_ϵ. If ϵ in $G(F)$ is semisimple then, following [**K1**], the stable conjugacy class of ϵ is

$$\{g^{-1}\epsilon g : g\sigma(g)^{-1} \in G_\epsilon, \sigma \in \Gamma\},$$

where $\Gamma = \operatorname{Gal}(\bar{F}/F)$. If Cent$(\epsilon, G)$ is connected then this coincides with the set of F-rational points in the conjugacy class of ϵ in $G(\bar{F})$. In general, an F-rational $\epsilon' = g^{-1}\epsilon g$ is stably conjugate to ϵ if and only if Int $g : G_{\epsilon'} \to G_\epsilon$ is an inner twist. If G is quasisplit over F then there is an ϵ' stably conjugate to ϵ such that $G_{\epsilon'}$ is quasisplit over F [**K1**, Lemma 3.3]. We now generalize the notion of image from [I, 1.3] (see also [**K2**]). It is convenient to use the notation $\gamma_H, \gamma, \gamma_G$ when γ_H is strongly G-regular and $\epsilon_H, \epsilon, \epsilon_G$ in general.

Suppose then that ϵ_H lies in the Cartan subgroup $G_H(F)$ of $H(F)$. Then we call ϵ_H a T_H-*image* of ϵ_G in $G(F)$ if for some admissible embedding $T_H \to T$ of T_H in G^* carrying ϵ_H to, say, ϵ there exists x in G^*, or just as well in G^*_{sc}, such that $\psi_x =$ Int $x \circ \psi$ has the properties that $\psi_x(\epsilon_G) = \epsilon$ and that both $T_G = \psi_x^{-1}(T)$ and $\psi_x : T_G \to T$ are defined over F. In varying T_H we obtain all *images* of ϵ_G. Observe that [**K2**, Lemma 10.2] shows that all images of a given ϵ_G are obtained by simply fixing one image ϵ_H, if it exists, and then taking all T_H-images for some T_H fundamental in H_{ϵ_H}.

1.3. The function Φ_f^H

Recall that to define transfer factors for (G, H) we may need to pass to a central extension of H. Call a central extension $\tilde{H}$ of H admissible if it is attached to a z-extension of G as in [I, 4.4] (although a wider class of extensions could be used [**K-S**]). The sequence

$$1 \to \tilde{Z}(F) \to \tilde{H}(F) \to H(F) \to 1$$

is then exact, where $\tilde{Z}$ is a central torus in $\tilde{H}$, and combinations of orbital integrals of functions on $G(F)$ are to be matched with those of functions on $\tilde{H}(F)$ that transform under $\tilde{Z}(F)$ according to a certain character $\tilde{\lambda}$ [I, 4.4].

Suppose ϵ_H is semisimple in $H(F)$ and $\tilde{\epsilon}_H$ lies in its preimage in $\tilde{H}(F)$. The factor $\Delta(\tilde{\gamma}_H, \gamma_G)$ has been defined for $\tilde{\gamma}_H$ strongly G-regular in $\tilde{H}(F)$, by which we mean that the image γ_H of $\tilde{\gamma}_H$ in $H(F)$ is strongly G-regular. We shall investigate the behavior of

$$\Phi_f^H(\tilde{\gamma}_H) = \sum_{\gamma_G} \Delta(\tilde{\gamma}_H, \gamma_G)\Phi(\gamma_G, f)$$

for $\tilde{\gamma}_H$ near $\tilde{\epsilon}_H$.

First, following [I, 4.3] it is easy to see that if ϵ_H is G-regular, but not strongly so, then Φ_f^H extends continuously to $\tilde{\epsilon}_H$ and that this extension is in fact smooth at $\tilde{\epsilon}_H$. Second, if ϵ_H is not the image of any semisimple element in $G(F)$ then no strongly G-regular element in $H_{\epsilon_H}(F)$ can be the image of an element in $G(F)$ and so Φ_f^H vanishes on the strongly G-regular elements in $\tilde{H}_{\tilde{\epsilon}_H}(F)$. In particular Φ_f^H vanishes for all $\tilde{\gamma}_H$ in a neighborhood of $\tilde{\epsilon}_H$ in $\tilde{H}(F)$.

We may then assume that ϵ_H is an image of an element ϵ_G in $G(F)$. There is an $\epsilon'_H = h^{-1}\epsilon_H h$ stably conjugate to ϵ_H such that $H_{\epsilon'_H}$ is quasisplit over F. If ϵ_H is a T_H-image of ϵ_G then we may multiply h by an element of $H_{\epsilon'_H}$ to assume that the homomorphism Int $h^{-1} : T_H \to H_{\epsilon'_H}$ is defined over F, that is, that $h \in \mathfrak{A}(T_H)$. Then h acts on the preimage $\tilde{T}_H$ of T_H in $\tilde{H}$ as an element of $\mathfrak{A}(\tilde{T}_H)$ and [I, 4.1.] implies that

$$\Phi_f^H(h^{-1}\tilde{\gamma}_H h) = \Phi_f^H(\tilde{\gamma}_H)$$

for all strongly G-regular $\tilde{\gamma}_H$ in $\tilde{T}_H(F)$. Thus we may replace ϵ_H by ϵ'_H and assume from now on that H_{ϵ_H} is quasisplit over F. Then H_{ϵ_H} is an endoscopic group for G_{ϵ_G} as we now explain in detail. We sometimes denote it by H_ϵ.

1.4. Descent for endoscopic data

We start then with semisimple ϵ_H in $H(F)$, an image of ϵ_G in $G(F)$, and H_{ϵ_H} quasisplit over F. Choose T_H such that ϵ_H is a T_H-image of ϵ_G. Let ϵ be the image of ϵ_H under some admissible embedding $T_H \to T$ of T_H in G^*. An argument as in the last paragraph allows us to choose the embedding so that G^*_ϵ is quasisplit over F. We will see that these choices are of no real importance, the essential data being ϵ_H, ϵ and ϵ_G and thus $H_{\epsilon_H}, G^*_\epsilon$ and G_{ϵ_G}.

To the endoscopic data $(H, \mathcal{H}, s, \xi)$ we shall attach an extension $\mathcal{H}_\epsilon$ of W_F by $\widehat{H}_{\epsilon_H}$ and an admissible embedding $\xi_\epsilon : \mathcal{H}_\epsilon \hookrightarrow {}^L G_{\epsilon_G}$ such that $H_{\epsilon_H}, \mathcal{H}_\epsilon, s$ and ξ_ϵ yield endoscopic data for G_{ϵ_G}. Further choices will be made, for example to specify $\widehat{G}_{\epsilon_G}$, but again they will not affect the isomorphism class of the endoscopic data. Moreover, all endoscopic data for G_{ϵ_G} will be so obtained up to isomorphism.

Suppose that $B_H \supset T_H$ in H and $B \supset T$ in G^* are Borel subgroups for which the already chosen $T_H \to T$ is the attached embedding. Also suppose $x \in G^*_{\mathrm{sc}}$ is such that $\psi_x(\epsilon_G) = \epsilon$, with both $T_G = \psi_x^{-1}(T)$ and $\psi_x : T_G \to T$ defined over F. Note that G^*_ϵ and ψ_x serve as quasisplit data for G_{ϵ_G}.

The embedding

$$T_H \to T \xleftarrow{\psi_x} T_G$$

is by definition dual to a diagram

$$\widehat{H} \leftarrow \mathcal{T}_H = \mathcal{T} \to \widehat{T} \xrightarrow{\psi_x} \widehat{T}_G.$$

This diagram allows us to identify $R(G, T_G)^\vee$ with $R(\widehat{G}, \mathcal{T})$ and then $R(G_{\epsilon_G}, T_G)^\vee$ with a subset of $R(\widehat{G}, \mathcal{T})$.

In fixing L-data $(\widehat{G}_\epsilon, \rho_\epsilon)$ and ${}^L G_\epsilon = \widehat{G}_\epsilon \rtimes W_F$ for G_{ϵ_G} or G^*_ϵ we may assume that $\widehat{G}_\epsilon$ contains $\mathcal{T}$ and that $R(\widehat{G}_\epsilon, \mathcal{T})$ coincides with $R(G_\epsilon, T_G)^\vee$ as a subset of $R(\widehat{G}, \mathcal{T})$. We set $B_\epsilon = B \cap G^*_\epsilon$ and let $\mathcal{B}_\epsilon$ be the Borel subgroup of $\widehat{G}_\epsilon$ generated by $\mathcal{T}$ and the $\mathcal{B}$-positive roots in $\mathcal{T}$ in $\widehat{G}_\epsilon$. The embedding $\widehat{T} \to \mathcal{T}$ of $\widehat{T}$ in $\widehat{G}_\epsilon$ provided by B_ϵ and $\mathcal{B}_\epsilon$ will be identified with $\mathcal{T} \to \widehat{T}$ above. The isomorphism $\widehat{T}_G \xrightarrow{\psi_x} \widehat{T} \longrightarrow \mathcal{T}$ embeds $\widehat{T}_G$ in $\widehat{G}_\epsilon$ and extends to an admissible embedding of ${}^L T_G$ in ${}^L G_\epsilon$ (see, for example, [I, 2.6]). The image, again denoted ${}^L T_G$, is independent of the choice of extension.

The element s lies in $\mathcal{T}$ and thus in $\widehat{G}_\epsilon$. Moreover $\widehat{H} = \text{Cent}(s, \widehat{G})^0$. We may take the dual $\widehat{H}_{\epsilon_H}$ of H_{ϵ_H} as the subgroup $\text{Cent}(s, \widehat{G}_\epsilon)^0$ of $\widehat{G}_\epsilon$; $\widehat{H}_{\epsilon_H}$ is normalized by ${}^L T_G$. We define $\mathcal{H}_\epsilon$ to be the subgroup of ${}^L G$ generated by $\widehat{H}_{\epsilon_H}$ and ${}^L T_G$, and ξ_ϵ to be the inclusion map. Observe that there is clearly a split exact sequence

$$1 \longrightarrow \widehat{H}_{\epsilon_H} \longrightarrow \mathcal{H}_\epsilon \longrightarrow W_F \longrightarrow 1$$

and that $(H_{\epsilon_H}, \mathcal{H}_\epsilon, s, \xi_\epsilon)$ *is a set of endoscopic data for* G_{ϵ_G}.

We also identify the embedding $\widehat{T}_H \longrightarrow \mathcal{T}$ given by $B_H \cap H_{\epsilon_H}$ and $\mathcal{B}_\epsilon \cap \widehat{H}_{\epsilon_H}$ with $\widehat{T}_H \to \mathcal{T}$ above. Then $T_H \to T$ *is an admissible embedding for both* (G, H) *and* $(G_{\epsilon_G}, H_{\epsilon_H})$*. Moreover any admissible embedding of a maximal torus of* H_{ϵ_H} *in* G^*_ϵ *is admissible as an embedding of a maximal torus of* H *in* G^* *and carries* ϵ_H *to* ϵ*.*

It remains to examine the effects of our choice. Suppose first that B and B_H are changed but that $T_H \to T$ remains fixed. Then $(\widehat{G}_\epsilon, \rho_\epsilon)$ is replaced by a pair $(\widehat{G}'_\epsilon, \rho'_\epsilon)$, Γ-isomorphic to it under a map that carries $R(\widehat{G}_\epsilon, \mathcal{T})$ to $R(\widehat{G}'_\epsilon, \mathcal{T})$ and the image of ${}^L T_G$ in ${}^L G_\epsilon$ to its image in ${}^L G'_\epsilon$. This isomorphism further fixes s and carries $\widehat{H}_{\epsilon_H}$ and $\mathcal{H}_\epsilon$ to the new $\widehat{H}'_{\epsilon_H}$ and $\mathcal{H}'_\epsilon$. Thus we obtain isomorphic endoscopic data for G_ϵ.

With T fixed, the choice of T_H and ψ_x does not affect endoscopic data. Now suppose we replace $T_H \to T$ by $\bar{T}_H \to \bar{T}$. Then ϵ_H lies in both $T_H, \bar{T}_H$ and ϵ in both $T, \bar{T}$. We may assume that the new Borel subgroups are obtained from B_H and B by conjugation in $\text{Cent}(\epsilon_H, H)$ and $\text{Cent}(\epsilon, G^*)$. Then again the new data are seen to be isomorphic. Note that if $\bar{T}_H \to \bar{T}$ is admissible for $(G_{\epsilon_G}, H_{\epsilon_H})$ then we may use conjugations in H_{ϵ_H} and G^*_ϵ, and the data are unchanged.

Finally it is straightforward to check that the choice of $(H, \mathcal{H}, s, \xi)$ within its equivalence class does not affect the class of $(H_{\epsilon_H}, \mathcal{H}_\epsilon, s, \xi_\epsilon)$ among data for G_{ϵ_G}. Moreover from any class of data for G_{ϵ_G} we can recover $s \in \mathcal{T}, \widehat{H} = \text{Cent}(s, \widehat{G})^0, \mathcal{H} = \langle \widehat{H}, {}^L T_G \rangle$ contained in ${}^L G$ and ϵ_H semisimple in $H(F)$ such that $(H_{\epsilon_H}, \mathcal{H}_\epsilon, s, \xi_\epsilon)$ lies in the class.

1.5. Descent for Φ_f^H

Continuing with ϵ_h, ϵ and ϵ_G, we assume that endoscopic data have been fixed once and for all by the choices of the last section. Since these choices will not be mentioned again we reserve no notation for them. In particular, $T_H \to T$ will be an arbitrary embedding of T_H in G_ϵ^* which is admissible for $(G_\epsilon^*, H_{\epsilon_H})$. To spare notation further we assume that G_ϵ^*, ψ are quasisplit data for G_{ϵ_G}. If then ϵ_H is a T_H-image of ϵ_G there is an $x \in (G_\epsilon^*)_{\text{sc}}$ such that $T_G = \psi_x^{-1}(T)$ and $\psi_x : T_G \to T$ are defined over F.

For $\tilde{\gamma}_H$ strongly G-regular in the preimage of $T_H(F)$ in $\tilde{H}(F)$ we calculate $\Phi_f^H(\tilde{\gamma}_H)$ as

$$\sum_{\gamma_G} \Delta(\tilde{\gamma}_H, \gamma_G) \Phi(\gamma_G, f)$$

where the sum is over representatives γ_G for the $G(F)$-conjugacy classes in the stable conjugacy class of the image γ_G^0 of $\tilde{\gamma}_H$ under

$$\tilde{T}_H \longrightarrow T_H \longrightarrow T \longrightarrow T_G.$$

We write γ_G as $w^{-1} \gamma_G^0 w, w \in \mathfrak{D}(T_G)$, and now describe a choice of representatives for $\mathfrak{D}(T_G)$.

Consider first the stable conjugacy class of ϵ_G in $G(F)$. Since $\text{Cent}(\epsilon_G, G)$ may be disconnected we pass to a z-extension of G and pick $\tilde{\epsilon} \in \tilde{G}(F)$ mapping to ϵ under $\tilde{G} \to G$. Suppose that $\{\tilde{\epsilon}_j = \tilde{w}_j^{-1} \tilde{\epsilon} \tilde{w}_j : 0 \leq j \leq n\}$ is a set of representatives for the conjugacy classes in the stable class of $\tilde{\epsilon}$, with $\tilde{w}_0 = 1$. Let w_j be the image of $\tilde{w}_j$ in $G(F)$, $\epsilon_j = w_j^{-1} \epsilon_G w_j$ and $G_j = \text{Cent}(\epsilon_j, G)^0$. Notice that the ϵ_j need not be distinct. We use (G_ϵ^*, ψ_j), where $\psi_j = \psi \circ \text{Int}\, w_j$, as quasisplit data for G_j. Define a subset $\mathcal{S}(T_H)$ of $\{0, 1, \cdots, n\}$ by $j \in \mathcal{S}(T_H)$ if and only if ϵ_j has ϵ_H as T_H-image *relative to* (G_j, H_{ϵ_H}), that is, if and only if there exists $h_j \in G_j$ such that $\text{Int}\, h_j^{-1} w_j^{-1}$ maps T_G to G_j over F. Then fix some such h_j and set $\hat{w}_j = w_j h_j, T_j = \hat{w}_h^{-1} T_G \hat{w}_j$. Passage to $\tilde{G}$ shows that

$$\{\hat{w}_j w' : j \in \mathcal{S}(T_H) \text{ and } w' \text{ a representative for } \mathfrak{D}(T_j, G_j)\}$$

is a set of representatives for $\mathfrak{D}(T_G, G)$.

Thus $\Phi_f^H(\tilde{\gamma}_H)$ is equal to

$$\sum_{j=0}^{n} \sum_{w' \in \mathfrak{D}(T_j, G_j)} \delta(j) \Delta(\tilde{\gamma}_H, w'^{-1} \gamma_G^j w', f) \Phi(w'^{-1} \gamma_G^j w', f) \tag{1.5.1}$$

where $\gamma_G^j = \hat{w}_j^{-1}\gamma_G^0\hat{w}_j, 0 \leq j \leq n$, and δ is the characteristic function of $S(T_H)$.

From the well-known construction of Harish-Chandra [**HC1**, Sect. 22; **HC2**, Part VI] we can find $f^j \in C_c^\infty(G_j(F)), 0 \leq j \leq n$, such that

$$\Phi(\delta, f) = \Phi(\delta, f^j)$$

for all regular semisimple δ in some neighborhood of ϵ_j in $G_{\epsilon_j}(F)$. It remains to relate transfer factors for (G, H) to those for $(G_{\epsilon_G}, H_{\epsilon_H})$.

1.6. Descent for transfer factors

To conserve notation $\mathcal{H}$ will be assumed an L-group, but $\mathcal{H}_\epsilon$ must of course remain arbitrary. Then $\tilde{H}_{\epsilon_H}$ will be an admissible central extension of H_{ϵ_H} (recall 1.3) and $\tilde{\epsilon}_H$ will be an element in the preimage of ϵ_H in $\tilde{H}_{\epsilon_H}(F)$. We suppose that ϵ_H is a $\bar{T}_H$-image as well as a T_H-image, and take $\tilde{\gamma}_H, \tilde{\bar{\gamma}}$ in $\tilde{H}_{\epsilon_H}(F)$ with strongly G-regular images $\gamma_H \in T_H, \bar{\gamma}_H \in \bar{T}_H$ in $H_{\epsilon_H}(F)$. Then the factors

$$\Delta = \Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$$

for (G, H) and

$$\Delta_\epsilon = \Delta(\tilde{\gamma}_H, \gamma_G; \tilde{\bar{\gamma}}_H, \bar{\gamma}_G)$$

for $(G_{\epsilon_G}, H_{\epsilon_H})$ are defined and non-zero whenever $\gamma_H, \bar{\gamma}_H$ are images with respect to $(G_{\epsilon_G}, H_{\epsilon_H})$ of $\gamma_G, \bar{\gamma}_G$ in $G_{\epsilon_G}(F)$. That will be our assumption on $\gamma_G, \bar{\gamma}_G$ throughout this section.

Let $\Theta = \Delta/\Delta_\epsilon$. Then Θ is naturally a product

$$\Theta_I\Theta_{II}\Theta_1\Theta_2\Theta_{IV},$$

the factors corresponding to those of Δ and Δ_ϵ [I, Sect. 3]

Keeping T_H and $\bar{T}_H$ fixed we consider $\tilde{\gamma}_H, \tilde{\bar{\gamma}}_H$ near $\tilde{\epsilon}_H$ with $\gamma_G, \bar{\gamma}_G$ both near ϵ_G. We will see in 3.1 that $\Theta_1(\tilde{\gamma}_H, \gamma_G; \tilde{\bar{\gamma}}_H, \bar{\gamma}_G) = 1$. A glance at the remaining factors convinces one that Θ extends continuously to $(\tilde{\epsilon}_H, \epsilon_G; \tilde{\epsilon}_H, \epsilon_G)$ taking a nonzero value there. The extension is then seen to be smooth.

Theorem 1.6.A.

$$\lim \Theta(\tilde{\gamma}_H, \gamma_G; \tilde{\bar{\gamma}}_H, \bar{\gamma}_G) = 1$$
$$\tilde{\gamma}_H, \tilde{\bar{\gamma}}_H \longrightarrow \tilde{\epsilon}_H$$
$$\gamma_G, \bar{\gamma}_G \longrightarrow \epsilon_G.$$

The proof will occupy Sections 3 to 6.

Suppose F is nonarchimedean. Then the theorem says that

$$\Theta(\tilde{\gamma}_H, \gamma_G; \tilde{\bar{\gamma}}_H, \bar{\gamma}_G) = 1$$

for $\tilde{\gamma}_H, \tilde{\bar{\gamma}}_H$ near $\tilde{\epsilon}_H$ and $\gamma_G, \bar{\gamma}_G$ near ϵ_G. Thus for the absolute factors $\Delta(\gamma_H, \gamma_G), \Delta(\bar{\gamma}_H, \bar{\gamma}_G)$ for G and $\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G), \Delta_\epsilon(\tilde{\bar{\gamma}}_H, \bar{\gamma}_G)$ for G_ϵ [I, 3.7] we have

$$\frac{\Delta(\gamma_H, \gamma_G)}{\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G)} = \frac{\Delta(\bar{\gamma}_H, \bar{\gamma}_G)}{\Delta_\epsilon(\tilde{\bar{\gamma}}_H, \bar{\gamma}_G)},$$

and so we may write

$$\Delta(\gamma_H, \gamma_G) = c\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G)$$

for $\tilde{\gamma}_H$ near $\tilde{\epsilon}_H$ and γ_G near ϵ_G, where c is a constant. Observe that for $\tilde{\gamma}_H$ near $\tilde{\epsilon}_H$ the factor $\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G)$ depends only on γ_H and γ_G (see [I, 4.4.A] and 3.5). We emphasize that the assertion of the theorem is that c is independent of the Cartan subgroup T_H containing γ_H. That this is crucial for the transfer of orbital integrals (and therefore characters) will be seen in Section 2.

In the archimedean case c is a function. This however presents no difficulties in our applications (2.4, 2.5).

1.7. Final formula

For the pair (G_j, H_{ϵ_H}) we write c_j in place of c. Then

(1.7.1) $$\Phi_f^H(\gamma_H) = \sum_{j=0}^{n} c_j \Phi_{fj}^{H_\epsilon}(\tilde{\gamma}_H)$$

for strongly G-regular $\tilde{\gamma}_H$ sufficiently close to $\tilde{\epsilon}_H$ in $\tilde{H}_{\epsilon_H}(F)$. Note that the characteristic function $\delta(j)$ of (1.5.1) has disappeared because by definition $\Phi_{fj}^{H_\epsilon}(\tilde{\gamma}_H) = 0$ for $j \notin \mathcal{S}(T_H)$ (recall 1.3).

§2. Consequences

2.1. Local transfer

We say that (G, H) *admits* Δ-*transfer* if for each $f \in C_c^\infty(G(F))$ there exists $f^{\tilde{H}} \in C_c^\infty(\tilde{H}(F),$ $\tilde{\lambda})$ notation of 1.3) such that f and $f^{\tilde{H}}$ have Δ-matching orbital integrals, that is,

(2.1.1) $$\Phi^{\mathrm{st}}(\tilde{\gamma}_H, f^{\tilde{H}}) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

for all strongly G-regular $\tilde{\gamma}_H$ in $\tilde{H}(F)$. If H is the quasisplit form of G, so that Δ is a constant, we often refer to stable transfer rather than Δ-transfer. Suppose that F is nonarchimedean. Then for $\tilde{\gamma}_H$ near

1 the factor $\Delta(\tilde{\gamma}_H, \gamma_G)$ depends only on the image γ_H of $\tilde{\gamma}_H$ in $H(F)$ (see [**I** 4.4] again); so we may denote it instead by $\Delta_{\mathrm{loc}}(\gamma_H, \gamma_G)$. We say that (G, H) admits *local Δ-transfer at the identity* if for any $f \in C_c^\infty(G(F))$ we can find $f^H \in C_c^\infty(H(F))$ such that

$$\Phi^{\mathrm{st}}(\gamma_H, f^H) = \sum_{\gamma_G} \Delta_{\mathrm{loc}}(\gamma_H, \gamma_G)\Phi(\gamma_G, f) \tag{2.1.2}$$

for all strongly G-regular γ_H near 1 in $H(F)$.

2.2. A characterization lemma

Throughout 2.2 and 2.3 we assume that F is nonarchimedean. Suppose that Φ^{st} is a stably-invariant function on the regular semisimple elements of $G(F)$ that is compactly supported modulo conjugation, viz., that vanishes along all conjugacy classes of regular semisimple elements that do not meet some fixed compact subset of $G(F)$. Then we call Φ^{st} a *local stable orbital integral* if for each semisimple element ϵ in $G(F)$ there exists $f_\epsilon \in C_c^\infty(G(F))$ such that $\Phi^{\mathrm{st}}(\gamma) = \Phi^{\mathrm{st}}(\gamma, f_\epsilon)$ for all *regular semisimple* γ near ϵ.

Lemma 2.2.A. *Let G be a quasisplit group. If Φ^{st} is a local stable orbital integral on $G(F)$ then there exists $f \in C_c^\infty(G(F))$ such that*

$$\Phi^{\mathrm{st}}(\gamma) = \Phi^{\mathrm{st}}(\gamma, f)$$

for all regular semisimple γ in $G(F)$.

Proof. By a simple passage to a z-extension ([**K1**]) we reduce immediately to the case that the derived group of G is simply connected.

If ϵ is semisimple in $G(F)$ we denote by Z_ϵ the center of G_ϵ and by Z'_ϵ the set of ϵ' in Z_ϵ at which D_G/D_{G_ϵ}, does not vanish. For ϵ' in Z'_ϵ we have $G_{\epsilon'} = G_\epsilon$ and $Z_{\epsilon'} = Z_\epsilon$, while if $\epsilon' \in Z_\epsilon - Z'_\epsilon$ then $G_{\epsilon'} \supsetneqq G_\epsilon$, so that $dim\, G_{\epsilon'} > dim\, G_\epsilon$, and $Z_{\epsilon'} \overset{\subset}{\neq} Z_\epsilon$. Notice that the group G_ϵ is Cent(Z_ϵ, G). Thus if $g \in G(\bar{F})$ then $g^{-1}\epsilon g$ is stably conjugate to ϵ if and only if Int $g^{-1} : Z_\epsilon \longrightarrow Z_{g^{-1}\epsilon g}$ is defined over F, that is, Z_ϵ is stably conjugate to $Z_{g^{-1}\epsilon g}$.

There are only finitely many stable conjugacy classes among groups Z_ϵ. We label representatives $Z_0, \cdots, Z_r$ for these classes so that Z_0 is the center of G, the group $G(\ell) = $ Cent(Z_ℓ, G) is quasisplit over F for each ℓ (using [**K1**, Lemma 3.3]), and so that $dim\, G(\ell) \leq dim\, G(k)$ if $k \leq \ell$. Notice that if $Z_\ell = Z_\epsilon$ then $G(\ell) = G_\epsilon$ and $Z'_\ell = Z'_\epsilon$.

It is sufficient to show for each $\ell = 0, \cdots, r$ that if a local stable orbital integral Φ^{st} vanishes on the regular semisimple elements in a neighborhood of $\bigcup_{k<\ell} Z_k(F)$ then there exists $f_\ell \in C_c^\infty(G(F))$ such that

$$\Phi^{\mathrm{st}}(\gamma) = \Phi^{\mathrm{st}}(\gamma, f_\ell) \tag{2.2.1}$$

for all γ near $\bigcup_{k\leq\ell} Z_k(F)$, for if ϵ' is stably conjugate to ϵ in this set then any regular γ' close to ϵ' is stably conjugate to a γ close to ϵ because G_ϵ is quasisplit. Thus (2.2.1) implies that

$$\Phi^{\mathrm{st}}(\gamma') = \Phi^{\mathrm{st}}(\gamma', f_\ell).$$

We then proceed inductively, replacing the original Φ^{st} by

$$\gamma \longrightarrow \Phi^{\mathrm{st}}(\gamma, f_0),$$

passing to Z_1, and so on.

Because G_{der} is simply connected, stable semisimple conjugacy classes are labelled by orbits of the Weyl group in a fixed Cartan subgroup T over F. Of course the orbits lie in $T(\bar{F})$, and not all such orbits label stable conjugacy classes. Suppose we are given a Galois-invariant metric on $T(\bar{F})$, one such orbit $\bar{\tau} = \{t_0, t_1, \cdots, t_s\}$, and a $\delta > 0$. Then the set of all $g \in G(F)$ such that the semisimple part of g is conjugate in $G(\bar{F})$ to an s such that $|s - t_i| < \delta$ for some i is an open subset of $G(F)$ that is stably invariant. Multiplying a given function f by the characteristic function of such a set, we concentrate its stable orbital integrals on the set $X(\tau, \delta)$ of regular γ such that γ is conjugate to s with $|s - t_i| < \delta$ for some δ.

Thus a stable conjugacy class that meets $Z'_\ell(F)$ will intersect $Z'_\ell(F)$ in a finite set $\{\epsilon_0, \cdots, \epsilon_r\}$, and mapping Z_ℓ into T over $\bar{F}$ we send the elements $\epsilon = \epsilon_0, \cdots, \epsilon_r$ to $t_0, \cdots, t_r$. We complete this set to a full orbit $\tau = \{t_0, \cdots, t_s\}$. Since G_ϵ is quasisplit, it is clear that if δ is sufficiently small then any stable conjugacy class in $X(\tau, \delta)$ meets any given neighborhood of ϵ in G_ϵ. In verifying this, we may suppose that $Z_\ell \subseteq T$ so that $t_0 = \epsilon$.

A stable conjugacy class in $X(\tau, \delta)$ is then represented by an $s \in T(\bar{F})$ such that $|s - \epsilon| < \delta$ and such that for any $\sigma \in \mathrm{Gal}(\bar{F}/F)$ there is a u_σ in the normalizer of T in $G(F)$ satisfying

$$\sigma(s) = u_\sigma^{-1} s u_\sigma. \tag{2.2.2}$$

Since

$$|\sigma(s) - \sigma(\epsilon)| = |\sigma(s) - \epsilon| = |u_\sigma^{-1} s u_\sigma - \epsilon|,$$

we can choose δ sufficiently small that (2.2.2) forces u_σ to centralize ϵ. Consequently, by Steinberg's Theorem, s determines a stable conjugacy class in $G_\epsilon(F)$. Moreover, if δ is small enough this class must meet the given neighborhood of ϵ in G_ϵ.

By assumption, there is an f_ℓ such that $\Phi^{\mathrm{st}}(\gamma) = \Phi^{\mathrm{st}}(\gamma, f_\ell)$ for γ in a neighborhood of ϵ in G_ϵ, and we conclude that this relation is valid on all of $X(\tau, \delta)$. A simple partition-of-unity argument completes the proof. Observe that we could have dispensed with the introduction of the groups Z_ℓ, if we had wished to do so.

2.3. Reduction to local transfer

Theorem 2.3.A *Suppose all pairs $(G_{\epsilon_G}, H_{\epsilon_H})$ have local Δ_ϵ-transfer at the identity. Then (G, H) has Δ-transfer.*

Proof. We first observe that if G is replaced by a z-extension $\tilde{G}$ the hypothesis of the theorem remains valid. Further it is then sufficient to prove the theorem for $\tilde{G}$. Thus we may as well assume $\mathcal{H}$ an L-group. By Lemma 2.2.A we have then just to show that

$$\Phi_f^H(\gamma_H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f),$$

defined so far for γ_H strongly G-regular in $H(F)$, is a local stable orbital integral on $H(F)$. We shall use the descent formula (1.7.1). Observe also that local Δ_ϵ-transfer at the identity for $(G_{\epsilon_G}, H_{\epsilon_H})$ implies local Δ_ϵ-transfer at ϵ_H for the same pair (see Lemma 3.5.A).

First, if ϵ_H is regular semisimple in $H(F)$, so that H_{ϵ_H} is a Cartan subgroup of H, then local transfer at ϵ_H for $(G_{\epsilon_G}, H_{\epsilon_H})$ implies by (1.7.1) that Φ_f^H extends smoothly to ϵ_H. Thus we have Φ_f^H defined on all regular semisimple elements of $H(F)$. It is further locally constant, stable, and compactly supported modulo conjugation.

For general ϵ_H we use (1.7.1) to obtain $f' \in C_c^\infty(H_{\epsilon_H}(F))$ and thus $f'' \in C_c^\infty(H(F))$ such that

$$\Phi_f^H(\gamma_H) = \Phi^{\mathrm{st}}(\gamma_h, f') = \Phi^{\mathrm{st}}(\gamma_H, f'')$$

for γ_H near ϵ_H. Thus Φ_f^H is a local stable orbital integral and the proof is complete.

2.4. Equisingular matching

We call ϵ_H and ϵ_G *equisingular* if H_{ϵ_H}, is an inner form of G_{ϵ_G}, that is, if ϵ_H is (G, H)-*regular* in the sense of [**K2**]. Assume that f, f^H have Δ-matching orbital integrals. For simplicity, we shall suppose that the derived group of G is simply connected so that neither H itself nor H_{ϵ_H} need be replaced by a central extension in the matchings. Note also that $\mathrm{Cent}(\epsilon_H, H)$ is connected [**K2**, Lemma 3.2]. Following [**K2**], and by the homogeneity of germs, we may expect a stable combination of the integrals of f^H along the conjugacy classes in the stable class of ϵ_H to match with some suitable combination of the integrals of f along the classes in the stable class of ϵ_G. We shall show this is true and so verify some conjectures in [**K2**].

The first step is to define a factor $\Delta(\epsilon_H, \epsilon_G)$. Let ϵ_H be a T_H-image of ϵ_G. Then we set

$$\Delta(\epsilon_H, \epsilon_G) = \lim \Delta(\gamma_H, \gamma_G),$$

where the limit is taken as $\gamma_H \longrightarrow \epsilon_H$ in $T_H(F)$ and $\gamma_G \longrightarrow \epsilon_G$, and γ_H is an image of γ_G. Suppose F nonarchimedean. If we consider also $\bar\gamma_H, \bar\gamma_G$ with $\bar\gamma_H$ near ϵ_H in another Cartan subgroup $\bar T_H(F)$ of $H_{\epsilon_H}(F)$ then $\Delta_\epsilon(\gamma_H, \gamma_G; \bar\gamma_H, \bar\gamma_G) \equiv 1$ since H_{ϵ_H} is the quasisplit inner form of G_{ϵ_G}. Thus Theorem 1.6.A asserts that

$$\begin{aligned}\Theta(\gamma_H, \gamma_G; \bar\gamma_H, \bar\gamma_G) &= \Delta(\gamma_H, \gamma_G; \bar\gamma_H, \bar\gamma_G)\\ &= \Delta(\gamma_H, \gamma_G)/\Delta(\bar\gamma_H, \bar\gamma_G)\\ &= 1\end{aligned}$$

if $\gamma_h, \bar\gamma_H$ near ϵ_H are images of $\gamma_G, \bar\gamma_G$ near ϵ. In other words, $\Delta(\gamma_H, \gamma_G)$ is a constant independent of the Cartan subgroup containing γ_H and $\Delta(\epsilon_H, \epsilon_G)$ equals this constant. If F is archimedean we still have $\Delta_\epsilon(\gamma_H, \gamma_G; \bar\gamma_H, \bar\gamma_G) \equiv 1$ but Theorem 1.6.A asserts only that

$$\lim \Delta(\gamma_H, \gamma_G)/\Delta(\bar\gamma_H, \bar\gamma_G) = 1.$$

We conclude nevertheless that $\Delta(\epsilon_H, \epsilon_G)$ is well defined, that is, independent of the choice of T_H.

We write $O(\epsilon_G, f)$ for the integral of f along the conjugacy class of ϵ_G, keeping in mind our convention for measures (1.1). A sign $e(G)$ is defined in [**K4**]. If we sum over representatives ϵ'_H for the conjugacy classes in the stable conjugacy class of ϵ_H then

$$O^{\mathrm{st}}(\epsilon_H, f^H) = \sum_{\epsilon'_H} e(H_{\epsilon'_H}) O(\epsilon'_H, f^H)$$

is a stable distribution [**K3, S5**].

Lemma 2.4.A *Suppose ϵ_H is (G,H)-regular then*

$$O^{\mathrm{st}}(\epsilon_H, f^H) = \sum_{\epsilon_G} e(G_{\epsilon_G})\Delta(\epsilon_H, \epsilon_G) O(\epsilon_G, f) \tag{2.4.1}$$

where the sum is over representatives ϵ_G for the conjugacy classes in $G(F)$ equisingular with ϵ_H.

Proof. (i) F nonarchimedean. Suppose γ_H is near ϵ_H in a fundamental Cartan subgroup $T_H(F)$ of $H_{\epsilon_H}(F)$. Then by descent (1.7),

$$\sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

coincides with

$$\sum_j \Delta(\gamma_H, \gamma_j)\Phi^{\mathrm{st}}(\gamma_j, f^j)$$

and thus with

$$\sum_j \Delta(\epsilon_H, \epsilon_j)\Phi^{\mathrm{st}}(\gamma_j, f^j).$$

The zero-degree term in the Shalika germ expansion of this expression is

$$\lambda \sum_j \Delta(\epsilon_H, \epsilon_j) e(G_j) f^j(\epsilon_j)$$

where λ is a constant depending only on ϵ_H (see [**K3**, §3]). Apart from λ, this is the right side of (2.4.1). At the same time,

$$\sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

coincides with $\Phi^{\mathrm{st}}(\gamma_H, f^H)$ which has $\lambda O^{\mathrm{st}}(\epsilon_H, f^H)$ as zero-degree term in its germ expansion. So (2.4.1) is proved.

(ii) F archimedean. Descent again yields

$$\sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f) = \sum_j \Delta(\gamma_H, \gamma_G)\Phi^{\mathrm{st}}(\gamma_G, f^j).$$

In place of germ expansions we use Harish-Chandra's limit formula [**HC3**, Lemma 17.5]. Again T_H is taken to be fundamental in H_{ϵ_H}. We write $\gamma_H \in T_H(F)$ as $\epsilon_H \exp X$ and multiply each side of the equation

$$\Phi^{\mathrm{st}}(\gamma_H, f^H) = \sum_j \Delta(\gamma_H, \gamma_j)\Phi^{\mathrm{st}}(\gamma_j, f^j)$$

by

$$\prod (\alpha(\epsilon_H) e^{\alpha(X)/2} - e^{-\alpha(X)/2}),$$

the product being over all positive roots of T_H in H. We then apply the operator $\varpi_{\epsilon_H} = \prod' H_\alpha$, the product being now over positive roots in H_{ϵ_H}, and take limits as $X \longrightarrow 0$. As a first step we obtain on the right side

$$\sum_j \Delta(\epsilon_H, \epsilon_j) \lim \varpi_{\epsilon_H} (\prod{}' e^{\alpha(X)/2} - e^{-\alpha(X)/2}) \Phi^{\mathrm{st}}(\gamma_j, f^j)$$

(see [**S5**] for the explicit form of $\Delta(\gamma_H, \gamma_j)$, especially the term $\Delta_2(\gamma_H, \gamma_j)$, and [**W**, p. 371] for a similar calculation).

We calculate this new limit by means of Harish-Chandra's formula as in [**S5**, §2.9], using the results of §37 of [**HC3**] to keep track of constants. The contribution of the right side is then

$$\sum_j \Delta(\epsilon_H, \epsilon_j) \lambda e(G_j) f^j(\epsilon_j) = \lambda \sum_j \Delta(\epsilon_H, \epsilon_j) e(G_j) O(\epsilon_j, f),$$

λ being a constant depending only on H_{ϵ_H}. The left side contributes $\lambda O^{\mathrm{st}}(\epsilon_H, f^H)$, and so the lemma is proved.

Observe that from our product formula for $\Delta(\gamma_H, \gamma_G)$ [**I**, §6.4] we obtain a product formula for $\Delta(\epsilon_H, \epsilon_G)$ as conjectured in [**K2**, §6.10].

2.5. Regular matching

Suppose δ_G is regular in $G(F)$, that is, that $\mathrm{Cent}(\delta_G, G)$ is of minimal dimension. Let $\delta_G = \epsilon_G u_G$ be the Jordan decomposition; both ϵ_G and u_G belong to $G(F)$. Assume ϵ_G has image ϵ_H in $H(F)$. As usual, we take H_{ϵ_H} quasisplit over F and then choose u_H regular unipotent in $H_{\epsilon_H}(F)$. Thus $\delta_H = \epsilon_H u_H$ is regular in $H(F)$. We shall use descent and the regular unipotent matching of [**I**, §5.5] to match integrals over the classes of δ_H and δ_G. For simplicity of notation we assume $\mathcal{H}, \mathcal{H}_\epsilon$ are L-groups and that $\mathrm{Cent}(\epsilon_H, H)$, $\mathrm{Cent}(\epsilon_G, G)$ are connected. The stable conjugacy classes of δ_H, δ_G are then the F-points in their $\bar{F}$-classes.

Set $\Phi^{\mathrm{st}}(\delta_H, f^H) = \sum \Phi(\delta'_H, f^H)$, where the sum is over representatives δ'_H for conjugacy classes in the stable conjugacy class of δ_H. Then descent to H_{ϵ_H} and Theorem 5.5.A of [**I**] show immediately that

$$\Phi^{\mathrm{st}}(\delta_H, f^H) = \lim_{\gamma_H \to \epsilon_H} D_{\epsilon_H}(\gamma_H) \Phi^{\mathrm{st}}(\gamma_H, f^H)$$

so that $f^H \to \Phi^{\mathrm{st}}(\delta_H, f^H)$ is a stable distribution.

At the same time we set

$$\Delta(\epsilon_H, \epsilon_G) = \lim_{\substack{\gamma_H \to \epsilon_H \\ \gamma_G \to \epsilon_G}} \Delta(\gamma_H, \gamma_G)/\Delta_\epsilon(\gamma_H, \gamma_G),$$

the limits taken as in 1.6, and

$$\Delta_\epsilon(\delta_G) = \lambda(\epsilon_G)\Delta(u_G)$$

in the notation of [**I**, 5.5], with λ, Δ calculated with respect to $(G_{\epsilon_G}, H_{\epsilon_H})$. Then we define

$$\Delta(\delta_H, \delta_G) = \Delta(\epsilon_H, \epsilon_G)\Delta_\epsilon(\delta_G).$$

Suppose f and f^H have Δ-matching orbital integrals. Descent and the regular-unipotent matching also imply easily that

$$\lim_{\gamma_H \to \epsilon_H} D_{\epsilon_H}(\gamma_H) \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f) = \sum_{\delta'_G} \Delta(\delta_H, \delta'_G)\Phi(\delta'_G, f),$$

and so we have the matching

$$\Phi^{\mathrm{st}}(\delta_H, f^H) = \sum_{\delta'_G} \Delta(\delta_H, \delta'_G)\Phi(\delta'_G, f).$$

Here we have dealt with both the nonarchimedean and archimedean cases. We also see that if $\delta_H = \epsilon_H u_H$ is regular in $H(F)$ and ϵ_H is not an image then

$$\Phi^{\mathrm{st}}(\delta_H, f^H) = 0.$$

Finally we remark that if $f \in C_c^\infty(G(F))$ is supported on the full regular set of $G(F)$ then we can find $f^H \in C_c^\infty(H(F))$ supported on the full regular set of $H(F)$ with Δ-matching integrals.

2.6. Archimedean transfer

Suppose F is archimedean. If $F = \mathbf{C}$ we can define transfer factors for G over $\mathbf{C}$ or for the group $\tilde{G} = \mathrm{Res}_{\mathbf{R}}^{\mathbf{C}} G$ over $\mathbf{R}$ obtained by restriction of scalars. On identifying $G(\mathbf{C})$ with $\tilde{G}(\mathbf{R})$ in the usual way we see that the transfer factors for a pair $(G(\mathbf{C}), H(\mathbf{C}))$ are the same as for $(\tilde{G}(\mathbf{R}), \tilde{H}(\mathbf{R}))$ (this is a special case of a fact used in 5.5). It is then sufficient to treat the case $F = \mathbf{R}$. Again there will be no harm in taking $\mathcal{H}$ to be an L-group.

For real groups a transfer factor, that we now denote $\Delta^{(\mathbf{R})}(\gamma_H, \gamma_G)$, was defined in [**S4**] using diagrams. It includes moreover an implicitly defined sign, (see [**S4**, Section 3.5]).

Theorem 2.6.A *There is a constant c such that*

$$\Delta(\gamma_H, \gamma_G) = c\Delta^{(\mathbf{R})}(\gamma_H, \gamma_G)$$

for all G-regular γ_H in $H(\mathbf{R})$.

Proof. By continuity we can assume γ_H strongly G-regular, and because both Δ and $\Delta^{(\mathbf{R})}$ satisfy the Local Hypothesis ([**I**, 4.2.B], [**L2**, Lemma 6.17]) we may assume that G is quasisplit over $\mathbf{R}$.

For an imaginary root α we may take the χ-datum χ_α as the character $z \longrightarrow z/|z|$ on $\mathbf{C}^\times$ and for the remaining roots we take χ_α trivial. Then inspection of the terms in each factor shows that

$$\Delta(\gamma_H, \gamma_G) = c(T_H)\Delta^{(\mathbf{R})}(\gamma_H, \gamma_G)$$

where $T_H = \mathrm{Cent}(\gamma_H, H)$. Up to a constant independent of $T_H, c(T_H)$ is either real or purely imaginary. To show that $c(T_H)$ is in fact independent of T_H, which is what the theorem asserts, we argue as follows.

By the matching of orbital integrals for $\Delta^{(\mathbf{R})}$ [**S4**] there is for each Schwartz function f on $g(\mathbf{R})$ a Schwartz function f^H on $H(\mathbf{R})$ (or an essentially Schwartz function if the embedding $\xi : {}^LH \hookrightarrow {}^LG$ is not of unitary type [**S3**]) such that

$$\Phi^{\mathrm{st}}(\gamma_H, f^H) = \sum_{\gamma_G} \Delta^{(\mathbf{R})}(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

for all strongly G-regular elements γ_H. We multiply both sides by $D_H(\gamma_H)$. The left side has a limit as $\gamma_H \longrightarrow 1$ through any Cartan subgroup and the limit is independent of the choice of Cartan subgroups. This follows from applying the Harish-Chandra jump conditions to stable orbital integrals [**S1**, Section 4]. The right side then has the same property. But so also does

$$\gamma_H \longrightarrow \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

by [**I**, Theorem 5.5.A]. Because G is quasisplit over $\mathbf{R}$ we can choose f such that the limit is nonzero. Thus we get a contradiction unless $c(T_H)$ is independent of T_H, and the theorem is proved.

We conclude now that for each Schwartz function f on $G(\mathbf{R})$ there exists an (essentially) Schwartz function f^H on $H(\mathbf{R})$ with Δ-matching orbital integrals. Then the Paley-Wiener results of [**C-D**] allow us to take $f^H \in C_c^\infty(H(\mathbf{R}), K_H)$ if $f \in C_c^\infty(G(\mathbf{R}), K)$, with K_H, K maximal compact subgroups of $H(\mathbf{R}), G(\mathbf{R})$ respectively and the notation indicating bi-K_H-finite or bi-K-finite functions.

Of course we have not used the Descent Theorem (1.6.A) in this proof of Δ-transfer. However the proof of $\Delta^{(\mathbf{R})}$-transfer that we have used is based on the Harish-Chandra jump conditions and these come from descent to centralizers of semiregular elements. Many of the arguments in [**S1**-**S4**] for $\Delta^{(\mathbf{R})}$-transfer are essentially special cases of results needed for 1.6.A. To prove Δ-transfer directly we may apply the results of Section 1 to verify the jump conditions of [**S1**] for Φ_f^H. Since we still need some of the arguments from [**S2**-**S4**] and overall the proof is not much shorter, we forgo the details.

§3. Comparison Lemmas

3.1. Reduction to quasisplit groups

Recall from 1.6 that

$$\Theta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) = \Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)/\Delta_\epsilon(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G),$$

where $\gamma_H, \bar{\gamma}_H$ are images of $\gamma_G, \bar{\gamma}_G$ relative to $(H_{\epsilon_H}, G_{\epsilon_G})$. We consider the limit of Θ as $\gamma_H, \bar{\gamma}_H$ approach ϵ_H through fixed Cartan subgroups denoted respectively $T_H, \bar{T}_H$, and $\gamma_G, \bar{\gamma}_G$ approach ϵ_G. Theorem 1.6.A, which we have to prove, states that this limit is 1.

Fix embeddings $T_H \longrightarrow T, \bar{T}_H \longrightarrow \bar{T}$ for $(H_{\epsilon_H}, G_\epsilon^*)$. Recall that we assume G_ϵ^*, ψ to be quasisplit data for G_{ϵ_G}. Write $\gamma, \bar{\gamma}$ for the images of $\gamma_H, \bar{\gamma}_H$, as usual. If we factor Θ as we factor Δ, Δ_ϵ then only $\Theta_1 = \Theta_{III_1}$ depends on $\gamma_G, \bar{\gamma}_G$ rather than on $\gamma, \bar{\gamma}$ alone. The next lemma allows us to replace G by G^*, ϵ_G by ϵ and $\gamma_G, \bar{\gamma}_G$ by $\gamma, \bar{\gamma}$.

Lemma 3.1.A.

$$\Theta_1(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) = 1.$$

Proof. We use the notation of [**I**, 3.4], putting a subscript ϵ on those objects attached to G_{ϵ_G}. We may take $u(\sigma)$ to be the image in G^*_{sc} of the element $u_\epsilon(\sigma)$ in $(G_\epsilon^*)_{\mathrm{sc}}$, and h and $\bar{h}$ to be the images of h_ϵ and $\bar{h}_\epsilon$. Then $v(\sigma)$ and $\bar{v}(\sigma)$ are the images of $v_\epsilon(\sigma)$ and $\bar{v}_\epsilon(\sigma)$. The cochains v_ϵ and $\bar{v}_\epsilon$ have the same coboundary and it takes values in the inverse image W (a finite group) of Z^*_{sc} in $(G_\epsilon^*)_{\mathrm{sc}}$. So we have a cocycle i with values in

$$V = T_{\epsilon-\mathrm{sc}} \times \bar{T}_{\epsilon-\mathrm{sc}}/W,$$

where we use a notational principle that admits several variants: the subscript $\epsilon - \mathrm{sc}$ indicates inverse image in $(G_\epsilon^*)_{\mathrm{sc}}$. The classes $\mathrm{inv}(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ and $\mathrm{inv}_\epsilon(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ are the images of the class of i under the homomorphisms induced by

$$
(3.1.1) \qquad \begin{tikzcd} & U \\ V \arrow[ur] \arrow[dr] & \\ & U_\epsilon. \end{tikzcd}
$$

Dual to (3.1.1) we have

$$
(3.1.2) \qquad \begin{tikzcd} & \widehat{U} \arrow[dl] \\ \widehat{V} & \\ & \widehat{U}_\epsilon. \arrow[ul] \end{tikzcd}
$$

To prove the lemma it is sufficient to show that the images of s_U and s_{U_ϵ} in $\widehat{V}$ are the same.

The L-data for G^*_ϵ (see 1.4) provide $\mathcal{T} \xrightarrow{\sim} \widehat{T}$ and $\mathcal{T} \xrightarrow{\sim} \widehat{\bar{T}}$. We have two commutative diagrams.

$$
\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ T_{\epsilon-\mathrm{ad}} \times \widehat{T}_{\epsilon-\mathrm{ad}} \arrow[r] & T_{\mathrm{ad}} \times \bar{T}_{\mathrm{ad}} \end{tikzcd}
$$

$$
\begin{tikzcd} V \arrow[r] \arrow[dr] & U_\epsilon \arrow[d] \\ & T_{\epsilon-\mathrm{ad}} \times \bar{T}_{\epsilon-\mathrm{ad}} \end{tikzcd}
$$

Using them we extend (3.1.2) to a commutative diagram

$$
\begin{tikzcd} & \widehat{U} \arrow[dl] & \widehat{T}_{\mathrm{sc}} \times \widehat{\bar{T}}_{\mathrm{sc}} \arrow[l] & \mathcal{T}_{\mathrm{sc}} \times \mathcal{T}_{\mathrm{sc}} \arrow[l, "\sim"'] \\ \widehat{V} & & & \\ & \widehat{U}_\epsilon \arrow[ul] & \widehat{T}_{\epsilon-\mathrm{sc}} \times \widehat{\bar{T}}_{\epsilon-\mathrm{sc}} \arrow[l] \arrow[uu] & \mathcal{T}_{\epsilon-\mathrm{sc}} \times \mathcal{T}_{\epsilon-\mathrm{sc}}. \arrow[l, "\sim"'] \arrow[uu] \end{tikzcd}
$$

We may write $\tilde{s} \in \mathcal{T}_{\mathrm{sc}}$ as the product of the image of $\tilde{s}_\epsilon$ in $\mathcal{T}_{\epsilon-\mathrm{sc}}$ with an element of the identity component $\mathcal{R}$ of the subgroup $\{x \in \mathcal{T}_{\mathrm{sc}} : \alpha^\vee(x) = 1, \alpha^\vee \in R(\widehat{G}_\epsilon, \mathcal{T})\}$. Embed $\mathcal{R}$ diagonally in $\mathcal{T}_{\mathrm{sc}} \times \mathcal{T}_{\mathrm{sc}}$ without change in notation. Then we have just to show that the image of $\mathcal{R}$ in $\widehat{V}$ is trivial. But $\mathcal{R}$ is connected and $\widehat{V} \longrightarrow \widehat{T}_{\epsilon-\mathrm{ad}} \times \widehat{\bar{T}}_{\epsilon-\mathrm{ad}}$ has finite kernel because $T_{\epsilon-\mathrm{sc}} \times \bar{T}_{\epsilon-\mathrm{sc}} \longrightarrow V$ does. Thus it is enough to show that

$$
\mathcal{R} \longrightarrow \widehat{V} \longrightarrow \widehat{T}_{\epsilon-\mathrm{ad}} \times \bar{T}_{\epsilon-\mathrm{ad}}
$$

is trivial. That, however, is immediate, and the lemma is proved.

3.2. Remarks and notation

We assume from now on that $G = G^*, \epsilon_G = \epsilon, \gamma_G = \gamma$ and $\bar{\gamma}_G = \bar{\gamma}$. Then $\Theta(\gamma_H, \gamma; \bar{\gamma}_H, \bar{\gamma})$ is a quotient $\Theta(\gamma_H, \gamma)/\Theta(\bar{\gamma}_H, \bar{\gamma})$. So also is each of $\Theta_I, \Theta_{II}, \Theta_2$ and Θ_{IV}. The embeddings $T_H \longrightarrow T, \bar{T}_H \longrightarrow \bar{T}$ being fixed, we may delete γ and $\bar{\gamma}$ from the notation. We may further fix Borel subgroups $B_H \supset B_{\epsilon_H} \supset T_H, B \supset B_\epsilon \supset T, \bar{B}_H$ and so on, for which $T_H \longrightarrow T, \bar{T}_H \longrightarrow \bar{T}$ are the attached embeddings. There is then a canonical identification of the roots of T in G with those of $\bar{T}$. It carries the B-positive roots to the $\bar{B}$-positive ones, the (positive) roots of T in G_ϵ to the (positive) roots of $\bar{T}$ in G_ϵ, and the roots from H to the roots from H. Thus we use the simpler notation $R = R(G), R_\epsilon = R(G_\epsilon), R_s = R(H)$, and so on, for root systems. Also $R(G) - R(G_\epsilon)$ will be abbreviated as $R(G/G_\epsilon)$ and $R(G) - (R(G_\epsilon) \cup R(H))$ as $R(G/G_\epsilon, H)$.

In this notation $\Theta_{IV}(\gamma_H)$ is written as

$$\prod_{\alpha \in R(G/G_\epsilon, H)} |\alpha(\gamma) - 1|^{1/2},$$

and so

$$\lim_{\gamma_H \longrightarrow \epsilon_H} \Theta_{IV}(\gamma_H) = \prod_{\alpha \in R(G/G_\epsilon, H)} |\alpha(\epsilon) - 1|^{1/2} = \lim_{\bar{\gamma}_H \longrightarrow \epsilon_H} \Theta_{IV}(\bar{\gamma}_H).$$

Thus Θ_{IV} contributes 1 to the limit and it remains to examine

$$\widehat{\Theta}(\gamma_H, \bar{\gamma}_H) = \frac{\Theta_I(\gamma_H)}{\Theta_I(\bar{\gamma}_H)} \cdot \frac{\Theta_{II}(\gamma_H)}{\Theta_{II}(\bar{\gamma}_H)} \cdot \frac{\Theta_2(\gamma_H)}{\Theta_2(\bar{\gamma}_H)}.$$

The First Lemma of Comparison will allow us to examine $\Theta_I(\gamma_H)/\Theta_I(\bar{\gamma}_H)$ and the Second to examine $\Theta_2(\gamma_H)/\Theta_2(\bar{\gamma}_H)$. For the remaining term observe for now that

$$\lim_{\gamma_H \longrightarrow \epsilon_H} \Theta_{II}(\gamma_H) = \prod_\alpha \chi_\alpha\Big(\frac{\alpha(\epsilon) - 1}{a_\alpha}\Big)$$

where the product is over representatives for the orbits in $R(G/G_\epsilon, H)$ under the Galois action for T. There is a similar formula for $\lim_{\bar{\gamma}_H \longrightarrow \epsilon_H} \Theta_{II}(\bar{\gamma}_H)$ involving the $\Gamma_{\bar{T}}$-orbits.

To compute Θ_I we need also to fix F-splittings $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ of G and $(\mathbf{B}_\epsilon, \mathbf{T}_\epsilon, \{Y_\beta\})$ of G_ϵ, but then we can use $\mathbf{B}, \mathbf{B}_\epsilon$ along with B, B_ϵ to identify T, T_ϵ with $\mathbf{T}$, and again identify all roots as roots of $\mathbf{T}$. Recall that we write Γ for $\mathrm{Gal}(\bar{F}/F), \Gamma_T$ for $\{\sigma_T : \sigma \in \Gamma\}$, for $\{\sigma = \sigma_{\mathbf{T}} : \sigma \in \Gamma\}$ and so on. In [**I**, Section 2] we identified Γ_T as a subgroup of $\Omega(G, \mathbf{T})\rtimes$. Now we find it convenient to work with

$\Omega(G,T) \rtimes \Gamma_T$ and to identify $\Gamma_{\bar{T}}$ as a subgroup of $\Omega_\epsilon \rtimes \Gamma_T$, where $\Omega_\epsilon = \Omega(G_\epsilon, T)$. Thus for $\sigma \in \Gamma$ we may write $\sigma_{\bar{T}}$ as $\omega \times \sigma_T$, where $\omega \in \Omega_\epsilon$. We shall often write $\bar{\sigma}$ for $\sigma_{\bar{T}}$ and σ for σ_T, so that $\bar{\sigma} = \omega \times \sigma$.

3.3. First Lemma of Comparison

The term $\Delta_I(\gamma_h)$ was defined in [**I**, 3.2] using a passage to G_{sc}. We can just as well take $\lambda = \lambda(T)$, which coincides with the image of $\lambda(T_{\mathrm{sc}})$ in $H^1(T)$, and pair this with the element $s_T \in \pi_0(\widehat{T}^\Gamma)$ defined by s. Then $\Theta_I(\gamma_H) = \langle \bar{\mu}, s_{\bar{T}} \rangle$ where $\mu = \lambda\lambda_\epsilon^{-1}$. Here again we use the subscript ϵ to indicate an object attached to G_ϵ. Similarly, $\Theta_I(\bar{\gamma}_H) = \langle \bar{\mu}, s_{\bar{T}} \rangle$, where $\bar{\mu} = \bar{\lambda}\bar{\lambda}_\epsilon^{-1}$ and the pairing is now that for $H^1(\bar{T})$ and $\pi_0(\widehat{\bar{T}}^\Gamma)$.

Let Int $h : \mathbf{T} \longrightarrow T$, Int $h_\epsilon : \mathbf{T}_\epsilon \longrightarrow T$ be the isomorphisms provided by $\mathbf{B}, B$ and $\mathbf{B}_\epsilon, B_\epsilon$. We now write $n(\sigma_T)$ for the element $n(\omega_T(\sigma))$ of [**I**, 2.3]. Then λ is represented by $hx(\sigma_T)n(\sigma_T)\sigma(g)^{-1}$ and λ_ϵ by $h_\epsilon x_\epsilon(\sigma_T)n_\epsilon(\sigma_T)\sigma(h_\epsilon)^{-1}$. Thus μ is represented by

$$hx(\sigma_T)n(\sigma_T)\sigma(h)^{-1}\sigma(h_\epsilon)n_\epsilon(\sigma_T)^{-1}x_\epsilon(\sigma_T)^{-1}h_\epsilon^{-1}.$$

There is a similar formula for $\bar{\mu}$.

Set

$$y(\sigma_T) = hx(\sigma_T)h^{-1} \cdot h_\epsilon x_\epsilon(\sigma_T)^{-1}h_\epsilon^{-1} = \prod_{\substack{a>0 \\ \sigma_T^{-1}a<0}} a_\alpha^{\alpha^\vee},$$

where the product is over $\alpha \in R(G/G_\epsilon)$.

On the other hand, for $\theta \in \Omega_\epsilon \rtimes \Gamma_T$ we have $n(\theta) \in \mathrm{Norm}(G,T)$ on regarding $\Omega_\epsilon \rtimes \Gamma_T$ as a subgroup of $\Omega \rtimes$ [**I**, 2.1] and $n_\epsilon(\theta) \in \mathrm{Norm}(G_\epsilon, T_\epsilon)$ if we regard $\Omega_\epsilon \rtimes \Gamma_T$ as a subgroup of $\Omega_\epsilon \rtimes_\epsilon$. Then

$$n(\theta_1)n(\theta_2) = t(\theta_1,\theta_2)n(\theta_1\theta_2)$$

$$n_\epsilon(\theta_1)n_\epsilon(\theta_2) = t_\epsilon(\theta_1,\theta_2)n_\epsilon(\theta_1\theta_2)$$

and if

$$\tau(\theta_1,\theta_2) = ht(\theta_1,\theta_2)h^{-1}h_\epsilon t_\epsilon(\theta_1,\theta_2)^{-1}h_\epsilon^{-1}$$

then

$$\tau(\theta_1,\theta_2) = \prod_{\substack{a>0 \\ \theta_1^{-1}a<0 \\ \theta_2^{-1}\theta_1^{-1}a>0}} (-1)^{\alpha^\vee},$$

with the product again over $\alpha \in R(G/G_\epsilon)$ (see [**I**, Lemma 2.1.A]).

The restriction of τ to Γ_T is also the coboundary of y [**I**, 2.2.a]. Since $\tau = \tau^{-1}$ we can write the more convenient:

$$\partial y = \tau^{-1} \tag{3.3.1}$$

(see also [**I**, 2.3]).

For $\omega \in \Omega_\epsilon$ we define $b(\omega) \in T$ by

$$hn(\omega)h^{-1} = b(\omega)h_\epsilon n_\epsilon(\omega)h_\epsilon^{-1}.$$

Finally, suppose $\mu(\sigma)$ represents the cohomology class μ. Then for $\omega \times \sigma \in \Omega_\epsilon \rtimes \Gamma_T$ we set

$$z(\omega \times \sigma) = b(\omega)\omega(y(\sigma)^{-1}\mu(\sigma))\tau(\omega, \sigma)^{-1}.$$

Note that $\omega = 1$ yields

$$\mu(\sigma) = y(\sigma)z(\sigma), \quad \sigma \in \Gamma_T.$$

Lemma 3.3.A. (First Lemma of Comparison)

$$\bar{\mu}(\bar{\sigma}) = \bar{y}(\bar{\sigma})z(\bar{\sigma}), \quad \bar{\sigma} \in \Gamma_{\bar{T}}.$$

Proof. Write $\bar{\sigma}$ as $\omega \times \sigma$. Then we calculate $\bar{\mu}(\bar{\sigma})$ as

$$\begin{aligned}
h\bar{x}(\bar{\sigma})n(\omega\sigma)\sigma(h^{-1}h_\epsilon)n_\epsilon(\omega\sigma)^{-1}\bar{x}_\epsilon(\bar{\sigma})^{-1}h_\epsilon^{-1}& \\
&= \bar{y}(\bar{\sigma})\tau(\omega, \sigma)^{-1}hn(\omega)n(\sigma)\sigma(h^{-1}h_\epsilon)n_\epsilon(\sigma)^{-1}n_\epsilon(\omega)^{-1}h_\epsilon^{-1} \\
&= \bar{y}(\bar{\sigma})\tau(\omega, \sigma)^{-1}hn(\omega)h^{-1}y(\sigma)^{-1}\mu(\sigma)h_\epsilon n_\epsilon(\omega)^{-1}h_\epsilon^{-1} \\
&= \bar{y}(\bar{\sigma})\tau(\omega, \sigma)^{-1}\omega(y(\sigma)^{-1}\mu(\sigma))b(\omega) \\
&= \bar{y}(\bar{\sigma})z(\bar{\sigma}),
\end{aligned}$$

and the lemma is proved.

Most of the time we will assume a condition that is satisfied, for example, if T_H is maximally split in H_{ϵ_H}. It is:

$$\begin{aligned}
&\Gamma_{\bar{T}} \subseteq \Omega_0 \rtimes \Gamma_G, \text{ where } \Omega_0 \text{ is the Weyl group} \\
&\text{for a root system } R_0 \subseteq R(H_{\epsilon_H}) \\
&\text{that has a base } \Sigma_0 \text{ stable under } \Gamma_T.
\end{aligned} \tag{3.3.2}$$

The assumption will be harmless because of the transitivity property in Lemma 4.1.A of [**I**].

Lemma 3.3.B. *Given (3.3.2) we may choose the cocycle $\mu(\sigma)$ representing μ such that*

$$\omega(\mu(\sigma)) = \mu(\sigma), \quad \omega \in \Omega_0, \quad \sigma \in \Gamma_T.$$

Proof. Multiplying h or h_ϵ by an appropriate element of T replaces $\mu(\sigma)$ by an arbitrary element in its class. So we need only verify that the class of μ lies in the image of $H^1(T^{\Omega_0}) \longrightarrow H^1(T)$, where T^{Ω_0} is the centralizer of Ω_0 in T. If $\Sigma_0 = \{\alpha_1, \cdots, \alpha_r\}$ then $t \longrightarrow (\alpha_1(t), \cdots, \alpha_r(t))$ yields an exact sequence $1 \longrightarrow T^{\Omega_0} \longrightarrow T \longrightarrow S \longrightarrow 1$ with S an induced torus. Since $H^1(S) = 1$, the lemma follows.

We suppose now that $\mu(\sigma)$ is fixed by Ω_0, inflate it to $\Omega_0 \rtimes \Gamma_T$ and then restrict to $\Gamma_{\bar{T}}$. Suppose the cocycle so obtained is ν. Then

$$\nu(\bar{\sigma}) = \mu(\sigma), \quad \sigma \in \Gamma.$$

Lemma 3.3.C. *Given (3.3.2) we have*

$$\langle \mu, s_T \rangle = \langle \nu, s_{\bar{T}} \rangle.$$

Proof. We identify $\bar{T}$ as T with Galois action $\Gamma_{\bar{T}}$ and thus $T \times \bar{T}$ as $T \times T$ with action $\Gamma_{T\times\bar{T}}$. The element $s_{\bar{T}}$ of $\pi_0(\widehat{\bar{T}}^\Gamma)$ is then identified with s_T and so we drop subscripts.

Working in $T \times \bar{T}$ we have

$$\langle \mu, s_T \rangle / \langle \nu, s_{\bar{T}} \rangle = \langle (\mu, \nu), (s, s^{-1}) \rangle.$$

If we define the F-torus A by

$$X_*(A) = \{(\lambda_1, \lambda_2) \in X_*(T \times \bar{T}) : \lambda_1 - \lambda_2 \in \langle \Sigma_0^\vee \rangle\},$$

with Galois action induced by $\Gamma_{T\times\bar{T}}$ then we have Galois homomorphisms

$$A \longrightarrow T \times \bar{T}, \quad \widehat{T} \times \widehat{\bar{T}} \longrightarrow \widehat{A}.$$

Since the image of (s, s^{-1}) under the map induced by $\widehat{T} \times \widehat{\bar{T}} \longrightarrow \widehat{A}$ is trivial we have only to show that (μ, ν) lies in the image of $H^1(A) \longrightarrow H^1(T \times \widehat{T})$. The diagonal embedding of T in $T \times \bar{T}$ factors through A but the induced homomorphism $T \longrightarrow A$ is not defined over F. However, by (3.3.2), if we restrict to T^{Ω_0} then we do get a map over F. Since μ takes values in T^{Ω_0} and has image (μ, ν) under $H^1(T^{\Omega_0}) \longrightarrow H^1(A) \longrightarrow H^1(T \times \bar{T})$ the lemma is proved.

Finally, continuing to assume (3.3.2), we set

$$v(\omega \times \sigma) = \tau(\omega, \sigma)\omega(y(\sigma))b(\omega)^{-1}\bar{y}(\bar{\sigma})^{-1}.$$

Then v is a 1-cocycle of $\Gamma_{\bar{T}}$ in $\bar{T} = T$. By the Lemma of Comparison it coincides with $\nu\bar{\mu}^{-1}$ and we conclude:

Lemma 3.3.D. *Under the condition (3.3.2) we have*

$$\Theta_I(\gamma_H)/\Theta_I(\bar{\gamma}_H) = \langle v, s_{\bar{T}} \rangle.$$

3.4. The Second Lemma of Comparison

The term $\Delta_2(\gamma_H)$ was defined in [**I**, 3.5] using the construction of [**I**, 2.6]. For once and for all, we assume $\mathcal{H}$ is an L-group, as we may without loss of generality. The choices of 3.2 provide $\widehat{T} \longrightarrow \mathcal{T}, \widehat{T}_H \longrightarrow \mathcal{T}$. As in [**I**, 2.6] we extend these to ${}^LT \hookrightarrow {}^LG, {}^LT_H \hookrightarrow {}^LH$ by defining

$$m(w) = r(w)n(\sigma) \times w,$$
$$m_s(w) = r_s(w)n_s(\sigma) \times w.$$

Here $w \longrightarrow \sigma$ under $W \longrightarrow \Gamma$, r and $n(\sigma)$ denote r_p and $n(w_T(\sigma))$ of [**I**, 2.6], and we indicate objects attached to H by the subscript s. For $\bar{T}, \bar{T}_H$ we have similarly $\bar{m}$ and $\bar{m}_s$. Then if $\xi : {}^LH \hookrightarrow {}^LG$ is the embedding provided by our endoscopic data we have

$$\xi(m_s(w)) = a(w)m(w)$$
$$\xi(\bar{m}_s(w)) = \bar{a}(w)\bar{m}(w).$$

On transporting $a, \bar{a}$ to $\widehat{T},\widehat{\bar{T}}$ we obtain $\mathbf{a} \in H^1(W, \widehat{T})$ and $\bar{\mathbf{a}} \in H^1(W,\widehat{\bar{T}})$. By definition,

$$\begin{aligned}\Theta_2(\gamma_H, \bar{\gamma}_H) &= \Delta_2(\gamma_H)/\Delta_2(\bar{\gamma}_H) \\ &= \langle \mathbf{a}, \gamma \rangle \langle \bar{\mathbf{a}}, \bar{\gamma} \rangle^{-1}.\end{aligned}$$

The definition of the cochain r appears in [**I**, 2.5]. Set

$$c(w) = r(w)r_s(w)^{-1}, \quad \bar{c}(w) = \bar{r}(w)\bar{r}_s(w)^{-1}.$$

The coboundaries of the cochains n, n_s defined on $\Omega(H) \rtimes \Gamma_T$ are denoted t, t_s (see [**I**, 2.1]). Set $\widehat{\tau} = tt_s^{-1}$. Finally for $\omega \in \Omega(H)$ we define $\hat{b}(\omega) \in \mathcal{T}$ by

$$n(\omega) = \hat{b}(\omega)n_s(\omega), \quad \omega \in \Omega(H).$$

We will freely transport objects among $\mathcal{T}, \widehat{T}$ and $\widehat{\bar{T}}$ without change in notation.

Lemma 3.4.A. (Second Lemma of Comparison). *Let $w \in W$ map to $\sigma \in \Gamma_T$ and to $\omega \times \sigma \in \Gamma_{\bar{T}}$. Then*

$$\bar{a}(w)^{-1} = \bar{c}(w)\hat{b}(\omega)\omega(c(w)^{-1}a(w)^{-1})\hat{\tau}(\omega,\sigma)^{-1}.$$

Proof.

$$\begin{aligned}\bar{m}(w) &= \bar{r}(w)n(\omega\sigma) \times w \\ &= \bar{r}(w)t(\omega,\sigma)^{-1}n(\omega)n(\sigma) \times w \\ &= \bar{r}(w)t(\omega,\sigma)^{-1}\hat{b}(\omega)n_s(\omega)r(w)^{-1}m(w) \\ &= \bar{r}(w)t(\omega,\sigma)^{-1}\hat{b}(\omega)\omega(r(w)^{-1}a(w)^{-1})n_s(\omega)\xi(m_s(w))\end{aligned}$$

and

$$\begin{aligned}\xi(\bar{m}_s(w)) &= \bar{r}_s(w)t_s(\omega,\sigma)^{-1}n_s(\omega)n_s(\sigma) \\ &= \bar{r}_s(w)t_s(\omega,\sigma)^{-1}\omega(r_s(w))n_s(\omega)\xi(m_s(w))\end{aligned}$$

so that

$$\bar{a}(w)^{-1} = \bar{c}(w)\hat{\tau}(\omega,\sigma)^{-1}\hat{b}(\omega)\omega(c(w)^{-1}a(w)^{-1}),$$

and the lemma is proved.

Let

$$z_1(\omega,w) = \hat{b}(\omega)\omega(c(w)^{-1}a(w)^{-1})\hat{\tau}(\omega,\sigma)^{-1}.$$

Then

$$a(w)^{-1} = c(w)z_1(1,w),$$

and

$$\bar{a}(w)^{-1} = \bar{c}(w)z_1(\omega,w).$$

Note the similarity to the First Lemma. The Second Lemma will be applied a little differently however. We shall give an example of the technique in the next section.

Note that $t \longrightarrow \omega(t)t^{-1}$ lifts to a homomorphism $\alpha_\omega : \widehat{T} \longrightarrow \widehat{T}_{\mathrm{sc}}$. Here, as elsewhere, $\widehat{T}_{\mathrm{sc}}$ denotes the inverse image of $\widehat{T}$ in the simply-connected covering $\widehat{G}_{\mathrm{sc}}$ of the derived group of $\widehat{G}$. Thus if we multiply $z_1(\omega,w)$ by $a(w)$ we obtain

$$\hat{z}(\omega,w) = \hat{b}(\omega)\omega(c(w)^{-1})\hat{\tau}(\omega,\sigma)^{-1}\omega(a(w))^{-1}a(w)$$

in the image of $\widehat{T}_{sc}$ in $\widehat{T}$. More precisely, $\hat{b}, c, \hat{\tau}$ may be constructed in $\widehat{G}_{\mathrm{sc}}$. Then

$$\hat{z}_{\mathrm{sc}}(\omega,w) = \hat{b}(\omega)\omega(c(w)^{-1})\hat{\tau}(\omega,\sigma)^{-1}\alpha_\omega(a(w)^{-1})$$

has image $\hat{z}(\omega, w)$ under $\widehat{G}_{\mathrm{sc}} \longrightarrow \widehat{G}$.

To calculate Θ_2 we also need Δ_2^ϵ which is attached to $(G_\epsilon, H_{\epsilon_H})$. For this we pass, if needed, to an admissible extension $\tilde{H}_{\epsilon_H}$ of H_{ϵ_H} (recall 1.3). We then have exact sequences

$$1 \longrightarrow \widehat{T} \longrightarrow \widehat{\tilde{H}} \longrightarrow \widehat{Z}_1 \longrightarrow 1,$$

and

$$1 \longrightarrow \widehat{\bar{T}} \longrightarrow \widehat{\tilde{\bar{T}}} \longrightarrow \widehat{Z}_1 \longrightarrow 1.$$

In place of $a_\epsilon, \bar{a}_\epsilon$ we have $\tilde{a}_\epsilon, \tilde{\bar{a}}_\epsilon$ that take values in $\widehat{\tilde{T}}, \widehat{\tilde{\bar{T}}}$. These two cocycles have the same projection on $\widehat{Z}_1$ [**I**, 4.4]. The Second Lemma of Comparison becomes

$$\tilde{\bar{a}}_\epsilon(w)^{-1} = \bar{c}_\epsilon(w) z_{1,\epsilon}(\omega, w)$$

and $\tilde{a}_\epsilon(w) z_{1,\epsilon}(\omega, w)$ is equal to

$$\hat{b}_\epsilon(\omega)\omega(c_\epsilon(w))^{-1}\hat{\tau}_\epsilon(\omega,\sigma)^{-1}\omega(\tilde{a}_\epsilon(w)^{-1})\tilde{a}_\epsilon(w)$$

which takes values in $\widehat{T}$ (or $\widehat{T}_{\mathrm{sc}}$ or $\widehat{T}_{\epsilon-\mathrm{sc}}$). The terms $\hat{b}_\epsilon, c_\epsilon, \hat{\tau}_\epsilon$ are $\hat{b}, c, \hat{\tau}$ for the group G_ϵ.

3.5. An application

The following was stated in [**I**] as Lemma 4.4.A but not proved in general.

Lemma 3.5.A. *There is a character λ on the center $Z(F)$ of $G(F)$ such that*

$$\Delta(z\gamma_H, z\gamma_G) = \lambda(z)\Delta(\gamma_H, \gamma_G), \quad z \in Z(F)$$

for all γ_H, γ_G.

This applies to arbitrary G but we reduce immediately to the quasisplit case for it is only Δ_2 that is affected by replacing γ_H, γ_G by $z\gamma_H, z\gamma_G$.

Proof All we need to show is that

$$\langle(\mathbf{a}, \bar{\mathbf{a}}), (z, z^{-1})\rangle = 1.$$

Define a torus S over F by

$$X^*(S) = \{(\lambda_1, \lambda_2) \in X^*(T) \times X^*(\bar{T}) : \lambda_1 - \lambda_2 \in X^*(T_{\mathrm{ad}})\}.$$

Here again we identify $\bar{T}$ with T over $\bar{F}$ and thus $X^*(\bar{T})$ with $X^*(T)$. We have Galois homomorphisms $T \times \bar{T} \longrightarrow S$ and $\widehat{S} \longrightarrow \widehat{T} \times \widehat{T}$. Since (z, z^{-1}) lies in the kernel of $T(F) \times \bar{T}(F) \longrightarrow S(F)$ it is enough to show that $(\mathbf{a}, \bar{\mathbf{a}})$ lies in the image of $H^1(W, \widehat{S}) \longrightarrow H^1(W, \widehat{T} \times \widehat{\bar{T}})$.

The torus $\widehat{S}$ is isomorphic to $\widehat{T} \times \widehat{\bar{T}}_{\mathrm{sc}}$. We obtain the factor $\widehat{T}$ by factoring the diagonal embedding $\widehat{T} \longrightarrow \widehat{T} \times \widehat{T} = \widehat{T} \times \widehat{\bar{T}}$ through $\widehat{S}$; so clearly $\widehat{T} \longrightarrow \widehat{S}$ is not compatible with the Galois action. On the other hand $\widehat{\bar{T}}_{\mathrm{sc}} \longrightarrow \widehat{S}$ is obtained from $X^*(\bar{T}_{\mathrm{ad}}) \longrightarrow \{0\} \times X^*(\bar{T}_{\mathrm{ad}}) \subset X^*(S)$, and so does respect Galois action.

By the Second Lemma of Comparison the cocycle $w \longrightarrow (a(w), \bar{a}(w))$ with values in $\widehat{T} \times \widehat{\bar{T}}$ is the image of the cochain

$$w \longrightarrow (a(w), \bar{c}(w)^{-1} \hat{z}_{\mathrm{sc}}(\omega, w)^{-1})$$

with values in $\widehat{S} = \widehat{T} \times \widehat{\bar{T}}_{\mathrm{sc}}$. We have to show that this cochain is a cocycle. It may be written as

$$(a(w), 1)(1, \bar{c}(w)^{-1})(1, \hat{z}_{\mathrm{sc}}(\omega, w)^{-1}).$$

The coboundary of the first term is

$$(w_1, w_2) \longrightarrow (1, \alpha_{\omega_1}(\sigma_1(a(w_2))))$$

if $w_i \longrightarrow \omega_i \times \sigma_i \in \Gamma_{\bar{T}}$; the coboundary of the second is $(1, \bar{\tau}(w_1, w_2))$. Note that $\bar{\tau}(w_1, w_2) = \bar{\tau}(\sigma_1, \sigma_2) = \hat{\tau}(\omega_1 \times \sigma_1, \omega_2 \times \sigma_2)$. Thus it remains to show that the coboundary of $\hat{z}_{\mathrm{sc}}$ is

$$(w_1, w_2) \longrightarrow \hat{\tau}(\omega_1 \times \sigma_1, \omega_2 \times \sigma_2)\alpha_{\omega_1}(\sigma_1(a(w_2))).$$

This is Lemma 4.2.A.

§4. Analysis of $b, \hat{b}$ and a reduction

4.1. Galois action

By definition,

$$b(\omega) = hn(\omega)h^{-1}h_\epsilon n_\epsilon(\omega)^{-1}h_\epsilon^{-1}, \quad \omega \in \Omega_\epsilon,$$

and in the dual setting,

$$\hat{b}(\omega) = n(\omega)n_s(\omega)^{-1}, \quad \omega \in \Omega(H) = \Omega_2.$$

We have immediately,

$$\begin{aligned} b(\omega_1)\omega_1(b(\omega_2))b(\omega_1\omega_2)^{-1} &= \tau(\omega_1, \omega_2) \\ \hat{b}(\omega_1)\omega_1(\hat{b}(\omega_2))\hat{b}(\omega_1\omega_2)^{-1} &= \hat{\tau}(\omega_1, \omega_2). \end{aligned} \tag{4.1.1}$$

Recall that these objects may be constructed in $G_{\rm sc}$ and $\widehat{G}_{\rm sc}$, the simply-connected forms for G and $\widehat{G}$. For G_ϵ and $\widehat{H} = \widehat{G}_s$ we may work in the simply-connected forms and project into $G_{\rm sc}$ or $\widehat{G}_{\rm sc}$. This will be the rule throughout, although in notation we may identify an element with its image in G or $\widehat{G}$.

To describe the effect of Γ_T on b and $\hat{b}$ we consider the two cases together but keep in mind that in the former we have a genuine Galois action and in the latter an algebraic action.

First define

$$f(\omega,\sigma) = ht(\sigma,\omega)t(\sigma\omega,\sigma^{-1})\sigma(\omega)(t(\sigma,\sigma^{-1})^{-1})h^{-1}$$

for $\omega \times \sigma \in \Omega_\epsilon \rtimes \Gamma_T$ and similarly $\hat{f}(\omega,\sigma)$ for $\omega \times \sigma \in \Omega_s \rtimes \Gamma_T$. Then we see easily that

$$h\mathbf{n}(\sigma)n(\omega)\mathbf{n}(\sigma)^{-1}h^{-1} = f(\omega,\sigma)hn(\sigma(\omega))h^{-1}$$

where $\mathbf{n}(\sigma) = n(\sigma) \times \sigma \in G_{\rm sc} \rtimes \Gamma$ (recall from 3.3 that we have changed slightly the notation of **[I]**). In the dual case we have

$$(n(\sigma) \times w)n(\omega)(n(\sigma) \times w)^{-1} = \hat{f}(\omega,\sigma)n(\sigma(\omega))$$

where $w \longrightarrow \sigma$ under $W \longrightarrow \Gamma$. Set $e = ff_\epsilon^{-1}$ and $\hat{e} = \hat{f}\hat{f}_s^{-1}$. We assume (3.3.2) for the first part of the next lemma.

Lemma 4.1.A.

$$(a) \qquad e(\omega,\sigma)b(\sigma(\omega)) = \sigma(\omega)(y(\sigma))y(\sigma)^{-1}\sigma(b(\omega)) \ \textit{for}\ \omega \times \sigma \in \Omega_\epsilon \rtimes \Gamma_T,$$

and

$$(b) \qquad \hat{e}(\omega,\sigma)\hat{b}(\sigma(\omega)) = \alpha_{\sigma(\omega)}(c(w)a(w))\sigma(\hat{b}(\omega)) \ \textit{for}\ w \longrightarrow \sigma \ \textit{and}\ \omega \times \sigma \in \Omega_s \rtimes \Gamma_T.$$

Proof. This is a straightforward calculation. For (a) we suppress h, h_ϵ from the equation in order to make the calculation more transparent. Then $n(\sigma(\omega)) = b(\sigma(\omega))n_\epsilon(\sigma(\omega))$ implies

$$\mathbf{n}(\sigma)n(\omega)\mathbf{n}(\sigma)^{-1} = e(\omega,\sigma)b(\sigma(\omega))\mathbf{n}_\epsilon(\sigma)n_\epsilon(\omega)\mathbf{n}_\epsilon(\sigma)^{-1}$$

and so it is enough to show that

$$\mathbf{n}(\sigma)n(\omega)\mathbf{n}(\sigma)^{-1} = \sigma(\omega)(y(\sigma))y(\sigma)^{-1}\sigma(b(\omega))\mathbf{n}_\epsilon(\sigma)n_\epsilon(\omega)\mathbf{n}_\epsilon(\sigma)^{-1}.$$

But

$$\begin{aligned}\mathbf{n}(\sigma)n(\omega)\mathbf{n}(\sigma)^{-1} &= \mathbf{n}(\sigma)b(\omega)n_\epsilon(\omega)\mathbf{n}(\sigma)^{-1} \\ &= \sigma(b(\omega))\mathbf{n}(\sigma)n_\epsilon(\omega)\mathbf{n}(\sigma)^{-1},\end{aligned}$$

and all we need is that

$$\mathbf{n}(\sigma)n_\epsilon(\omega)\mathbf{n}(\sigma)^{-1} = \sigma(\omega)(y(\sigma))y(\sigma)^{-1}\mathbf{n}_\epsilon(\sigma)n_\epsilon(\omega)\mathbf{n}_\epsilon(\sigma)^{-1}.$$

Suppressing h, h_ϵ we deduce from the equation

$$hx(\sigma)n(\sigma)\sigma(h)^{-1} = \mu(\sigma)h_\epsilon x_\epsilon(\sigma)n_\epsilon(\sigma)\sigma(h_\epsilon)^{-1}$$

that $\mathbf{n}(\sigma)$ acts on $n_\epsilon(\omega)$ as

$$\mu(\sigma)x_\epsilon(\sigma)x(\sigma)^{-1}\mathbf{n}_\epsilon(\sigma) = \mu(\sigma)y(\sigma)^{-1}\mathbf{n}_\epsilon(\sigma).$$

Since $\sigma(\omega)$ fixes $\mu(\sigma)$ we are done.

(b) is similar. Because $\xi : {}^LH \hookrightarrow {}^LG$ is the identity on $\widehat{H} = \widehat{G}_s$, we may write $(n(\sigma) \times w)n(\omega)(n(\sigma) \times w)^{-1}$ as

$$\hat{e}(\omega, \sigma)\hat{b}(\sigma(\omega))\xi(n_s(\sigma) \times w)n_s(\omega)\xi(n_s(\sigma) \times w)^{-1}.$$

Then all we need is

$$(n(\sigma) \times w)n_s(\omega)(n(\sigma) \times w)^{-1} = \alpha_{\sigma(\omega)}(a(w)c(w))\xi(n_s(\sigma) \times w)n_s(\omega)\xi(n_s(\sigma) \times w)^{-1}.$$

But $r(w)n(\sigma) \times w$, an element of ${}^LG$, acts on $n_s(\omega)$ as

$$a(w)^{-1}r_s(w)\xi(n_s(\sigma) \times w).$$

Thus $n(\sigma) \times w$ acts as $a(w)^{-1}c(w)^{-1}\xi(n_s(\sigma) \times w)$ and (b) follows.

4.2. Calculation of a coboundary

Recall the cochain $\hat{z}$ on $\Omega_s \rtimes W_T$:

$$\hat{z}(\omega, w) = \hat{b}(\omega)\omega(c(w)^{-1})\hat{\tau}(\omega, \sigma)^{-1}\alpha_\omega(a(w)^{-1}).$$

Lemma 4.2.A. *The coboundary of $\hat{z}$ is*

$$\hat{\tau}_a : (\omega_1 w_1; \omega_2 w_2) \longrightarrow \hat{\tau}(\omega_1\sigma_1, \omega_2\sigma_2)\alpha_{\omega_1}(\sigma_1(a(w_2))).$$

Proof. Because $\hat{\tau}_a$ is a 2-cocycle it defines an extension of $\Omega_s \rtimes W_T$ by $\widehat{T}_{\mathrm{sc}}$. This extension is generated by $\widehat{T}_{\mathrm{sc}}$ and elements $s(\eta), \eta \in \Omega_s \rtimes W_T$, with

$$s(\eta)ts(\eta)^{-1} = \eta(t)$$

and

$$s(\eta_1)s(\eta_2) = \hat{\tau}_a(\eta_1, \eta_2)s(\eta_1\eta_2).$$

We prove the lemma by showing that

$$\eta \longrightarrow \hat{z}(\eta)^{-1}s(\eta) = s_1(\eta)$$

splits the extension.

Let $w \longrightarrow \sigma \in \Gamma_T$. Then

$$\begin{aligned} s_1(\omega w) &= \hat{b}(\omega)^{-1}\omega(c(w))\hat{\tau}(\omega,\sigma)\alpha_\omega(a(w))s(\omega w) \\ &= \hat{b}(\omega)^{-1}\omega(c(w))\hat{\tau}_a(\omega, w)s(\omega w) \\ &= \hat{b}(\omega)^{-1}s(\omega)c(w)s(w) \\ &= s_1(\omega)s_1(w). \end{aligned}$$

Moreover,

$$\begin{aligned} s_1(\omega_1\omega_2) &= \hat{b}(\omega_1\omega_2)^{-1}s(\omega_1\omega_2) \\ &= \omega_1(\hat{b})\omega_2))^{-1}\hat{b}(\omega_1)^{-1}\hat{\tau}(\omega_1,\omega_2)\hat{\tau}_a(\omega_1,\omega_2)^{-1}s(\omega_1)s(\omega_2) \\ &= \omega_1(\hat{b}(\omega_2))^{-1}\hat{b}(\omega_1)^{-1}s(\omega_1)s(\omega_2) \\ &= s_1(\omega_1)s_1(\omega_2). \end{aligned}$$

Finally,

$$\begin{aligned} s_1(w_1w_2) &= c(w_1w_2)^{-1}s(w_1w_2) \\ &= c(w_1)^{-1}w_1(c(w_2))^{-1}\hat{\tau}(w_1,w_2)\hat{\tau}_a(w_1,w_2)^{-1}s(w_1)s(w_2) \\ &= s_1(w_1)s_1(w_2). \end{aligned}$$

To show that

$$s_1(\omega_1 w_1)s_1(\omega_2 w_2) = s_1(\omega_1\sigma_1(\omega_2)w_1w_2)$$

and thus complete the proof of the lemma we need only verify that

$$s_1(\sigma(\omega)) = s_1(w)s_1(\omega)s_1(w)^{-1}. \tag{4.2.1}$$

The right side is

$$\hat{z}(w)^{-1}\sigma(\hat{z}(\omega))^{-1}\sigma(\omega)(\hat{z}(w))s(w)s(\omega)s(w)^{-1}$$

which equals

$$\sigma(\omega)(c(w)^{-1})c(w)\sigma(\hat{b}(\omega))^{-1}s(w)s(\omega)s(w)^{-1}$$

and $s(w)s(\omega)s(w)^{-1}$ is $s(\sigma(\omega))$ times

$$\hat{\tau}_a(w,\omega)\hat{\tau}_a(w\omega,\omega^{-1})\sigma(\omega)(\hat{\tau}_a(w,w^{-1})^{-1}).$$

It follows readily from the definitions that this last product is equal to

$$\hat{e}(\omega,\sigma)\alpha_{\sigma(\omega)}(\sigma(a(w^{-1})))$$

or, since $a(w)$ is a cocycle, to

$$\hat{e}(\omega,\sigma)\alpha_{\sigma(\omega)}(a(w)^{-1}).$$

Thus the right side of (4.2.1) equals

$$\sigma(\hat{b}(\omega))^{-1}\alpha_{\sigma(\omega)}(c(w)a(w))^{-1}\hat{e}(\omega,\sigma)s(\sigma(\omega)).$$

Since the left side is $\hat{b}(\sigma(\omega))^{-1}s(\sigma(\omega))$ we need only appeal to part (b) of Lemma 4.1.A to finish the proof.

Recall the definition of the cochain z on $\Omega_\epsilon \rtimes \Gamma_T$:

$$z(\omega,\sigma) = b(\omega)\omega(y(\sigma)^{-1})\tau(\omega,\sigma)^{-1}.$$

A similar, but simpler, argument establishes the next lemma.

Lemma 4.2.B. *z has coboundary τ.*

4.3. Explicit form

In the setting of [**I**, 2.1] suppose that β is a positive root. Let $\beta = \nu\beta_0$ where $\nu \in \Omega$ and β_0 is simple. Write ω_0 for ω_{β_0}.

Lemma 4.3.A.

$$n(\omega_\beta) = \delta_\nu(\beta)n(\nu)n(\omega_0)n(\nu)^{-1}$$

where

$$\delta_\nu(\beta) = \prod(-1)^{\alpha^\vee}$$

and the product is over roots α for which

$$\alpha > 0,\ \ \omega_\beta\alpha < 0 \ \text{ and } \ \nu^{-1}\omega_\beta\alpha > 0.$$

Proof. $n(\omega_\beta) = n(\nu\omega_0\nu^{-1})$, and this is the product of

$$t(\nu\omega_0, \nu^{-1})^{-1}t(\nu, \omega_0)^{-1}\omega_\beta(t(\nu, \nu^{-1}))$$

and $n(\nu)n(\omega_0)n(\nu)^{-1}$. Moreover $t(\nu, \omega_0) = 1$ since $\alpha > 0, \nu^{-1}\alpha < 0$ and $\omega_0^{-1}\nu^{-1}\alpha > 0$ implies that $\nu^{-1}\alpha = -\beta_0$ and hence that $\alpha = -\beta$, contradicting $\alpha, \beta > 0$.

It remains then to show that

$$\delta_\nu(\beta) = t(\nu\omega_0, \nu^{-1})^{-1}\omega_\beta(t(\nu, \nu^{-1})).$$

The right side is $\prod(-1)^{\alpha^\vee}$, the product being taken over those α for which $\alpha > 0, \nu^{-1}\omega_\beta\alpha < 0, \omega_\beta\alpha > 0$ and those for which $\omega_\beta\alpha > 0, \nu^{-1}\omega_\beta\alpha < 0$. This coincides with the product over $\alpha < 0, \nu^{-1}\omega_\beta\alpha < 0, \omega_\beta\alpha > 0$ and thus with $\delta_\nu(\beta)$; so the lemma is proved.

Let $R_\beta = \{\alpha \in R : \alpha > 0, \omega_\beta\alpha < 0, \alpha \neq \beta\}$. Then a root α lies in R_β if and only if $-\omega_\beta\alpha$ does and then the two elements are distinct. We can therefore choose a subset R_β^+ of R_β such that R_β is the disjoint union of R_β^+ and $-\omega_\beta(R_\beta^+)$. Then

$$\delta_G(\beta) = \prod_{\alpha \in R_\beta^+} (-1)^{\alpha^\vee}$$

is well determined up to a factor $(\pm 1)^{\beta^\vee}$ because $-\omega_\beta\alpha^\vee = -\alpha^\vee + \langle\alpha^\vee, \beta\rangle\beta^\vee$. Notice that we may take $R_\beta^+ = \{\alpha \in R_\beta : \nu^{-1}\omega_\beta\alpha > 0\}$ since $\nu^{-1}\omega_\beta(-\omega_\beta\alpha) = -\nu^{-1}\alpha$ and $\nu^{-1}\omega_\beta\alpha = \omega_{\beta_0}(\nu^{-1}\alpha)$ have opposite signs. Thus

$$n(\omega_\beta) = \delta_G(\beta)(\pm 1)^{\beta^\vee} n(\nu)n(\omega_0)n(\nu)^{-1}. \tag{4.3.1}$$

Clearly this also holds when β is negative.

We return to the setting of 4.1, working in G_{sc} and $\hat{G}_{\mathrm{sc}}$. Since (3.3.2) is in force, there exists $R_0 \subset R(H_{\epsilon_H})$ with Γ_T-stable base Σ_0. Then $\Gamma_{\bar{T}} \subseteq \Omega_0 \rtimes \Gamma_T$, Ω_0 being the Weyl group generated by Σ_0. We are interested in $b(\omega), \hat{b}(\omega)$ for $\omega \in \Omega_0$. By (4.1.1) we need only consider $b(\omega_\beta), \hat{b}(\omega_\beta)$ with $\beta \in \Sigma_0$. Set

$$\delta(\beta) = \delta_G(\beta)\delta_{G_\epsilon}(\beta)^{-1}$$

and

$$\hat{b}(\beta) = \delta_{\widehat{G}}(\beta^\vee)\delta_{\widehat{H}}(\beta^\vee)^{-1}.$$

Lemma 4.3.A

There exist $b_\beta \in \bar{F}^\times, \hat{b}_\beta \in \mathbf{C}^\times$ *such that*

$$b(\omega_\beta) = b_\beta^{\beta^\vee} \delta(\beta)$$

and

$$\hat{b}(\omega_\beta) = \hat{b}_\beta^{\beta} \hat{\delta}(\beta).$$

Proof. The first relation is immediate from (4.3.1) once we recall that $n(\omega_\beta) = b(\omega_\beta)n_\epsilon(\omega_\beta)$, and note that both $n = n(\nu)n(\omega_0)n(\nu)^{-1}$ and $n_\epsilon = n_\epsilon(\nu_\epsilon)n_\epsilon(\omega_{0,\epsilon})n_\epsilon(\nu_\epsilon)^{-1}$ lie in the image of $SL(2)$ in G determined by β, so that $n = \alpha^{\beta^\vee} n_\epsilon$, with $a \in \bar{F}^\times$. The second relation is proved in the same way.

Note that $\omega_\beta \longrightarrow b_\beta^{\beta^\vee}$ has a natural extension to a 1-coboundary b_ω of Ω_0 in T_{sc} as follows. Choose $t \in T_{\mathrm{sc}}$ such that $\beta(t) = b_\beta, \beta \in \Sigma_0$. Then clearly $\omega \longrightarrow t\omega(t)^{-1}$ has the property that $\omega_\beta \longrightarrow b_\beta^{\beta^\vee}$. It is independent of the choice of t, provided we take t in the image of $\bar{F}^\times \otimes \langle \Sigma_0^\vee \rangle$. Define $\delta(\omega), \omega \in \Omega_0$, by

$$b(\omega) = b_\omega \delta(\omega)$$

and the dual $\hat{\delta}(\omega)$ similarly. Note that $\partial\delta = \tau, \partial\hat{\delta} = \hat{\tau}$.

4.4. Root types

To prove Theorem 1.6.A we use $R(H)$ and $R(G_\epsilon)$ to partition the roots. The notation will be as follows.

type(a):$R(H) \cap R(G_\epsilon) = R(H_{\epsilon_H}) = R^{(a)}$

type(b):$R(H) - R(G_\epsilon) = R(H/H_{\epsilon_H}) = R^{(b)}$

type(c):$R(G_\epsilon) - R(H) = R(G_\epsilon/H_{\epsilon_H}) = R^{(c)}$

type(d):$R - (R(H) \cup R(G_\epsilon)) = R(G/H, G_\epsilon) = R^{(d)}$.

This also gives a classification of the Γ_T-orbits $\mathcal{O}$ and the $\Gamma_{\bar{T}}$-orbits $\bar{\mathcal{O}}$ in R. Observe that

$$\Theta_{II}(\gamma_H) = \prod_{\mathcal{O} \subseteq R^{(d)}} \chi_\alpha \left(\frac{\alpha(\gamma) - 1}{a_\alpha} \right).$$

with a similar formula for $\Theta_{II}(\bar{\gamma}_H)$.

In the next section we so arrange some choices that roots of type (b) or (c) contribute nothing to $\Theta_I(\gamma_H, \bar{\gamma}_H)$.

4.5. Analysis of Θ_I, Θ_2

Recall that if

$$v(\bar{\sigma}) = \tau(\omega, \sigma)\omega(y(\sigma))b(\omega)^{-1}\bar{y}(\bar{\sigma})^{-1},$$

where $\bar{\sigma} = \omega \times \sigma$, then we evaluate $\Theta_I(\gamma_H, \bar{\gamma}_H)$ by pairing the class of the cocycle v with $s_{\bar{T}}$. By definition, $\tau(\omega, \sigma) = \prod(-1)^{\alpha^\vee}$, where the product is over roots α for which $\alpha > 0, \omega^{-1}\alpha < 0, \sigma^{-1}\omega^{-1}\alpha > 0$ and $\alpha \in R(G/G_\epsilon)$. Such α are of type (b) or of type (d) and we consider the products $\tau^{(b)}, \tau^{(d)}$ over roots of each type to define $\tau = \tau^{(b)}\tau^{(d)}$. The same can be done with y and $\bar{y}$. Then to write v as $v^{(b)}v^{(d)}$ it remains to factor b as $b^{(b)}b^{(d)}$. To describe the contribution from roots of type (b) we recall that these are the roots of H outside H_{ϵ_H}. But b was attached to the pair (G, G_ϵ). Now we use the fact that $\Sigma_0 \subset R(H_{\epsilon_H})$ in the assumption (3.3.2) to observe that we can attach b^H to (H, H_{ϵ_H}) in the same manner. From (4.3) we have $b^H(\omega) = b^H_\omega \delta^H(\omega), \omega \in \Omega_0$, along with $b(\omega) = b_\omega \delta(\omega)$. Recall that there is some freedom of choice in δ^H, δ. For $\beta \in \Sigma_0$ we may arrange that $\delta^H(\omega_\beta)$ is the same as the contribution $\delta^{(b)}(\omega_\beta)$ to $\delta(\omega_\beta)$ from roots of type (b). Then we define $\delta^{(b)}(\omega) = \delta^H(\omega)$ for all $\omega \in \Omega_0$ and $\delta^{(d)} = \delta/\delta^{(b)}$. Note that $\partial\delta^{(b)} = \tau^H = \tau^{(b)}$ and $\partial\delta^{(d)} = \tau^{(d)}$. Set $b^{(b)}_\omega = b^H_\omega$ and $b^{(d)}_\omega = b_\omega/b^{(b)}_\omega$. Finally, $b^{(d)} = b^{(d)}_\omega \delta^{(d)}$ has coboundary $\tau^{(d)}$, while $b^{(b)} = b^{(b)}_\omega \delta^{(b)}$ coincides with b^H. This yields a factoring $v = v^{(b)}v^{(d)}$, and $v^{(b)}, v^{(d)}$ are cocycles. Thus

$$\begin{aligned} \langle v, s_{\bar{T}} \rangle &= \langle v^{(b)}, s_{\bar{T}} \rangle \langle v^{(d)}, s_{\bar{T}} \rangle \\ &= \langle v^{(d)}, s_{\bar{T}} \rangle \end{aligned}$$

because s central in $\mathcal{H}$ implies that

$$\langle v^{(b)}, s_{\bar{T}} \rangle = \langle v^H, s_{\bar{T}} \rangle = 1.$$

Observe that if there are no roots of type (d) then $v^{(d)}$ is trivial and $\Theta_I(\gamma_H, \bar{\gamma}_H) = 1$.

We shall use a similar but dual argument for Θ_2. Here we have roots of types (c) and (d) to deal with. The term $\Delta_2(\gamma_H)/\Delta_2(\bar{\gamma}_H)$ is obtained by pairing the cocycle

$$\begin{aligned} (a(w), \bar{a}(w)) &= (a(w), a(w)\bar{c}(w)^{-1}\hat{b}(\omega)^{-1}\hat{\tau}(\omega, \sigma)\omega(c(w))\alpha_\omega(a(w))) \\ &= (a(w), a(w)\bar{c}(w)^{-1}\hat{z}(\omega, w)^{-1}), \end{aligned}$$

of W in $\widehat{T} \times \widehat{\bar{T}}$ with the element $(\gamma, \bar{\gamma}^{-1}$ of $T(F) \times \bar{T}(F)$. We may thicken $\widehat{T}$ to $\widehat{\tilde{T}}$ as at the end of Section 3.4 in order to compare directly with $\Delta_{2,\epsilon}(\gamma_H)/\Delta_{2,\epsilon}(\bar{\gamma}_H)$. In notation we will not distinguish between T and $\tilde{T}$, or $\bar{T}$ and $\tilde{\bar{T}}$. Thus for $\Theta_2(\gamma_H)/\Theta_2(\bar{\gamma}_H)$ we have to pair the cocycle

$$(a(w), \bar{a}(w))(a_\epsilon(w), \bar{a}_\epsilon(w)^{-1}) = (a(w)a_\epsilon(w)^{-1}, \bar{a}(w)\bar{a}_\epsilon(w)^{-1})$$

with $(\gamma, \bar{\gamma}^{-1})$.

Now $\hat{b}$ is attached to (G,H). Similarly we have $\hat{b}_\epsilon$ attached to $(G_\epsilon, H_{\epsilon_H})$. Factor $\hat{b}(\omega)$ as $\hat{b}^{(c)}(\omega)\hat{b}^{(d)}(\omega) = \hat{b}_\epsilon(\omega)\hat{b}^{(d)}(\omega)$, as we did $b(\omega)$. The factoring of $\bar{c}, \hat{\tau}$ and c is immediate, again as before. Set $a^{(c)}(w) = a_\epsilon(w)$ and $a^{(d)}(w) = a(w)/a_\epsilon(w)$. Then clearly $\bar{a}(w)$ factors as $\bar{a}^{(c)}(w)\bar{a}^{(d)}(w)$ where $\bar{a}^{(c)}(w) = \bar{a}_\epsilon(w)$ and $\Theta_2(\gamma_H, \bar{\gamma}_H)$ equals

$$\langle (a^{(d)}(w), \bar{a}^{(d)}(w)), (\gamma, \bar{\gamma}^{-1})\rangle,$$

or, more explicitly,

$$\langle (a^{(d)}(w), a^{(d)}(w)\bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)), (\gamma, \bar{\gamma}^{-1})\rangle,$$

with

$$\hat{z}^{(d)}(\omega, w) = \hat{b}^{(d)}(\omega)\hat{\tau}^{(d)}(\omega,\sigma)^{-1}\omega(c^{(d)}(w)^{-1})\alpha_\omega(a^{(d)}(w)^{-1}).$$

Observe that in the case that there are no roots of type (d) our proof of Theorem 1.6.A is complete.

§5. Final Reductions

5.1. Introduction

To complete the proof of Theorem 1.6.A we have to show that

$$\Theta_I^{(d)}(\gamma_H; \bar{\gamma}_H) = \langle v^{(d)}, s_{\bar{T}}\rangle$$

and

$$\Theta_2^{(d)}(\gamma_H; \bar{\gamma}_H) = \langle (a^{(d)}(w), \bar{a}^{(d)}(w)), (\gamma, \bar{\gamma}^{-1})\rangle$$

satisfy

5.1. *The limit as $\gamma_H, \bar{\gamma}_H$ approach ϵ_H of*

$$\Theta_I^{(d)}(\gamma_H; \bar{\gamma}_H)\Theta_2^{(d)}(\gamma_H; \bar{\gamma}_H)$$

is equal to

$$\prod_{\mathcal{O}\in \mathrm{Orb}^{(d)}} \chi_\alpha\left(\frac{a_\alpha}{\alpha(\epsilon)-1}\right) \prod_{\bar{\mathcal{O}}\in\overline{\mathrm{Orb}}^{(d)}} \chi_{\bar{\alpha}}\left(\frac{\bar{\alpha}(\epsilon)-1}{a_{\bar{\alpha}}}\right).$$

Here $\mathrm{Orb}^{(d)}$ denotes the collection of orbits in $R^{(d)}$, and $\overline{\mathrm{Orb}}^{(d)}$ has a similar meaning. It is moreover understood that α is a representative of $\mathcal{O}$, and $\overline{\alpha}$ of $\overline{\mathcal{O}}$. In addition, for an asymmetric orbit $\mathcal{O}$ we may choose, when convenient, the χ-data and the a-data trivial, but then $a_\alpha = -1$ for $\alpha \in -\mathcal{O}$. We propose to verify (5.1.1) in part by induction on the dimension of G_{der}.

Suppose R_0 satisfies the condition (3.3.2). For example, R_0 could be $R(H_\epsilon)$ itself. We choose μ so that the condition of Lemma 3.3.B is satisfied. Suppose moreover that $R_0 \subseteq R_1 \subseteq R$, where R_1 is also Γ_T-invariant. Then the dual system $R_1^\vee$ may be taken as $\{\alpha^\vee : \alpha \in R_1\}$, and we may construct a quasi-split group G_1 containing Cartan subgroups identical to T and $\bar{T}$ (and identified with them), and with R_1 as its root system. The group G_1 need not be isomorphic to a subgroup of G but it will have an endoscopic group attached to s_T in $\widehat{T}$ or to $\bar{s}_T$ in$\widehat{\bar{T}}$. Denote this endoscopic group by H_1. The groups T and $\bar{T}$ have images in H_1.

Let $\Sigma_T = \mathbf{Z}_2 \times \Gamma_T$. Taking the non-trivial element of $\mathbf{Z}_2$ to act as -1 we obtain an action of Σ_T and of $\Omega_0 \times \Sigma_T$ on R and on $R_{\sim 1} = R - R_1$. Let Λ be a set on which Σ_T acts, and denote the image of $\lambda \in \Lambda$ under the non-trivial element of $\mathbf{Z}_2$ by $-\lambda$. Suppose that $-\lambda$ is never equal to λ. Finally, extend the action on Λ to $\Omega_0 \times \Sigma_T$ by letting Ω_0 act trivially.

The critical lemma for the reduction is the following.

Lemma 5.1.A. *If there is a mapping from $R_{\sim 1}$ to Λ compatible with the action of $\Omega_0 \times \Gamma_T$ then Assertion 5.1.1 is true with respect to $G, T, \bar{T}, \epsilon$ if it is true with respect to $G_1, T, \bar{T}, \epsilon$.*

Before proving this lemma it is as well to remind ourselves what it means. We have endoscopic data for four groups $G, G_\epsilon, G_1, G_{\epsilon,1}$ all of which share tori $T, \bar{T}$. However the factors Δ_2 are defined on covering groups $T, \bar{T}, T_\epsilon, \bar{T}_\epsilon, T_1, \bar{T}_1, T_{\epsilon,1}, \bar{T}_{\epsilon,1}$ defined by central extensions of the four groups. For G itself we have of course made the necessary extensions at the very beginning, so that for G the tori $T, \bar{T}$ are covered by themselves. We must also choose $\tilde{\epsilon}$ in T_ϵ, mapping to ϵ and therefore common to $\bar{T}_\epsilon$, as well as $\epsilon_1 \in T_1 \cap \bar{T}_1, \tilde{\epsilon}_1 \in T_{\epsilon,1} \cap \bar{T}_{\epsilon,1}$ with similar properties. Thus the Assertion 5.1.1 is to be understood as applying not literally to $G_{1,\epsilon}$ but to the covering group of G_1 and to ϵ_1 in it. Of course, only the limit of Θ_2 could possibly be affected by the various choices, and it is not.

5.2. Beginning of the proof of critical lemma

We may as well assume that the mapping from $R_{\sim 1} \longrightarrow \Lambda$ is surjective. Observe that the classification of roots in R_1 into types is the same in R_1 as in R. The choise of a-data and χ-data remains at our disposal.

It is clear that, copying the definitions of §2.1 and §2.5 of **[I]**, we may define fields $F_\lambda, F_{\pm\lambda}, \lambda \in \Lambda$, and, associated to Λ, collections of a-data, $\{a_\lambda\}$, and χ-data, $\{\chi_\lambda\}$. Since the mapping from $R_{\sim 1}$ is compatible with the actions of Γ_T and $\Gamma_{\bar{T}}$ we choose $a_\alpha = a_{\bar{\alpha}} = a_\lambda$ and $\chi_\alpha = \chi_\lambda \circ Nm_{F_\alpha/F_\lambda}, \chi_{\bar{\alpha}} = \chi_\lambda \circ Nm_{F_{\bar{\alpha}}/F_\lambda}$ if $\alpha \longrightarrow \lambda$. Inside R_1 we choose the same a-data and χ-data for G as for G_1.

If $\mathrm{Orb}_{\sim 1}^{(d)}, \overline{\mathrm{Orb}}_{\sim 1}^{(d)}$ denote the orbits outside R_1 it is easy to show that with these choices of a-data and of χ-data the second expression in (5.1.1) divided by the analogous expression for G_1 is equal to

$$\prod_{\mathcal{O}\in\, \mathrm{Orb}_{\sim 1}^{(d)}} \chi_\alpha\left(\frac{a_\alpha}{\alpha(\epsilon)-1}\right) \prod_{\overline{\mathcal{O}}\in\overline{\mathrm{Orb}}_{\sim 1}^{(d)}} \chi_{\overline{\alpha}}\left(\frac{\overline{\alpha}(\epsilon)-1}{a_{\overline{\alpha}}}\right). \tag{5.2.1}$$

To prove Lemma 5.1.A we show that these same choices lead to

$$\Theta_I^{(d)}(\gamma_H, \bar{\gamma}_H) = \Theta_I^{(d)}(\gamma_H, \bar{\gamma}_H)_1, \tag{5.2.2}$$

$$\lim \Theta_2^{(d)}(\gamma_H, \bar{\gamma}_H) = \lim \Theta_2^{(d)}(\gamma_H, \bar{\gamma}_H)_1. \tag{5.2.3}$$

The subscript 1 on the right indicates that we are calculating with respect to G_1.

If $\Gamma_\alpha, \Gamma_\lambda$ consist of the elements in the Galois group Γ_T fixing α, λ respectively, then

$$\chi_\alpha\left(\frac{a_\alpha}{\alpha(\epsilon)-1}\right) = \chi_\lambda\left(\prod_{\Gamma_\lambda/\Gamma_\alpha} \frac{a_\lambda}{\sigma\alpha(\epsilon)-1}\right).$$

Thus the first product on the left side of (5.2.1) is the product over a set of representatives for the orbits of the image of $R_{\sim 1}$ in Λ of

$$\chi_\lambda\left(\prod_{\alpha\longrightarrow\lambda} \frac{a_\lambda}{\alpha(\epsilon)-1}\right).$$

The same calculation barred yields the identical result, so that (5.2.1) is 1.

The left side of (5.2.2) is obtained by pairing the cocycle $v^{(d)}$ with $s_{\bar{T}}$. We shall factor $v^{(d)}$ as $v_1^{(d)}$, the cocycle attached to G_1, times $v_{\sim 1}^{(d)}$ and then show that $v_{\sim 1}^{(d)}$ is a coboundary. The relation (5.2.2) follows immediately.

Since $v^{(d)}$ is a product,

$$v^{(d)}(\sigma) = \tau^{(d)}(\omega,\sigma)\omega(y^{(d)}(\sigma))b^{(d)}(\omega)^{-1}\bar{y}^{(d)}(\overline{\sigma})^{-1},$$

we easily factor it by factoring each term. The first two and the last are given as products over the roots in $R^{(d)}$, which we may decompose as products over $R_1^{(d)}$ and $R^{(d)} - R_1^{(d)} = R_{\sim 1}^{(d)}$. Finally we define $b_{\sim 1}^{(d)}(\omega)$ by the equation

$$b^{(d)}(\omega) = b_1^{(d)}(\omega)b_{\sim 1}^{(d)}(\omega).$$

Since R_1 and $R_{\sim 1}$ are invariant under Ω_0, we may define

$$y_{\sim 1}^{(d)}(\omega\sigma) = \prod_{\substack{\alpha>0 \\ \alpha\in R_{\sim 1} \\ \omega^{-1}\sigma^{-1}\alpha<0}} a_\alpha^{\alpha^\vee}$$

on all of $\Omega_0 \times \Gamma_T$. The a-data have been so chosen that $\overline{y}_{\sim 1}^{(d)}$ is obtained by restricting $y_{\sim 1}^{(d)}$ to $\Gamma_{\bar{T}}$. Moreover, it follows from Lemma 2.2.A of **[I]** that the coboundary of $y_{\sim 1}^{(d)}$ is the inverse of $\tau_{\sim 1}^{(d)}$. Thus

$$\tau_{\sim 1}^{(d)}(\omega, \sigma) = y_{\sim 1}^{(d)}(\omega)^{-1}\omega(y_{\sim 1}^{(d)}(\sigma))^{-1}y_{\sim 1}^{(d)}(\omega\sigma)$$

and (5.2.2) is a consequence of the next lemma.

Lemma 5.2.A. *We may so choose* $b^{(d)}(\omega)$ *and* $b_1^{(d)}(\omega)$ *that*

$$b_{\sim 1}^{(d)}(\omega) = y_{\sim 1}^{(d)}(\omega)^{-1}.$$

In addition to $b_1^{(d)}$, we may define $b_1^{(b)}$ and b_1, and then $b_{\sim 1}^{(b)}, b_{\sim 1}$. Moreover

$$b_{\sim 1}(\omega) = b_{\sim 1}^{(b)}(\omega) b_{\sim 1}^{(d)}(\omega).$$

Lemma 5.2.A will follow from the next lemma, applied first to the pair G, G_1 and then to the pair H, H_1, the group H_1 being defined by $R_1 \cap R(H)$.

Lemma 5.2.B. *We may so choose* $b(\omega)$ *and* $b_1(\omega)$ *that*

$$b_{\sim 1}(\omega) = y_{\sim 1}(\omega)^{-1}.$$

Proof. Both side of this equation have the same coboundary, so that it is sufficient to verify that it can be satisfied for $\omega = \omega_\beta, \beta \in \Sigma_0$. Recall from Lemma 4.3.A that

$$b(\omega_\beta) = b_\beta^{\beta^\vee}\delta(\beta). \tag{5.2.4}$$

A similar equation is valid in G_1:

$$b_1(\omega_\beta) = b_{1,\beta}^{\beta^\vee}\delta_1(\beta). \tag{5.2.5}$$

The factors $\delta(\beta)$ and $\delta_1(\beta)$ are defined by sets R_β^+ and $R_{1,\beta}^+$. We may suppose that $R_\beta^+ \cap R_1 = R_{1,\beta}^+$. To define $R_\beta^+ \cap (R_{\sim 1}) = R_{\sim 1,\beta}^+$ choose a set Λ^+ of representatives for the orbits of $\mathbf{Z}_2$ in Λ and agree that $\alpha \in R_\beta^+ \cap (R_{\sim 1})$ if and only if $\alpha \longrightarrow \lambda \in \Lambda^+$. Observe that if $\alpha \longrightarrow \lambda$ then $-\omega_\beta\alpha \longrightarrow -\lambda$.

From (5.2.4) and (5.2.5) we obtain a factorization

$$b_{\sim 1}(\omega_\beta) = b^{\beta^\vee}_{\sim 1,\beta}\delta_{\sim 1}(\beta)$$

with

$$\delta_{\sim 1}(\beta) = \prod_{\alpha \in R^+_{\sim 1,\beta}} (-1)^{\alpha^\vee}.$$

The expression on the right in Lemma 5.2.B is

$$\prod_{\substack{\alpha \in R_{\sim 1} \\ \alpha > 0, \omega^{-1}\alpha < 0}} a_\alpha^{-\alpha^\vee}, \tag{5.2.6}$$

when $\omega = \omega_\beta$. Putting α and $-\omega\alpha$ together, and noting that

$$\begin{aligned} a_\alpha^{\alpha^\vee} a_{-\omega\alpha}^{-\omega\alpha^\vee} &= \alpha_\alpha^{\alpha^\vee}(-a_\alpha)^{-\omega\alpha^\vee} = a_\alpha^{\langle\alpha^\vee,\beta\rangle\beta^\vee}(-1)^{\omega\alpha^\vee}, \\ (-1)^{\omega\alpha^\vee} &= (-1)^{\alpha^\vee}(-1)^{\langle\alpha^\vee,\beta\rangle\beta^\vee}, \end{aligned}$$

we see that (5.2.6) is equal to

$$\delta_{\sim 1}(\beta) \prod_{\substack{\alpha \in R^+_{\sim 1,\beta} \\ \alpha > 0, \omega^{-1}\alpha < 0}} a_{-\alpha}^{-\langle\alpha^\vee,\beta\rangle\beta^\vee}.$$

To show that

$$\prod_{\substack{\alpha \in R^+_{\sim 1,\beta} \\ \alpha > 0, \omega^{-1}\alpha < 0}} a_{-\alpha}^{-\langle\alpha^\vee,\beta\rangle} \tag{5.2.7}$$

is a possible choice for $b_{\sim 1,\beta}$ we apply the next lemma. To state it, set

$$e_{\sim 1}(\omega,\sigma) = e(\omega,\sigma)e_1(\omega,\sigma)^{-1}$$

with

$$e(\omega,\sigma) = \tau(\sigma,\omega)\tau(\sigma\omega,\sigma^{-1})\sigma(\omega)(\tau(\sigma,\sigma^{-1}))^{-1}$$

as in (4.1).

Lemma 5.2.C. *Suppose that for each $\beta \in \Sigma_0$ we are given $c(\beta) \in \bar{F}^\times$. We may choose $b_{\sim 1,\beta} = c(\beta)$ for all β if and only if the following equations are valid for all $\sigma \in \Gamma_T$:*

$$\sigma(c(\beta))^{\sigma\beta^\vee}\sigma(\delta_{\sim 1}(\beta)) = y_{\sim 1}(\sigma)\sigma(\omega)(y_{\sim 1}(\sigma)^{-1})e_{\sim 1}(\omega,\sigma)c(\sigma\beta)^{\sigma\beta^\vee}\delta_{\sim 1}(\sigma\beta).$$

If we define $c(\beta)$ by (5.2.7) this equation is simply

$$\sigma(y_{\sim 1}(\omega_\beta)^{-1}) = y_{\sim 1}(\sigma)\sigma(\omega_\beta)(y_{\sim 1}(\sigma)^{-1})e_{\sim 1}(\omega_\beta,\sigma)y_{\sim 1}(\omega_{\sigma\beta})^{-1},$$

for in the lemma ω is ω_β. This equation is better written as

$$e_{\sim 1}(\omega_\beta,\sigma) = y_{\sim 1}(\sigma)^{-1}\sigma(y_{\sim 1}(\omega_\beta))^{-1}y_{\sim 1}(\sigma(\omega_\beta))\sigma(\omega_\beta)(y_{\sim 1}(\sigma)).$$

Inserting the factors of

$$y_{\sim 1}(\sigma\omega_\beta)y_{\sim 1}(\sigma\omega_\beta)^{-1}\sigma(\omega_\beta)(y_{\sim 1}(\sigma^{-1}))^{-1}\sigma(\omega_\beta)(\sigma(y_{\sim 1}(\sigma^{-1}))) = 1$$

at suitable places and recalling that the boundary of $y_{\sim 1}$ is $\tau_{\sim 1}^{-1}$ we transform the equation to

$$e_{\sim 1}(\omega_\beta,\sigma) = \tau_{\sim 1}(\sigma,\omega_\beta)\tau_{\sim 1}(\sigma\omega_\beta,\sigma^{-1}(\sigma(\omega_\beta)(\tau_{\sim 1}(\sigma,\sigma^{-1}))^{-1}$$

which is true by definition.

Lemma 5.2.C is verified by applying the definitions and the following lemma for G_1 as well as for G.

Lemma 5.2.D. *Suppose that for each $\beta \in \Sigma_0$ we are given $c(\beta) \subset \bar{F}^\times$. We may choose $b_\beta = c(\beta)$ for all β if and only if the following equations are valid for all $\sigma \in \Gamma_T$:*

$$\sigma(c(\beta))^{\sigma\beta^\vee}\sigma(\delta(\beta)) = y(\sigma)\sigma(\omega)(y(\sigma)^{-1})e(\omega,\sigma)c(\sigma\beta)^{\sigma\beta^\vee}\delta(\sigma\beta).$$

Proof. The necessity, namely the equation

$$\sigma(b(\omega)) = y(\sigma)\sigma(\omega)(y(\sigma)^{-1})e(\omega,\sigma)b(\sigma(\omega)),$$

is the first part of Lemma 4.1.A.

To prove the sufficiency we observe that what the conditions of the lemma determine are the quotients $(\sigma(c(\beta))c(\sigma\beta)^{-1})^{\langle\lambda,\beta^\vee\rangle}$, where λ is any weight. Thus it permits multiplication of any given collection by a collection $\{d(\beta)\}$ satisfying

$$(\sigma(d(\beta))d(\sigma\beta)^{-1})^{\sigma\beta^\vee} = 1.$$

On the other hand, we do not destroy the condition of Lemma 3.3.B if we multiply h by an element t such that $\beta(t\sigma(t^{-1}))^{\beta^\vee} = 1$ for all $\beta \in \Sigma_0$. Replacing β by $\sigma\beta$, we rewrite this condition as

$$\sigma\beta(t)^{\sigma\beta^\vee} = \sigma(\beta(t))^{\sigma\beta^\vee} \quad \forall \beta \in \Sigma_0.$$

Since the change in h replaces $b(\omega_\beta)$ by $\beta(t)b(\omega_\beta)$ the equality (5.2.2) is proved.

5.3. The term $\Theta_2^{(d)}$.

The relation (5.2.3) causes the most difficulty. For the calculations we must pass explicitly to a z-extension G_1' of G_1. The associated quantities will be denoted with a prime. All tori $\widehat{T}$,$\widehat{\bar{T}}$, $\widehat{T}_H = \widehat{T}_s$,$\widehat{\bar{T}}_H = \widehat{\bar{T}}_s$, as well as $\bar{T}_1$,$\widehat{\bar{T}}_1$, $\widehat{T}_{s,1}$,$\widehat{\bar{T}}_{s,1}$ were identified. Thus we have embeddings $\widehat{T} \hookrightarrow \widehat{T}_1'$, $\widehat{T}_s \hookrightarrow \widehat{T}_{s,1}'$ and so on. We transfer cocycles with values on one group to another group in which it is imbedded without change in notation. Suppose ϵ' in T' maps to ϵ. The left side of (5.2.3) is obtained by pairing $(a(w)a_\epsilon(w)^{-1}, \bar{a}(w)\bar{a}_\epsilon(w)^{-1})$ with $(\epsilon, \epsilon^{-1})$ or, passing to $\widehat{T}'$, with $(\epsilon', \epsilon'^{-1})$. The right side is obtained by pairing $(a_1(w)a_{\epsilon,1}(w)^{-1}, \bar{a}_1(w)\bar{a}_{\epsilon,1}(w)^{-1})$ with $(\epsilon', \epsilon'^{-1})$. Thus it suffices to show that $(a(w)a_1(w)^{-1}, \bar{a}(w)\bar{a}_1(w)^{-1})$ yields 1 upon pairing with $(\epsilon', \epsilon'^{-1})$ and that $(a_\epsilon(w)a_{\epsilon,1}(w)^{-1}$, $\bar{a}_\epsilon(w)\bar{a}_{\epsilon,1}(w)^{-1})$ also yields 1.

We shall prove the first assertion which is a statement about G and G_1. The second follows from it upon substitution of G_ϵ for G, and $G_{\epsilon,1}$ for G_1. Set $a_2(w) = a(w)a_1(w)^{-1}, \bar{a}_2(w) = \bar{a}(w)\bar{a}_1(w)^{-1}$. We want to show that

(5.3.1) $$\langle (a_2, \bar{a}_2), (\epsilon', \epsilon'^{-1}) \rangle = 1.$$

Let Y be the span over **Z** of R_0, and define a torus S by the relation

$$X_*(\widehat{S}) = \{(\lambda, \mu) | \lambda \in X_*(\widehat{T}'), \mu \in X_*(\widehat{\bar{T}}'), \lambda - \mu \in Y\}.$$

Notice that $X_*(\widehat{T}')$ and $X_*(\widehat{\bar{T}}')$ are, if desired, identified, so that the locations of λ and μ are specified only to make the Galois action clear.

The inclusion $X_*(\widehat{S}) \longrightarrow X_*(\widehat{T}') \oplus X_*(\widehat{\bar{T}}')$ defines $\widehat{S} \longrightarrow \widehat{T}' \times \widehat{\bar{T}}'$ and $T' \times \bar{T}' \longrightarrow S$. Under the latter, $(\epsilon', \epsilon'^{-1}) \longrightarrow 1$. Thus to establish (5.3.1) it suffices to show that $(a_2, \bar{a}_2)$ is the image of a cocycle with values in $\widehat{S}$. The decomposition

$$X_*(\widehat{S}) = X_*(\widehat{T}') \oplus Y : (\lambda, \mu) \sim (\lambda, \lambda - \mu)$$

yields an isomorphism $\widehat{S} \simeq \widehat{T}' \times \widehat{R}$, if $\widehat{R}$ is defined by $X_*(\widehat{R}) = Y$. It is not an isomorphism of Galois modules. Nontheless if a_2' is a cocycle with values in $\widehat{T}'$ then $w \longrightarrow a_2'(w) \times 1 = a_3(w)$ does take values in $\widehat{S}$. There are two points to verify.

(5.3.2) The cocycle a_2' can be so chosen that a_3 is a cocycle.

(5.3.3) It can at the same time be so chosen that the image of a_3 is in the same class as $(a_2, \bar{a}_2)$.

This will take some effort.

The tori $\widehat{T}$,$\widehat{\bar{T}}$, $\widehat{T}_H = \widehat{T}$, are identified in a fixed way with $\mathcal{T}$. The normalizer of $\mathcal{T}$ in ${}^LG$ projects modulo $\mathcal{T}$ itself to $\Omega(G) \rtimes \Gamma_T$. Denote the inverse image of $\Omega_0 \rtimes \Gamma_T$ by ${}^LM$. Since Ω_0 is contained in $\Omega(G_1), \Omega(M)$ and $\Omega(M_1)$ we may define ${}^LM_1', {}^LM_s, {}^LM_{s,1}'$ in the same way, the kernel of ${}^LM_1 \longrightarrow \Omega_0 \rtimes \Gamma_T$ or of ${}^LM_{s,1} \longrightarrow \Omega_0 \rtimes \Gamma_T$ being $\widehat{T}'$.

The cocycles used to define a_2 are defined by means of homomorphisms attached to the χ-data:

$$\begin{array}{ll} \xi : {}^LT \longrightarrow {}^LM \subseteq {}^LG; & \xi_1' : {}^LT' \longrightarrow {}^LM_1' \subseteq {}^LG_1'; \\ \xi_s : {}^LT \longrightarrow {}^LM_s \subseteq {}^LG_s; & \xi_{s,1}' : {}^LM_{s,1}' \subseteq {}^LG_{s,1}'; \end{array}$$

and to imbeddings:

$$\eta : {}^LH \hookrightarrow {}^LG; \;\; \eta_1' : {}^LH_1' \hookrightarrow {}^LG_1'.$$

The cocycle $\bar{a}_2$ is defined in a similar manner by $\bar{\xi}, \bar{\xi}_1', \bar{\xi}_s, \bar{\xi}_{s,1}'$.

We shall construct homomorphisms

$$\varphi : {}^LM \longrightarrow {}^LM_1', \varphi_s : {}^LM_s \longrightarrow {}^LM_{s,1}' \tag{5.3.4}$$

with the following properties:

(a) They are compatible with the projections to $\Omega_0 \rtimes \Gamma_T$.

(b) Let $\pi, \bar{\pi}$ be the natural homomorphisms

$$\pi : {}^LT \longrightarrow {}^LT', \quad \bar{\pi} : {}^L\bar{T} \longrightarrow {}^L\bar{T}'$$

defined by the imbedding $\widehat{T} \hookrightarrow \widehat{T}'$,$\widehat{\bar{T}} \hookrightarrow \widehat{\bar{T}}'$. Then

$$\xi_1' \circ \pi = \varphi \circ \xi, \;\; \bar{\xi}_1 \circ \bar{\pi} = \varphi \circ \bar{\xi}, \;\; \xi_{s,1}' \circ \pi = \varphi \circ \xi_s, \;\; \bar{\xi}_{s,1}' \circ \bar{\pi} = \varphi \circ \bar{\xi}_s.$$

(c) There is a $t \in \widehat{T}'$ such that

$$\varphi \circ \eta = \mathrm{ad} t \circ \eta_1' \circ \varphi_s,$$

on $\widehat{M}_s$, the inverse image of Ω_0 in ${}^LM_s$.

These conditions are clarified by a diagram

$$\begin{array}{ccccc} {}^LT & \xrightarrow{\xi} & {}^LM & \xrightarrow{\varphi} & {}^LM_1' \\ \| & & \uparrow\eta & & \uparrow\eta_1' \\ {}^LT & \underset{\xi_s}{\longrightarrow} & {}^LM_S & \underset{\varphi_s}{\longrightarrow} & {}^LM_{s,1}' \end{array}$$

For technical reasons we need variants of φ, φ_s. Let $\widetilde{T}$ be the inverse image of $\mathcal{T}_s$ in the simply-connected covering $\widetilde{G}$ of the derived group of $\widehat{G}$ with the action of the Galois group defined by T. Define $\widetilde{\bar{\bar{T}}}$ in a similar fashion with $\Gamma_{\bar{T}}$ replacing Γ_T. Let $\widetilde{H}$ be the connected centralizer of s in $\widetilde{G}$, and let $\widetilde{G}_1$ be the connected S-group associated to the group with root system R_1 and Cartan subgroups dual to $\widetilde{T}$. Let $\widetilde{H}_1$ be the connected centralizer of s in $\widetilde{G}_1$.

We have imbeddings

$$\tilde{\eta} : \widetilde{H} \hookrightarrow \widetilde{G}, \quad \tilde{\eta} : \widetilde{H}_1 \hookrightarrow \widetilde{G}_1.$$

Let $\widetilde{M}$ be the inverse image of Ω_0 in the normalizer of $\widetilde{T}$ in $\widetilde{G}$, and define $\widetilde{M}_s, \widetilde{M}_1, \widetilde{M}_{s,1}$ in a similar fashion. There are obvious maps

$$\widetilde{M} \longrightarrow {}^LM, \widetilde{M}_s \longrightarrow L_s^M, \widetilde{M}_1 \longrightarrow {}^LM_1', \widetilde{M}_{s,1} \longrightarrow {}^LM_{s,1}'.$$

We shall also construct

$$\tilde{\varphi} : \widetilde{M} \longrightarrow \widetilde{M}_1, \ \ \tilde{\varphi}_s : \widetilde{M}_s \longrightarrow \widetilde{M}_{s,1}$$

so that the diagrams

$$\begin{array}{ccccccccc} \widetilde{M} & \xrightarrow{\tilde{\varphi}} & \widetilde{M}_1 & & \widetilde{M}_s & \xrightarrow{\tilde{\varphi}_s} & \widetilde{M}_{s,1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}^LM & \underset{\varphi}{\longrightarrow} & {}^LM_1' & & {}^LM_s & \underset{\varphi_s}{\longrightarrow} & {}^LM_{s,1}', \end{array} \tag{5.3.5}$$

are commutative. Observe that ${}^LM$ acts on $\widetilde{M}, {}^LM_1'$ on $\widetilde{M}_1$, and so on.

Moreover there will be a lifting of t to $\tilde{t}$ in $\widetilde{T}$ such that

$$\tilde{\varphi} \circ \tilde{\eta} = \text{ ad } \tilde{t} \circ \tilde{\eta}_1 \circ \tilde{\varphi}_s.$$

Finally, if $m \in {}^LM, m_s \in {}^LM_s, \tilde{m} \in \widetilde{M}_1, \widetilde{M}_s \in \widetilde{M}_s$, then

$$\tilde{\varphi}(m(\tilde{m})) = \varphi(m)(\tilde{\varphi}(\tilde{m})), \ \tilde{\varphi}_s(m_s(\tilde{m}_s)) = \varphi_s(m_s)(\tilde{\varphi}_s(\tilde{m}_s)). \tag{5.3.6}$$

Granting the constructing of $\varphi, \varphi_s, \tilde{\varphi}, \tilde{\varphi}_s$ we verify (5.3.2) and (5.3.3). Set

$$\psi_1 = \varphi \circ \eta, \ \ \psi_2 = \text{ad} t \circ \eta_1' \circ \varphi_s, \ \ \tilde{\psi}_1 = \ \ \tilde{\varphi} \circ \tilde{\eta}, \ \ \tilde{\psi}_2 = \text{ad}\tilde{t} \circ \tilde{\eta}_1 \circ \tilde{\varphi}_s$$

and let

$$\psi_1(m) = a_2'(w)\psi_2(m)$$

if $m \in {}^L M_s$ projects to w. It is of course condition (c) that guarantees that a_2' is a function of this projection alone. Since

$$a(w)\xi(m) = \eta \circ \xi_s(m), \qquad m \in {}^L T, m \longrightarrow w,$$

and

$$a_1(w)\xi_1(m) = \eta_1 \circ \xi_{s,1}(m), \qquad m \in {}^L T, m \longrightarrow w,$$

we conclude from condition (b) that

$$a_2'(w) = a(w)a_1^{-1}(w)t\sigma(t)^{-1},$$

if $w \to \sigma \in \Gamma_T$. Thus a_2 and a_2' lie in the same class; so we replace a_2 by a_2' in (5.3.1).

The same calculation is however, valid for $\bar{a}$ and $\bar{a}_1$. Thus we conclude that we may replace $\bar{a}_2$ by a_2' in (5.3.1). With this choice of a_2' the condition (5.3.3) is clear.

Passing to the condition (5.3.2) we first note that if $\omega \in \Omega_0$ then the homomorphism $\omega - 1$ takes $X_*(\widehat{T}')$ to Y and thus defines a homomorphism $\alpha_\omega : \widehat{T}' \to \widehat{R}$. Under the isomorphism $\widetilde{S} \simeq \widehat{T}' \times \widehat{R}$, the action of $\sigma \in \Gamma$ sends $(\hat{t}, 1)$ to $(\sigma_T(\hat{t}), \alpha_\omega(\sigma_T(\hat{t}))^{-1})$ if $\sigma_{\bar{T}} = \omega \times \sigma_T$. Thus the boundary of $w \to (a_2'(w), 1)$ is

$$(a_2'(w_1), 1)(w_1(a_2'(w_2)), \alpha_{\omega_1}(\sigma(a_2')(w_2)))^{-1}(a_2'(w_1w_2)^{-1}, 1),$$

which is simply $\alpha_{\omega_1}(\sigma_1(a_2'(w_2)))^{-1}$ or $\sigma_1(\alpha_{\sigma_1^{-1}(\omega_1)}(a_2'(w_2))^{-1})$. Thus we have to show that for all $\omega \in \Omega_0$ and all w in W

$$\alpha_\omega(a_2'(w)) = 1. \tag{5.3.7}$$

Since $\alpha_{\omega_1\omega_2}(\hat{t}) = \omega_1 \circ \alpha_{\omega_2}(\hat{t})\omega_1(\hat{t})$ it suffices to verify this for $\omega = \omega_\beta, \beta \in \Sigma_0$. Then if $\widehat{R}_\beta$ is defined by $X_*(\widehat{R}_\beta) = \mathbf{Z}\beta$, the homomorphism obviously factors through $\widehat{T}' \to \widehat{T} \to \widehat{R}_\beta$. Since $\mathbf{Z}\beta \subseteq X_*(\widehat{T})$, there is a homomorphism $\lambda_\beta : \widehat{R}_\beta \to \widehat{T}$. If it were injective we could verify (5.3.7) by establishing the relation

$$\omega(a_2'(w)) = a_2'(w),$$

for the composition $\lambda_\beta \circ \alpha_\omega$ takes $\tilde{t}$ to $\omega(\tilde{t})\tilde{t}^{-1}$.

Unfortunately the homomorphism λ_β is not always injective. If, however, we replace $\widehat{T}$ by $\widetilde{T}$ then it becomes so. Of course $a_2'(w)$ is not defined in $\widetilde{T}$, but let $\tilde{a}_2'(w)$ lie in $\widetilde{T}$ and have an image in $\widetilde{T}$, that is congruent to $a_2'(w)$ modulo the center of $\widetilde{G}_1'$. Then $\alpha_\omega(a_2'(w))$ is the image of $\alpha_\omega(\tilde{a}_2'(w))$ and

$$\alpha_\omega(\tilde{a}_2'(w)) = \omega(\tilde{a}_2'(w))\tilde{a}_2'(w)^{-1}.$$

Choose $m \in M_s$ mapping to $\sigma^{-1}(\omega)$, where $w \to \sigma \in \Gamma_T$. Then

$$\psi_1(w)(\tilde{\psi}_1(m)) = \tilde{\psi}_1(w(m)) = \tilde{\psi}_2(w(m)).$$

On the other hand,

$$\begin{aligned}\psi_1(w)(\tilde{\psi}_1(m)) &= \tilde{a}_2'(w)(\psi_2(w)(\tilde{\psi}_2(m)))\tilde{a}_2'(w)^{-1}\\ &= \tilde{a}_2'(w)\tilde{\psi}_2(w(m))\tilde{a}_2'(w)^{-1}.\end{aligned}$$

We conclude that

$$1 = \tilde{a}_2'(w)\tilde{\psi}_2(w(m))\tilde{a}'(w)^{-1}\tilde{\psi}_2(w(m))^{-1} = \tilde{a}_2'(w)\omega(\tilde{a}_2'(w))^{-1}.$$

5.4. Construction of φ and φ_s

We begin with a general system $\mathcal{R}$, as in §2 of **[I]**, and assume further that we have a surjective map $\mathcal{R} \to \Lambda$, Λ having the properties of (5.1). We suppose that χ-data are given on Λ and that if $\alpha \to \lambda$ then $\chi_\alpha = \chi_\lambda \circ Nm_{F_\alpha/F_\lambda}$.

Let p be a given gauge on $\mathcal{R}$. On Λ we can construct a gauge p_Λ by choosing in each orbit a representative λ, as well as coset representatives $\sigma_1, \cdots, \sigma_r$ for $\Lambda_\pm/\Gamma$ and then defining $p_\Lambda(\sigma_i^{-1}\lambda)$ to be 1. Define, in the notation of §2.5 of **[I]**, r_{p_Λ} by

$$r_{p_\Lambda}(w) = \prod_{i=1}^{n} \chi_\lambda(v_0(u_i(w)))^{\lambda_i'},$$

where

$$\lambda_i' = \sum_{\alpha \to \lambda_i} \alpha, \quad \lambda_i = \sigma_i^{-1}\lambda.$$

Set $p'(\alpha) = p_\Lambda(\lambda(\alpha))$, and define a gauge p_0' on $\mathcal{R}$ in the same way as p_Λ was defined on Λ.

Lemma 5.4.A. *The cochains $w \to s_{p/p'}(w) r_{p_\Lambda}(w)$ and $w \to s_{p/p'_0}(w) r_{p'_0}(w)$ differ by a coboundary.*

It follows readily from §2.5 of **[I]** that the quotient of those two cochains is a cocycle. It is enough to prove this lemma when the module X has $\{\alpha \in \mathcal{R} \mid p'(\alpha) = 1\}$ as a basis over $\mathbf{Z}$, $\mathcal{R}$ is a single orbit under the group Σ introduced in §2.5 of **[I]**, and for each $\lambda \in \Lambda$,

$$\lambda = \sum_{\alpha \to \lambda} \alpha.$$

To prove the lemma we have to prove that the quotient factors through a Galois group, and that if $\alpha \in \mathcal{R}$ then the projection on the one-parameter subgroup $\widehat{R}_\alpha$ associated to $\mathbf{Z}_\alpha$ of the restriction of the cocycle to $W_{\pm\alpha}$ is a coboundary. Fix one $\alpha \in \mathcal{R}$, and let $\lambda = \lambda(\alpha)$.

Take the Weil group W to be $W_{L/F}$ where L is a large, finite, Galois extension of F. We first observe that for $x \in L^\times \subseteq W_{L/F}$,

$$r_{p_\Lambda}(x) = r_{p'_0}(x). \tag{5.4.1}$$

The right side has been computed in §2.5**[I]** as

$$\prod_{\sigma \in \Gamma_{\pm a} \backslash \Gamma} \chi_\alpha(Nm^L_{F_\alpha} \sigma x)^{\sigma^{-1}\alpha} = \prod_{p'_0(\beta)=1} \chi_\beta \left(\prod_{\Gamma_\beta} \sigma(x) \right)^\beta,$$

the equality following from the relation

$$\chi_{\sigma\beta}(\sigma x) = \chi_\beta(x).$$

Since $\chi_{-\beta} = \chi_\beta^{-1}$, the gauge p'_0 can be relaced by p'. If $\alpha \to \lambda$ then in the same way the left side is

$$\prod_{p_\Lambda(\mu)=1} \chi_\mu \left(\prod_{\sigma \in \Gamma_\mu} \sigma(x) \right)^\mu$$

which is equal to

$$\prod_{p_\Lambda(\mu)=1} \prod_{\beta \to \mu} \chi_\mu \left(\prod_{p \in \Gamma_\mu \backslash \Gamma_\beta} \prod_{\sigma \in \Gamma_\beta} \rho\sigma(x) \right)^\beta$$

or

$$\prod_{p_\Lambda(\mu)=1} \prod_{\beta \to \mu} \chi_\beta \left(\prod_{\sigma \in \Gamma_\beta} \sigma(x) \right)^\beta.$$

Thus (5.4.1) is clear.

Since the restrictions of $s_{p/p'}$ and s_{p/p'_0} to $W_{\pm\alpha}$ always have trivial projections to $\widehat{R}_\alpha$, it is only the quotient $r_{p_\Lambda}(w)r_{p'_0}(w)^{-1}$ that need be considered. Taking the projection of the restriction also allows us to suppose that $\Lambda = \{\pm\lambda\}, \Gamma = \Gamma_{\pm\lambda}$, with $p_\Lambda(\lambda) = 1$.

Choose $v_0 = 1 \in W_{\pm\lambda}$, and if λ is symmetric $v_1 = v \in W_{\pm\lambda} - W_{+\lambda}$. If $w \in W_{\pm\lambda}$ let $w = x(w)v_i, x(w) \in W_{+\lambda}$. Then $\chi = \chi_\lambda$ may be regarded as a character of $W_{+\lambda} = W_{F_{+\lambda}/F} = W$, and

$$r_{p_\lambda}(w) = \chi(x(w))^\lambda.$$

Projecting on $\widehat{R}_\alpha$ we obtain $\chi(x(w))^\lambda$.

Since $r_{-p'_0} = r_{p'_0}$, we may assume that $p'_0 = 1$. It is convenient to consider three cases separately, although in all of them the conclusion will be that $r_{p'_0}(w)$ projects to $\chi(x(w))^\alpha$.

(i) α asymmetric, λ asymmetric. We calculate with the notation of §2.5 of **[I]**, noting that with our choice of χ_α,

$$\chi_\alpha(u_i(w)) = \chi(w_i w w_i^{-1}).$$

Taking, as we may, $w_1 = 1$, so that $w_{1'}$ is also 1 for $w \in W_{+\alpha} = W_{\pm\alpha}$, we see that the projection of

$$r_{p'_0}(w) = \prod \chi(w_{i'} w w_i^{-1})^{\alpha_i}$$

on $\widehat{R}_\alpha$ is $\chi(w_1 w w_1^{-1})^\alpha = \chi(w)^\alpha$.

(ii) α asymmetric, λ symmetric. Let $v \to \tau \in \Gamma$ and let $\sigma_1, \cdots, \sigma_r$ be a set of representatives for $\gamma_{+\alpha}\backslash\Gamma_{+\gamma}$. Then $\{\sigma_1, \cdots, \sigma_r, \sigma_1\tau, \cdots, \sigma_r\tau\}$ is a set of representatives for $\Gamma_{\pm\alpha}\backslash\Gamma$. Lift σ_i to w_i in W, and set $\bar{x}(w) = vx(w)v^{-1}$. We take $\sigma_1 = 1, w_1 = 1$. If $w = x(w)$ then

$$r_{p'_0}(w) = \prod_i \chi(x_i x(w) w_{i'}^{-1})^{\alpha_i} \prod_i \chi(w_i \bar{x}(w) w_{j'}^{-1})^{\tau^{-1}\alpha_i},$$

where $j' = j'(i, w)$ is defined by $\sigma_i\bar{\sigma} = \rho_i(\bar{\sigma})\sigma_{j'}$ if $\bar{x}(w) \to \bar{\sigma}$. If $w \in \Gamma_{\pm\alpha}$ then the projection of the right hand side on $\widehat{R}_\alpha$ is $\chi(x(w))$ because α is asymmetric. If w were $x(w)v$ and σ the image of w in Γ then $\sigma\alpha$ would be α and $\sigma\lambda$ would be λ. This is impossible.

(iii) α symmetric, λ symmetric. We take the representatives of $\Gamma_{\pm\alpha}\backslash\Gamma$ to lie in Γ_λ and we take v in $\Gamma_{\pm\alpha}\backslash\Gamma_{+\alpha}$. If $w = x(w)$ lies in $W_{+\alpha}$ then $r_{p'_0}(w)$ is calculated as in (i). If $w = x(w)v$ lies in $W_{\pm\alpha}\backslash W_\alpha$ then the projection of $v_{p'_0}$ on $\widehat{R}_\alpha$ is again $\chi(x(w))^\alpha$.

Returning to our special situation we choose p_Λ on Λ, p'_0 on $R_{\sim 1}$, and p^1_0 on R_1. Together p^1_0 and p'_0 define p_0 on R. We have natural factorizations

$$\begin{aligned} s_{p/p_0}(w) &= s_{p/p_{0,1}}(w) s_{p/p_0,\sim 1}(w) = s_{p/p^1_0}(w) s_{p/p'_0}(w) \\ r_{p_0}(w) &= r_{p_0,1}(w) r_{p_0,\sim 1}(w) = r_{p^1_0}(w) r_{p'_0}(w) \\ r_p(w) &= r_{p,1}(w) r_{p,\sim 1}(w). \end{aligned}$$

There are similar factorizations for the barred quantities.

Recall from §2.6 of **[I]** that

$$\begin{aligned} \xi(w) &= r_p(w) n(\omega_T(\sigma)) \times w \\ &= s_{p/p_0,1}(w) r_{p_0,1}(w) s_{p/p_0,\sim 1}(w) r_{p_0,\sim 1}(w) n(\omega_T(\sigma)) \times w. \end{aligned}$$

In view of the lemma, we are free to replace $s_{p/p_0,\sim 1} r_{p_0,\sim 1}$ by $s_{p/p'} r_{p_\Lambda}$, obtaining a homomorphism that we still denote by ξ,

$$\xi(w) = s_{p/p_0,1}(w) r_{p_0,1}(w) s_{p/p'}(w) r_{p_\Lambda}(w) n(\omega_T(\sigma)) \times w.$$

We modify $\bar{\xi}(w)$ in the exactly the same fashion.

The homomorphism φ is defined on $\widehat{T}$ as the imbedding of $\widehat{T}$ into $\widehat{T}'$. To define it on all of ${}^L M$, we must define $\varphi(n(\omega\omega_T(\sigma)) \rtimes w), \omega \in \Omega_0, w \in W, w \to \sigma$. Since $s_{p/p'}$ is defined on all of $\Omega_0 \rtimes \Gamma_T$ we may set

$$\varphi(n(\omega\omega_T(\sigma)) \rtimes w) = s^{-1}_{p/p'}(\omega\sigma) r^{-1}_{p_\Lambda}(w) n_1(\omega\omega_T(\sigma)) \rtimes w.$$

There are three conditions to verify in order to show that φ is a homomorphism. For simplicity, write $n(\omega_T(\sigma)) \rtimes w \in {}^L G$ as $n(w)$.

(i) $\varphi(n(\omega_1))\varphi(n(\omega_2)) = t(\omega_1, \omega_2)\varphi(n(\omega_1\omega_2)), \omega_1, \omega_2 \in \Omega_0$

(ii) $\varphi(n(w_1))\varphi(n(w_2)) = t(\sigma_1, \sigma_2)\varphi(n(w_1 w_2)), w_i \in W_T, w_i \to \sigma_i$

(iii) $\varphi(n(w))\varphi(n(\omega))\varphi(n(w))^{-1} = e_t(\omega, \sigma)\varphi(n(\sigma(\omega)), \omega \in \Omega_0, \sigma \in w_T, w \to \sigma$.

Here

$$e_t(\omega, \sigma) = t(\omega, \sigma) t(\sigma\omega, \sigma^{-1}) \sigma(\omega) (t(\sigma, \sigma^{-1}))^{-1}. \tag{5.4.2}$$

The relation (i) amounts to

$$s^{-1}_{p|p'}(\omega_1)\omega_1(s^{-1}_{p|p'}(\omega_2)) t_1(\omega_1, \omega_2) s_{p|p'}(\omega_1\omega_2) = t(\omega_1, \omega_2). \tag{5.4.3}$$

The p that occurs here is really the restriction of p on R to $R_{\sim 1}$, and in this sense $tt_1^{-1} = t_p$. Since $t_{p'}(\omega) = 1$ for $\omega \in \Omega_0$ because p' is invariant under Ω_0 the relation (5.4.3) is clear from Lemma 2.4.A of **[I]**.

The relation (ii) is valid for a similar reason. One has only to observe that

$$r_{P_\Lambda}^{-1}(w_1)\sigma_1(r_{p_\Lambda}^{-1}(w_2))r_{p_\Lambda}(w_1w_2) = t_{p'}(\sigma_1, \sigma_2).$$

The relation (iii) amounts to the equality of

$$s_{p/p'}^{-1}(\sigma)\sigma(\omega)(s_{p/p'}(\sigma))e_{t_1}(\omega, \sigma)\sigma(s_{p/p'}^{-1}(\omega))$$

and

$$e_t(\omega, \sigma)s_{p/p'}^{-1}(\sigma(\omega)),$$

if e_{t_1} is defined by the analogue of (5.4.2), G_1 replacing G and t_1 therefore replacing t. If $e_{\sim 1}$ is defined as in (5.2.8), the element ω_β being replaced by ω then $e_{\sim 1} = e_t e_{t_1}^{-1}$. On the other hand, the boundary of $s_{p/p'}$ being $t_p/t_{p'}$ we have

$$s_{p/p'}^{-1}(\sigma)\sigma(\omega)(s_{p/p'}(\sigma))\sigma(s_{p/p'}^{-1}(\omega))s_{p/p'}^{-1}(\sigma(\omega)) = e_{t_p}^{-1}e_{t_{p'}}.$$

Clearly

$$e_{\sim 1} = e_{t_p}.$$

Moreover

$$t_{p'}(\omega, \sigma) = 1, t_{p'}(\sigma\omega, \sigma^{-1}) = t_{p'}(\sigma, \sigma^{-1}) = \sigma(\omega)(t_{p'}(\sigma, \sigma^{-1})),$$

because p' is invariant under Ω_0. Thus

$$e_t e_{t_1}^{-1} = e_{t_p} = e_{t_p}^{-1}e_{t_{p'}}.$$

(The inelegant appeal to the fact that we are dealing with cochains of order two is entailed by an infelicitous definition of ξ in §2.6 of **[I]**).

The homomorphism φ_s is defined in a similar fashion. Condition (a) is of course manifest, and condition (b) follows easily from the definitions. To verify (c) we have to prove the existence of $t \in \widehat{T}$ such that

(5.4.4) $$\varphi(\eta(n_s(\omega_\beta))) = \text{ ad } t(\eta_1(\varphi_s(n_s(\omega_\beta)))), \quad \beta \in \Sigma_0.$$

Suppose we can prove the existence of $t_1 \in \widehat{T}$ and for each $\beta \in \Sigma_0$ of $\lambda_\beta \in \mathbf{C}^\times$ such that

$$\begin{aligned}\varphi(\eta(n_s(\omega_\beta))) &= \lambda_\beta^\beta \,\text{ad}\, t_1(\eta_1(\varphi_s(n_s(\omega_\beta)))) \\ &= \lambda_\beta^\beta t_1 \omega_\beta(t_1)^{-1} \eta_1(\varphi_s(n_s(\omega_\beta))).\end{aligned} \tag{5.4.5}$$

Then we choose t_2 such that $\beta^\vee(t_2) = \lambda_\beta$ for all $\beta \in \Sigma_0$, and (5.4.4) follows with $t = t_1 t_2$.

Observe first of all that if $\omega = \omega_\beta$ then

$$\eta(n_s(\omega)) = \hat{b}^{-1}(\omega) n(\omega),$$

so that

$$\varphi(\eta(n_s(\omega))) = \hat{b}^{-1}(\omega) s_{p/p'}^{-1}(\omega) n_1(\omega).$$

On the other hand,

$$\eta_1(\varphi_s(n_s(\omega))) = s_{p_s/p'_s}^{-1}(\omega)\eta_1(n_{s,1}(\omega)) = s_{p_s/p'_s}^{-1}(\omega)\hat{b}_1^{-1}(\omega) n_1(\omega).$$

Thus the pertinent factor is

$$s_{p/p'}^{-1}(\omega) s_{p_s/p'_s}^{-1}(\omega) \hat{b}^{-1}(\omega)\hat{b}_1(\omega).$$

Since p' is invariant under Ω_0,

$$s_{p/p'}^{-1}(\omega) s_{p_s/p'_s}(\omega) = \prod_{\substack{a>0,\omega^{-1}\alpha<0,p'(\alpha)=1 \\ \alpha \in R-(R_s \cup R_1)}} (-1)^\alpha,$$

with $R_s = R(H)$. Observing that $p'(-\omega^{-1}\alpha) = -p'(\alpha)$, we write this as

$$t_1\omega(t_1)^{-1} \prod_{\alpha \in R^+ - (R_s^+ \cup R_1^+)} (-1)^\alpha = t_1 \omega(t_1)^{-1} \hat{\delta}(\omega)\hat{\delta}_a^{-1}(\omega)$$

with

$$t_1 = \prod_{\substack{\alpha \in R^+ - R_{s,1}^+ \\ p'(\alpha)=-1}} (-1)^\alpha.$$

Equation (5.4.5) now follows from part (b) of Lemma 4.3.A.

The coroots α that appear in the definition of $s_{p/p'}$ and r_{p_Λ} are also coweights of $\widetilde{T}$, so that we may interpret the expression defining them as giving functions $\tilde{s}_{p/p'}$ and $\tilde{r}_{p_\Lambda}$ with values in $\widetilde{T}$. Then $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(n(\omega\omega_T(\sigma)) \times w) = \tilde{s}_{p/p'}^{-1}(\omega\sigma)\tilde{r}_{p_\Lambda}^{-1}(w) n_1(\omega\omega_T(\sigma)) \times w,$$

and even on a group containing $\widetilde{M}$ that we could call ${}^L\widetilde{M}$. That it is a homomorphism is proved in exactly the same way as for φ. The homomorphism $\tilde{\varphi}_s$ is defined in a similar fashion, and the diagrams (5.3.5) are clearly commutative. That the element t can be also lifted to $\tilde{t}$ so that $\tilde{\varphi} \circ \tilde{\eta} = \text{ ad } \tilde{t} \circ \tilde{\eta}_1 \circ \tilde{\varphi}$, is also clear. Finally (5.3.6) is valid because we can define $\tilde{\varphi}$ on ${}^L\widetilde{M}$ and $\tilde{\varphi}_s$ on ${}^L\widetilde{M}_s$.

5.5. Reducing the dimension of G_{der}

Since we are arguing by induction, we can exclude from consideration any group G with subgroups $G^1, \cdots, G^r$ such that $dim\, G^i_{\mathrm{der}} < dim\, G_{\mathrm{der}}$ for all i and such that the truth of the statement (5.1.1) for all of the G^i implies its truth for G. Since that statement is clearly invariant under z-extensions $[\mathbf{K}_1]$ and since any two z-extensions are covered by a common z-extension, we can immediately suppose that G_{der} is simple over F, and indeed that G is obtained by restriction of scalars from a group over a larger field. Then a simple argument that we prefer to omit allows us to assume that G_{der} is absolutely simple.

There are two obvious ways of constructing a root system R_1 between $R_0 = R(H_\epsilon)$ and R invariant under Γ_T and with a map to Λ satisfying the conditions of the critical Lemma 5.1.A. The first is to take $R_1 = \langle \alpha \mid \alpha(\epsilon)^2 = 1\rangle, \Lambda = \{\alpha(\epsilon)\} \subset \bar{F}^\times$ and the map $\alpha \to \alpha(\epsilon)$. The second is to take $R_1 = \langle \alpha \mid \alpha^\vee(s)^2 = 1\rangle, \Lambda = \{\alpha^\vee(s)\} \subset \mathbf{C}^\times$ with Γ_T acting trivially and the map $\alpha \to \alpha^\vee(s)$. Since we can always use the transitivity of Lemma 4.1.A of **[I]** to choose $T = T_{\epsilon_H}$, we can also suppose that (3.3.2) is satisfied. Thus we are reduced to the case that $\alpha(\epsilon) = \pm 1$ and $\alpha^\vee(s) = \pm 1$ for all α.

In general let us call $(G, \epsilon, s, T, \bar{T})$ primitive if

(i) the element ϵ is not central in G, and s is not central in $\widehat{G}$,

(ii) G_{der} is absolutely simple,

(iii) the set $R^{(d)}$ is not empty,

(iv) there is no system $R_1 \subseteq R$ satisfying the conditions of the critical lemma, so that in particular ϵ^2 is central in G and s^2 central in $\widehat{G}$.

We now assume that $(G, \epsilon, s, T, \widehat{T})$ is primitive, and see what this implies, although we shall see that the critical lemma allows us to impose even further constraints on the problem.

On p. 708 of $[\mathbf{L}_2]$ a diagram, either a Dynkin diagram or an extended Dynkin diagram, together with an action of the Galois group was attached to the pair $(\widehat{T}, s)$ (there denoted $({}^LT_0, \kappa)$). In the present case the diagram $D^\vee$ (there denoted D) is the union of $\mathfrak{X}^\vee_0, \mathfrak{X}^\vee_1$ (there denoted $\mathfrak{X}_0, \mathfrak{X}_1$). The same construction can be appliec to T, ϵ yielding $D, \mathfrak{X}_0, \mathfrak{X}_1$.

Lemma 5.5.A. *If $(G, \epsilon, s, T, \bar{T})$ is primitive then both diagrams $D, D^\vee$ are extended Dynkin diagrams, and $\mathfrak{X}_1, \mathfrak{X}_1^\vee$ both consist of a single simple root, whose coefficient in the expansion of the largest root is 2.*

It suffices to treat the diagram D. Then $\mathfrak{X}_0$ is just a set of simple roots for G_ϵ. If it were a subset of a set of simple roots for G then G_ϵ would be the Levi factor of a parabolic subgroup of G. This parabolic subgroup need not be defined over F. Take $R_1 = R(G_\epsilon)$ and let $\lambda(\alpha)$ be the restriction of α to the center of G_ϵ. It is clear that $\lambda(\alpha) \neq \lambda^{-1}(\alpha)$ for $\alpha \in R(G/G_1)$. Thus $\alpha \to \lambda(\alpha)$ satisfies the condition of Lemma 5.1.A, contradicting the assumption of primitivity.

We conclude in particular that D is an extended diagram, and that $\mathfrak{X}_0$ contains the negative of the largest root α_0. Let $\alpha_1, \cdots, \alpha_\ell$ be the simple roots. By construction every root α such that $\alpha(\epsilon) = -1$ is the sum of a single element of $\mathfrak{X}_1$ and an integral linear combination with non-negative coefficients of the elements of $\mathfrak{X}_0$. If $\alpha_j \in \mathfrak{X}_1$ then $-\alpha_j$ assumes the value -1 at ϵ. Thus for some α_k,

$$-\alpha_j = \alpha_k + \sum_{i \neq j,k} c_i \alpha_i - c_0 \alpha_0, \tag{5.5.1}$$

with $c_i \geq 0, c_i = 0$ if $\alpha_i \notin \mathfrak{X}_0$. Since

$$\alpha_0 = \sum_{i=1}^{\ell} b_i \alpha_i \tag{5.5.2}$$

with $b_i > 0$ for all i, the equation (5.5.1) is only possible if $j \neq k$ and $\mathfrak{X}_1 = \{\alpha_j, \alpha_k\}, c_0 = b_j = b_k = 1$ or $j = k$ and $\mathfrak{X}_1 = \{\alpha_j\}, cb_j = 2$. If $\mathfrak{X}_1 = \{\alpha_j\}$ and $b_j = 1$ then (5.5.2) implies that $\alpha_0(\epsilon) = -1$, which is out of the question. Thus to prove the lemma, we need only exclude the possibility that $\mathfrak{X}_1 = \{\alpha_j, \alpha_k\}$.

The action of G_ϵ on the quotient $\mathfrak{g}/\mathfrak{g}_\epsilon$ is a direct sum of distinct irreducible representations ρ. Thus to each $\alpha \in R(G/G_s)$ we can associate the $\rho = \rho(\alpha)$ in which it appears. Clearly Γ acts on the set of these ρ and $\sigma(\rho(\alpha)) = \rho(\sigma\alpha)$ while $\rho(-\alpha) = \tilde{\rho}$, the contragredient of ρ. We extend the action of $\Gamma = \Gamma_T$ to $\Sigma = \Sigma_T$ by demanding that the non-trivial element of $\mathbf{Z}_2 \subseteq \Sigma$ send ρ to $\tilde{\rho}$.

We claim that

$$\mathfrak{g}_1 = \mathfrak{g}_\epsilon + \sum_{\rho \simeq \tilde{\rho}} \mathfrak{g}_\rho,$$

$\mathfrak{g}_\rho$ being the space of ρ, is a subalgebra. The root system of G_1 is obviously a possible R_1, for we can take $\Lambda = \{\rho \mid \rho \simeq \tilde{\rho}\}$.

Each ρ has a minimal weight and by the definition of $\mathfrak{X}_0, \mathfrak{X}_1$ this weight lies in $\mathfrak{X}_1$. Let it be $\alpha_j = \alpha_j(\rho)$. Then $\mathfrak{g}_\rho \simeq \mathfrak{g}_{\tilde\rho}$ if and only if

$$-\alpha_j = \alpha_j + \sum_{\alpha_i \in \mathfrak{X}_0} c_i\alpha_i - c_0\alpha_0, \quad c_i > 0.$$

Comparing this equation with equation (5.5.1) we see that $\mathfrak{g}_1 = \mathfrak{g}$ if $\mathfrak{X}_1$ consists of a single element, and that $\mathfrak{g}_1 = \mathfrak{g}_\epsilon$ if it consists of two elements. We infer first that it is in both cases a subalgebra, and then that $\mathfrak{X}_1$ consists of a single element, that we denote α_j.

The connected components of $\mathfrak{X}_0$ decompose it into the disjoint union of connected diagrams permuted amongst themselves by Γ_T and, in the same way, $\Gamma_{\bar T}$, for we could as well define $\mathfrak{X}_0, \mathfrak{X}_1$ starting with $\bar T$ rather than T. The result is the same. Suppose that $\mathfrak{X}_0$ is the disjoint union of two non-empty subdiagrams $\mathfrak{X}_0', \mathfrak{X}_0''$, each invariant under Γ_T and each the union of connected components of $\mathfrak{X}_0$ itself. We shall use the transitivity of Lemma 4.1.A of **[I]** and the Critical Lemma to show that this case too may be reduced to that of a group of lower dimension, so that we may impose one further restriction,

(v) $\mathfrak{X}_0$ is not the disjoint union of two non-empty subdiagrams $\mathfrak{X}_0', \mathfrak{X}_0''$ each invariant under Γ_T and each the union of connected components of $\mathfrak{X}_0$ itself. Moreover $\mathfrak{X}_0^\vee$ satisfies the same condition.

Supposing that such a decomposition exists we let $\mathfrak{X}_0'$ be the subset containing the negative of the largest root. Since $\mathfrak{X}_0$ is the Dynkin diagram of $\mathfrak{g}_\epsilon$ this leads to a direct sum decomposition,

$$\mathfrak{g}_\epsilon = \mathfrak{g}' \oplus \mathfrak{g}''. \tag{5.5.3}$$

To be definite, we put the center of $\mathfrak{g}_\epsilon$ in $\mathfrak{g}'$. It is pertinent to observe that $\mathfrak{g}_\epsilon$ has the same rank as $\mathfrak{g}$ because $\mathfrak{X}_1$ consists of a single element. We have seen, moreover, that the representation ρ of $\mathfrak{g}$ on $\mathfrak{g}/\mathfrak{g}_\epsilon$ is irreducible.

The decomposition (5.5.3) implies a tensor-product decomposition $\rho \simeq \rho' \otimes \rho''$. Since $\rho \simeq \tilde\rho$ we have $\rho' \simeq \tilde\rho', \rho'' \simeq \tilde\rho''$. Moreover every element α of $R(G/G_\epsilon)$ may be represented as $\alpha = (\alpha', \alpha'')$, where (α', α'') are the weights of ρ' and ρ''.

Lemma 5.5.B (i) *α' is never zero;* (ii) *There is an α for which α'' is not zero. Moreover α'' is zero if and only if α is a rational linear combination of roots in $\mathfrak{X}_0'$.*

Every root of $\mathfrak{X}_0'$ is orthogonal to every root of $\mathfrak{X}_0''$. Thus $\mathfrak{X}_0', \mathfrak{X}_0''$ span mutually orthogonal subspaces of $X^*(T_{\mathrm{der}}) \otimes \mathbf{R}$ whose sum is $X^*(T_{\mathrm{der}}) \otimes \mathbf{R}$. The components α', α'' of α may be identified with its components in the two summands. Thus α'' is zero if and only if α is a (necessarily rational) linear combination of roots in $\mathfrak{X}'_0$ and α' if and only if it is a linear combination of roots in $\mathfrak{X}''_0$. Since $\mathfrak{X}''_0$ is contained in the set of roots simple with respect to a suitable order, a root that is a rational linear combination of the elements of $\mathfrak{X}''_0$ is necessarily an integral linear combination. The first assertion of the lemma follows.

If α'' were 0 for all α then ρ'' would be trivial and the roots in $\mathfrak{X}''_0$ orthogonal to α_j and to all the roots in $\mathfrak{X}'_0$. This contradicts the assumed absolute simplicity.

Take $R_0 = R(H_\epsilon)$, and let R'_0 be the set of $\alpha \in R_0$ that are roots in $\mathfrak{g}'$ and R''_0 the set of $\alpha \in R_0$ that are roots in $\mathfrak{g}''$. If Ω', Ω'' are the Weyl groups of $\mathfrak{g}', \mathfrak{g}''$ then $\Omega_0 = \Omega'_0\Omega''_0$ with $\Omega'_0 = \Omega_0 \cap \Omega', \Omega''_0 = \Omega_0 \cap \Omega''$. In the same way H_ϵ factors as $H'_\epsilon \cdot H''_\epsilon$.

If

$$\Gamma_{\bar{T}} = \{\omega(\sigma)\sigma \mid \sigma \in \Gamma_T\}$$

and if $\omega(\sigma) = \omega'(\sigma)\omega''(\sigma), \omega'(\sigma) \in \Omega'_0, \omega''(\sigma) \in \Omega''_0$ then

$$\Gamma_{T'} = \{\omega'(\sigma)\sigma \mid \sigma \in \Gamma_T\}$$

is a subgroup of $\Omega_0 \rtimes \Gamma_T$ and

$$\Gamma_{\bar{T}} = \{\omega''(\sigma)\sigma' \mid \sigma' \in \Gamma_{T'}\}.$$

As the notation implies, $\Gamma_{T'}$ gives the Galois action for a group T' that could be substituted either for $\bar{T}$ or for T. By Lemma 4.1.A of [**I**] it suffices to prove Theorem 5.1.A for the pairs (T, T') and $(T', \bar{T})$. To deal with the pair (T, T') we apply the critical lemma, taking the R_0 that appears there to be R'_0 and R_1 to be the union of $R(G_\epsilon)$ and

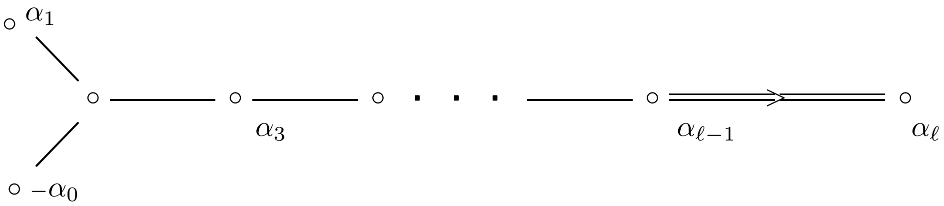
$$\{\alpha \in R(G/G_\epsilon) \mid \alpha'' = 0\}$$

and introducing yet a third Cartan subgroup, whose projection on H''_ϵ is stably conjugate to the projections of T or T', but whose projection on H'_ϵ is maximal split. Then λ is defined on $R(G/G_1)$ by $\lambda(\alpha) = \alpha''$. For the pair $T', \bar{T}$ we take the R_0 of the Critical Lemma to be R''_0. The set R_1 is $R(G_\epsilon)$ and λ is defined on $R(G/G_1)$ by $\lambda(\alpha) = \alpha'$. It is again necessary to introduce a third supplementary torus and to use transitivity.

Lemma 5.5.C. *Suppose that $(G, \epsilon, s, T, \bar{T})$ is primitive and that in addition the condition (v) is satisfied. Then all roots in $R(G)$ are of the same length.*

We examine the possibilities for groups with roots of unequal length, using the appendices of [**B**].

(i) B_ℓ: The diagram is



The dual diagram is

and

$$\alpha_0 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_\ell, \beta_0 = 2\beta_1 + \ldots 2\beta_{\ell-1} + \beta_\ell.$$

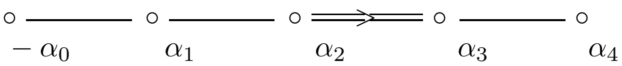
Condition (v) and Lemma 5.5.A imply that $\mathfrak{X}_1 = \{\alpha_\ell\}$, that $\ell = 2k$ is even, that $\mathfrak{X}_1^\vee = \{\beta_k\}$ and that Γ_T acts nontrivially on the diagram. We conclude that $\alpha(\epsilon) = 1$ if α is long and that $\alpha(\epsilon) = 1$ if α is short. On the other hand, $\alpha^\vee(s) = -1$ only if $\alpha^\vee$ is short, although $\alpha^\vee(s)$ can be 1 for some short roots. Since

$$R^{(d)} = \{\alpha \mid \alpha(\epsilon) = -1, \alpha^\vee(s) = -1\},$$

it is empty, and groups G of type B_ℓ are excluded.

(ii) C_ℓ: By the symmetry of the conditions, groups of type C_ℓ are also excluded.

(iii) F_4: The diagram is

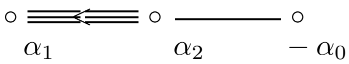


and

$$\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Then $\mathfrak{X}_1 = \{\alpha_4\}$. Thus, with the notation of Table VIII of [**B**], $\alpha(\epsilon) = -1$ if and only if $\alpha = \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$, and therefore, in particiular, only if α is short. By duality $\alpha^\vee(s) = -1$ only if α is long, so that $R^{(d)}$ is empty.

(iv) G_2: The diagram is



and

$$\alpha_0 = 3\alpha_1 + 2\alpha_2.$$

Thus $\mathfrak{X}_1 = \{\alpha_2\}$. This however, is incompatible with condition (v).

§6. The Order-Two Case

6.1. Introduction

According to the last section it remains only to prove (5.1.1) under the following assumptions:

$$\alpha(\epsilon) = \pm 1, \alpha^\vee(s) = \pm 1, \alpha \in R; \tag{6.1.1}$$

and

$$\text{all roots in } R \text{ are the same length.} \tag{6.1.2}$$

In this case the roots of type (d) are those $\alpha \in R$ for which $\alpha(\epsilon) = -1 = \alpha^\vee(s)$.

We begin by describing formulas for $\Theta_1^{(d)}$ and $\lim \Theta_2^{(d)}$. Let $T_{(s)}$ be the F-torus with dual $\widehat{T}_{(s)} = \widehat{\bar{T}}_{\text{ad}}/\{1, s_{\bar{T}}\}$ endowed with the action induced by $\Gamma_{\bar{T}}$. Then dual to

$$1 \longrightarrow \{1, s_{\bar{T}} \longrightarrow \widehat{\bar{T}}_{\text{ad}} \longrightarrow \widehat{T}_{(s)} \longrightarrow 1$$

we have

$$1 \longrightarrow A \longrightarrow T_{(s)} \longrightarrow \bar{T}_{\text{sc}} \longrightarrow 1$$

where A has order exactly two. Each of the terms in $v^{(d)}$, and hence also $v^{(d)}$ itself, can be constructed in $\bar{T}_{\text{sc}}$. We do so without change in notation. A lifting $\tilde{v}$ of $v^{(d)}$ to a cochain in $T_{(s)}(\bar{F})$ will be described in (6.2). The coboundary $\partial\tilde{v}$ takes values in A and so defines an element ϵ_{I} of $H^2(\Gamma, A) = \{\pm 1\}$. Then

$$\Theta_{\text{I}}^{(d)}(\gamma_H, \bar{\gamma}_H) = \epsilon_1. \tag{6.1.3}$$

To check this we note that the inclusion $\{1, s_{\bar{T}}\} \hookrightarrow \widehat{\bar{T}}_{\text{ad}}$ yields

$$H^{-1}(X_*(\bar{T}_{\text{sc}})) \longrightarrow H^{-1}(\mathbf{Z}/2\mathbf{Z}).$$

We have a commutative diagram

$$\begin{array}{ccc} H^1(\bar{T}_{\text{sc}}) & \longrightarrow & H^2(A) \\ \wr| & & \wr| \\ H^{-1}(X_*(\bar{T}_{\text{sc}})) & \longrightarrow & H^{-1}(\mathbf{Z}/2\mathbf{Z}) \end{array}$$

(see [**L1**] or [**M**] and since $\Theta_{\mathrm{I}}^{(d)}(\gamma_H, \bar{\gamma}_H)$ is given by evaluating $v^{(d)}$, as an element of $H^{-1}(X_*(\bar{T}_{\mathrm{sc}}))$, on s, (6.1.3) follows.

The term $\lim \Theta_2^{(d)}$ is handled similarly. Following (4.5) it is given as

$$\langle (a^{(d)}(w), a^{(d)}(w)\bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1}), (\epsilon, \epsilon^{-1})\rangle,$$

the pairing being that for $H^1(W, \widehat{T} \times \widehat{\bar{T}})$ and $T(F) \times \bar{T}(F)$. We shall move to the torus $\widehat{S} = \widehat{T} \times \widehat{\bar{T}}_{\mathrm{sc}}$ from (3.5). It has the Galois action

$$\bar{\sigma} : (t_1, t_2) \longrightarrow (\sigma(t_1), \alpha_\omega(\sigma(t_1))\omega\sigma(t_2)), \tag{6.1.4}$$

where $\bar{\sigma} = \omega \times \sigma$. We also have $T \times \bar{T} \to S$ over F and dually $\widehat{S} \to \widehat{T} \times \widehat{\bar{T}}$ (see (3.5) for definitions). Recall that

$$w \longrightarrow (a^{(d)}(w), \bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1})$$

is a cocycle in $\widehat{S}$ with image

$$w \longrightarrow (a^{(d)}(w), a^{(d)}(w)\bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1})$$

under

$$H^1(W, \widehat{S}) \longrightarrow H^1(W, \widehat{T} \times \widehat{\bar{T}}).$$

More precisely, this was shown in (3.5) for cochains without the superscript (d), but a factoring argument as in (4.5) allows us to consider just the contributions of type (d). We may therefore compute $\lim \Theta_2(\gamma_H, \bar{\gamma}_H)$ as

$$\langle (a^{(d)}(w), \bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1}), \epsilon_S\rangle$$

where ϵ_S is the image of $(\epsilon, \epsilon^{-1})$ under $T(F) \times \bar{T}(F) \to S(F)$. Note that $\epsilon_S = (1, \epsilon_{\mathrm{ad}})$ where $\epsilon_{\mathrm{ad}} = \epsilon_{\mathrm{ad}}^{-1}$ is the image of ϵ under $\bar{T} \to \bar{T}_{\mathrm{ad}}$.

Next define a torus $T_{(\epsilon)}$ over F by

$$1 \longrightarrow \{1, \epsilon_{\mathrm{ad}}\} \longrightarrow \bar{T}_{\mathrm{ad}} \longrightarrow T_{(\epsilon)} \longrightarrow 1.$$

Then we have

$$1 \longrightarrow B \longrightarrow \widehat{T}_{(\epsilon)} \longrightarrow \widehat{\bar{T}} \longrightarrow 1,$$

with B of order 2, and

$$1 \longrightarrow 1 \times B \longrightarrow \widehat{T} \times \widehat{T}_{(\epsilon)} \longrightarrow \widehat{T} \times \widehat{\bar{T}}_{\mathrm{sc}} \longrightarrow 1,$$

or, more simply,

$$1 \longrightarrow B \longrightarrow \widehat{T} \times \widehat{T}_{(\epsilon)} \longrightarrow \widehat{S} \longrightarrow 1.$$

In $\widehat{T} \times \widehat{T}_{(\epsilon)}$ the Galois action is given by (6.1.4) with α_ω now taking values in $\widehat{T}_{(\epsilon)}$, as is possible because the roots of Σ_0 lie in $X_*(\widehat{T}_{(\epsilon)})$. Dual to this is

$$1 \longrightarrow \{1, \epsilon_S\} \longrightarrow S \longrightarrow T \times T_{(\epsilon)} \longrightarrow 1.$$

In the next section we will define a lifting of $\bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1} \in \widehat{\overline{T}}_{sc}$ to $\widehat{T}_{(\epsilon)}$ and thus a lifting of the cocycle

$$u^{(d)}(w) = (a^{(d)}(w), \bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1})$$

to a cochain $\tilde{u}(w)$ with values in $\widehat{T} \times \widehat{T}_{(\epsilon)}$. The coboundary $\partial\tilde{u}$ takes values in B and factors through $W \to \Gamma$. It then defines an element ϵ_2 of $H^2(\Gamma, B) = \{\pm 1\}$ and

$$\lim \Theta_2(\gamma_H, \bar{\gamma}_H) = \epsilon_2. \tag{6.1.5}$$

This follows from the commutativity of the diagram below (see the observations at the end of Section 6.5).

$$\begin{array}{ccccc} H^1(W_F, \widehat{T} \times \widehat{T}_{(\epsilon)}) & \longrightarrow & H^1(W_F, \widehat{S}) & \longrightarrow & H^2(\Gamma, B) \\ \wr| & & \wr| & & \wr| \\ (T \times T_{(\epsilon)})(F)^* & \longrightarrow & S(F)^* & \longrightarrow & \{1, \epsilon_S\}^*. \end{array}$$

We observe that $A \simeq B \simeq \mu_2$ and that the homomorphism

$$H^2(\Gamma, \mu_2) \longrightarrow H^2(\Gamma, \mu_4)$$

is injective. This is essential in all that follows.

6.2. Liftings

Fix $i = i_{\bar{F}} \in \bar{F}^\times$ such that $i^2 = -1$. We write $v^{(d)}(\bar{\sigma})$ as $\bar{y}^{(d)}(\bar{\sigma})^{-1}v_*(\omega \times \sigma)$, where

$$v_*(\omega \times \sigma) = \tau^{(d)}(\omega, \sigma)\omega(y^{(d)}(\sigma))b^{(d)}(\omega)^{-1}$$

with $\bar{\sigma} = \omega \times \sigma$, and lift term by term. First

$$\bar{y}^{(d)}(\bar{\sigma}) = \prod_{\bar{\mathcal{O}} \subseteq R^{(d)}} \bar{y}_{\bar{\mathcal{O}}}(\bar{\sigma})$$

where

$$\bar{y}_{\bar{\mathcal{O}}}(\bar{\sigma}) = \prod_{\substack{\alpha \in \bar{\mathcal{O}} \\ a > 0 \\ \bar{\sigma}^{-1}a < 0}} a_\alpha^{\alpha^\vee}.$$

We shall define $\tilde{\tilde{y}}_{\bar{\mathcal{O}}}(\bar{\sigma}) \in T_{(s)}$ and then set $\tilde{y}^{(d)}(\bar{\sigma}) = \prod_{\bar{\mathcal{O}}} \tilde{\tilde{y}}_{\bar{\mathcal{O}}}(\bar{\sigma})$.

To fix a-data for all asymmetric orbits, choose one orbit, say $\bar{\mathcal{O}}$, from each pair $\pm\bar{\mathcal{O}}$ and set $a_\alpha = -1$ for $\alpha \in \bar{\mathcal{O}}, a_\alpha = 1$ for $\alpha \in -\bar{\mathcal{O}}$. Define

$$\tilde{\tilde{y}}_{\bar{\mathcal{O}}}(\bar{\sigma}) = \prod_{\substack{a \in \bar{\mathcal{O}} \\ a>0 \\ \bar{\sigma}^{-1}a<0}} i^{2\alpha^\vee}$$

and

$$\tilde{\tilde{y}}_{-\bar{\mathcal{O}}}(\bar{\sigma}) = 1.$$

On the other hand, if $\bar{\mathcal{O}}$ is symmetric fix $\alpha > 0$ in $\bar{\mathcal{O}}$ and then representatives $\sigma_1 = 1, \sigma_2, \cdots, \sigma_n$ for $\Gamma_{\pm\alpha}\backslash\Gamma$ such that $\alpha_j = \bar{\sigma}_j^{-1}\alpha > 0$ for each j. Thus $\bar{\mathcal{O}} = \{\pm\alpha_j : 1 \leq j \leq n\}$. Fix $\sqrt{a_\alpha} \in \bar{F}^\times$ and define $\sqrt{a_{\alpha_j}} = \sigma_j^{-1}\sqrt{a_\alpha}$. We have

$$\bar{y}_{\bar{\mathcal{O}}}(\bar{\sigma}) = \prod_{\substack{j \\ \bar{\sigma}^{-1}\alpha_j<0}} a_{\alpha_j}^{\alpha_j^\vee}.$$

Set

$$\tilde{\tilde{y}}_{\bar{\mathcal{O}}}(\bar{\sigma}) = \prod_{\substack{j \\ \bar{\sigma}^{-1}a_j>0}} (\sqrt{a_{\alpha_j}})^{2\alpha_j^\vee}.$$

It remains to lift the terms in $v_*(\omega \times \sigma)$. We define $\tilde{y}^{(d)}(\sigma)$ in the same way we did $\tilde{\tilde{y}}^{(d)}(\bar{\sigma})$ and then lift $\omega(y^{(d)}(\sigma))$ as $\omega(\tilde{y}^{(d)}(\sigma))$. The term $\tau^{(d)}(\omega, \sigma)$ is a product of elements $(-1)^{\alpha^\vee}$ over certain α of type (d). Lift $\tau^{(d)}(\omega, \sigma)$ to the corresponding product $\tilde{\tau}^{(d)}(\omega, \sigma)$ of elements $i^{2\alpha^\vee}$. Recall from (4.5) that $b^{(d)}(\omega) = b_\omega^{(d)}\delta^{(d)}(\omega)$. The factor $\delta^{(d)}(\omega)$ is a product of terms $(-1)^{\alpha^\vee}$ each of which we lift to $i^{2\alpha^\vee}$. Recall from (4.3) that $b_\omega^{(d)}$ is of the form $\prod_k x_k^{\beta_k^\vee}$, where $x_k \in \bar{F}^\times, \beta_k \in \Sigma_0$. Such an element is naturally lifted to $T_{(s)}$ (by the same formula) since $\Sigma_0 \subset X^*(T_{(s)})$.

On the dual side we have to lift the cocycle

$$(a^{(d)}(w), \bar{c}^{(d)}(w)^{-1}\hat{z}^{(d)}(\omega, w)^{-1}) = (1, \bar{c}^{(d)}(w)^{-1})u_*(\omega \times w)$$

with

$$\begin{aligned} u_*(\omega \times w) &= (a^{(d)}(w), \hat{z}^{(d)}(\omega, w)^{-1}) \\ &= (a^{(d)}(w), \hat{\tau}^{(d)}(\omega, \sigma)\omega(c^{(d)}(w))\alpha_\omega(a^{(d)}(w))\hat{b}^{(d)}(\omega)^{-1}). \end{aligned}$$

Now $i = i_{\mathbf{C}}$ will denote a square root of -1 in $\mathbf{C}$. We start with $\bar{c}^{(d)}(w)$, lifting it to $\tilde{\bar{c}}^{(d)}(w)$ in $\widehat{T}_{(\epsilon)}$. Then $(1, \bar{c}(w)^{-1})$ is to be lifted to $(1, \tilde{\bar{c}}^{(d)}(w)^{-1})$.

The term $\bar{c}^{(d)}(w)$ is a product $\prod r_{\pm\bar{\mathcal{O}}}(w)$ over pairs $\pm\bar{\mathcal{O}}$ of orbits of type (d). We proceed term by term in this product. If $\bar{\mathcal{O}}$ is asymmetric then we take χ-data for $\pm\bar{\mathcal{O}}$ to be trivial. There is still a nontrivial contribution to $r_{\pm\bar{\mathcal{O}}}$, namely the term $s_{p/q}$ of [**I**, 2.4]. Here p is the gauge on $\pm\bar{\mathcal{O}}$ defined by the fixed order on the roots and q is given by $q(\alpha) = 1$ if and only if $\alpha \in \bar{\mathcal{O}}$. Thus ([**I**, 2.4])

$$s_{p/q}(\sigma) = \prod_{\substack{\alpha>0 \\ \bar{\sigma}^{-1}\alpha<0 \\ \alpha\in\bar{\mathcal{O}}}} (-1)^{\alpha}$$

and $r_{\pm\bar{\mathcal{O}}}(w) = s_{p/q}(\sigma)$ if $w \to \sigma$ under $W \to \Gamma$. We define $\tilde{r}_{\pm\bar{\mathcal{O}}}(w)$ to be the product $i^{2\alpha^\vee}$ over the same roots.

If $\bar{\mathcal{O}}$ is symmetric the contribution $s_{p/q}$ is trivial because we have arranged that $p = q$. On the other hand, the χ-data $\{\chi_\alpha\}$ now are nontrivial. Following [**I**, 2.5] we write

$$r_{\bar{\mathcal{O}}}(w) = \prod_{j=1}^{n} \chi_\alpha(v_0(u_j(w)))^{\alpha_j} = \prod_{j=1}^{n} s(u_j(w))^{\alpha_j}.$$

Fix some square root of the complex number $s(u_j(w))$, denoting it by $\sqrt{s}(u_j(w))$, and then set

$$\tilde{r}_{\bar{\mathcal{O}}}(w) = \prod_{j=1}^{n} \sqrt{s}(u_j(w))^{2\alpha_j}.$$

It remains to lift $u_*(\omega \times w)$. Again we proceed term by term. We lift $c^{(d)}(w)$ as we did $\bar{c}^{(d)}(w), \hat{\tau}^{(d)}(\omega,\sigma)$ as we did $\tau^{(d)}(\omega,\sigma)$, and $\hat{b}^{(d)}(\omega)$ as we did $b^{(d)}(\omega)$. We shall regard α_ω as taking values in $\widehat{T}_{(\epsilon)}$. Then

$$\tilde{u}_*(\omega \times w) = (a^{(d)}(w), \tilde{\hat{\tau}}^{(d)}(\omega,\sigma)\omega(\tilde{c}^{(d)}(w))\alpha_\omega(a^{(d)}(w))\tilde{\hat{b}}^{(d)}(\omega)^{-1}).$$

6.3. Some coboundaries

For α of type (d) the element $(-1)^{2\alpha^\vee}$ of A is nontrivial and all such elements coincide. Similarly $B = \{(\pm 1)^{2\alpha}; \alpha \text{ type } (d)\}$. We identify both A and B with $\mu_2 = \{\pm 1\}$. At the same time we identify $i_{\mathbf{C}}$ and $i_{\bar{F}}$, and then the subgroup B' of $\widehat{T}_{(\epsilon)}$ generated by $\{i^{2\alpha} : \alpha \text{ type } (d)\}$ with the subgroup A' of $T_{(s)}$ generated by $\{i^{2\alpha^\vee} : \alpha \text{ type } (d)\}$ (recall that all roots have the same length); this of course does not respect the action of Γ_T or $\Gamma_{\widehat{T}}$. According to (6.1.3) and (6.1.5) we have to compute the 2-cocycle $\partial\tilde{v}\partial\tilde{u}$ with values in μ_2. This coincides with $\partial\tilde{v}/\partial\tilde{u}$ which is more convenient for calculations. Where needed, we inflate cocycles of Γ to W without mentioning it in notation.

In this section we investigate the contributions to $\partial\tilde{v}/\partial\tilde{u}$ from $\tilde{v}_*$ and $\tilde{u}_*$. First, v_* and $\tilde{v}_*$ are well defined on $\Omega_0 \rtimes \Gamma_T$. From Lemma 4.2.B we have

$$\partial v_*(\omega_1\rho, \omega_2\sigma) = \tau^{(d)}(\omega_1\rho, \omega_2\sigma)^{-1}$$

and so if we lift $\tau^{(d)}$ to $\tilde{\tau}^{(d)}$ by replacing each term $(-1)^{\alpha^\vee}$ by $i^{2\alpha^\vee}$ then we conclude that $\partial\tilde{v}_*\tilde{\tau}^{(d)}$ takes values in $A = \mu_2$. Similarly u_* and $\tilde{u}_*$ are well defined on $\Omega_0 \rtimes W$ and Lemma 4.2.A shows that $\partial\tilde{u}_*\tilde{\hat{\tau}}^{(d)}$ takes values in $B = A = \mu_2$. Here $\omega \times \sigma \in \Omega_0 \rtimes W$ acts on $\widehat{T} \times \widehat{T}_{(\epsilon)}$ as in (6.1.4). Hence $\partial\tilde{v}_*, \partial\tilde{u}_*$ take values in $B' = A'$, and $\partial\tilde{v}_*/\partial\tilde{u}_*$ takes values in A since $\tilde{\hat{\tau}}^{(d)}$ is identified with $\tilde{\tau}^{(d)}$. The cochain $\partial\tilde{v}_*/\partial\tilde{u}_*$ is not in general a cocycle (since the operator ∂ in the numerator is that for the Galois action on $T_{(s)}$ and the operator in the denominator is for the dual algebraic action). We calculate its coboundary as the coboundary of $\partial\tilde{v}_*\tilde{\tau}^{(d)}/\partial\tilde{u}_*\tilde{\hat{\tau}}^{(d)}$ and thus as

$$(\omega_1\rho, \omega_2\sigma, \omega_3\tau) \longrightarrow \rho\tilde{\tau}^{(d)}(\omega_2\sigma, \omega_3\tau)/\rho\tilde{\hat{\tau}}^{(d)}(\omega_2\sigma, \omega_3\tau)$$

which equals

$$\prod_{\substack{\alpha<0, \alpha\in R^{(d)} \\ \sigma^{-1}\omega_2^{-1}\alpha<0 \\ \tau^{-1}\omega_3^{-1}\sigma^{-1}\omega_2^{-1}\alpha>0}} \left(\frac{\rho i}{i}\right)^{2\alpha^\vee}.$$

When A is identified as μ_2 this becomes

$$(\omega_1\rho, \omega_2\sigma, \omega_3\tau) \longrightarrow \left(\frac{\rho i}{i}\right)^{N(\omega_2\sigma, \omega_3\tau)},$$

where $N(\omega_2\sigma, \omega_3\tau)$ is the number of roots α appearing in the product above.

We embed μ_2 in $\mu_4 = \mu_4(\bar{F})$, the group of fourth roots of unity in $\bar{F}$. The group $\Omega_0 \rtimes W$ acts on μ_4 through Γ. Fix $\xi \in \bar{F}^\times$ such that $\xi^2 = i$ and consider

$$\mathcal{M}(\omega_1\rho, \omega_2\sigma) = \left(\frac{\xi}{\rho\xi}\right)^{N(\omega_2\sigma)},$$

where $N(\omega_2\sigma)$ is the number of α of type (d) for which $\alpha > 0, \sigma^{-1}\omega_2^{-1}\alpha < 0$. The cochain $\mathcal{M}$ takes values in μ_4. Observing that

$$N(\omega_1\sigma) + N(\omega_2\tau) - N(\omega_1\sigma\omega_2\tau) = 2N(\omega_1\sigma, \omega_2\tau)$$

we find that $\partial\mathcal{M}$ is the inverse of (6.3.1). Let $\theta = \mathcal{M}\partial\tilde{v}^*/\partial\tilde{u}^*$. Then we have proved:

Lemma 6.3.A.

θ is a 2-cocycle of $\Omega_0 \rtimes \Gamma_T$ with values in μ_4.

Lemma 6.3.B.

$$\theta(\omega_1 \times \rho, \omega_2 \times \sigma) = \theta(\rho, \sigma).$$

Proof. Because θ is a 2-cocycle it defines an extension of $\Omega_0 \rtimes \Gamma_T$ by μ_4. The extension is generated by μ_4 and elements $\hat{\eta}, \eta \in \Omega_0 \rtimes \Gamma_T$, with

$$\hat{\eta} x \hat{\eta}^{-1} = \eta(x), \hat{\eta}_1 \hat{\eta}_2 = \theta(\eta_1, \eta_2)(\eta_1\eta_2)\hat{}.$$

We have to show that

$$(\omega_1\rho)\hat{}(\omega_2\sigma)\hat{} = \theta(\rho, \sigma)(\omega_1\rho\omega_2\sigma)\hat{}.$$

For this it is sufficient to verify

$$\hat{\omega}_1\hat{\omega}_2 = (\omega_1\omega_2)\hat{} \tag{6.3.2}$$

$$\hat{\omega}\hat{\sigma} = (\omega\sigma)\hat{} \tag{6.3.3}$$

and

$$\hat{\sigma}\hat{\omega}\hat{\sigma}^{-1} = (\sigma\omega\sigma^{-1})\hat{}. \tag{6.3.4}$$

Moreover (6.3.4) need only be verified for $\omega = \omega_\beta, \beta \in \Sigma_0$. Indeed, assume that (6.3.4) holds for ω_1 and ω_2. Then

$$\begin{aligned} \hat{\sigma}(\omega_1\omega_2)\hat{}\hat{\sigma}^{-1} &= \hat{\sigma}\hat{\omega}_1\hat{\omega}_2\hat{\sigma}^{-1} \\ &= \hat{\sigma}\hat{\omega}_1\hat{\sigma}^{-1}\hat{\sigma}\hat{\omega}_2\hat{\sigma}^{-1} = (\sigma\omega_1\sigma^{-1})\hat{}(\sigma\omega_2\sigma^{-1})\hat{} \\ &= (\sigma\omega_1\omega_2\sigma^{-1})\hat{} \end{aligned}$$

so that (6.3.4) is also valid for $\omega_1\omega_2$.

If all these conditions are satisfied then

$$(\omega_1\rho)\hat{}(\omega_2\sigma)\hat{} = \hat{\omega}_1\hat{\rho}\hat{\omega}_2\hat{\sigma} = \hat{\rho}(\rho^{-1}\omega_1\rho)\hat{}\hat{\omega}_2\hat{\sigma}$$

or

$$\hat{\rho}\hat{\sigma}(\sigma^{-1}\rho^{-1}\omega_1\rho\omega_2\sigma)\hat{} = \theta(\rho, \sigma)(\rho\sigma)\hat{}((\rho\sigma)^{-1})\hat{}(\omega_1\rho\omega_2\rho^{-1})\hat{}(\rho\sigma)\hat{}$$

which is

$$\theta(\rho, \sigma)(\omega_1\rho\omega_2\sigma)\hat{}$$

and the lemma follows.

The three conditions may be rewritten as

$$\theta(\omega_1, \omega_2) = 1 \tag{6.3.5}$$

$$\theta(\omega, \sigma) = 1 \tag{6.3.6}$$

$$\theta(\sigma, \omega)\sigma(\omega)(\theta(\sigma, \sigma^{-1}))^{-1}\theta(\sigma\omega, \sigma^{-1}) = 1. \tag{6.3.7}$$

First observe that $\mathcal{M}(\omega_1, \omega_2) = \mathcal{M}(\omega, \sigma) = 1$. In addition, we calculate $\mathcal{M}(\sigma, \omega)\sigma(\omega)(\mathcal{M}(\sigma, \sigma^{-1})^{-1})$ $\mathcal{M}(\sigma\omega, \sigma^{-1})$ as

$$\left(\frac{\xi}{\sigma\xi}\right)^{N(\omega)} \left(\frac{\xi}{\sigma\xi}\right)^{-N(\sigma^{-1})} \left(\frac{\xi}{\sigma\xi}\right)^{N(\sigma^{-1})} = \left(\frac{\xi}{\sigma\xi}\right)^{N(\omega)}.$$

But $N(\omega)$ is twice the number of roots of type (d) in R_β^+; so we may rewrite this expression as $\left(\frac{i}{\sigma i}\right)^{[R_\beta^+(d)]}$.

Turning now to $\partial\tilde{v}_*$ and $\partial\tilde{u}_*$, each taking values in A', and $\mathcal{L} = \partial\tilde{v}_*/\partial\tilde{u}_*$ taking values in $A = \mu_2$, we have

$$\mathcal{L}(\omega_1, \omega_2) = \partial\tilde{\tilde{b}}(\omega_1, \omega_2)/\partial\tilde{b}(\omega_1, \omega_2) = 1$$

and $\mathcal{L}(\omega, \sigma)$ is the quotient of

$$\tilde{b}(\omega)^{-1}\omega(\tilde{y}(\sigma))\tilde{\tau}(\omega, \sigma)^{-1}\omega(\tilde{y}(\sigma))^{-1}\tilde{b}(\omega)$$

by

$$\tilde{\tilde{b}}(\omega)^{-1}\omega(\tilde{c}(w))\alpha_\omega(a(w))\tilde{\tilde{\tau}}(\omega, \sigma)^{-1}\omega(\tilde{c}(w))^{-1}\alpha_\omega(a(w))^{-1}\tilde{\tilde{b}}(\omega),$$

and this is

$$\tilde{\tau}(\omega, \sigma)^{-1}\tilde{\tilde{\tau}}(\omega, \sigma) = 1.$$

Here, and below, the superscript (d) has been omitted from the notation. Similarly we find

$$\mathcal{L}(\sigma\omega, \sigma^{-1})\sigma(\omega)(\mathcal{L}(\sigma, \sigma^{-1})) = 1,$$

and so it remains to compute $\mathcal{L}(\sigma, \omega)$ in the case $\omega = \omega_\beta, \beta \in \Sigma_0$. We obtain $\mathcal{L}(\sigma, \omega)$ as the quotient of

$$\tilde{y}(\sigma)\sigma(\tilde{b}(\omega)^{-1})\tilde{\tau}(\sigma(\omega), \sigma)^{-1}\sigma(\omega)(\tilde{y}(\sigma))^{-1}\tilde{b}(\sigma(\omega))$$

by

$$\tilde{c}(w)\sigma(\tilde{\hat{b}}(\omega))^{-1}\tilde{\hat{\tau}}(\sigma(\omega),\sigma)^{-1}\sigma(\omega)(\tilde{c}(w)^{-1})\tilde{\hat{b}}(\sigma(\omega))\tilde{\alpha}_{\sigma(\omega)}(a(w))^{-1}.$$

We may cancel $\tilde{\tau}$ and $\tilde{\hat{\tau}}$ and then insert $\tilde{e}(\omega,\sigma)$ in the numerator and $\tilde{\hat{e}}(\omega,\sigma)$ in the deonominator. Recall that $e(\omega,\sigma)$ and $\hat{e}(\omega,\sigma)$ were defined in (4.1). We consider only the contribution from roots of type (d) and lift to $\tilde{e}$ and $\tilde{\hat{e}}$ by lifting each term $(-1)^{\alpha^\vee}$ or $(-1)^{\alpha}$ to $i^{2\alpha^\vee}=i^{2\alpha}$.

We also factor $\tilde{b}(\omega)$ as $\tilde{b}_\omega\tilde{\delta}(\omega)$ and similarly $\tilde{\hat{b}}(\omega)$. This yields

$$\mathcal{L}(\sigma,\omega)=E_1E_2E_3(\widehat{E}_1\widehat{E}_2\widehat{E}_3)^{-1}$$

where

$$\begin{aligned}
E_1 &= \tilde{b}_{\sigma(\omega)}\sigma(\tilde{b}_\omega^{-1}),\\
E_2 &= \tilde{y}(\sigma)\sigma(\omega)(\tilde{y}(\sigma))^{-1}\\
E_3 &= \tilde{\delta}(\sigma(\omega))\sigma(\tilde{\delta}(\omega))^{-1}\tilde{e}(\omega,\sigma)\\
\widehat{E}_1 &= \tilde{\hat{b}}_{\sigma(\omega)}\sigma(\tilde{\hat{b}}_\omega{}^{-1})\\
\widehat{E}_2 &= \tilde{c}(w)\sigma(\omega)(\tilde{c}(w))^{-1}\alpha_\omega(a(w)^{-1})
\end{aligned}$$

and

$$\widehat{E}_3=\tilde{\hat{\delta}}(\sigma(\omega))\sigma(\tilde{\hat{\delta}}(\omega))^{-1}\tilde{\hat{e}}(\omega,\sigma).$$

Recall that $\omega=\omega_\beta$. To simplify this expression for $\mathcal{L}(\sigma,\omega)$ we need some preparation.

Lemma 6.3.C.

There exists $\eta_\beta=\pm 1$ such that

$$\delta(\sigma(\omega_\beta))\sigma(\delta(\omega_\beta))^{-1}e(\omega_\beta,\sigma)=\eta_\beta^{\sigma\beta^\vee}$$

and

$$\widehat{\delta}(\sigma(\omega_\beta))\sigma(\widehat{\delta}(\omega_\beta))^{-1}\hat{e}(\omega_\beta,\sigma)=\eta_\beta^{\sigma\beta}.$$

Proof. This is a long calculation in which we repeatedly use the fact that

$$(-1)^{\alpha^\vee}(-1)^{-\omega_{\sigma\beta}(\alpha^\vee)}=(-1)^{\langle\alpha^\vee,\sigma\beta\rangle\sigma\beta^\vee}.$$

First, $e(\omega,\sigma)=\tau(\sigma,\omega)\tau(\sigma\omega,\sigma^{-1})\sigma(\omega)(\tau(\sigma,\sigma^{-1})^{-1})$. The last two factors are respectively the product over $\alpha>0,\omega^{-1}\sigma^{-1}\alpha<0,\sigma\omega^{-1}\sigma^{-1}\alpha>0$ and that over $\sigma\omega^{-1}\sigma^{-1}\alpha>0,\omega^{-1}\sigma^{-1}\alpha>0$ of $(-1)^{\alpha^\vee}$. When multiplied together they yield the product over $\alpha<0,\omega^{-1}\sigma^{-1}\alpha<0,\sigma\omega^{-1}\sigma^{-1}\alpha>0$. The first factor is the product over $\alpha > 0, \sigma^{-1}\alpha < 0, \omega^{-1}\sigma^{-1}\alpha > 0$ of $(-1)^{\alpha^\vee}$. For $e(\omega, \sigma)$ we then have the products over

$$\alpha > 0, \omega^{-1}\sigma^{-1}\alpha > 0, \sigma^{-1}\alpha > 0, \sigma\omega^{-1}\sigma^{-1}\alpha < 0 \tag{1}$$

and

$$\alpha > 0, \omega^{-1}\sigma^{-1}\alpha > 0, \sigma^{-1}\alpha < 0, \sigma\omega^{-1}\sigma^{-1}\alpha > 0 \tag{2}$$

of $(-1)^{\alpha^\vee}$.

The contributions of $\sigma(\delta(\omega_\beta)), \delta(\sigma(\omega_\beta))$ to the left side of the first equation are the products over, respectively,

$$\sigma^{-1}\alpha > 0, \omega^{-1}\sigma^{-1}\alpha < 0, \sigma^{-1}\alpha \in R_\beta^+ \tag{3}$$

and

$$\alpha > 0, \sigma\omega^{-1}\sigma^{-1}\alpha < 0, \alpha \in R_\beta^+ \tag{4}$$

of the same term. Here $\omega = \omega_\beta$. The left side of the second equation is an exactly analogous product, roots replacing coroots in the exponents.

Consider the contribution of $\{\pm\alpha, \pm\alpha'\}$, where $\alpha' = -\sigma\omega^{-1}\sigma^{-1}\alpha$. If α and α' have opposite signs then these can be contributions only to (2) and (3). Taking $\alpha > 0$ and supposing that $\omega^{-1}\sigma^{-1}\alpha$ and $\sigma^{-1}\alpha$ have opposite signs, we see immediately that these contributions are:

$$\begin{aligned}
&\sigma^{-1}\alpha > 0, \sigma^{-1}\alpha \in R_\beta^+ : (-1)^{-\alpha'^\vee}(-1)^{\alpha^\vee};\\
&\sigma^{-1}\alpha > 0, \sigma^{-1}\alpha \notin R_\beta^+ : (-1)^{-\alpha'^\vee}(-1)^{\alpha'^\vee};\\
&\sigma^{-1}\alpha < 0, -\sigma^{-1}\alpha \in R_\beta^+ : (-1)^{\alpha'^\vee}(-1)^{-\alpha^\vee};\\
&\sigma^{-1}\alpha < 0, -\sigma^{-1}\alpha \notin R_\beta^+ : (-1)^{\alpha^\vee}(-1)^{-\alpha'^\vee}.
\end{aligned}$$

Thus we are done with the case of opposite signs. Observe that explicit calculations were not really necessary. It suffices to observe that the contribution to each of (2) and (3) is $(-1)^{\gamma^\vee}, \gamma \in \{\pm\alpha, \pm\alpha'\}$.

If α, α' have the same sign, then $\{\pm\alpha, \pm\alpha'\}$ can contribute to (1), (3) or (4). There must be a contribution from exactly one root to (4). There is a contribution from at most one root to (3). It occurs if and only if there is no contribution to (1). Because we may identify each root with its coroot the same argument applies on the dual side to yield the second equation. The lemma is thus proved.

We now recall the equations in Lemma 4.1.A. The first,

$$e(\omega,\sigma)b(\sigma(\omega)) = \sigma(\omega)(y(\sigma))y(\sigma)^{-1}\sigma(b(\omega)),$$

is an equation in G_{sc}. It may be written as

$$b_{\sigma(\omega)}\sigma(b_\omega^{-1})y(\sigma)\sigma(\omega)(y(\sigma))^{-1} = [\delta(\sigma(\omega))\sigma(\delta(\omega))^{-1}e(\omega,\sigma)]^{-1}.$$

But $b_\omega = b_\beta^{\beta^\vee}, b_{\sigma(\omega)} = b_{\sigma\beta}^{\alpha\beta^\vee}$ because $\omega = \omega_\beta$, and

$$y(\sigma)\sigma(\omega)(y(\sigma))^{-1} = \prod a_\alpha^{\alpha^\vee}\prod a_\alpha^{-\omega\alpha^\vee} = \left(\prod a_\alpha^{\langle\alpha^\vee,\beta\rangle}\right)^{\beta^\vee},$$

the product being over α such that $\alpha > 0, \sigma^{-1}\alpha < 0$. Thus the equation may be rewritten as

$$\left[b_{\sigma\beta}\sigma(b_\beta)^{-1}\prod a_\alpha^{\langle\alpha^\vee,\beta\rangle}\right]^{\sigma\beta^\vee} = \eta_\beta^{\sigma\beta^\vee}$$

or, since we are in G_{sc}, as

$$b_{\sigma\beta}\sigma(b_\beta)^{-1}\prod a_\alpha^{\langle\alpha^\vee,\beta\rangle} = \eta_\beta.$$

Now we pass to $T_{(s)}$, obtaining

$$\left[b_{\sigma\beta}\sigma(b_\beta)^{-1}\prod a_\alpha^{\langle\alpha^\vee,\beta\rangle}\right]^{\sigma\beta^\vee} = \eta_\beta^{\sigma\beta^\vee}$$

in this torus. The left side is E_1E_2 and so

$$E_1E_2 = \eta_\beta^{\sigma\beta^\vee}.$$

Similarly on the dual side, we find

$$\widehat{E}_1\widehat{E}_2 = \eta_\beta^{\sigma\beta}.$$

But $\eta_\beta^{\sigma\beta^\vee}$ is identified with $\eta_\beta^{\sigma\beta}$. We therefore cancel E_1E_2 with $\widehat{E}_1\widehat{E}_2$ and conclude that

$$\mathcal{L}(\sigma,\omega) = E_3/\widehat{E}_3.$$

This quotient is simply $\frac{\sigma(\tilde{\tilde{\delta}}(\omega))}{\sigma(\tilde{\delta}(\omega))}$ or $\prod\limits_{\alpha\in R_\beta^+(d)}(\frac{i}{\sigma(i)})^{2\alpha^\vee}$. In μ_2 this is $(\frac{i}{\sigma(i)})^{[R_\beta^+(d)]}$. Hence

$$\mathcal{L}(\sigma,\omega)\mathcal{L}(\sigma\omega,\sigma^{-1})\sigma(\omega)(\mathcal{L}(\sigma,\sigma^{-1}))^{-1}$$

cancels with

$$\mathcal{M}(\sigma\omega)\mathcal{M}(\sigma\omega,\sigma^{-1})\sigma(\omega)(\mathcal{M}(\sigma,\sigma^{-1}))^{-1}$$

and the proof of Lemma 6.3.B is complete.

We now return to $\partial\tilde{v}/\partial\tilde{u}$ and write it as a product of cocycles

$$(\mathcal{M}\partial\tilde{\bar{y}}/\partial\tilde{\bar{c}})^{-1} \cdot (\mathcal{M}\partial\tilde{v}_*/\partial\tilde{u}_*).$$

Lemma 6.3.B says that we may calculate the second cocycle as $\mathcal{M}\partial\tilde{y}/\partial\tilde{c}$ using the action of Γ_T rather than $\Gamma_{\bar{T}}$. Factor $\mathcal{M}$ as $\Pi_{\mathcal{O}}\mathcal{M}_{\mathcal{O}}$ where

$$\mathcal{M}_{\mathcal{O}}(\rho,\sigma) = \left(\frac{\xi}{\rho\xi}\right)^{N_{\mathcal{O}}(\sigma)},$$

$N_{\mathcal{O}}(\sigma)$ being the number of α in $\mathcal{O}$ for which $\alpha > 0$ and $\sigma^{-1}\alpha < 0$. Then we have

$$\mathcal{M}\partial\tilde{y}/\partial\tilde{c} = \prod_{\pm\mathcal{O}} \lambda_{\pm\mathcal{O}}$$

where

$$\lambda_{\pm\mathcal{O}} = \mathcal{M}_{\mathcal{O}}\mathcal{M}_{-\mathcal{O}}\partial\tilde{y}_{\mathcal{O}}\partial\tilde{y}_{-\mathcal{O}}/\partial\tilde{r}_{\pm\mathcal{O}}$$

if $\mathcal{O}$ is asymmetric. By now familiar arguments show that $\partial\tilde{y}_{\mathcal{O}}\partial\tilde{y}_{-\mathcal{O}}/\partial\tilde{r}_{\pm\mathcal{O}}$ ($\mathcal{O}$ asymmetric) and $\partial\tilde{y}_{\mathcal{O}}/\partial\tilde{r}_{\mathcal{O}}$ ($\mathcal{O}$ symmetric) take values in μ_2 and that $\lambda_{\pm\mathcal{O}}$ is a 2-cocycle with values in μ_4.

Similarly,

$$\mathcal{M}\partial\tilde{\bar{y}}/\partial\tilde{\bar{c}} = \prod_{\pm\bar{\mathcal{O}}} \bar{\lambda}_{\pm\bar{\mathcal{O}}}$$

where now

$$\mathcal{M}_{\bar{\mathcal{O}}}(\bar{\rho},\bar{\sigma}) = \left(\frac{\xi}{\rho\xi}\right) N_{\bar{\mathcal{O}}}(\bar{\sigma}).$$

We conclude:

Theorem 6.3.C.

$$\partial\tilde{v}/\partial\tilde{u} = \left(\prod_{\pm\mathcal{O}} \lambda_{\pm\mathcal{O}}\right)\left(\prod_{\pm\bar{\mathcal{O}}} \bar{\lambda}_{\pm\bar{\mathcal{O}}}\right)^{-1}.$$

6.4. Remaining steps

To complete the proof of (5.1.1) we will show:

(6.4.1) $\lambda_{\pm\mathcal{O}}$ is trivial for $\pm\mathcal{O}$ asymmetric,

(6.4.2) $\mathrm{inv}_F\lambda_{\mathcal{O}} = \chi_\alpha(a_\alpha)$ for $\mathcal{O}$ symmetric,

and

$$\prod_{\mathcal{O}} \chi_\alpha(-2) = \prod_{\bar{\mathcal{O}}} \chi_{\bar\alpha}(-2). \tag{6.4.3}$$

Here inv_F denotes the isomorphism $H^2(\Gamma, \mu_4) \cong \mathbf{Z}_4$. Since we have chosen $i \in \mathbf{C}^\times$ and $i \in \bar{F}^\times$ we may identify this $\mathbf{Z}_4$ with $\mu_4(\mathbf{C})$. In this section we prove (6.4.1) and (6.4.3).

Suppose that $\mathcal{O}$ is asymmetric. Then

$$(\partial\tilde{y}_{\mathcal{O}}\partial\tilde{y}_{-\mathcal{O}}/\partial\tilde{r}_{\pm\mathcal{O}})(\rho,\sigma) = \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0 \\ \alpha\in\mathcal{O}}} \frac{\rho i}{i},$$

and so

$$\lambda_{\pm\mathcal{O}}(\rho,\sigma) = \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0 \\ \alpha\in\mathcal{O}\cup-\mathcal{O}}} \frac{\xi}{\rho\xi} \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0 \\ \alpha\in\mathcal{O}}} \frac{\rho i}{i}.$$

Let $\xi/\rho\xi = \epsilon$. Then $\epsilon^4 = 1, \rho(i)/i = \epsilon^{-2}$ and

$$\lambda_{\pm\mathcal{O}}(\rho,\sigma) = \left(\prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0 \\ \alpha\in\mathcal{O}}} \epsilon\cdot\epsilon^{-2} \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0 \\ \alpha\epsilon-\mathcal{O}}} \epsilon \right) = (\epsilon^{-1})^{N_1}\epsilon^{N_2},$$

where N_1 is the number of terms in the first product and N_2 is the number in the second. We observe that $N_1 = N_2$ and thus prove (6.4.1).

Let Φ be the group of all automorphisms φ of $R^{(d)}$ such that $\varphi(-\alpha) = -\varphi(\alpha)$ for all α. Fix α and let Φ_α be the group fixing α and $\Phi_{\pm\alpha}$ the group fixing the set $\{\pm\alpha\}$. Let θ_i $i = 1, 2$, be the characters of order two of $\Phi_{\pm\alpha}/\Phi_\alpha$ and set

$$\rho_i = \mathrm{Ind}_{\Phi_{\pm\alpha}}^{\Phi}\theta_i.$$

To be specific, let θ_1 be the trivial character. For some subset S of $R^{(d)}$ such that $R^{(d)}$ is the disjoint union of S and $-S$ let $N(\varphi)$ be the number of α in S such that $\varphi^{-1}(\alpha) \in -S$. Then

$$\det\ \rho_2(\varphi)/\det\ \rho_1(\varphi) = (-1)^{N(\varphi)}.$$

The group $\Omega_0 \rtimes \Gamma_T$ is imbedded in Φ and it is easily seen that $N(\omega)$ is even for $\omega \in \Omega_0$. Thus if σ in Γ_T corresponds to $\bar\sigma$ in $\Gamma_{\bar{T}}$ then

$$\det\ \rho_2(\sigma)/\det\ \rho_1(\sigma) = \det\ \rho_2(\bar\sigma)/\det\ \rho_1(\bar\sigma). \tag{6.4.4}$$

Lemma 6.4.A. *Suppose $\sigma \in \Gamma_T$ corersponds to $x \in F^\times$. Then*

$$\prod_{\mathcal{O}} \chi_\beta(x) = \det\ \rho_2(\sigma)/\det\ \rho_1(\sigma).$$

Since the lemma will be valid for $\bar{T}$ just as well as for T, it together with (6.4.4) implies (6.4.3).

Proof. We note that ρ_i is the direct sum over a set of representatives φ for the double cosets $\Phi_{\pm\alpha}\backslash\Phi/\Gamma_T$ of the representations

$$\rho_i^\varphi = \mathrm{Ind}_{\Gamma_T\cap\varphi^{-1}\Phi_{\pm\alpha}\varphi}^{\Gamma_T}\theta_i \circ \text{ ad } \varphi.$$

If $\varphi^{-1}(\alpha) = \beta$ then

$$\Gamma_T \cap \varphi^{-1}\Phi_{\pm\alpha}\varphi = \Gamma_{\pm\beta}$$

and $\theta_2 \circ$ ad φ is the character χ_β regarded as a character of $\mathrm{Gal}(\bar{F}/F_{\pm\beta}) = \Gamma_{\pm\beta}$.

If $\sigma_1, \cdots, \sigma_r$ is a set of representatives for $\Gamma_{\pm\beta}\backslash\Gamma$ and $\sigma_j\sigma = \beta_j(\sigma)\sigma_{j'}$ then

$$\det\ \rho_2^\varphi(\sigma)/\det\rho_1^\varphi(\sigma) = \prod_j \chi_\beta(\beta_j(\sigma)),$$

and, by local class-field theory, this is $\chi_\beta(x)$. Since

$$\det\ \rho_2(\sigma)/\det\ \rho_1(\sigma) = \prod_\varphi \det\ \rho_2^\varphi(\sigma)/\det\ \rho_1^\varphi(\sigma),$$

the lemma follows.

6.5. Symmetric orbits

Throughout this section $\mathcal{O}$ will be a symmetric orbit. Recall that $\mathcal{O} = \{\pm\alpha_j : 1 \le j \le n\}$, where $\alpha_j = \sigma_j^{-1}\alpha > 0$ and $\sigma_1 = 1, \sigma_2, \cdots, \sigma_n$ are representatives for $\Gamma_{\pm\alpha}$. Define $\alpha_j(\rho), \alpha_{j'}(\sigma) \in \Gamma_{\pm\alpha}$ by $\sigma_j\rho = \alpha_j(\rho)\sigma_{j'}$ and $\sigma_{j'}\sigma = \alpha_{j'}(\sigma)\sigma_{j''}$. For $\sigma \in \Gamma_{\pm\alpha}$ define $\delta(\sigma) = 0$ if $\sigma\alpha = \alpha$ and $\delta(\sigma) = 1$ if $\sigma\alpha = -\alpha$.

Lemma 6.5.A.

$$\partial\tilde{y}_{\mathcal{O}}(\rho,\sigma) = \prod_j \sigma_j^{-1}\left[\left[\frac{\sqrt{a_\alpha}}{\alpha_j(\rho)\sqrt{a_\alpha}}\right]^{\delta(\alpha_{j'}(\sigma))}\right]^{2\alpha_j^\vee}.$$

Proof. Let θ be the character on $\Gamma_{\pm\alpha}$ given by $\theta(\sigma) = \pm1$ according as $\sigma\alpha = \pm\alpha$. Then we find

$$\tilde{y}_{\mathcal{O}}(\rho) = \prod_{j=1}^n \sigma_j^{-1}\left\{\begin{matrix} 1 & \text{if } \theta(\alpha_j(\rho)) = 1 \\ \sqrt{a_\alpha} & \text{if } \theta(\alpha_j(\rho)) = -1 \end{matrix}\right\}^{2\alpha_j^\vee}$$

$$\rho(\tilde{y}_{\mathcal{O}}(\sigma)) = \prod_{j=1}^n \sigma_j^{-1}\left\{\begin{matrix} 1 & \text{if } \theta(\alpha_{j'}(\sigma)) = 1 \\ \alpha_j(\rho)(\sqrt{a_\alpha})^{\theta(\alpha_j(\rho))} & \text{if } \theta(\alpha_j'(\sigma)) = -1 \end{matrix}\right\}^{2\alpha_j^\vee}.$$

and

$$\tilde{y}_{\mathcal{O}}(\rho\sigma)^{-1} = \prod_{j=1}^{n} \sigma_j^{-1} \left\{ \begin{array}{ll} 1 & \text{if } \theta(a_j(\rho))\theta(\alpha_{j'}(\sigma)) = 1 \\ (\sqrt{a_\alpha})^{-1} & \text{if } \theta(\alpha_j(\rho))\theta(\alpha_{j'}(\sigma)) = -1 \end{array} \right\}^{2\alpha_j^\vee}.$$

Thus

$$\partial\tilde{y}_{\mathcal{O}}(\rho,\sigma) = \prod_j \sigma_j^{-1} A_j^{2\alpha_j^\vee}$$

where A_j is given by the following table:

$\theta(\alpha_j(\rho))$	$\theta(\alpha_{j'}(\sigma))$	A_j
1	1	1
1	-1	$\alpha_j(\rho)\sqrt{a_\alpha}/\sqrt{a_\alpha}$
-1	1	1
-1	-1	$\sqrt{a_\alpha}/\alpha_j(\rho)\sqrt{a_\alpha}$

Since $\alpha_j(\rho)\sqrt{a_\alpha} = \pm\sqrt{a_\alpha}$ if $\alpha_j(\rho)\alpha = \alpha$, that is, if $\theta(\alpha_j(\rho)) = 1$, the lemma follows.

Recall that to define $r_{\mathcal{O}}$ and $\tilde{r}_{\mathcal{O}}$ we have chosen w_j mapping to σ_j under $W \to \Gamma$, writing $w_j w = u_j(w)w_{j'}, W_\pm = W_+v_0 \cup W_+v_1, v_0u = v_0(u)v_k$ with $k = 0$ or 1 and $s(u) = \chi_\alpha(v_0(u)), u \in W_\pm$. Then

$$\tilde{r}_{\mathcal{O}}(w) = \prod_j \sqrt{s}(u_j(w))^{2\alpha_j},$$
$$w\tilde{r}_{\mathcal{O}}(w') = \prod_r \sqrt{s}(u_{j'}(w'))^{\theta(\alpha_j(\rho))2\alpha_j}$$

where $w \to \rho$ under $W \to \Gamma$, and

$$\tilde{r}_{\mathcal{O}}(ww')^{-1} = \prod_j \sqrt{s}(u_j(w)u_{j'}(w'))^{-2\alpha_j}.$$

Hence

Lemma 6.5.B.

$$\partial\tilde{r}_{\mathcal{O}}(w,w') = \prod_j B_j^{2\alpha_j}$$

where

$$B_j = \frac{\sqrt{s}(u_j(w))\sqrt{s}(u_{j'}(w')))^{\theta(\alpha_j(\rho))}}{\sqrt{s}(u_j(w)j_{j'}(w'))}.$$

From these two lemmas we conclude that

$$\lambda_{\mathcal{O}}(w,w') = \prod_j \left(\frac{\sigma_j^{-1}(A_j)}{B_j}\right)\left(\frac{\xi}{\rho\xi}\right)^{\delta(\alpha_{j'}(\sigma))}.$$

Let

$$C_j = \left(\frac{\xi}{\alpha_j(\rho)\xi}\right)^{\delta(\alpha_{j'}(\sigma))},$$

and define λ' by

$$\lambda_{\mathcal{O}} = \left(\prod_j \sigma_j^{-1}\left(\frac{A_j C_j}{B_j}\right)\right)\lambda'.$$

Lemma 6.5.C.

λ' is a coboundary.

Proof. Because $\sigma_j^{-1}A_j/A_j = \sigma_j^{-1}B_j/B_j$ we may write λ' as

$$\prod_j \frac{\sigma_j^{-1}A_j}{B_j}\sigma_j^{-1}C_j^{-1}\left(\frac{\xi}{\rho\xi}\right)^{\delta(\alpha_{j'}(\sigma))}.$$

From the definition of A_j we see that

$$\begin{aligned}\frac{\sigma_j^{-1}A_j}{B_j} &= \left(\frac{\sigma_j^{-1}(i)}{i}\right)^{\delta(\alpha_j(\rho))\delta(\alpha_{j'}(\sigma))}\\ &= \eta_j^{2\delta(\alpha_j(\rho))\delta(\alpha_{j'}(\sigma))}\end{aligned}$$

where $\eta_j = \sigma_j^{-1}(\xi)/\xi$. We also find that

$$(\sigma_j^{-1}C_j^{-1})(\xi/\rho\xi)^{\delta(\alpha_{j'}(\sigma))}$$

coincides with $(\rho\eta_{j'}/\eta_j)^{\delta(\alpha_{j'}(\sigma))}$. Thus

$$\lambda' = \prod_j \eta_j^{2\delta(\alpha_j(\rho))\delta(\alpha_{j'}(\sigma))}(\rho\eta_{j'}/\eta_j)^{\delta(\alpha_{j'}(\sigma))}.$$

On the other hand, the coboundary of $\prod\limits_j \eta_j^{\delta(\alpha_j(\sigma))}$ is

$$\prod_j \left(\eta_j^{\delta(\alpha_j(\rho))}(\rho\eta_{j'})^{\delta(\alpha_{j'}(\sigma))}\eta_j^{-\delta(\alpha_j(\rho)\alpha_{j'}(\sigma))}\right).$$

Because

$$\delta(\alpha_j(\rho)) + \delta(\alpha_{j'}(\sigma)) - \delta(\alpha_j(\rho)\alpha_{j'}(\sigma)) = 2\delta(\alpha_j(\rho))\delta(\alpha_{j'}(\sigma)),$$

this coboundary coincides with λ', and the lemma is proved.

We discard the term λ' from $\lambda_{\mathcal{O}}$ leaving

$$\prod_j \sigma_j^{-1}\left(\frac{A_j C_j}{B_j}\right)$$

which equals

$$\prod_j \sigma_j^{-1}[(\xi\sqrt{a_\alpha}/\alpha_j(\rho)(\xi\sqrt{a_\alpha}))^{\delta(\alpha_{j'}(\sigma))}(\sqrt{s}(u_j(w))\sqrt{s}(u_{j'}(w'))^{\theta(\alpha_j(\rho))}/\sqrt{s}(u_j(w)u_{j'}(w')))^{-1}].$$

Consider the cocycle λ_α of $W_{\pm\alpha}$ in μ_4 given by

$$(w,w') \to (\xi\sqrt{a_\alpha}/\rho(\xi\sqrt{a_\alpha}))^{\delta(\sigma)}(\sqrt{s}(w)(\sqrt{s}(w')^{\theta(\rho)}/\sqrt{s}(ww'))^{-1}$$

where $w, w' \to \rho, \sigma$ under $W_{\pm\alpha} \to \Gamma_{\pm\alpha}$. Then $\lambda_{\mathcal{O}}$ is the image of λ_α under the corestriction homomorphism from $H^2(W_{\pm\alpha}, \mu_4)$ to $H^2(W, \mu_4)$. Since

$$\mathrm{inv}_F\lambda_{\mathcal{O}} = \mathrm{inv}_{F_{\pm\alpha}}\lambda_\alpha,$$

to prove (6.4.2) it is sufficient to prove that

$$\text{(6.5.1)} \qquad \mathrm{inv}\lambda_\alpha = \chi_\alpha(a_\alpha).$$

If $R_1 \to R_2$ is an isogeny of tori over a local field F with kernel D then local class field theory ([**M**], Chap. 1) yields two sequences in duality

$$\text{(6.5.2)} \qquad D(F) \longrightarrow R_1(F) \longrightarrow R_2(F) \longrightarrow H^1(F, D)$$

$$\text{(6.5.3)} \qquad H^2(F,\widehat{D}) \longleftarrow H^1(W,\widehat{R}_1) \longleftarrow H^1(W,\widehat{R}_2) \longleftarrow H^1(F,\widehat{D}).$$

The pairing is in all cases to $\mathbf{C}^\times$ or a subgroup of it, and

$$\widehat{D} = \mathrm{Hom}(D, \mu_\infty(\bar{F}) \otimes \mu_\infty(\mathbf{C})).$$

We use in two different ways the compatibility of the two sequences with the pairing, for the commutativity of diagrams (6.6.1) and (6.6.2) are special cases of it. Since we use the compatibility at both ends, we need to pay attention to signs. Moreover we know of no reference for the compatibility, although it follows from standard results. So we include here some very brief remarks, based on the constructions in [$\mathbf{L}_3$].

First of all, when proving the compatibility, one can confine attention to elements of $H^1(F, \widehat{R}_i)$ and thus of $H^1(K/F, \widehat{R}_i)$ for some large K (notice the proof that λ_α factors through $\Gamma_{\pm\alpha}$ in the next section). Then $\alpha \in H^1(K/F, \widehat{R}_i)$ when paired with the cup product $\beta \cup \gamma$ of $\beta \in H^{-2}(K/F, X_*(R_i))$ and the fundamental class of K/F yields $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$. The pairing between $\alpha' \in H^i(F, \widehat{D})$ and $\beta' \in H^j(F, D)$ is given by

$$\beta' \cup \alpha' \in H^2(F, D \otimes \widehat{D}) \longrightarrow H^2(F, \mu_n(\bar{F}) \otimes \mu_n(\mathbf{C})) \simeq \mu_n(\mathbf{C}).$$

Here n is sufficiently large and $\beta' \cup \alpha' : \rho, \sigma \to \beta'(\rho) \otimes \rho\alpha'(\sigma)$.

Choosing n so that $nX_*(R_2) \subseteq X_*(R_1) \subseteq X_*(R_2)$ we see that it is enough to treat the case that $nX_*(R_2) = X_*(R_1)$. Then

$$1 \longrightarrow D \longrightarrow R_1 \longrightarrow R_2 \longrightarrow 1$$

is obtained by tensoring

$$1 \longrightarrow \mu_n(\bar{F}) \longrightarrow \bar{F}^\times \longrightarrow \bar{F}^\times \longrightarrow 1 \tag{6.5.4}$$

with $X_*(R_1)$.

Suppose $\alpha' \in H^1(F, \widehat{D})$ has image α and β' is the image of $\beta \cup \gamma$. Then by Proposition 5 of [**Se**, Chap. VIII, Sect. 3,] $\beta' = \beta \cup \delta\gamma, \delta$ being the map $H^2(F, \bar{F}^\times) \to H^3(F, \mu_n(\bar{F}))$ attached to (6.5.4). Thus

$$\beta' \cup \alpha' = \beta \cup \delta\gamma \cup \alpha' = -\beta \cup \alpha' \cup \delta\gamma.$$

Choose n divisible by $[K : F]$. Then

$$H^2(K/F, \mu_n(K)) \longrightarrow H^2(K/F, K^\times),$$

is an isomorphism as is

$$H^2(K/F, \mu_n(K)) \longrightarrow H^2(K/F, \mu_m(K))$$

if $n \mid m$.

The product $\beta \cup \alpha' = \gamma_1$ is a class in $H^{-2}(K/F, \mu_n(\mathbf{C}))$. Lifting γ, γ_1 to $\tilde{\gamma}, \tilde{\gamma}_1$ with values in $\mu_{n^2}(\bar{K}), \mu_{n^2}(\mathbf{C})$ we obtain a product of chains $\tilde{\gamma}_1 \cup \tilde{\gamma}$ that projects to $\gamma_1 \cup \gamma$. Since

$$\partial(\tilde{\gamma}_1 \cup \tilde{\gamma}) = \partial\tilde{\gamma}_1 \cup \gamma + \gamma_1 \cup \partial\tilde{\gamma}$$

must be trivial in $H^1(K/F, \mu_n(K) \otimes \mu_n(\mathbf{C}))$, and $\partial\tilde{\gamma} = \delta\gamma, \partial\tilde{\gamma}_1 = \langle\beta, \alpha\rangle$, we conclude that

$$\beta' \cup \alpha' = \langle\beta, \alpha\rangle\gamma.$$

This is one of the compatibilities.

For the other, take $\alpha \in H^1(K/F, \widehat{D})$, and $\alpha' = \delta\alpha, \beta' = \beta \cup \gamma$. Then

$$\beta \cup \gamma \cup \delta\alpha = \beta \cup \delta\alpha \cup \gamma = \delta(\beta \cup \alpha) \cup \gamma,$$

and $\delta(\beta \cup \alpha) = \langle\beta, \alpha\rangle$.

6.6. Final calculations

First we observe that

$$B = \sqrt{a}(w)(\sqrt{s}(w'))^{\theta(\rho)}(\sqrt{s}(ww'))^{-1}, w, w' \in W_{\pm},$$

is given by the following table. The elements t, t' lie in W_+.

w	w	B
t	t'	$\sqrt{s}(t)\sqrt{s}(t')\sqrt{s}(tt')^{-1}$
t	$t'v$	$\sqrt{s}(t)\sqrt{s}(t')\sqrt{s}(tt')^{-1}$
tv_1	t'	$\sqrt{s}(t)\sqrt{s}(t')^{-1}\sqrt{s}(t\bar{t}')^{-1}$
tv_1	$t'v_1$	$\sqrt{s}(t)\sqrt{s}(t')^{-1}\sqrt{s}(t\bar{t}'v_1^2)^{-1}$

The proof of (6.5.1) will be divided into the following cases:

(i) $i \in F_{\pm\alpha}$;

(II) $i \in F_\alpha - F_{\pm\alpha}$;

(III) $i \notin F_\alpha$.

We shall delete the subscript α from notation. Thus $a_\alpha, \chi_\alpha, \lambda_\alpha, F_{\pm\alpha}$ become $a, \chi, \lambda, F_\pm$; we write F_+ for F_α.

To verify directly that λ factors through $\Gamma_\pm$, we choose an open subgroup U of finite index in $F^\times$ that does not contain -1. Then $V = U^2$ is also open and of finite index and $u \to v = u^2$ is a topological isomorphism between the two groups. Define a character μ of V by $\mu(v) = \chi(u)$. Then μ^2 is equal to χ on V. We may suppose U and V are invariant under $\Gamma_\pm$. Then V is a normal subgroup of $W_\pm$ and we may so choose $\sqrt{s}(t), t \in W_+$, that

$$\sqrt{s}(vt) = \mu(v)\sqrt{s}(t).$$

Since $\mu(v\bar{v}) = \chi(u\bar{u}) = 1, v \in V$, it follows easily from the table that B factors through $V\backslash W_\pm$ and thus through $\Gamma_\pm$. We now prove (6.5.1).

Case I.

We may assume F nonarchimedean. Let L be the cyclic quartic extension $F_+(\xi\sqrt{a})$ of $F_\pm$, τ be the generator $\xi\sqrt{a} \to i\xi\sqrt{a}$ and μ_1 be the character on $\mathrm{Gal}(L/F_\pm)$ given by $\mu_1(\tau) = i^{-1}$. We pull μ_1 back to a character on $\Gamma_\pm$ and observe that

$$\frac{\xi\sqrt{a}}{\rho(\xi\sqrt{a})} = \mu_1(\rho), \quad \rho \in \Gamma_\pm.$$

Because $\chi(-1) = \chi(i^2) = 1$ we may choose a character μ_2 on $F_+^\times$ such that $\mu_2^2 = \chi$. Regard μ_2 as a character on W_+ and set $\sqrt{s} = \mu_2$. As usual, let $w, w' \to \rho, \sigma$ under $W_\pm \to \Gamma_\pm$. Then in the table of values for B we obtain $1, 1, \mu_2(t'\bar{t}')^{-1}, \mu_2(t'\bar{t}'v_1^2)^{-1}$, so that B is given by

$$(w, w') \longrightarrow [\mu_2 \circ \ trans \ (w')]^{-\delta(\rho)}$$

where $trans$ is the transfer homomorphism $W_\pm^{\mathrm{ab}} \to W_+^{\mathrm{ab}}$. Since $\mu_2 \circ \ trans$ corresponds to restriction of μ_2 to $F_\pm^\times$ we may write this as

$$(w, w') \longrightarrow \mu_2(\sigma)^{-\delta(\rho)}$$

on identifying this restriction as a character on $\Gamma_\pm$.

Observe next that on $F_\pm^\times$ or $\Gamma_\pm$ we have $\mu_1^2 = \mu_2^2 = \chi$ so that $\theta = \mu_2\mu_1^{-1}$ is of order two. Also $(\rho, \sigma) \mapsto \mu_2(\sigma)^{-\delta(\rho)}$ is cohomologous to $(\rho, \sigma) \to \mu_2(\rho)^{\delta(\sigma)}$, for they differ by the boundary of $\rho \to \mu_2(\rho)^{\delta(\rho)}$. We conclude that λ is the cocycle

$$(\rho, \sigma) \longrightarrow \theta(\rho)^{\delta(\sigma)}.$$

We interpret this as the cup-product of θ in $H^1(F_\pm, \mu_2(\mathbf{C}))$ and δ in $H^1(F_\pm, \mu_2(\bar{F}))$.

In general the diagram,

$$(6.6.1) \qquad \begin{array}{ccccccc} \mu_n(F_\pm)^* & \longleftarrow & (F_\pm^\times)^* & \longleftarrow & (F_\pm^\times)^* & \longleftarrow & H^1(F_\pm, \mu_n(\bar{F}))^* \\ \wr & & \wr & & \wr & & \wr \\ H^2(F_\pm, \mu_n(\mathbf{C})) & \longleftarrow & H^1(W_\pm, \mathbf{C}^\times) & \longleftarrow & H^1(W_\pm, \mathbf{C}^\times) & \longleftarrow & H^1(F_\pm, \mu_n(\mathbf{C})) \end{array}$$

defined by $x \to x^n, \bar{F}^\times \to \bar{F}^\times$, is commutative. Thus

$$\mathrm{inv}\ \lambda = \theta(a^2) = \mu_2(a^2)\mu_1^{-1}(a^2) = \chi(a)\mu_1^{-1}(a^2).$$

However, the norm of $\xi\sqrt{a}$ is

$$i \cdot (-1) \cdot (-i) \cdot \xi^4 a^2 = a^2,$$

so that $\mu_1(a^2) = 1$. Thus the relation (6.5.1) is valid in this case.

Observe that slight variations of the preceding arguments allow one to verify readily in all cases that if (6.5.1) is valid for one choice of a and χ then it is valid for all.

Case II.

We may choose $a = -i, \sqrt{a} = \xi^{-1}$. Then λ is given by B^{-1}. The diagram (6.6.1) is now to be replaced by the analogous diagram for the group R with the twisted action $\rho : x \to \rho(x)^{\theta(\rho)}$. Then $\mathbf{C}^\times$ is replaced by $\widehat{R}$. We take $n = 4$, and then the kernel of $x \to x^4, \widehat{R} \to \widehat{R}$ is, because of the twisted action and because $i \in F_+ - F_\pm$, isomorphic to $\mu_4(\bar{F}_\pm)$. The analogue of (6.6.1) is

$$\text{(6.6.2)} \qquad \begin{array}{ccccccc} \mathbf{Z}_4^* & \leftarrow & R^* & \leftarrow & R^* & \leftarrow & H^1(F_\pm, \mathbf{Z}_4)^* \\ \wr| & & \wr| & & \wr| & & \wr| \\ H^2(F_\pm, \mu_4(\bar{F}_\pm)) & \leftarrow & H^1(W_\pm, \widehat{R}) & \leftarrow & H^1(W_\pm, \widehat{R}) & \leftarrow & H^1(F_\pm, \mu_4(\bar{F}_\pm)) \end{array}$$

It is again commutative.

The element B^{-1} lies in $H^2(F_\pm, \mu_4(F_\pm))$, and comes as the boundary of a 1-cochain on $W_\pm$ with values in $\widehat{R}$, namely $w \to \sqrt{s}(w)^{-1}$. Taking the fourth power, we obtain the cocycle $r : w \to s(w)^{-2}$.

The element inv λ is obtained, after our identification, by pairing B with $a^{-1} = i$ in $\mathbf{Z}_4$ (or $\mu_4(\bar{F})$). Thus, by commutativity of the diagram, it is obtained as the value of the character ν associated to the cocycle r on a^{-1}. In general if $z \in \widehat{R}$, thus if $z \in F_+$ and $z\bar{z} = 1$, then $z = y\bar{y}^{-1}$ and $\nu(z) = \chi(y)^{-2}$ because the function s^2 restricted to W_+ is χ^2. For $z = i$ we have $y = 1 + i$ and

$$\chi(y)^{-2} = \chi(2i)^{-1} = \chi(-i) = \chi(a),$$

because 2 is the norm of y.

Case III.

We have not been able in Case III to deal directly with fields of even residual characteristic. They can, however, be handled by a global argument. Suppose, for present purposes, that F_+ is a quadratic extension of the number field $F_\pm$, that $a \in F_+$ and $\bar{a} = -a$, and that χ is an idèle-class character of I_{F_+} whose restriction to $I_{F_\pm}$ is the character $\theta_{F_+/F_\pm}$ associated to the quadratic extension. The construction of λ in Section 6.5 can be carried out globally. At a place in $F_\pm$ that does not split in F_+, the global construction is compatible with the local. However, λ can also be restricted to the local Weil group at a place v that splits in F_+. We claim that the relation (6.5.1) is valid at this local place. Thus

$$\text{inv } \lambda_v = \chi_v(a) = 1.$$

To see this observe first that $\chi_v(-1) = 1$ so that χ_v is a square. Since $\theta(\rho) = 1$ if ρ lies in the decomposition group, the denominator of the expression defining λ_v is 1. Since $\delta(\sigma) = 0$ if σ is in the decomposition group, the numerator is also 1.

We conclude that if (6.5.1) is valid at all but one place then it is valid at the remaining place. It is certainly valid at the archimedean places, for if they are not split they fall under Case II. Moreover, given local data at one place, we can extend these local data to global data and in such a way that at any prescribed finite set of places not containing the original one the extension splits. Therefore it suffices to treat the case of odd residual characteristic.

Since $i \notin F_+$ we have a diagram of fields

$$\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[dr, dash] & \\
F_+ \arrow[dr, dash, "\text{ramified}"'] & & E = F_\pm(i) \arrow[dl, dash, "\text{unramified}"] \\
 & F_\pm &
\end{tikzcd}$$

All intermediate extensions are quadratic and $[K : F_\pm] = 4$. Let q be the number of elements in the residue field, so that $q \equiv 3 (\mathrm{mod}\, 4)$ and $q^2 - 1 \equiv 0 (\mathrm{mod}\, 8)$. Let m be the largest power of 2 dividing $q^2 - 1$ and let ζ be a primitive m^{th} root of unity in E. We may suppose that

$$\zeta^{(q^2-1)/8} = \xi, \;\; \zeta^{(q^2-1)/4} = i.$$

Since $F_+/F_\pm$ is ramified, we may choose a so that $a^2 = \varpi$ is a uniformizing parameter in $F_\pm$. Then $K(\sqrt[4]{a})$ and $K(\sqrt[4]{\zeta})$ are linearly disjoint and Galois over $F_\pm$.

Thus we may enlarge our diagram of fields to

$$\begin{tikzcd}
K(\sqrt[4]{a}) & & K(\sqrt{\zeta a}) & & K(\sqrt[4]{\zeta}) \\
 & & K \arrow[ull, dash] \arrow[u, dash] \arrow[urr, dash] \arrow[dll, dash] \arrow[drr, dash] & & \\
F_+ & & & & E \\
 & & F_\pm \arrow[ull, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

There exists a $\tau \in \Gamma_\pm$ such that

$$\tau a = -a, \quad \tau(\sqrt[4]{a}) = \xi\sqrt[4]{a}, \tau(\sqrt[4]{\zeta}) = \sqrt[4]{\zeta} \tag{6.6.3}$$

We choose $1, \tau$ as representatives for $\Gamma_\pm \backslash \Gamma_+$.

The group of units $O_\pm^\times$ in $O_\pm$ is a product $\{\pm 1\}U$, where U is the set of all a in $O_\pm^\times$ whose image in the residue field has odd order. We define χ on $F_\pm^\times$ by the following conditions:

$$\chi \mid U \equiv 1; \quad \chi(-1) = -1, \quad \chi(-a^2) = 1.$$

Then we extend it to $F_+^\times$, obtaining a character χ of order 4. This character defines a cyclic quartic extension of F_+ that evidently contains K because χ^2 is unramified. Thus it is the quadratic extension of K associated to the character $\nu(x) = \chi(x\bar{x})$. Since ν is not trivial on units it is ramified, and thus of the form $K(\sqrt{\gamma a})$, where γ is a unit. Then $1 = \nu(-\gamma a) = \nu(\gamma)\chi(a^2) = -\nu(\gamma)$, so that $\nu(\gamma) = -1$. Consequently we may take $\gamma = \zeta$, and the field is $K(\sqrt{\zeta a})$.

Consider the element δ of $H^1(F_+, \mu_4(\bar{F}_+))$ given by $\rho \to \sqrt[4]{a}/\rho(\sqrt[4]{a})$ and the element θ of $H^1(F_+, \mu_4(\mathbf{C}))$ given by χ. According to the diagram (6.6.1) the invariant of their cup product is $\chi(a)$.

To complete the proof of (6.5.1) in Case III and thus of Theorem 1.6.A it remains to show that λ is in the class of the constriction of $\delta \cup \theta$. The cup product itself is given by

$$(\rho\sigma) \longrightarrow (\sqrt[4]{a}/\rho(\sqrt[4]{a}))^{\text{ ord }(\sigma)},$$

if we identify, as usual, $\mu_4(\mathbf{C})$ with $\mathbf{Z}_4$ and set $\chi(\sigma) = i \text{ ord } (\sigma)$.

We calculate the corestriction with the coset representatives $(1, \tau)$ for $\Gamma_+ \backslash \Gamma_\pm$ obtaining a 2-cocycle μ. We construct the extension of $\Gamma_\pm$ by $\mu_4(\bar{F})$ defined by μ, letting $\hat{\sigma}$ be the representative of $\sigma \in \Gamma_\pm$ in it. Then $\hat{\rho} \cdot \hat{\sigma} = \mu_{\rho,\sigma}(\rho\sigma)\hat{}$. We want to choose $a_\sigma \in \mu_4(\hat{F})$ so that if $\tilde{\sigma} = a_\sigma\hat{\sigma}$ then $\tilde{\rho}\tilde{\sigma} = \lambda_{\rho,\sigma}(\rho\sigma)$.

We first examine the restriction of the cocycle μ to Γ_+. There it is given by

$$\mu_{\rho,\sigma} = (\sqrt[4]{a}\ \rho(\sqrt[4]{a})^{-1})^{\text{ord}(\sigma)}\tau^{-1}(\sqrt[4]{a}\ \tau\rho\tau^{-1}(\sqrt[4]{a})^{-1})^{\text{ord}(\tau\sigma\tau^{-1})}$$

We claim that $\text{ord}(\tau\sigma\tau^{-1}) = -\text{ord}(\sigma)$. It is enough to verify this on an element σ such that $\sigma(\sqrt{\zeta a}) = \pm i\sqrt{\zeta a}$. Then, by (6.6.3),

$$\tau\sigma\tau^{-1}(\sqrt{\zeta a}) = \tau\sigma(-i\sqrt{\zeta a}) = \tau(\pm\sqrt{\zeta a}) = \mp i\sqrt{\zeta a} = \sigma^{-1}(\sqrt{\zeta a}).$$

Thus

$$\mu_{\rho,\sigma} = (\sqrt[4]{a}\ \rho(\sqrt[4]{a})^{-1}\tau^{-1}(\sqrt[4]{a})^{-1}\rho\tau^{-1}(\sqrt[4]{a}))^{\mathrm{ord}(\sigma)} = (\xi\rho(\xi)^{-1})^{\mathrm{ord}(\sigma)}.$$

On the other hand, the numerator of the quotient defining λ is trivial on Γ_+ since $\delta(\sigma) = 0$ for $\sigma \in \Gamma_+$. The denominator is easily calculated since it is just the pullback to Γ_+ through χ of the 2-cocycle of the extension

$$1 \longrightarrow \mu_2(\mathbf{C}) \longrightarrow \mu_8(\mathbf{C}) \longrightarrow \mu_4(\mathbf{C}) \longrightarrow 1.$$

If we choose σ_1 such that $\mathrm{ord}(\sigma_1) = 1$ then for $0 \leq a, b < 4$ this is just

$$\lambda_{\sigma_1^a,\sigma_1^b} = \begin{cases} 1 & a+b<4 \\ -1 & a+b \geq 4. \end{cases}$$

Thus both cocycles are defined on $\mathrm{Gal}(K(\sqrt{\zeta a})/F_+)$ and if we take

$$\tilde{\sigma}_1 = \hat{\sigma}_1,\ (\sigma_1^2)^{\sim} = \xi\sigma_1(\xi)^{-1}(\sigma_1^2)^{\wedge},\quad (\sigma_1^3) = \xi\sigma_1(\xi)^{-1}(\sigma_1^3)^{\wedge}$$

then $\tilde{\rho}\cdot\tilde{\sigma} = \lambda_{\rho,\sigma}(\rho\sigma)$ on Γ_+, because $\sigma_1^2(\xi) = \xi^{q^2} = \xi$.

If σ belongs to Γ_+ then $1\cdot\tau = \tau, \tau\cdot\sigma = \tau\sigma\tau^{-1}\cdot\tau, \tau\cdot\tau = \tau^2$, so that

$$\mu_{\tau,\sigma} = \tau^{-1}(\sqrt[4]{a}\ \tau^2(\sqrt[4]{a})^{-1})^{\mathrm{ord}\sigma} = \xi^{-2\mathrm{ord}(\sigma)} = i^{-\mathrm{ord}(\sigma)}.$$

On the other hand,

$$\lambda_{\tau,\sigma} = (\sqrt{\chi(1)})^{-1}\sqrt{\chi(\sigma)}\sqrt{\chi(\tau\sigma\tau^{-1})} = \sqrt{\chi(\sigma)}\sqrt{\chi(\tau\sigma\tau^{-1})},$$

and we have chosen $\sqrt{\chi(\sigma_1^a)} = \sqrt{e^{2\pi ia/4}} = e^{2\pi ia/8}$ for $0 \leq a < 4$. Thus

$$\lambda_{\tau,\sigma_1^a} = e^{2\pi ia/8}e^{2\pi i(4-a)/8} = -1,\ 0 < a < 4,$$

$$\lambda_{\tau,1} = 1.$$

Then we set $\tilde{\tau} = \beta\hat{\tau}$, where β is yet to be determined, and define $(\tau\sigma)^{\sim}$ so that

$$(\tau\sigma)^{\sim} = \lambda_{\tau,\sigma}^{-1}\tilde{\tau}\tilde{\sigma} = \lambda_{\tau,\sigma}^{-1}\beta\tau(a_\sigma)\hat{\tau}\hat{\sigma} = \lambda_{\tau,\sigma}^{-1}\beta\tau(a_\sigma)\mu_{\tau,\sigma}(\tau\sigma)^{\wedge}$$

This done, we have to verify that

$$\tilde{\sigma}\tilde{\tau} = \lambda_{\sigma,\tau}(\sigma\tau)^{\sim}. \tag{6.6.4}$$

If this equation is valid then

$$\tilde{\sigma}\cdot(\tau\rho)^{\sim}=\sigma(\lambda_{\tau,\rho})^{-1}\tilde{\sigma}\tilde{\tau}\tilde{\rho}=\sigma(\lambda_{\tau,\rho})^{-1}\lambda_{\sigma,\tau}(\sigma\tau)^{\sim}\tilde{\rho}$$

which equals

$$\sigma(\lambda_{\tau,\rho})^{-1}\lambda_{\sigma,\tau}\lambda^{-1}_{\tau,\tau^{-1}\sigma\tau}\tau(\lambda_{\tau^{-1}\sigma\tau,\rho})\lambda_{\tau,\tau^{-1}\sigma\tau\rho}(\sigma\tau\rho)^{\sim}=\lambda_{\sigma,\tau\rho}(\sigma\tau\rho)^{\sim}.$$

Moreover, if (6.6.4) is valid for ρ as well as σ then

$$(\rho\sigma)^{\sim}\tilde{\tau}=\lambda^{-1}_{\rho,\sigma}\tilde{\rho}\tilde{\sigma}\tilde{\tau}=\lambda^{-1}_{\rho,\sigma}\rho(\lambda_{\sigma,\tau})\lambda_{\rho,\sigma\tau}(\rho\sigma\tau)^{\sim}=\lambda_{\rho\sigma,\tau}(\rho\sigma\tau)^{\sim}.$$

Thus it enough to verify it for σ_1.

The left side is equal to $\sigma_1(\beta)\mu_{\sigma_1,\tau}(\sigma\tau)^{\widehat{}}$; the right side is

$$\lambda_{\sigma_1,\tau}\lambda^{-1}_{\tau,\sigma_1^{-1}}\beta\tau(a_{\sigma_1^{-1}})\mu_{\tau,\sigma_1^{-1}}(\sigma\tau)^{\widehat{}}.$$

Since $\mathrm{ord}(\tau^2)=2, 1\cdot\tau=\tau, \tau\sigma_1=(\tau\sigma_1\tau^{-1})\tau$,

$$\mu_{\sigma_1,\tau}=\tau^{-1}(\sqrt[4]{a}\tau^2(\sqrt[4]{a})^{-1})^{-1}=i,$$

while

$$\lambda_{\sigma_1,\tau}=\sqrt{e^{2\pi i/4}}^{-1}\sqrt{e^{2\pi i/4}}=1.$$

Thus we must have

$$\sigma_1(\beta)\beta^{-1}=-\tau(\xi\sigma_1(\xi)^{-1})=-\xi\sigma_1(\xi)^{-1}=-\xi^{1-q}.$$

Since $\sigma_1(i)=-1$, we may take

$$\beta=\xi^{-1}i=\xi.$$

It is easy to verify that $(\tau\rho)^{\sim}\tilde{\sigma}=\lambda_{\tau\rho,\sigma}(\tau\rho\sigma)^{\sim}$ if $\rho,\sigma\in\Gamma_+$, and if

$$\tilde{\tau}^2=\lambda_{\tau,\tau}(\tau^2)^{\sim}\tag{6.6.5}$$

it is easy to verify that $(\rho\tau)^{\sim}(\sigma\tau)^{\sim}=\lambda_{\rho\tau,\sigma\tau}(\rho\tau\sigma\tau)^{\sim}, \rho,\sigma\in\Gamma_+$. Thus the proof of Theorem 1.6.A will be complete once (6.6.5) is established. It is clear that $\mu_{\tau,\tau}=1$ and that $\mathrm{ord}(\tau^2)=2$. Therefore

$$\tilde{\tau}^2=(\beta\hat{\tau})^2=\beta^2\hat{\tau}^2=\beta^2(\tau^2)^{\widehat{}}=-\beta^2(\tau^2)^{\sim}.$$

On the other hand,

$$\lambda_{\tau,\tau} = \sqrt{\zeta a}\ \tau(\sqrt{\zeta a})^{-1} = \xi^{-2} = -\xi^2 = -\beta^2,$$

and we are done.

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